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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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1975

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OUTER PRODUCT EXPANSIONS AND THEIR USES IN DIGITAL IMAGE PROCESSING

HARRY C. ANDREWS AND CLAUDE L. PATTERSON

Introduction. The two separate disciplines of digital image processing and numerical analysis often merge when one models imaging techniques in linear or matrix formalisms. This paper develops such a merger by utilizing pictures or images as an aid in discussing the series rank-one matrix expansions obtained when the numerical techniques of singular value decomposition (SVD) are explored. However, the use of SVD as a tool in digital image processing is more than a novel teaching aid, and, in fact, represents one of the methods under investigation for both image representation as well as image restoration.

Inherent in these aspects of digital image processing is computer storage, in digital form, of a two-dimensional array of numbers representing individual brightness values taken from an original photograph, scene, or camera tube. However, a sampled and quantized image is merely a matrix of nonnegative numbers (albeit, possibly a quite large matrix), which may then be manipulated in a digital computer by a large class of linear and nonlinear operations. Digital image processing can then be reduced to analysis of large scale matrix manipulations for many of the operations known as transformations, expansions, representations, and restorations. This paper investigates some of these techniques in the context of singular value decomposition.

Consider the matrix G as an image that has been sampled and quantized in space, where the ith row and jth column correspond to the x and y spatial coordinates of a scene $g(x, y)$. Assume that G has the dimensions n by n. Techniques involving nonsquare matrices have been explored in much of the literature; however, no discussion will be offered herein as it does not contribute significantly to the philosophy of this paper.

The concept of singular value decomposition (SVD) [**1**–**3**] of the image G can be shown to be particularly useful when interpreted as a type of eigenvector expansion for storage or bandwidth savings. The concept of the eigenvalue map, condition number, and rank of an image are then introduced as guides to potential computer storage savings.

After the discussions of images and their SVD for efficient computer representation purposes, an imaging model is presented for restoration purposes. Here the objective is the removal or inversion of undesirable effects caused by such imperfect imaging circumstances as defocus, motion blur, noise removal, and many other degradations. These degradations are modeled by a matrix H which is referred to as the point spread function matrix and describes the linear relation of a point of light into the system and the spread of that light out of the system. Many restoration techniques are discussed, all of which attempt the inversion of H to form

H^{-1}. The SVD approach will become useful for handling the possible singularity of H and the possible efficient description of H for many different models. In addition the pseudoinversion of H using the SVD becomes intriguing for image restoration purposes.

Images and their singular value decompositions. Any matrix G may be expressed as

$$G = UDV^t, \tag{1}$$

where U and V are arbitrary unitary matrices and D is a matrix comprised of the coefficients of expansion of the rows and columns of our image G. In other words,

$$d_{ij} = \boldsymbol{u}_i^t G \boldsymbol{v}_j, \tag{2}$$

where d_{ij} are the elements of the matrix D, and $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$ are the columns of U and V respectively.

With proper choice of U and V the matrix becomes diagonal, and we have a particularly useful expansion known as a singular value decomposition. If D is diagonal of rank r (i.e., if D has r positive diagonal terms d_i) then

$$G = \sum_{i}^{r} d_i \boldsymbol{u}_i \boldsymbol{v}_i^t. \tag{3}$$

The d_i are known as the singular values of G and $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ as the singular vectors. The singular values are the square roots of the eigenvalues of G^tG, since if

$$G = U\Lambda^{\frac{1}{2}}V^t, \tag{4}$$

then

$$GG^t = U\Lambda U^t \tag{5a}$$

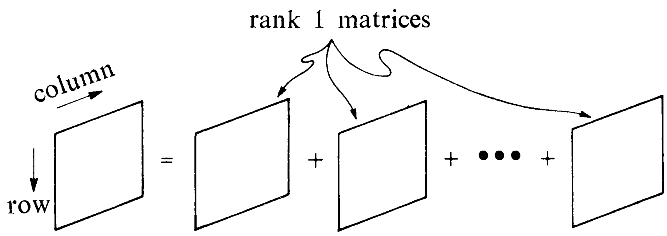
$$G^tG = V\Lambda V^t \tag{5b}$$

where Λ is the diagonal matrix of eigenvalues of GG^t, the columns of U are the eigenvectors of GG^t, and the columns of V are the eigenvectors of G^tG. The symmetry and squareness of GG^t and G^tG, guarantee the realness of the λ_i and the orthogonal property of the eigenvector sets $\{\boldsymbol{u}_i\}$ and $\{\boldsymbol{v}_i\}$, respectively. From (3) it is evident that the smaller r is, the fewer the degrees of freedom defining the image G; by ordering the eigenvalues in monotonic decreasing order, we obtain the most efficient mean-square representation of the image in the smallest (truncated) set of retained components:

$$\{d_i, \boldsymbol{u}_i, \boldsymbol{v}_i\}.$$

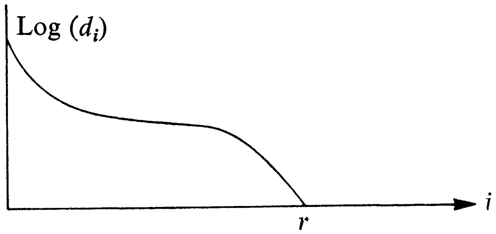
Equation (3) implies that the image can be represented by a sum of weighted outer product matrices $(\boldsymbol{u}_i\boldsymbol{v}_i^t)$ each of rank 1. This summation is pictorially represented

in Figure 1 as stating that the image G is the sum of basis or eigenimages $\boldsymbol{u}_i\boldsymbol{v}_i^t$ which are weighted according to their respective singular values.



$$G = d_1\boldsymbol{u}_1\boldsymbol{v}_1^t + d_2\boldsymbol{u}_2\boldsymbol{v}_2^t + \cdots + d_r\boldsymbol{u}_r\boldsymbol{v}_r^t$$

(a) Singular Value Decomposition of G



(b) Monotonically Ordered Singular Value Map (See image of Figure 2)
(The logarithm is used for plotting convenience)

FIG. 1. Matrix Expansion in Outer Products

The condition number of the image is of interest as a guide in the potential efficient representation of the image in terms of its eigenimages. Specifically,

$$C(G) = \frac{d_{\max}}{d_{\min}} = \frac{d_1}{d_r} \tag{6a}$$

when the singular values are monotonic non-increasingly ordered. In practice it often is not clear what value r should take because of experimental measurement and computer round off errors. This means that the theoretical rank of the image may in fact be n and the condition number consequently quite large. Thus truncation of the sum in (3) might be tolerable and a means of determining the truncation point might utilize a modified condition number

$$C_k(G) = d_1/d_k. \tag{6b}$$

When $C_k(G)$ and $C_{k+1}(G)$ are large and differ widely we may assume that error is dominating and truncation is appropriate. Using the SVD as a means of represen-

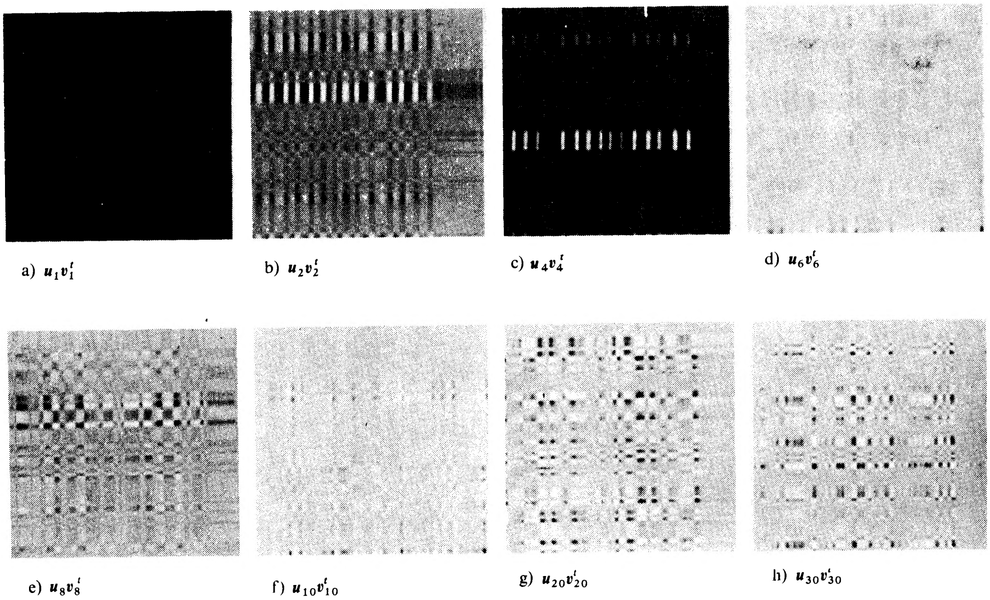


FIG. 2. Selected Singular Value Outer Product Eigenimages of "Text"

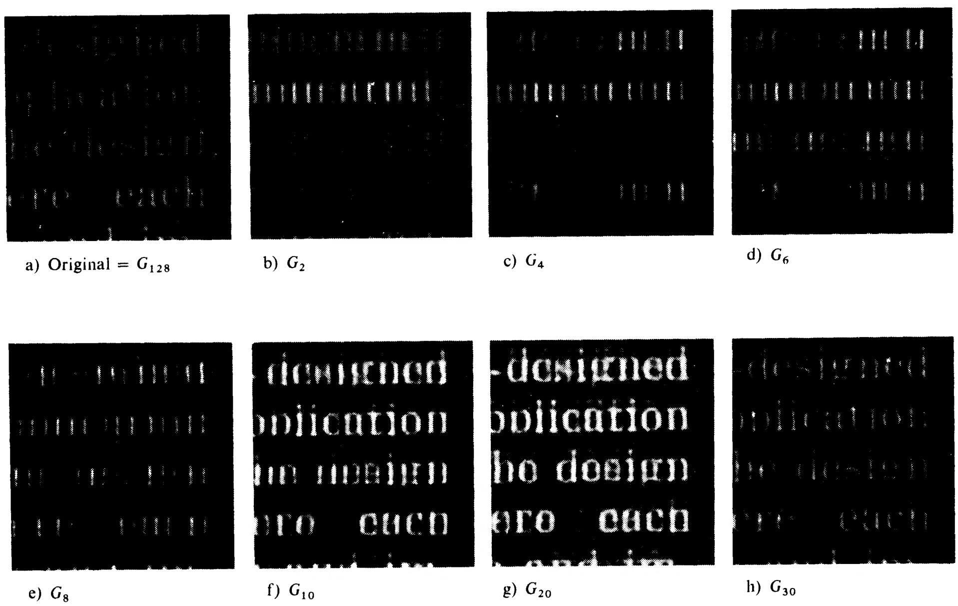


FIG. 3. Selected Partial Sums of SVD Expansions of "Text"

tation (retaining only k such outer products), we find that

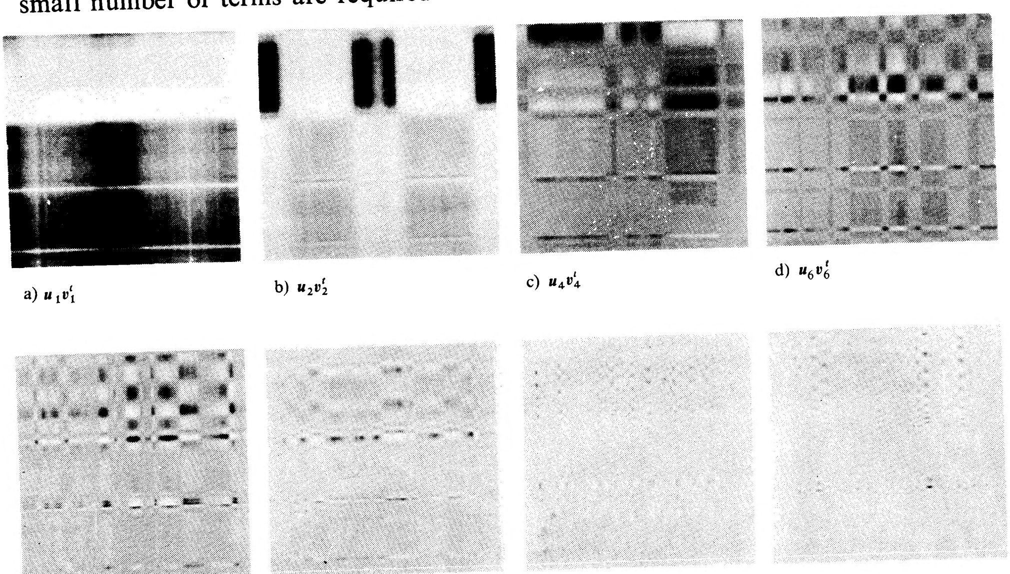
$$G_k = \sum_{i=1}^{k} d_i u_i v_i^t \tag{7}$$

and the norm between G and G_k is

$$\| G - G_k \|^2 = \sum_{i=k+1}^{r} d_i^2. \tag{8}$$

Thus, a monotonic decreasing order of the eigenvalues minimizes norm (or mean-square) truncation error.

In Figures 2 through 5, the SVD's of images are illustrated. Selected individual singular vector outer product images are presented (in componentwise magnitude form) to illustrate the decomposition of imagery into its basic two-dimensional components. Along with individual singular vector matrices, various partial summations of these matrices are included to demonstrate the cumulative effect of the individual outer products on the reconstruction of the entire image. Obviously, only a small number of terms are required for retention of significant image information.



a) $u_1 v_1^t$ b) $u_2 v_2^t$ c) $u_4 v_4^t$ d) $u_6 v_6^t$

e) $u_8 v_8^t$ f) $u_{10} v_{10}^t$ g) $u_{20} v_{20}^t$ h) $u_{30} v_{30}^t$

FIG. 4. Selected Singular Value Outer Product Eigenimages of "Satellite"

It is of interest to observe the eigenimages of Figures (2) and (4). It is evident that these eigenimages are uniquely matched to their respective scenes (text and satellite) and provide basis spaces which are optimal for scene representation. The local-global nature of the basis images provides a unique representation not available with other more traditional expansions. The fact that the first and second eigenimages

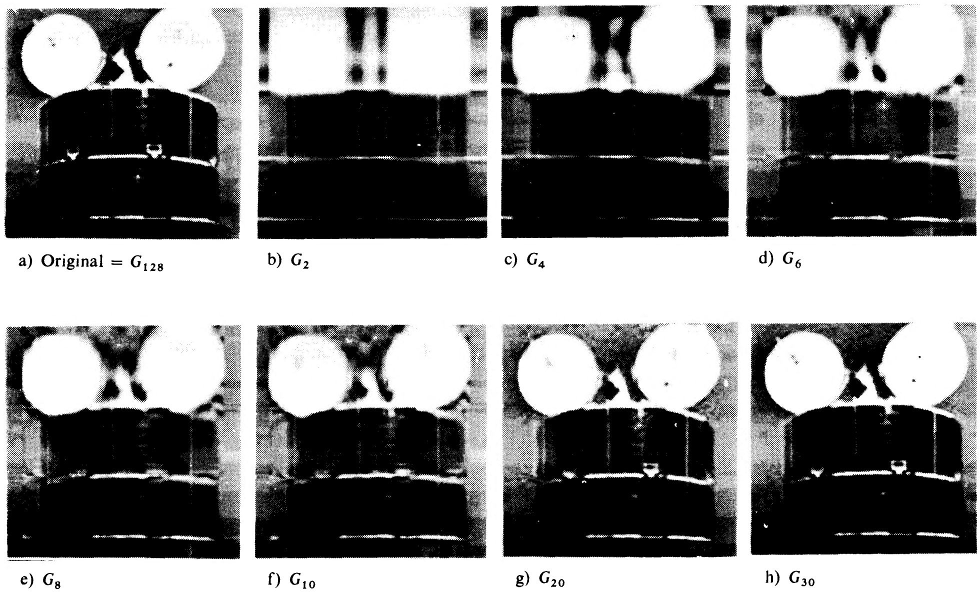
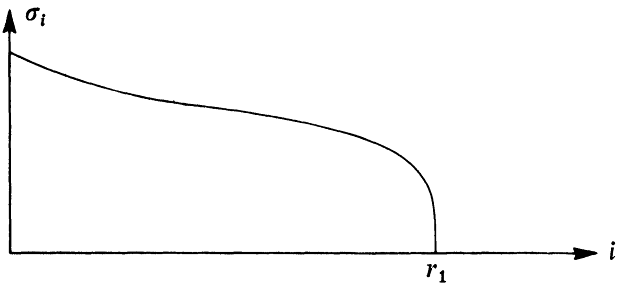
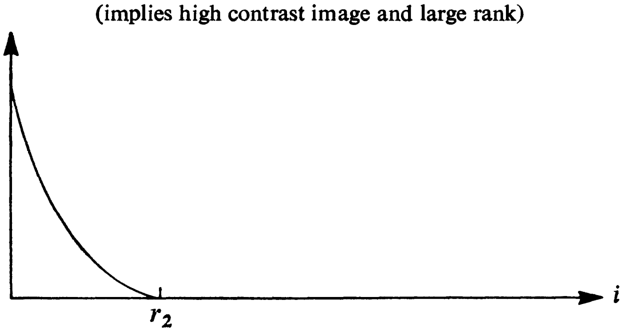


FIG. 5. Selected Partial Sums of SVD Expansions of "Satellite"



(a) Poor Bandwidth Reduction Potential
(implies high contrast image and large rank)



(b) Good Bandwidth Reduction Potential
(implies low contrast image and small rank)

FIG. 6. Eigenvalue Map for Image Representation ($r_1 \gg r_2$)

contain both high and low frequency regions matched to the original image is unobtainable in any other type of expansion (i.e., Fourier, Walsh, etc.). Indeed, filtering the weights of the singular values is an intriguing possibility left for future pursuits.

By investigating the actual singular (or eigen) values of the image, we can measure quantitatively the effect of each individual outer product matrix upon the entire reconstructed image. The eigenvalue map (a plot of eigenvalue vs index in monotonic decreasing order) provides an intuitive feel for the effective possible bandwidth reduction. Figure 6 presents two extreme cases for possible image representations. The smaller the rank and the more concentrated the energy in lower eigenvalues, the better the potential savings. Note also that the SVD technique requires storage of only $(2n + 1)r$ scalar values for full image representation, whereas the general orthogonal expansion in separable kernels requires n^2 scalar values stored (i.e., the d_{ij}) [**4**]. If we choose to absorb the d_i as a scalar factor into the singular vectors, then we require only $2nr$ storage locations.

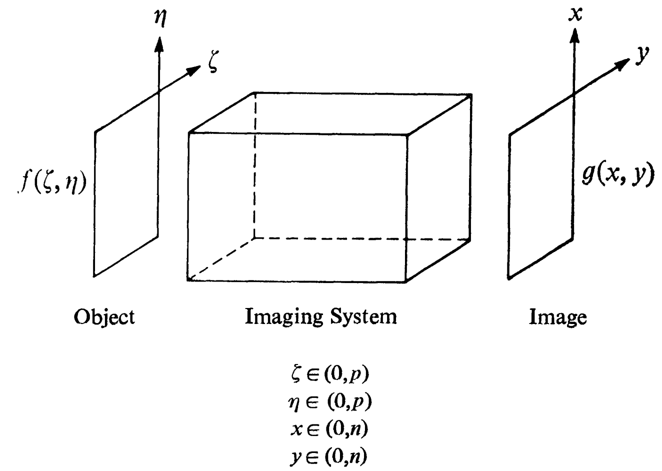


FIG. 7. A Linear Imaging System Model

Point spread functions and their singular value decompositions. As mentioned, it is possible to use the algebra of matrices and outer products as a tool in restoration processes where the point spread function matrix H is now of specific concern. Toward development of approaches in that direction, consider a model of a linear imaging system to be given in continuous notation as

$$g(x,y) = \int_{-P/2}^{P/2}\int_{-P/2}^{P/2} f(\zeta,\eta)h(x,y,\zeta,\eta)d\zeta d\eta, \tag{9}$$

where the object $f(\zeta,\eta)$ is imaged through the space variant point spread function (SVPSF) $h(x,y,\zeta,\eta)$ and Equation (9) represents a Fredholm integral equation of the first kind. Figure 7 provides such a model. By using a scanning or stacking

operator [5–7], we can represent the linear system of (9) as

$$g = Hf, \tag{10}$$

where g (the image) is n^2 by 1, f (the object) is p^2 by 1, and H (the point spread function) is n^2 by p^2. We assume we have analytic knowledge of H and desire recovery of f. A valuable reference for the SVD use in the solution of this type of equation can be found in [8]. From a modeling viewpoint, we would like the imaging system to conserve energy, and, because the scalar elements of f and g are themselves energy measurements, we have:

a. *Condition* 1

$$\sum_{i=1}^{p^2} f_i = \sum_{i=1}^{n^2} g_i. \tag{11a}$$

Similarly, an impulse or Dirac-Delta function (point source of light) anywhere in f should result in the same amount of energy in g independent of its position in f. Therefore,

b. *Condition* 2

$$\sum_{i=1}^{n^2} h_{ij} = 1. \tag{11b}$$

Because of the nature of energy sensing devices, we also have

c. *Condition* 3

$$\begin{aligned} f_i &\geqq 0 \\ g_i &\geqq 0 \\ h_{ij} &\geqq 0. \end{aligned} \tag{11c}$$

Let $H = U \operatorname{diag}(\sigma_i) V^t$ be the singular value decomposition of H. Because the H matrix is nonnegative, we know from the Perron-Frobenius Theorem [10] the following is componentwise nonnegative:

d. *Condition* 4

$$H_1 = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^t \geqq 0. \tag{11d}$$

It is now possible to discuss the singular value decomposition (SVD) of H as

$$H = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^t \tag{12a}$$

and an approximation to the point spread function matrix as

$$H_k = \sum_{i=1}^{k} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^t . \tag{12b}$$

Because of the relative smoothness of H, i.e., the lack of drastic changes in the point spread function as it moves both in the object plane (the column axis) and the image plane (the row axis), the rank is expected to be relatively low. Thus, fairly close approximations, in the norm, to the very large matrix H may be afforded by storage of a very few singular values and singular vectors.

Further simplification of the matrix H often comes from imaging systems that have the property of space invariant point spread functions (SIPSF), separable point spread functions, or both. For the SIPSF system

$$h(x, y, \zeta, \eta) = h(x-\zeta, y-\eta). \tag{13a}$$

The concept of the point spread function, an optical term, is the same as that of an impulse response, an engineering term, and is the kernel of the Fredholm equation (9). Referring to Figure 7, we see that physically the function $h(x, y, \zeta, \eta)$ is the output of a point source (Dirac-Delta function) of light located at (ζ, η) in the input. If that output function changes its shape as the point source explores the (ζ, η) plane, we refer to the system as space variant or a space variant point spread function (SVPSF). If the output of the point source of light retains its shape but is only a function of the coordinate difference between input and output plane, we refer to the system as space invariant (SIPSF) and Equation (13a) holds. The Fredholm equation then becomes a convolution and, in matrix notation, we have a partitioned matrix

$$H = \begin{array}{cc} & \xrightarrow{\quad} \zeta, \eta \\ \Big\downarrow \; x, y & \left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \end{array}\right] \end{array}, \tag{13b}$$

where each block is a Toeplitz matrix. A matrix H is a Toeplitz matrix if h_{ij} is a function of only $i-j$. Such point spread function matrices are referred to as block or vector Toeplitz forms [**10**].

The second simplification of the point spread function matrix comes when that function is separable:

$$h(x, y, \zeta, \eta) = a(x, \zeta) b(y, \eta), \tag{14a}$$

implying that the horizontal imperfections are modeled separately and independently of the vertical imperfections. Certain x-ray machines have this property. Under this assumption, the matrix H can then be equated with the Kronecker product [**11**] of smaller core matrices, A and B, both of size n by p,

$$H = A \oplus B, \tag{14b}$$

where $\oplus$ is the Kronecker product operand [**12**].

If both separability and space invariance properties hold, then

$$h(x, y, \zeta, \eta) = a(x-\zeta)b(y-\eta), \tag{15a}$$

and consequently,

$$H = A \oplus B, \tag{15b}$$

where both A and B are Toeplitz matrices.

The advantage of the separability assumption is that, when using the concepts of SVD, we can determine the SVD of the core matrices A and B, and simultaneously obtain the singular values of the larger matrix H. Since the eigenvalues and eigenvectors of the Kronecker matrix are the pairwise combination of eigenvalues and Kronecker combination of eigenvectors of the core matrices **[11]**, then

$$A = \sum_{i=1}^{r} \alpha_i \boldsymbol{a}_i \boldsymbol{c}_i^t \qquad B = \sum_{i=1}^{s} \beta_i \boldsymbol{b}_i \boldsymbol{e}_i^t .$$

The $\{\sigma_i, \boldsymbol{u}_i, \boldsymbol{v}_i\}$ set associated with the SVD of H becomes

$$\begin{aligned} &\{\sigma_i = \alpha_j \beta_k \quad \forall j, k\} \\ &\{\boldsymbol{u}_i = \boldsymbol{a}_j \oplus \boldsymbol{b}_k \ \forall j, k\} \\ &\{\boldsymbol{v}_i = \boldsymbol{c}_j \oplus \boldsymbol{e}_k \ \forall j, k\}. \end{aligned}$$

Consequently, it is only necessary to store $(r + s)(n + p + 1)$ scalar values compared to $(np)^2$ values if the full H matrix is stored.

One result of the space invariance assumption is that the resulting vector Toeplitz (or core matrix Toeplitz) forms have a semi-symmetric property when the matrices are square. In other words, if A is Toeplitz, then $a_{ij} = a_{i-j}$ and the matrix is symmetric about the minor diagonal (i.e., the diagonal which is at 90 degrees to the traditional diagonal). In addition, a good approximation to many Toeplitz forms is the circulant matrix C, whose eigenvectors are well known to be the trigonometric waveforms or roots of unity. Thus,

$$A \approx C = F^* \Lambda F, \tag{16}$$

where F is the Fourier matrix, while Λ is a diagonal matrix, whose entries are the Fourier transform of the first row of C. Consequently, circular convolution often becomes a computationally efficient substitute for more exacting Toeplitz eigenvector computations.

One interesting extremal condition for the SVD representation of H occurs in the perfect image situation. In this case,

$$H = I,$$

and the norm between H and H_k becomes

$$\|H - H_k\| = \sum_{i=k+1}^{n^2} \sigma_i^2,$$

but all the eigenvalues of H are unity, and consequently,

$$\|H - H_k\| = n^2 - k.$$

Note that, due to the multiplicity of eigenvalues, there is no unique eigenvector outer product expansion. This, unfortunately, is a very poor approximation as function of k, and indeed, in the perfect imaging limit, SVD does not appear attractive. At the other extreme, where H is defined by the all-one's matrix to within a scale factor of $(n^2)^{-1}$, we have

$$h_{ij} = 1 \,\forall\, ij \text{ and } \|H - H_1\| = 0,$$

because

$$H = \mathbf{1}\mathbf{1}^t.$$

In other words, H has rank $r = 1$ and is perfectly represented by one singular value and its associated outer product. It is believed that a type of continuum exists in transversing between these two extreme conditions on H. Also, the worse the imaging system, the better the use of SVD techniques, which is intuitively appealing because in poor imaging there is little left of the object for restoration.

In the problem of restoring imagery from linear systems in the absence of noise, the objective is to find an inverse (pseudo or true) to H to achieve a better estimate of the object f. Thus

$$\hat{f} = H^+ g, \tag{17}$$

(where H^+ is interpreted as either the pseudo or true inverse) is the objective of noise free restoration. Because of storage and computational considerations, we have concentrated on the use of the SVD as a mechanism for efficiently handling and approximating H^+. If H is of rank r, then

$$H^+ = \sum_{i=1}^{r} \sigma_i^{-1} \mathbf{v}_i \mathbf{u}_i^t. \tag{18}$$

However, we must realize that a small error in the norm between H and H_k may not necessarily reflect a small difference between H^{-1} (if such exists) and H_k^{-1}. In fact, the condition number of H yields a clue as to the quality of such an inversion [**13**]. Noting (18), we see that the small singular values produce large elements in the pseudoinverse. Moreover, small singular values may result simply from measurement and roundoff error. Thus we must proceed with caution.

The object f can be formed from

$$\hat{f} = H^+ g = \sum_{i=1}^{r} \sigma_i^{-1} (\mathbf{u}_i^t \mathbf{g}) \mathbf{v}_i. \tag{19}$$

The inner product of $(u_i^t g)$ is a scalar weighting (along with σ_i^{-1}) on the singular vector v_i, and the kth estimate of f becomes

$$\hat{f}_k = \hat{f}_{k-1} + \sigma_k^{-1}(u_k^t g)v_k,$$

where a partial sum formulation was used to suggest a convergence to the true object if H is truly consingular. In addition, the recursive computation of the set $\{\sigma_i, u_i, v_i\}$, as well as of $\hat{f}$, implies that simultaneous computer storage of the entire set of images and point spread function matrices is not necessary. Finally, the nonlinear condition that f be componentwise nonnegative can be added, and implementation of the constraint recursively in analogy to the modified Van Cittert technique is a possibility [**14**]. Due to Condition 4, the Perron-Frobenius Theorem, at least the first estimate $\hat{f}$ is componentwise nonnegative without any other conditions imposed.

Conclusions. This paper has been written as an aid to the understanding of basic digital image manipulations within a general purpose computer. The vehicle of matrix theory, and specifically singular value decomposition is used to interpret image expansions and representation algorithms. Particular emphasis is placed on singular value outer product expansions as an efficient means of image matrix storage and potential point spread function matrix pseudo-inversion. Estimates of objects that have experienced specific point spread function degradations can be obtained by pseudo-inversion of the impulse response matrix. When implemented sequentially, an efficient computational tool results. It is hoped that further research along these lines might lead to powerful digital image manipulations with various computing power machines at considerable savings in storage, computations, and associated problems involved in operations on extremely large scale images.

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References

1. G. H. Golub and C. Reinsch, Singular value decomposition and least squares solutions, Numer. Math., 14 (1970) 403–420.

2. A. Albert, Regression and the Moore-Penrose Pseudoinverse, Academic Press, New York, 1972.

3. G. E. Forsythe and C. B. Moler, Computer Solutions of Linear Algebraic Systems, Prentice-Hall, Englewood Cliffs, N.J., 1967.

4. J. D. Kennedy, S. J. Clark, and W. A. Parkyn, Jr., Digital Imagery Data Compression Techniques, McDonnell Douglas Corporation Report No. MDC-GO-402, January 1970.

5. M. P. Ekstrom, A numerical algorithm for identifying spread functions of shift-invariant imaging systems, IEEE Trans. on Computers, no. 4, C–22 (April 1973), 322–328.

6. ———, Computational Aspects of Solving Planar Integral Equations by Regularization, Proc. Sixth Asilomar Conference on Circuits and Systems, Pacific Grove, California, (November 15–17, 1972) 489–493.

7. D. Nissen, A note on the variance of a matrix, Econometrics, 36 (1968) 603–604.

8. J. M. Varah, On the numerical solution of ill-conditioned linear systems with applications to ill-posed problems, SIAM Journal of Numerical Analysis, No. 2, 10 (1973) 257.

9. J. Todd, Survey of Numerical Analysis, Chapter 8, McGraw-Hill, New York, 1958.

10. M. P. Ekstrom, An Iterative Improvement Approach of the Numerical Solution of Vector Toeplitz Systems, IEEE Transactions, C–23 (March 1974) 320–325.

11. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.

12. H. C. Andrews, Computer Techniques in Image Processing, Chapter 5, Academic Press, New York, 1972.

13. M. P. Ekstrom, A spectral characterization of the ill-conditioning in numerical deconvolution, submitted to IEEE Trans. Audio and Electroacoustics, (September 1972).

14. P. A. Jansson, R. H. Hunt, and E. K. Plyler, Resolution enhancement of spectra, JOSA, no. 5, 60 (May 1970) 596–599.

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FOUR PANEL TALKS ON PUBLISHING*

I. WHY PUBLISH? — BY LEONARD GILLMAN

There are several reasons why one should publish.

1. It is good for the *soul.* It makes you feel that you have done something. We all have an urge to communicate, but some people never get around to putting things down on paper. Many of these now say they wish they had.

2. It is good for the *psyche.* People recognize your name. They come up to you at meetings and say they have seen your paper. They may even say they have read your paper. *They* feel you have done something.

3. It is good for the *mind.* It forces you to solidify your ideas, to organize them and think them through. You can learn a great deal that way. You feel *you* have done something. And you fulfill an obligation to your profession to communicate and share knowledge.

4. It is good for the *body.* It leads to recognition by your dean in the form of material rewards. You feel *they* have done something.

* Presented at Annual MAA Meeting, Dallas Texas, January 1973.

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II. WHAT TO PUBLISH — BY P. R. HALMOS

When I accepted membership on this panel, I talked about it with almost every mathematician I saw and I asked for advice and help — in effect I conducted a sort of one-man opinion research poll. Some of the comments I received were not helpful. "Boy, you've got yourself into a corner!", they said. "That's an impossible subject!"

The first question, clearly, is "What is the question?" A proper understanding of the meaning of "What to publish?" depends on at least an approximate answer to "Why publish?". That answer varies with the self-awareness, the honesty, and the idealism of the one who gives it. It could, for instance, be to improve one's financial, academic, and social standing, or co-operatively to push the frontiers of knowledge ever more forward.

Another preliminary examination of the meaning of the question "What to publish?" should ask "Who is asking?" Is an author at the beginning of his career asking for advice on how to tell wheat from chaff? Is a referee or an editor asking which papers to accept? Is a publisher or a mathematical society asking which subjects are commercially or scientifically profitable? Authors, editors, and publishers do in fact ask these questions, every day, and they answer them; they make a decision in each case, somehow. The somehow seems helter-skelter, unsystematic. Is it perhaps the purpose of panel discussions such as this one to point to some guiding principles that can be used in every case? Or is it wrong to think that there *are* any guiding principles, and is it, if there are any, wrong to try to enforce them?

My opinion research poll revealed some hostility to the very asking of the question. Some mathematicians resent it, even before it leads to any detailed discussion of standards. Emphasis on the question implies to them that standards that they do not share might be applied, that somebody might interfere with their own publication program — it all smacks of elitism.

(I digress to comment how curiously pejorative the word "elitism" has become in recent years. My dictionary defines the "elite" as the "best or most skilled members of a given social group". To be against elitism would seem therefore like saying that "good" means "bad". But that's how it is; to insist nowadays that something — almost anything — be good — be it automobiles, the air we breathe, high school education, supreme court justices, the council of the American Mathematical Society, or published theorems — is elitist, and therefore bad.)

Only a few of my informants were, however, hostile to the question; a somewhat larger number took a pathetically friendly attitude. They were happy about the ques-

tion; they almost said "I thought you'd never ask!"; and they gave positive answers. What to publish? By all means more expository papers (not a surprising answer); that folk theorem that is "well known" but not accessible anywhere; and more book reviews (I wasn't expecting that one!).

Let me remind you that most laws (with the exception only of the regulatory statutes that govern traffic and taxes) are negative. Consider, as an example, the Ten Commandments. When Moses came back from Mount Sinai, he told us what to be by telling us, eight out of ten times, what *not* to do. It may therefore be considered appropriate to say what *not* to publish. I warn you in advance that all the principles that I was able to distill from interviews and from introspection, and that I'll now tell you about, are a little false. Counterexamples can be found to each one — but as directional guides the principles still serve a useful purpose.

First, then, do not publish fruitless speculations; do not publish polemics and diatribes against a friend's error. Do not publish the detailed working out of a known principle. (Gauss discovered exactly which regular polygons are ruler-and-compass constructible, and he proved, in particular, that the one with 65537 sides — a Fermat prime — is constructible; please do not publish the details of the procedure. It's been tried.)

Do not publish in 1975 the case of dimension 2 of an interesting conjecture in algebraic geometry, one that you don't know how to settle in general, and then follow it by dimension 3 in 1976, dimension 4 in 1977, and so on, with dimension $k - 3$ in 197k. Do not, more generally, publish your failures: I tried to prove so-and-so; I couldn't; here it is — see?!

Adrian Albert used to say that a theory is worth studying if it has at least three distinct good hard examples. Do not therefore define and study a new class of functions, the ones that possess left upper bimeasurably approximate derivatives, unless you can, at the very least, fulfill the good graduate student's immediate request: show me some that do and show me some that don't.

A striking criterion for how to decide not to publish something was offered by my colleague John Conway. Suppose that you have just finished typing a paper. Suppose now that I come to you, horns, cloven hooves, forked tail and all, and ask: if I gave you $1000.00, would you tear the paper up and forget it? If you hesitate, your paper is lost — do not publish it. That's part of a more general rule: when in doubt, let the answer be no.

As I went around asking people's opinion about what to publish, the view that I heard expressed most often is that too much is now being published. The active, talented mathematicians, both the beginners and the established ones, complain about the flood of junk in the journals; their estimates of how much of it should have been published vary from a generous 50% to a stringent 2%. The less active, less motivated members of the community complain (in my opinion rightly so) about the great pressure that is used on them to publish, publish, publish. They are frequently people of good taste, and they do not want to contribute to the flood of junk.

Almost everybody's answer to "What to publish?" can be expressed in either one word — "less" — or two words — "good stuff".

The trouble, of course, is how to define "good stuff" — how to establish canons of taste. G. H. Hardy's criteria are easy to say (is it true?, is it new?, is it interesting?), but are they easy to apply?

The newness of a paper can manifest itself in various ways: it can contain a new fact, a new proof, or a new method. Usually, of course, the three are mixed together, but there are striking cases when they occur in pure form. A paper by Morrison and Brillhart on the factorization of $2^{2^7}+1$ in the *Bulletin of the AMS* (1971) is nothing but a brutal fact, but both the authors and the editors of the Bulletin were right to decide that the fact was sufficiently new and that it sufficiently satisfied a pre-existing curiosity for the mathematical world as a whole to receive it with interest. Landau's *tour-de-force* proof of the irreducibility of cyclotomic polynomials (*Math. Z.*, 1929) reveals no new facts and no new methods, but if someone can do in one paragraph what, till then, many others have done in many laborious pages, he is right to think that the rest of us will think his proof interesting. When Cantor proved the existence of transcendental numbers (*Crelle's J.*, 1874), he didn't tell the world any fact that the world didn't already know; and he didn't improve Liouville's proof (*J. de Math.*, 1851) — indeed not — Cantor's result was much less sharp — but, undeniably, he introduced into mathematics a new method, the method of proving the existence of something by transfinite arithmetic, that has become a standard part of every graduate student's mathematical toolkit.

It's not much good to say "publish your *deep* results only", because depth is no easier to define than interest. It is of some use, just the same, because even the rough approximations to a definition may be suggestive of the direction in which the truth lies. Is a theorem deep when its proof is complicated? Yes, often, but by no means always. The theorem that 1 plus aleph null is aleph null, whose proof is short and trivial, is in my opinion one of the deepest in mathematics. Does "deep" mean the same as "surprising"? Yes, sometimes, but by no means always. No one finds the Jordan curve theorem surprising, but if a long and messy proof is any criterion, it sure is deep. One other test of depth deserves mention, namely breadth of contact. If a theorem about number theory uses the methods of complex function theory in its proof and has applications to the topological problem of determining the homotopy groups of spheres, it is probably deep.

Scientific publication has always been a fiercely competitive activity, but nowadays it is much more so than before, perhaps because it has become a question of many people's livelihood — of money. Graduate students compete with each other, assistant professors vie for promotion, and the full professors try to fight off the bright youngsters who are threatening them from below.

In any case the question — "What to publish?" — is not an operational one. If I knew the answer and revealed it, would anyone do what I told him to? What people will publish is not what anyone tells them to, but what the current social, political,

military, financial, academic, and perhaps even mathematical atmosphere calls for. The thing that discussion can best clarify is not what should but what will happen — we may not prescribe, but, possibly, we can predict. My own perhaps too optimistic prediction is that the junk will not continue to increase. The flood crest was brought onto us by an unexpected and unresisted increase in the availability of funds, and that is now over. In the coming decades no one, I think, will go into mathematics unless he wants to. Those of us who are in already will not be as rich as we used to be — a lot of the newly founded journals will go broke — the quantity of publication will decrease. The pressure to publish, no matter what, will decrease. The result will not be Utopia, not by any means, but, I predict, the problems we'll face will be different — and, whatever they are, they will not precipitate panel discussions on "Why, What, and How to Publish?".

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III. PUBLICATION RELATED TO TEACHING — BY HARLEY FLANDERS

Allow me to cover three topics very briefly, then concentrate on lower division texts. First is material in mathematics education. I can say nothing at all about professional research in education, but I suspect there are professional journals for it. Obvious outlets for reports on experiments, suggestions on how to introduce specific topics, curriculum changes, and the like are the Mathematical Education section of the MONTHLY, the TWO-YEAR COLLEGE MATHEMATICS JOURNAL, the MATHEMATICS TEACHER, and the MATHEMATICAL GAZETTE.

Second are short subject matter articles. The best outlets are the Classroom Notes section of the MONTHLY, the MATHEMATICS MAGAZINE, THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL, and the GAZETTE.

Third is upper division texts. Generally these have a small market; they are priced high, perhaps 6 to 8 cents per page, so the publisher can recover his investment from library plus some individual sales. In many ways, the publication of most texts at this level is like that of graduate texts and monographs.

Lower division texts. These compete with each other in a fierce market. They must be priced accordingly, often around 1.5 cents to 2 cents per page. With a necessarily expensive production job, the publishers may have to turn over 15–20 thousand copies to break even.

Why write a lower division text? First there is always a need for good new texts in every subject. Second is the profit motive. You are a professional, and textbook writing is use of your professional skills, for which you can expect compensation.

However, do not write a text simply on the basis of one brainstorm. You must write every section of your textbook with the same skill and enthusiasm, not just one

section for which you feel inspired. For instance, after years of teaching trig, you may decide the "right way" to develop the trigonometric functions is to start with the cotangent (because all the functions can be expressed rationally in terms of cot and because of higher authority from the Weierstrass factorization). This inspiration might make an acceptable Classroom Note, but it is inadequate for a whole trig text.

How to write. For me, this is synonymous with how to teach, except I have more time to think when writing and to correct errors. Good teaching implies a constant awareness of your students' understanding. So as you write, you yourself must be a student reader, an average one at that. And you must be critical. Is every single chapter, section, paragraph, sentence, word, figure clear? You must pay infinite attention to detail, to proofreading. And to exercises; are there enough rote problems? do they fit the text? are they graded? are they workable? are the answers correct? Don't hire a graduate student to prepare the answer manual. He has no vested interest in perfection in your book, and like most students, he is probably satisfied with part credit.

Working with a co-author can be useful provided his skills complement your own, each of you can give and take sharp criticism, and you really work cooperatively on everything in the book. The arrangement where one author is responsible for certain chapters and the other for the remainder, may work for a treatise, but is usually a disaster in a text.

How to publish. Publisher and author have somewhat different motives. The publisher wants to maximize his shareholders' profits, to improve his image in the industry, and to lengthen his list of publications. The author wants to write an excellent book he can be proud of, to have an attractive production job, to maximize his personal profit, at least to get a reasonable return for his invested time, and to minimize aggravation.

How do you pick your publisher? Probably no one knows the answer. I do think that if none of the larger, established houses is interested in your proposed text, then forget it, because you do not have a salable commodity.

When the publishers want your manuscript, they dish out a certain amount of soft soap, not to be taken at face value. For instance "We'll really help you write and produce your book," "We have the best marketing organization in the industry," "We sold $n!$ copies of———."

Unless you are an established author, when you make your proposal to a publisher, do so only with a good sized chunk of manuscript. Every publisher knows that most books started are never finished, so unless you have about a third or more of your book to show, the publisher will not really believe you.

If possible, have a complete manuscript before you negotiate a contract. Then you may have the happy situation of two or more publishers competing for it. I say

"negotiate" rather than "sign "because there is no reason whatsoever why you should simply sign a contract the publisher's attorney has written. After all, your book contract may turn out to be the single biggest business deal of your life, so why not get professional advice yourself? At least discuss it with some experienced colleagues. And do not rely on any verbal agreement — everything should be in the contract, the financial arrangements, who does what in production, how the book will be marketed.

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IV. HOW TO PUBLISH — BY BEATRICE SHUBE

To write a book is a labor of love, usually undertaken by someone who feels a desire and an obligation to make the knowledge of a particular subject accessible in one place. Writing is a lonely and difficult task, yet it provides great satisfaction. Publishing is a business, but it is also a communications service that contributes to the advancement of knowledge and education. The combination of an author and publisher is an unholy partnership with all the aspects of a love-hate relationship. I want to talk about that partnership in the hope that I can show why it should be one of mutual trust. I will not presume to tell you what to write by telling you what mathematics should be all about, although I will hint at approaches. I will not give you the mechanics of putting a manuscript together. Harley Flanders has written an excellent article on this subject for the January 1971 issue of the AMERICAN MATHEMATICAL MONTHLY. In addition, all of you should have a copy of Swanson's excellent book on *Mathematics into Type*, which is published by the Mathematical Society.

The best way to do what I want to do is to outline how a book becomes.

Whether the idea for a book orginates with the author or the editor, the editor consults first with his advisors in the field concerning its merits. (As an aside, I should say, very often an editor will also seek to persuade an outstanding mathematician to write a book on a topic of his — the mathematician's choice.) If the idea is the author's and it is approved, the editor negotiates an agreement with the author. If the idea is the editor's and there is no author in sight but rather the need is to find one, this becomes a quest that can continue for months and sometimes years, provided, of course, that the idea remains current and valuable, and not fulfilled by another publisher. In either case the manuscript is not yet written, and this takes considerable time and effort not only on the part of the author but also the editor who must serve as a goad and a support. This simple statement does not take into account all the corporate and personal expense and frustration that is usually a part of the initial arrangement, discussion, and development of a manuscript. Although there is agreement on the type of book, the approach to the subject, the number of printed pages, the number and kind of illustrations the finished book will contain, and the date on which the master manuscript will be delivered for final review and, hopefully, if all

goes well, for editing and manufacture, this agreement is subject to all the frailties of human nature. (As an aside, if you find your friendly editor chewing Gelusil tablets, just remember ulcers are an occupational hazard.) The time from the inception of the idea to the final manuscript has been known to vary from a year to many, and we have seen some manuscripts written over a twenty-year period.

Finally, the manuscript is ready for copy editing; that is, checking for clarity, consistency, and style and marking for the printer. The author sees the edited manuscript to add what last minute changes are required and to check the copy editor's work. He may not, however, take full advantage of this opportunity, which then leads to some of the antagonisms that an author has on occasion exhibited toward his publisher. The next time the author sees his work it is in type — in galley proof — and it looks entirely different to him from his typewritten manuscript, and 'oh those fool publishers what have they done to it'. Incidentally, before the manuscript went to the printer an author may have seen sample pages prepared from a random selection of the more difficult and representative material in his manuscript to show what format and typefaces are to be used. Again the author may have deferred corrections he was well aware needed to be made until galleys on the assumption that it is easier to make corrections at that stage. It is not. Extensive corrections cause delay and are costly. Publishers are at the mercy of their suppliers, and compositors work for many publishers. Any need for extensive correction causes a delay in the publication schedule. We publishers queue up for our work, and if we miss a date we are penalized in respect to both time and cost, for costs rise steadily. Still more costly are corrections in page proof. Here the only corrections that should be made are typographical or printer's errors. The only time major changes are permissible at this stage are those that pertain to the usefulness of the book and the needs of the market, and are mutually agreed on by the author and publisher.

All these steps can at any turn cause antagonism between the author and publisher. But, this is as nothing when it comes to prices. Here the publisher is an unconscionable robber baron. The price of a book is always too high. But prices are based not only on composition, paper, and printing and binding; there are also the costs of editing, design, promotion, marketing, distribution; billing; shipping, storage, returns, and so on. With mathematics books, the expense of composition is not an inconsiderable part of costs. It is the most complex and costly kind of composition, and, because of its nature, mathematics requires more physical space.

Earlier I said I would not tell you what topics you should be writing on; only that I would hint at approaches. A book is a tool; it is an instrument for conveying knowledge. Therefore it should not be written for your peers alone, but for many people who need to know and apply mathematical knowledge. Mathematics is basic to all knowledge, and all of you should be flexible in applying it to real-world problems. Meaningful communication should be established with other disciplines, government, and industry. Applications should be emphasized not alone because the word applied is fashionable just now, but because applications do tend to stimulate

new areas for investigation and broaden the market. Neither the pure nor the applied aspects of any science can stand alone. Each contributes to the other more than some would care to admit. I have stressed the importance of mathematics; may I now say that we should make it interesting as well as fun to do.

At this point you may be wondering why a publisher continues. Well he has a dream, and he wants to be a part of the world of knowledge. He knows that a scientific book is a tool, and it must be made available, despite the limited audience to which it is addressed. He is also aware that the branching and twigging effect that has occurred in science and technology is reflected in lower sales. The publisher needs to break even and to make a profit so that he will be there to publish your current book and your succeeding books.

JOHN WILEY AND SONS, 605 THIRD AVENUE, NEW YORK, NY 10016.

REPRESENTATIONS OF SL $(2, p)$

J. E. HUMPHREYS

1. Introduction. The **special linear group SL** $(2,p^n)$ consists of all 2×2 matrices of determinant 1 with entries from the field of p^n elements, where p is a prime. (These matrices do form a group under matrix multiplication, thanks to the product rule for determinants). The matrices $\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ commute with all others, and thus form a normal subgroup Z, and the quotient groups **PSL** $(2,p^n) =$ **SL** $(2,p^n)/Z$ turn out to be **simple** when $p^n \neq 2,3$, i.e., they have no proper normal subgroups (cf. Dornhoff [**6**, Part A, §35]). This is just the first of a number of infinite families of simple groups which arise from Lie theory by way of matrix groups over finite fields. In fact, if we add to the list so obtained, the alternating groups A_n $(n \geqq 5)$, we get all but 22 or so of the presently known (nonabelian) simple groups. So it is a good idea for anyone interested in finite groups and their applications (cf. Waall [**27**]) to find out more about these remarkable families. Here we shall limit ourselves to the groups $\mathbf{SL}(2,p^n)$ when $n = 1$, to avoid some technical details involving tensor products. But most of the ideas carry over to the case of arbitrary n — and even to the other families of Lie type.

If G is any finite group, F any field, a **representation** of G is by definition a homomorphism ρ: $G \to \mathbf{GL}(n,F)$ (= group of all invertible $n \times n$ matrices over F).

The number n is called the **degree** of ρ. A representation of degree 1 is essentially just a homomorphism of G into the multiplicative group F^* of F. As the name implies, a representation provides a sort of picture of G: in place of abstract group elements, multiplied abstractly, we get concrete matrices, multiplied in a familiar way. But the picture may be a poor likeness of the original; for example, let ρ send all elements of G to 1 (this is called the **1-representation** of G and denoted 1_G). Even so, the study of *all* possible representations of G often yields a very good "composite" picture of G (cf. Klemm [**17**]). In the next three sections we shall survey some of the theory of representations for arbitrary G and then see what can be said about $\mathbf{SL}(2,p)$.

The reader may object that $\mathbf{SL}(2,p)$ is already given concretely enough (in its "natural" representation of degree 2 over the field of p elements). But in fact the nicest results are obtained by studying representations of a group over the field $\mathbf{C}$ of complex numbers.

2. Group representations. In §2–§4, G is an arbitrary finite group, and all representations are over $\mathbf{C}$. The theory to be summarized here was developed around 1900, largely by Frobenius, Schur, and Burnside. As general references we suggest Curtis and Reiner [**5**], Dornhoff [**6**, Part A].

In order to sort out the representations of G, it is necessary to decide first when two of them are to be viewed as essentially the same. Given $\rho: G \to \mathbf{GL}(n,\mathbf{C})$, each $x \in G$ is represented by a matrix $\rho(x)$, which in turn describes a linear transformation of the vector space $V = \mathbf{C}^n$ relative to the usual basis. So G acts (via ρ) on V, which we express by saying that V is a **G-module.** This *action* of G on V really does not depend in any essential way on the basis chosen for V, although a change of basis would lead to different representing matrices. Indeed, if $A \in \mathbf{GL}(n,\mathbf{C})$ describes a change of basis, then the new matrices are of the form $A\rho(x)A^{-1}$ $(x \in G)$. This leads us to say that ρ is **equivalent** to another representation $\rho': G \to \mathbf{GL}(n,\mathbf{C})$ (same degree n) if there exists $A \in \mathbf{GL}(n,\mathbf{C})$ such that $\rho'(x) = A\rho(x)A^{-1}$ for all $x \in G$.

Consider, as the first nonabelian example, the symmetric group S_3 of order $6 = 3!$. S_3 is generated by the 2-cycle $x = (12)$ and the 3-cycle $y = (123)$, subject only to the relations: $x^2 = 1 = y^3$, $yx = xy^2$. To construct a representation of S_3, we just have to specify two matrices $\rho(x)$, $\rho(y)$ satisfying the same relations. If we identify 1×1 matrices with scalars, we can write down a couple of obvious representations of degree 1: the 1-representation $\rho(x) = 1 = \rho(y)$, and the *sign* $\rho'(x) = -1$, $\rho'(y) = 1$ ($+1$ for an even permutation, -1 for an odd permutation). These two representations are *not* equivalent. (Exercise: They are the only possibilities of degree 1.)

Turning to degree 2, we propose

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho(y) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$

where ω is a primitive cube root of 1 in $\mathbf{C}$. It is easy to check that these matrices satisfy the required relations. On the other hand, we could take

$$\rho'(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho'(y) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

But ρ' is equivalent to ρ: choose $A = \begin{pmatrix} \omega & 1 \\ 1 & \omega \end{pmatrix}$.

What about degree 3? Here S_3 has a "natural" representation: to a permutation of $\{1,2,3\}$ corresponds the linear transformation which permutes the usual basis (e_1, e_2, e_3) of $\mathbf{C}^3$ in the same way. For instance,

$$(123) \quad \text{goes to} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

But notice that the vector $e_1 + e_2 + e_3$ is fixed by all permutations of the subscripts, while the complementary subspace spanned by $e_1 - e_3$, $e_2 - e_3$, is stable under all the representing transformations. Changing the basis to $(e_1+e_2+e_3,\ e_1-e_3,\ e_2-e_3)$, we get an equivalent representation under which

$$(12) \quad \text{goes to} \quad \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \quad \text{and } (123) \text{ goes to} \quad \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right].$$

What we have here is the **direct sum** of the 1-representation and the representation of degree 2 constructed above. So nothing new has been obtained.

Conversely, we can build up representations of arbitrarily large degree by combining known ones in this way. The crucial thing is therefore to find those representations (called **indecomposable**) which cannot be broken down further into direct sums. The Krull-Schmidt Theorem (for G-modules) assures us in advance that each representation is a direct sum of indecomposable ones, the summands being unique up to ordering and equivalence. But it gives us no guidance in finding them!

Here a stronger condition comes into play: $\rho\colon G \to \mathbf{GL}(n, \mathbf{C})$ is called **irreducible** if no proper subspace of $\mathbf{C}^n$ is stable under all $\rho(x)$, $x \in G$. (Irreducible implies indecomposable, but not vice versa.) The reader can verify, for example, that the above representation of degree 2 of S_3 is irreducible, as is any representation of degree 1. Another basic result in algebra (Jordan-Hölder Theorem) assures us that each G-module has a "composition series" with irreducible "factors" which are essentially unique. In matrix language, this means that ρ is equivalent to some ρ', where all $\rho'(x)$ have the form:

$$\begin{bmatrix} \rho_1(x) & & & & * \\ & \rho_2(x) & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \rho_k(x) \end{bmatrix}$$

the ρ_i being irreducible representations and the $*$ entries being unspecified.

Fortunately, we do not have to choose between the Krull-Schmidt approach and the Jordan-Hölder approach, thanks to

(2.1) MASCHKE'S THEOREM: *Every representation of G is equivalent to a direct sum of irreducible ones.*

The remaining task is to describe the irreducible representations of G. Again, we are fortunate, because there are not "too many" of them:

(2.2) *The number of (inequivalent) irreducible representations of G is equal to the number of conjugacy classes of G.*

This result is somewhat peculiar, because examples show that there is (in general) no natural 1-1 correspondence between the representations and the classes. In our example, S_3 has three conjugacy classes: $\{1\}$, $\{(12),(13),(23)\}$, $\{(123),(132)\}$, one for each type of cycle structure. So the three irreducible representations already constructed must be the only ones, and we can fairly claim to have determined (up to equivalence) *all* possible representations of S_3. Notice that the degrees 1, 1, 2 obey the following general rule:

(2.3) *Let $\rho_1, \cdots, \rho_s$ be the distinct irreducible representations of G, n_i = degree of ρ_i. Then $\sum n_i^2 = |G|$ (the order of G). Moreover, n_i divides $|G|$.*

The reason for $|G|$ to appear here becomes plainer if we introduce the **regular representation** of G. The reader may recall Cayley's Theorem, which identifies an abstract group with a subgroup of the symmetric group $S_{|G|}$ ($x \in G$ permutes the elements of G by right multiplication). In turn, $S_{|G|}$ may be represented by matrices obtained by permuting columns of the identity matrix (as was done above for S_3). This yields the desired representation ρ_G of G, of degree $|G|$. The equality in (2.3) then comes from the more precise fact:

(2.4) *The regular representation ρ_G is the direct sum of the various ρ_i, each ρ_i occurring n_i times.*

3. Characters. So far, the theory looks very satisfactory. But when G is a large or "complicated" group (and simple groups tend to be quite complicated!), the

actual construction of irreducible representations is no routine matter. In fact, we usually have to settle for something less explicit.

Recall from linear algebra that the **trace** tr M of a square matrix M is the sum of its diagonal entries, and that this does not change if M is replaced by the similar matrix AMA^{-1} (thanks to the fact that $\operatorname{tr} XY = \operatorname{tr} YX$). Given a representation $\rho: G \to \mathbf{GL}(n, \mathbf{C})$, the function $\chi(x) = \operatorname{tr} \rho(x)$ from G to $\mathbf{C}$ is therefore the same for any representation equivalent to ρ. We call χ the **character** of ρ. Note that the character of a direct sum is just the sum of the characters. If x is conjugate to y in G (say $x = zyz^{-1}$), then $\rho(x) = \rho(z)\rho(y)\rho(z)^{-1}$ is similar to $\rho(y)$ and thus has the same trace. So χ may also be thought of as a $\mathbf{C}$-valued function on the set of conjugacy classes of G. This suggests that we write down an $s \times s$ **character table,** the rows indexed by the characters $\chi_1, \cdots, \chi_s$ of the irreducible representations $\rho_1, \cdots, \rho_s$, and the columns by representatives $x_1, \cdots, x_s$ of the distinct conjugacy classes, with the value $\chi_i(x_j)$ appearing in the (i,j) position. It is customary to put the class $\{1\}$ first, so the first column contains the numbers $\chi_i(1)$, which are clearly just the degrees n_i. For S_3, see Table 1.

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

TABLE 1. Character table of S_3

$\mathbf{C}$-valued functions on conjugacy classes of G may be added or multiplied by scalars, so they form an s-dimensional vector space CF(G) over $\mathbf{C}$. (What is the most obvious basis?) This vector space has a natural (unitary) inner product, the bar denoting complex conjugation:

$$(3.1) \qquad (\chi,\phi)_G = |G|^{-1} \sum_{x \in G} \chi(x)\overline{\phi(x)} = |G|^{-1} \sum_{j} h_j \chi(x_j)\overline{\phi(x_j)}$$

(x_j = *class representative,* h_j = *number of conjugates of* x_j).

Remarkably enough:

(3.2) *The irreducible characters* $\chi_1, \cdots, \chi_s$ *form an orthonormal basis for* $CF(G)$.

This is one of the basic **orthogonality relations.** The other involves the columns of the character table:

$$(3.3) \qquad \sum_{i=1}^{s} \chi_i(x_j)\chi_i(x_k) = \delta_{jk}|G|/h_k .$$

These relations are often used to get information about unknown characters from

information about known ones. (For instance, the reader should be able to fill in the third row of Table 1 once the first two rows are known.) Moreover, (3.2) shows that the character χ, which at first sight seems to contain only partial information about ρ, actually determines ρ uniquely:

(3.4) *Two representations of G having the same character are equivalent. If χ is a character of G, then χ is irreducible if and only if* $(\chi,\chi)_G = 1$.

4. How to construct characters. G may have a subgroup H whose characters we already know. Say ϕ is one of these. It is easy enough to extend ϕ to a **C**-valued function $\dot{\phi}$ on G, by decreeing that $\dot{\phi}(x) = 0$ if $x \notin H$. But $\dot{\phi}$ has little chance of being a character — for one thing, ϕ may take distinct values at elements of H which are not conjugate in H but are conjugate in G. So we try something fancier:

$$\phi^G(x) = |H|^{-1} \sum_{y \in G} \dot{\phi}(yxy^{-1}) \qquad (x \in G). \tag{4.1}$$

A moment's scrutiny should make it clear that ϕ^G is at least a class function on G. Indeed:

(4.2) ϕ^G *is a character of G (called an* **induced character**)).

The reason *why* ϕ^G should be a character cannot be well understood without going into some technical details about tensor products. To see that some sort of "product" is involved, observe that:

(4.3) *The degree* $\phi^G(1)$ *is the product of* $\phi(1)$ *and the index* $[G\colon H]$.

If H has fairly small index in G (i.e., if H is big), and if ϕ is irreducible, then there is at least a chance that ϕ^G may also be irreducible. Take, for example, $G = S_3$, H the cyclic subgroup generated by (123). H has an obvious representation of degree 1 sending (123) to ω (= primitive cube root of 1). The induced character given by (4.1) then has degree 2 and in fact occupies the third row of Table 1.

There is another useful technique, based in a different way on tensor products:

(4.4) *If χ, ϕ are characters of G, then so is their product*: $(\chi\phi)(x) = \chi(x)\phi(x)$.

The degree of $\chi\phi$ being the product $\chi(1)\,\phi(1)$, there is usually little hope that the product will be irreducible even if χ and ϕ are. But $\chi\phi$ *may* contain new irreducible constituents, and we *may* be able to sort them out.

5. The groups SL(2,p). From now on $G = \mathbf{SL}(2,p)$ unless otherwise specified. Since we are dealing with a whole family of groups, not just a single group, the question arises at once: Why should we expect the representations of (say) **SL**(2, 17) to have anything at all to do with those of (say) **SL**(2,71)? The common origin of these groups in Lie theory does not at first seem to have any connection with their

representations over **C**. One of our purposes, then, will be to explain what connection exists.

It is an observed fact that many phenomena surrounding these groups are (in some sense) independent of p, or else vary "smoothly" with p. For example:

(5.1) *The order of* $\mathbf{SL}(2,p)$ *is* $p^3 - p = p(p+1)(p-1)$.

This formula is easy to derive (cf. Dornhoff [**6**, Part A, Lemma 35.2]). It shows that the order of a p-sylow subgroup of G is p. One subgroup of this size consists of the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$; we denote it by U. Let T be the subgroup of all diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ in G. This is cyclic of order $p-1$ (being isomorphic to the multiplicative group of the field of p elements.) The product $B = TU$ is the group of upper triangular matrices in G, and has order $p(p-1)$, hence index $p+1$.

From now on *we shall always assume that p is an odd prime*, to avoid a few awkward statements. The case $p = 2$ is essentially done, anyway, since $\mathbf{SL}(2,2)$ happens to be isomorphic to S_3. (To see this, notice that $\mathbf{SL}(2,2)$ has order 6, by (5.1), and permutes the three distinct subspaces of dimension 1 in 2-dimensional space over the field of 2 elements.)

The next two sections are based on the exposition in Dornhoff [**6**, Part A, §38], which in turn rests on Schur [**19**].

6. Conjugacy classes of SL(2, p). To find the irreducible representations of G (or at least their characters), we have to know how many to look for, i.e., how many classes G has (2.2). The answer is a polynomial in p, namely, $p+4$. (If 4 is interpreted as the square of the order of the center of G, the same formula will work for $p = 2$.)

The best way to survey the classes is to introduce a factorization valid in any finite group (for a given prime p). Call $x \in G$ **p-regular** if its order is relatively prime to p, **p-singular** if its order is a power of p.

(6.1) *Let* $x \in G$. *Then there exist unique elements* $y, z \in G$ *satisfying the conditions*: $x = yz = zy$, y *p-regular*, z *p-singular*.

This boils down to an assertion about the cyclic group generated by x, which the reader can readily check.

In our case, G is given as a matrix group (over the field of p elements), so we can ask what meaning this factorization has in terms of *linear algebra*. A *p-singular* element of G is a matrix whose eigenvalues are both 1 (why?). On the other hand, a matrix has order prime to p if and only if it is diagonalizable (either over the prime field or over a quadratic extension). By adapting Jordan normal forms, one can show without much trouble that there are p classes of p-regular elements, including the class of 1, two other classes of p-singular elements, and two "mixed"

classes. There are essentially two types of p-regular elements: those conjugate already in G to a diagonal matrix and those which only become diagonal over a field of p^2 elements.

In Table 2 we list class representatives, along with the number of elements in the class (= index in G of the centralizer in G of an element in the class). Here

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, z = -1, \quad c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix},$$

Representative	1	z	a^l	b^m	c	d	zc	zd
# Conjugates	1	1	$p(p+1)$	$p(p-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$

TABLE 2. Class representatives of $\mathbf{SL}(2, p)$

where v generates the multiplicative group of the field of p elements. Also, b denotes an element of order $p+1$ which is not diagonalizable over the prime field. The non-conjugate powers are: a^l $(1 \leqq l \leqq (p-3)/2)$, b^m $(1 \leqq m \leqq (p-1)/2)$, plus of course $1 = a^0 = b^0$, $z = a^{(p-1)/2} = b^{(p+1)/2}$. As an exercise, the reader might verify that the number of classes exhibited is $p+4$ (as claimed) and that the total number of elements is $p^3 - p$ (5.1).

A group of order 2, called the **Weyl group,** makes its appearance here. It brings about, for example, the conjugacy of a^k and a^{-k} (and similarly for powers of b). *This group W is the same no matter what p is,* which turns out to be a key factor in the uniformity of the results which follow. Strictly speaking, $W = N_G(T)/T$, where the normalizer of the diagonal group T consists of all matrices with exactly one nonzero entry in each row and each column (check this). W should be thought of as the symmetric group S_2.

7. Characters of SL(2,'p). The character table of $\mathbf{SL}(2, p)$ was first obtained by Frobenius; shortly afterwards, Schur [**19**] and (independently) H. Jordan [**16**] found the characters of $\mathbf{SL}(2, p^n)$. We are following Dornhoff's version of Schur's paper.

It turns out that the irreducible characters of G occur in two "series." The first of these is very easy to construct by the induced character technique sketched in Section 4. (In this case one easily gets the *representations*, not just the characters.) Begin with T, the cyclic group of order $p-1$ generated by a. There are $p-1$ distinct characters of T having degree 1 (cf. (2.2) and (2.3)), given as follows: Let $\tau \in \mathbf{C}$ be a primitive $(p-1)$-th root of 1, and define $\lambda_i(a^t) = \tau^{it}$. We could induce these characters to G, but since T has rather large index in G this would be unwise. Instead, we extend λ_i first to a character (again called λ_i) of the triangular group $B = TU$ by requiring that $\lambda_i(x) = 1$ for $x \in U$. (Why is this legitimate?) Now $[G: B] = p+1$,

so the induced character $\zeta_i = \lambda_i^G$ has degree $p+1$ (4.3). From (4.1) the values of ζ_i on class representatives can be computed without much labor (Table 3).

1	z	a^l	b^m	c	d	zc	zd
$p+1$	$(-1)^i(p+1)$	$\tau^{il}+\tau^{-il}$	0	1	1	$(-1)^i$	$(-1)^i$

TABLE 3. Values of ζ_i

A glance at Table 3 shows that not all ζ_i are distinct; so we can limit our attention to the indices $0 \leqq i \leqq (p-1)/2$. The next question is: Which (if any) of the ζ_i are irreducible? A straightforward calculation using (3.1) shows that if $1 \leqq i \leqq (p-3)/2$, then $(\zeta_i, \zeta_i)_G = 1$, so (3.4) says that these ζ_i are indeed irreducible. (Since we are looking for a total of $p+4$, our task is roughly half completed.) On the other hand, $(\zeta_0, \zeta_0)_G = 2$. This means (cf. (3.2)) that ζ_0 is the sum of *two* irreducible characters. It can be shown that one of these is 1_G, so the other one (denoted ψ and called the **Steinberg character**) has degree p = highest power of p dividing G (cf. Steinberg [**23**]). Finally, when $i = (p-1)/2$, we again get $(\zeta_i,\zeta_i)_G = 2$. In this case ζ_i can be shown to split into a sum $\xi_1 + \xi_2$, where each ξ_i has degree $(p+1)/2$. (But the values of these characters are a bit tricky to compute.)

The series of characters just constructed "corresponds" in some sense to the family of p-regular classes represented by powers of a (this in spite of our remark following (2.2)!). So it is natural to turn to the cyclic group S of order $p+1$ generated by b for another series. Let $\sigma \in \mathbf{C}$ be a primitive $(p+1)$-th root of 1, and define characters of S by $\phi_i(b^t) = \sigma^{it}$. So far, so good. Unfortunately, S (like T) has large index in G but fails (unlike T) to sit inside a larger group to which we can trivially extend ϕ_i. The induced character $\phi_i{}^G$ has degree $p(p-1)$, and therefore could not be irreducible (cf. (2.3)).

At this point, the second technique in Section 4 for constructing characters is invoked. Consider the class function:

$$\theta_i = \zeta_i\psi - \zeta_i - \phi_i{}^G \qquad (1 \leqq i \leqq (p+1)/2). \tag{7.1}$$

The values of θ_i are not hard to compute (Table 4). But the reason for picking θ_i in the first place is certainly obscure; for now, just note the presence of the Steinberg character ψ. At any rate, when $1 \leqq i \leqq (p-1)/2$, $(\theta_i, \theta_i)_G = 1$ (and $\theta_i(1) > 0$),

1	z	a^l	b^m	c	d	zc	zd
$p-1$	$(-1)^i(p-1)$	0	$-(\sigma^{im}+\sigma^{-im})$	-1	-1	$(-1)^{i+1}$	$(-1)^{i+1}$

TABLE 4. Values of θ_i

which guarantees that θ_i is a bona fide irreducible character of G. As to $\theta_{(p+1)/2}$, it splits into a sum $\eta_1 + \eta_2$ of two irreducible characters of degree $(p-1)/2$. Once the values of η_1, η_2 are pinned down (with the aid of the orthogonality relations (3.2), (3.3)), we are in possession of $p+4$ distinct irreducible characters, so our task is done (2.2). The results are shown in Table 5, where $\varepsilon = (-1)^{(p-1)/2}$. For

	1	z	a^l	b^m	c	d
1_G	1	1	1	1	1	1
ψ	p	p	1	-1	0	0
ζ_i	$p+1$	$(-1)^i(p+1)$	$\tau^{il}+\tau^{-il}$	0	1	1
ξ_1	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^l$	0	$\frac{1}{2}(1+\sqrt{\varepsilon p})$	$\frac{1}{2}(1-\sqrt{\varepsilon p})$
ξ_2	$\frac{1}{2}(p+1)$	$\frac{1}{2}\varepsilon(p+1)$	$(-1)^l$	0	$\frac{1}{2}(1-\sqrt{\varepsilon p})$	$\frac{1}{2}(1+\sqrt{\varepsilon p})$
θ_i	$p-1$	$(-1)^i(p-1)$	0	$-(\sigma^{im}+\sigma^{-im})$	-1	-1
η_1	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$
η_2	$\frac{1}{2}(p-1)$	$-\frac{1}{2}\varepsilon(p-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}(-1-\sqrt{\varepsilon p})$	$\frac{1}{2}(-1+\sqrt{\varepsilon p})$

TABLE 5. Characters of **SL** $(2, p)$ (classes of zc, zd omitted)

brevity, the classes of zc and zd are omitted: for any character χ, $\chi(zc) = \chi(z)\chi(1)^{-1}\chi(c)$, and similarly for zd. It is an amusing exercise to verify directly from this table the formula (2.3) for $|G|$.

The characters θ_i are rather mysterious, having been constructed in a roundabout and seemingly arbitrary way. Moreover, the *representations* to which they belong are nowhere in sight. (It is a difficult matter to construct them explicitly, cf. Tanaka [**25**], Silberger [**20**], Gel'fand [**7**].) In recent years a more systematic approach to the characters of groups of Lie type has been formulated by Harish-Chandra [**9**] (see Springer [**21**]), making clear why two series are to be expected here. On the other hand, Green [**8**] was already able around 1955 to compute explicitly the characters of the finite "general linear" groups, thus going well beyond **SL**$(2, p)$. But there are still a lot of unanswered questions.

One striking fact about Table 5 is that most of the irreducible characters have degree either $p+1$ or $p-1$, these being polynomials in p with highest term p ($=$ highest power of p dividing $|G|$). Equally striking is the fact that the actual character values on p-regular classes (which account for most classes) are so simple and involve numbers such as $\tau^{il} + \tau^{-il}$ which are in some sense "symmetric" relative to the Weyl group W discussed in Section 6. These phenomena occur for other groups of Lie type as well, so it is reasonable to look for some further explanation of them in Lie theory. This we do next.

8. Irreducible modular representations. For the moment let G be any finite group, p any prime dividing $|G|$, and K an algebraically closed field of characteristic p. As alleged in Section 1, the representations of G over K are not so well-behaved as those over **C**. Indeed, the main results listed in Sections 2 and 3 break down com-

pletely. The "modular" theory therefore seems at first very unpromising. But since the late 1930's Brauer and others (cf. Brauer and Nesbitt [2]), have made it a valuable tool in the study of "ordinary" representations (i.e., those over $\mathbf{C}$). General references for the modular theory are Curtis and Reiner [5], Dornhoff [6, Part B].

The first interesting fact, which generalizes (2.2), is:

(8.1) *The number of (inequivalent) irreducible representations of G over K is equal to the number of p-regular conjugacy classes of G.*

However, (2.3) fails to hold over K, and the characters are not very useful: e.g., the first part of (3.4) fails, and we cannot make sense of the inner product (3.1) when p and hence $|G|$ is 0 in K. To get around this, Brauer associated with an irreducible representation $\rho: G \to \mathbf{GL}(n, K)$ a $\mathbf{C}$-valued function ϕ on G_{reg} (= set of p-regular elements of G), nowadays called the **Brauer character.** To define ϕ, notice that for $x \in G_{\mathrm{reg}}$, the eigenvalues of $\rho(x)$ are certain roots of unity in K, of order relatively prime to p, and their sum is the trace of $\rho(x)$ (the usual character). We just replace these eigenvalues by corresponding complex roots of unity and call the sum $\phi(x)$. If r = number of p-regular classes, denote by $\phi_1, \cdots, \phi_r$ the corresponding irreducible Brauer characters. The usefulness of these functions is indicated by:

(8.2) *$\phi_1, \cdots, \phi_r$ form a basis of the vector space (over $\mathbf{C}$) of $\mathbf{C}$-valued class functions on G_{reg}. An (ordinary) character of G, restricted to G_{reg}, is a nonnegative integral linear combination of the ϕ_i.*

In particular, the restrictions to G_{reg} of the (ordinary) irreducible characters $\chi_1, \cdots, \chi_s$ of G can be expressed as:

$$\chi_i = \sum_{j=1}^{r} d_{ij}\phi_j \qquad (d_{ij} \in \mathbf{Z}^+). \tag{8.3}$$

The integers d_{ij} form an $s \times r$ matrix D, called the **decomposition matrix** of G (relative to p). In terms of actual representations, (8.3) reflects the fact that the representation with character χ_i may be "reduced modulo p" to obtain a representation over K whose composition factors have the indicated Brauer characters with multiplicities d_{ij}.

The point of all this is that a knowledge of D and of the ϕ_j would enable us to write down that portion of the character table of G corresponding to G_{reg}. When $G = \mathbf{SL}(2, p)$, virtually all classes are p-regular, so this would be a very large portion. Of course, it seems at this stage no easier to find D and the Brauer characters than to find the χ_i. (We mention, in passing, that in principle D itself can be found if all χ_i and ϕ_j are known. This is quite hard to do in practice — cf. Srinivasan [22]. And, of course, it tends to defeat the purpose of the theory!)

Now let $G = \mathbf{SL}(2, p)$ again (and let the prime in question be p). Lie theory actually provides a systematic procedure for constructing all the irreducible modular

representations of G. We view G as acting on a 2-dimensional vector space over K, with basis (e_1, e_2), and we extend that action in a natural way to the space of homogeneous polynomials of degree $\lambda \geqq 0$ in e_1 and e_2 (viewed as indeterminates). This space of polynomials has dimension $\lambda + 1$, with basis $(e_1^\lambda, e_1^{\lambda-1}e_2, \cdots, e_2^\lambda)$, and is denoted M_λ. For instance,

$$a = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

sends e_1 to ve_1 and e_2 to $v^{-1}e_2$, so it must send $e_1^3e_2$ to $(ve_1)^3(v^{-1}e_2) = v^2(e_1^3e_2)$. It is not too hard to verify that the resulting G-module is irreducible provided $0 \leqq \lambda < p$. (For $\lambda = 0$, we just get 1_G.) So we obtain irreducible Brauer characters of degrees $1, 2, \cdots, p$. Since there are just p p-regular classses (Section 6), (8.1) insures that we need look no further.

The case $\lambda = p - 1$ is especially interesting. Here the reader can verify that the Brauer character agrees on G_{reg} with the *Steinberg character* ψ constructed in Section 7. In fact, the representation M_{p-1} is precisely the "reduction modulo p" of the ordinary representation whose character is ψ.

9. Lie algebra representations. The construction of G-modules in Section 8 is straightforward, but does not fully reveal the influence of Lie theory. So we shall give a slightly more abstract version, based on the **Lie algebra** $\mathfrak{g} = \mathfrak{sl}(2, K)$, which is by definition the set of all 2×2 matrices over K having trace 0. $\mathfrak{g}$ is closed under addition and scalar multiplication, so it is a vector space over K. One basis consists of

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\mathfrak{g}$ is also closed under a bilinear (but not associative) "product" operation: $[u, v] = uv - vu$.

A **representation** of $\mathfrak{g}$ is just a linear transformation ρ: $\mathfrak{g} \to \mathbf{M}(n, K)$ (= space of of all $n \times n$ matrices over K) such that $\rho[u, v] = \rho(u)\rho(v) - \rho(v)\rho(u)$. As for group representations, there is a natural notion of "equivalence," and K^n can be viewed as a "$\mathfrak{g}$-module."

The analogous Lie algebra over $\mathbf{C}$ has a well-known series of irreducible representations, which in dimensions $\leqq p$ adapt at once to $\mathfrak{g}$. Letting $(e_0, \cdots, e_\lambda)$ be a basis for $K^{\lambda+1}$, we simply prescribe:

$$\text{(9.1)} \qquad \begin{aligned} \rho(h)e_i &= (\lambda - 2i)e_i, \\ \rho(x)e_i &= (\lambda - i + 1)e_{i-1} \qquad (e_{-1} = 0), \\ \rho(y)e_i &= (i + 1)e_{i+1} \qquad (e_{\lambda+1} = 0). \end{aligned}$$

The reader can check that this recipe yields a representation of $\mathfrak{g}$ (of degree

$\lambda + 1$). What is more, the G-module M_λ of Section 8 can in a sense be identified with this $\mathfrak{g}$-module, via "exponentiation." We give a rough idea of how this works. If A is any square matrix, then $\exp A = 1 + A + A^2/2! + A^3/3! + \cdots$ may or may not be a well-defined matrix over K. Since division by p is impossible in K, one clearly has to require that $A^p = 0$. For instance, $\exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ is defined and equals $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. It turns out that we can exponentiate the matrix $\rho(x)$ or $\rho(y)$ given by (9.1) and thereby get matrices representing elements of G on M_λ.

This needs to be made more precise, of course, but it will suffice to indicate how $\mathfrak{g}$ is brought into the picture. For more details, consult Curtis [**4**], Steinberg [**24**].

10. Principal indecomposable modules. In Section 8 we pointed out the desirability of finding the decomposition matrix D (without first knowing $\chi_1, \cdots, \chi_s$). It happens that D is related in a remarkable way to another matrix of integers, which we now explain.

Consider briefly the case of an arbitrary finite group G. Maschke's Theorem (2.1) breaks down rather badly in characteristic p, when p divides $|G|$, leaving us with a vast assortment of indecomposable representations other than the irreducible ones. (Since $\mathbf{SL}(2,p)$ has cyclic p-sylow subgroups, it is actually possible in this case to classify them to some extent, cf. Janusz [**14**]. But this seems beyond reach for other groups of Lie type.) Fortunately, some of these are both useful and manageable. As in Section 2, we can construct the regular representation of G (over K) and decompose it via the Krull-Schmidt Theorem into a direct sum of indecomposables. These are called the **principal indecomposable modules** (PIM's for short). Two pleasant facts emerge:

(10.1) *There is a natural* 1-1 *correspondence between PIM's and irreducible modular representations of G, each of the latter occurring as unique "top" composition factor of its PIM. Moreover, a PIM occurs as many times in the regular representation as the degree of the associated irreducible representation.*

(10.2) *The degree of a PIM is divisible by the highest power of p dividing* $|G|$.

Denoting by η_i the Brauer character of the PIM corresponding to ϕ_i ($1 \leqq i \leqq r$), we can write (thanks to (8.2)):

$$\eta_i = \sum_{j=1}^{r} c_{ij}\phi_j \quad (c_{ij} \in \mathbf{Z}^+). \tag{10.3}$$

The $r \times r$ matrix $C = (c_{ij})$ is called the **Cartan matrix** of G (relative to p). It satisfies:

(10.4) $C = D^tD$ (D^t = *transpose of* D); *in particular, C is symmetric.*

(10.5) *The determinant of C is a certain (predictable) power of p.*

C can therefore be found once D is known. (Brauer also showed how to compute C — in principle, but rarely in practice — if only $\phi_1, \cdots, \phi_r$ are known.) Conversely, D can sometimes be reconstructed from a knowledge of C and this turns out to be the case for $\mathbf{SL}(2, p)$!

An example may help to fix the ideas. Take $G = S_3$, $p = 2$. There are two p-regular classes (those of 1 and (123)), hence two irreducible representations over K: the 1-representation and the (reduction modulo 2 of the) degree 2 representation constructed in Section 2. Table 6 gives the Brauer characters. Comparison with Table 1 shows that

	1	(123)
ϕ_1	1	1
ϕ_2	2	-1

TABLE 6. Brauer characters of $S_3, p = 2$

$$D^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, $C = D^tD = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (and $\det C = 2$).

The general set-up may still seem to be hopelessly complicated (perhaps the reader wishes we had quit while we were ahead, at the end of Section 7). But the strategy is a simple one: to "trap" the ordinary characters χ_i of a group G between the η_i and the ϕ_i. Indeed, (10.4) comes essentially from the fact that each η_i is a sum of certain χ_j (or their restrictions to G_{reg}), while each χ_j is a sum of certain ϕ_i in a *reciprocal* fashion.

Henceforth, $G = \mathbf{SL}(2, p)$. Denote by R_λ the PIM corresponding to M_λ $(0 \leqq \lambda < p)$, as in (10.1). Since $\dim M_\lambda = \lambda + 1$, (10.2) makes it clear that M_λ is strictly smaller than R_λ except possibly when $\lambda = p - 1$. In fact:

(10.6) $\qquad R_{p-1} = M_{p-1}$ *(Steinberg representation).*

11. PIM's for the Lie algebra. Now the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,K)$ re-enters the picture. It happens that $\mathfrak{g}$ too has PIM's. The "regular representation" here involves the so-called "restricted universal enveloping algebra," an algebra of dimension p^3 over K which we shall not attempt to describe further. (10.1) carries over verbatim:

(11.1) *There is a natural* 1-1 *correspondence between PIM's Q_λ and irreducible modular representations M_λ $(0 \leqq \lambda < p)$, Q_λ occurring as often as the degree of M_λ.*

Pollack [**18**] studied these PIM's in detail and found (cf. (10.2), (10.6)):

(11.2) $Q_{p-1} = M_{p-1}$ *(Steinberg representation). If $0 \leqq \lambda < p - 1$, then Q_λ has degree $2p$; its composition factors are M_λ and $M_{p+2-\lambda}$, each repeated twice.*

These results were explained (and generalized) by Humphreys [**10**] (cf. Verma [**26**]), in a way which makes plain the role of the *Weyl group*. (For example, the 2 in $2p$ results from the fact that W has order 2.) Here W appears as a group of order 2 acting on the subspace $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$: the nontrivial element of W sends h to $-h$. W acts equally on the numbers λ, which in Lie theory are viewed as linear functions on $\mathfrak{h}$ and called **weights**: λ is sent to $\pm\lambda$. The weight 1 plays a special role and gets the special name δ. Now define weights λ, μ to be **linked** if $w(\lambda+\delta) \equiv (\mu+\delta) \pmod p$ for some $w \in W$. This is an equivalence relation, with equivalence classes:

(11.3) $$\{0, p-2\},\ \{1, p-3\}, \cdots, \{p-1\}.$$

It is no accident that these pairs (or the single weight $p-1$) also occur in (11.2):

(11.4) *If M_λ and M_μ occur as composition factors of any indecomposable representation of $\mathfrak{g}$, then λ and μ must be linked.*

In effect, then, each PIM Q_λ ($\lambda \neq p-1$) involves a single linkage class, repeated twice. And in each such case, $\dim M_\lambda + \dim M_{p-2-\lambda} = p$.

12. Comparison of PIM's. The results of Section 11 are pleasant, but of course one has to ask what bearing they have on G. Recent work of Jeyakumar [**15**], Humphreys [**11, 12**], Humphreys and Verma [**13**], reveals the following pattern: G and $\mathfrak{g}$ share the same irreducible representations M_λ. They *almost* share the same PIM's — but not quite, since the regular representation of G has degree p^3-p while that of $\mathfrak{g}$ has degree p^3 (cf. (10.1), (11.1)). In fact Q_λ may be constructed as a summand of a suitable "tensor product" $M_\mu \otimes M_{p-1}$; the latter is also a representation of G, and it turns out that Q_λ is stable under the action of G. What is more:

(12.1) *Q_λ (viewed as a representation of G) involves R_λ as a summand. In particular,* $\dim R_\lambda \leqq \dim Q_\lambda \leqq 2p$.

By counting dimensions and using the last assertion of (10.1), the reader will see quickly that all but one R_λ must coincide with Q_λ (viewed as a representation of G). The odd case turns out to be: $Q_0 = R_0 + R_{p-1}$.

The tensor product involved here begins to shed light on the mysterious formula (7.1), and points up the pivotal role of the Steinberg representation (which is unique in being both irreducible and a PIM). On a more practical level, the tensor product construction yields precise information about R_λ, obtained earlier only by resort to Brauer's theory or a knowledge of ordinary characters (cf. [**11**], [**15**]):

(12.2) *The composition factors of R_λ ($0 \leqq \lambda < p-1$) are: M_λ, $M_{p-1-\lambda}$, M_λ, $M_{p-3-\lambda}$ (the last omitted if $p-3-\lambda < 0$).*

In particular, the matrix C can now be written down explicitly (for any given p). Evidently p plays no essential role in the general pattern: there are three diagonal

blocks, one the 1×1 matrix (1) corresponding to the Steinberg representation, the other two of the form illustrated in Table 7 (entries not shown being 0).

$$\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & 1 & 2 & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & \cdot & & & \\ & & & & 2 & 1 & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 3 \end{bmatrix}$$

TABLE 7. A block of C

With a little effort, the reader can use the equation (10.4) to reconstruct the matrix D, whose entries are all 0 or 1 (cf. Brauer and Nesbitt [**2**, p. 590]). The fact that D can be recovered in one and only one way from C is a special fact about $\mathbf{SL}(2, p)$, which fails for most groups (but does seem to persist for some other groups of Lie type). At any rate, without using the results of Section 7, most of the character table of G for a given p can in principle be written down using just the modular theory! We shall see below how well it can be done in practice.

It has to be added that Brauer, Dade, and others have been able to derive these results about the modular theory of $\mathbf{SL}(2, p)$ in another (very different) way, by means of a deep general theory (cf. Dornhoff [**6**, Part B, §71], Alperin and Janusz [**1**]). But this general theory does *not* apply to other groups of Lie type, so we avoid discussing it here.

13. The role of the Weyl group. The calculation of C (and then D) sketched in Section 12 does not yet explain adequately the regularities encountered earlier.

A closer look at D (cf. (12.2)) reveals that M_λ, $M_{p+1-\lambda}$ (resp. M_λ, $M_{p+3-\lambda}$) occur together in the decomposition of some ordinary character of degree $p+1$ (resp. $p-1$). It is suggestive to view each pair of weights $\{\lambda,\ p-1-\lambda\}$ or $\{\lambda,\ p-3-\lambda\}$ as a **deformation** of the linkage class $\{\lambda,\ p-2-\lambda\}$. These two deformations can be assigned in a precise way, *independent of p*, to the two elements (or conjugacy classes) of W, as follows. Say $W = \{e, w\}$, $w^2 = e$. Set $\delta_e = 0$, $\delta_w = \delta\ (= 1)$. Then:

(13.1) *The element e deforms the linkage class* $\{\lambda,\ p-2-\lambda\}$ *by adding* $e(\delta_e)=0$ to λ and $e(\delta_w)=1$ *to* $p-2-\lambda$. *The element w deforms this class by adding* $w(\delta_e)=0$ *to* λ *and* $w(\delta_w)=-1$ *to* $p-2-\lambda$.

From this perspective, the dimension polynomial $p=\dim M_\lambda+\dim M_{p-2-\lambda}$ is being "deformed" to yield either $p+1$ or $p-1$. In defense of this formulation, which appears at first sight artificial and overly elaborate, we remark that (a) it describes the observed facts, (b) it generalizes to some other groups of Lie type, (c) it ties in with the suspicion on other grounds that the "series" of ordinary characters found in Section 7 are somehow connected with the conjugacy classes of W. (The classes of W are known in any case to lead to the two families of p-regular classes described in Section 6.)

The above pairs of weights also tie in neatly with the simple form taken by the actual *character values* on p-regular classes (Table 5). An example should make this clear. Take $p=13$, $\lambda=8$. Then

$$a=\begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix}$$

is represented on M_λ by a diagonal matrix whose diagonal entries are the following powers of v: 8, 6, 4, 2, 0, -2, -4, -6, -8. Replacing v by τ, a primitive $(p-1)$-th root of 1 in $\mathbf{C}$, we see that the Brauer character of M_λ assigns to a the number: $\tau^8+\tau^6+\cdots+\tau^{-8}$. Since $\tau^{12}=1$, the exponents here can equally well be thought of as: 8, 6, 4, 2, 0, 10, 8, 6, 4. Now the other weight in the pair $\{\lambda, p-1-\lambda\}$ is 4, and similar reasoning leads to the exponents: 4, 2, 0, 10, 8. So the Brauer character of M_8+M_4 assigns to a the corresponding sum of powers of τ. But each even exponent from 0 to 10 occurs in the combined list exactly twice, except that 8 and 4 each occur three times. Since τ is a root of $X^{12}-1$, but not of X^2-1, τ is also a root of

$$(X^{12}-1)/(X^2-1)=X^{10}+X^8+X^6+X^4+X^2+1.$$

So most terms in the Brauer character add up to 0, leaving just: $\tau^8+\tau^4$ $(=\tau^{-4}+\tau^4)$.

On the other hand, the Brauer character of M_8 assigns to the element b a sum of powers of σ, a primitive $(p+1)$-th root of 1, the exponents (modulo 14) being: 8, 6, 4, 2, 0, 12, 10, 8, 6. For M_4 the exponents are: 4, 2, 0, 12, 10. Here the combined list contains each even exponent from 0 to 12 exactly twice. But σ is a root of $X^{12}+X^{10}+\cdots+1$, so the Brauer character assigns to b the value 0. This is not unexpected (if we know Table 5), since the preceding paragraph already showed that the *ordinary character* in question must be ζ_4.

The other pair to which the weight 8 belongs is $\{8, 2\}$. Here a is assigned the list of exponents (modulo 12): 8, 6, 4, 2, 0, 10, 8, 6, 4 and 2, 0, 10. So the Brauer character has value 0 at a. On the other hand, b is assigned exponents (modulo 14): 8, 6 ,4, 2, 0

12, 10, 8, 6 and 2, 0, 12. Here each even exponent occurs twice, except that 10 and 4 only occur once. So the Brauer character assigns to b the value $-(\sigma^{10}+\sigma^4)$. We recognize the ordinary character in question as θ_4.

14. Conclusion. The reader who has persisted this far might want to look at some of the cited references in order to see how the actual *proofs* go, and how to treat the groups $\mathbf{SL}(2, p^n)$ as well. Admittedly, there is a lot of general theory mixed into the study of these particular representations. But, in compensation, there are still many fascinating open questions about the other groups of Lie type, posed within this same framework. The reader may want to have a hand in settling them.

References

1. J. L. Alperin, G. J. Janusz, Resolutions and periodicity, Proc. Amer. Math. Soc., 37 (1973) 403–406.

2. R. Brauer, C. Nesbitt, On the modular characters of groups, Ann. of Math., 42 (1941) 556–590.

3. R. W. Carter, Simple Groups of Lie Type, Interscience, New York, 1972.

4. C. W. Curtis, Representations of Lie algebras of classical type with application to linear groups, J. Math. Mech., 9 (1960) 307–326.

5. C. W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras Interscience, New York, 1962.

6. L. Dornhoff, Group Representation Theory, M. Dekker, New York, 1971 (Part A): 1972 (Part B).

7. S. I. Gel'fand, Representations of the full linear group over a finite field, Math. USSR-Sb., 12 (1970) 13–39.

8. J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc., 80 (1955) 402–447.

9. Harish-Chandra, Eisenstein series over finite fields, in: Functional Analysis and Related Fields, ed. F. E. Browder, Springer, New York, 1970.

10. J. E. Humphreys, Modular representations of classical Lie algebras and semisimple groups, J. Algebra, 19 (1971) 51–79.

11. ———, Projective modules for SL $(2,q)$, J. Algebra, 25 (1973) 513–518.

12. ———, Some computations of Cartan invariants for finite groups of Lie type, Comm. Pure Appl. Math., 26 (1973) 745–755.

13. J. E. Humphreys, D.-N. Verma, Projective modules for finite Chevalley groups, Bull. Amer. Math. Soc. 79 (1973) 467–468.

14. G. J. Janusz, Indecomposable modules for finite groups, Ann. of Math., 89 (1969) 209–241.

15. A. V. Jeyakumar, Principle indecomposable representations for the group SL $(2,q)$, J. Algebra, 30 (1974) 444–458.

16. H. Jordan, Group characters of various types of linear groups, Amer. J. Math., 29 (1907) 387–405.

17. M. Klemm, Charakterisierung der Gruppen PSL $(2,p^f)$ und PSU $(3,p^{2f})$ durch ihre Charaktertafel, J. Algebra, 24 (1973) 127–153.

18. R. D. Pollack, Restricted Lie algebras of bounded type, Bull. Amer. Math. Soc., 74 (1968) 326–331.

19. I. Schur, Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math., 132 (1907) 85–137.

20. A. J. Silberger, An elementary construction of the representations of SL(2, $GF(q)$), Osaka J. Math., 6 (1969) 329–338.

21. T. A. Springer, Cusp forms for finite groups; Characters of special groups; in: Seminar on Algebraic Groups and Related Finite Groups, Lect. Notes in Math. 131, Springer, New York, 1970.

22. B. Srinivasan, On the modular characters of the special linear group SL $(2, p^n)$, Proc. London Math. Soc. (3), 14 (1964) 101–114.

23. R. Steinberg, Prime power representations of finite linear groups II, Canad. J. Math., 9 (1957) 347–351.

24. ———, Representations of algebraic groups, Nagoya Math. J., 22 (1963) 33–56.

25. S. Tanaka, Construction and classification of irreducible representations of the special linear group of the second order over a finite field, Osaka J. Math., 4 (1967) 65–84.

26. D.-N. Verma, Role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, in: Proc. 1971 Summer School on Group Representations, ed. I.M. Gel'fand, to appear.

27. R. W. van der Waall, Remarks on the Artin L-functions of the groups $GL_2(F_3)$ and $SL_2(F_3)$, Indag. Math., 35 (1973) 41–48.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answer should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

17. Can anyone supply the Monthly with a complete list of programmed or individualized self-instruction texts in arithmetic, algebra, geometry and calculus for use in developmental classes in two or four-year colleges?

18. A. B. Willcox. Can anyone supply the name and address of a firm that manufactures mathematical models for educational use in university courses? Until the middle 60's such models were available from a German firm, Rudolf Stoll KG, but this firm seems to have gone out of business.

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061.

COUNTING TREES IN A CERTAIN CLASS OF GRAPHS

D. J. KLEITMAN AND B. GOLDEN

Abstract. The number of trees contained in a certain class of graphs is obtained by a simple argument. The graphs can be represented by choosing the integers from 1 to n as vertices and connecting vertices whose difference mod n is one or two. The answer is n times the square of the nth Fibonacci number.

A tree is a graph that is connected, and contains no cycles.

A famous theorem of Cayley states that there are n^{n-2} different trees on n labelled vertices. One can phrase the question to which this statement is an answer in the following manner: How many trees on the same set of vertices are contained in the complete graph on n labelled vertices? In this form, one can raise the same question about any graph: Given a graph G how many trees on its (labelled) vertices does it contain? The same number tells how many distinct bases (maximal linearly independent sets) there are in the matroid defined by the graph.

The number in question can be expressed in several convenient ways. Thus if one writes the negative of the adjacency matrix for G, with the degree of the ith vertex as the ith diagonal, the number is equal to any cofactor of the resulting matrix. However, computation of this number for any particular graph represents a challenge. Moon has gathered and described seven or eight different tractable approaches to the Cayley problem. There are at least as many potential approaches to any other specific problem, and finding one that can be readily applied for a given graph can be an interesting exercise.

In this note we raise this question about the graph whose vertices are the integers from one to n and whose edges are all pairs of integers whose difference (mod n,) is one or two. [The result obtained below was conjectured by Bedrosian.] We show that the number, X_n of labelled trees contained in this graph on n vertices is given by n times the square of the nth Fibonacci number, f_n, at least for $n \geqq 5$. For $n < 5$ a similar result can be obtained, but for a "graph" with multiple edges.

One such graph for n even and one for n odd are illustrated in Figures 1 and 2.

In contrast with phrasing the problem in terms of the cofactor of a matrix, the approach followed below yields a relatively easy evaluation of X_n.

We first compute the number of trees in a simpler graph, namely that for which, on the same vertices, we allow only edges between pairs whose differences (not

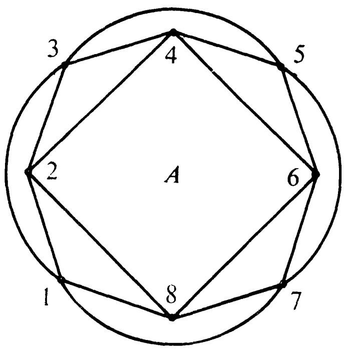

FIG. 1: $n = 8$

B

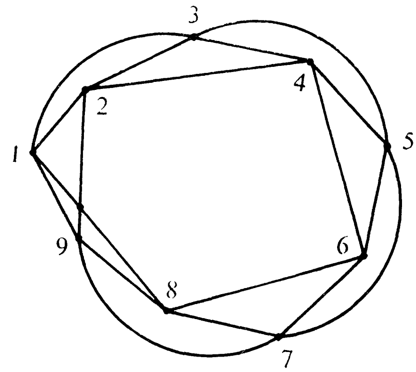

FIG. 2: $n = 9$

mod n) are one or two. Such a graph is illustrated in Figure 3 for $n = 10$. We will refer below to graphs defined by this property as "strip graphs." We show that the number, $F(n)$, of labelled trees in a strip graph on n vertices is the $(2n-2)$nd Fibonacci number, f_{2n-2}. We then show that the answer to the "strip graph" question supplies an answer to our original question.

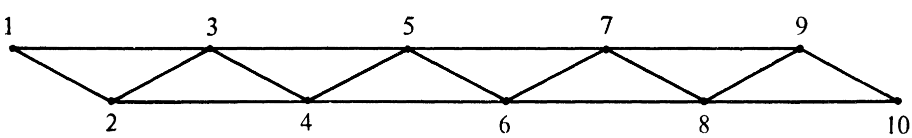

FIG. 3: $n = 10$ strip graph

LEMMA 1: $F(n) = f_{2n-2}$.

Proof. Let $G(n)$ be the number of labelled trees in the n vertex "strip graph" that contain the edge $(n-1, n)$. Since every tree must connect the vertex (n) to the rest of the graph, if a tree fails to contain the edge $(n-1, n)$ it must contain the edge $(n-2, n)$ and must contain a tree on the vertices other than n. Thus we have $F(n) = G(n) + F(n-1)$, where the first term counts the trees containing $(n-1, n)$ and the second counts the rest.

A tree containing the edge $(n-1, n)$ that fails to contain $(n-2, n)$ will likewise be a tree on the $n-1$ vertices other than n. On the other hand, a tree containing both $(n-1, n)$ and $(n-2, n)$ will look, as far as the first $n-1$ vertices are concerned, just like a tree containing the edge $(n-2, n-1)$. We therefore have

$$G(n) = F(n-1) + G(n-1),$$

where the first term counts trees not containing $(n-2, n)$ and the second trees containing it.

The F's and G's together arranged in the sequence $\cdots F(n), G(n+1), F(n+1), G(n+2), F(n+2), G(n+3) \cdots$ therefore obey the well-known Fibonacci recursion, $f_n = f_{n-1} + f_{n-2}$, and by examining the small n cases ($F(2) = G(2) = 1$) we can deduce that $F(n)$ is the $(2n-2)$nd Fibonacci number f_{2n-2}, (here $f_1 = f_2 = 1$), while $G(n)$ is the $(2n-3)$rd.

We now relate this result, in the even n case, to the desired tree counting for our original class of graphs. We show that for even n, X_n the number of trees in our original n vertex graph, is given by

$$n(F_n + F_{n-2} + \cdots + F_2).$$

LEMMA 2. $X_n = n(F_n + F_{n-2} + \cdots + F_2) = n(f_2 + f_6 + \cdots + f_{2n-2})$.

Proof. By definition no tree can contain a cycle. Thus, in Figure 1, given any tree T there must be a path in the plane from the interior of the diagram (A) to the outside region (B) which intersects no edge of T. Consider any two such A-B paths. If we reverse one and join their ends at (A) and also their ends at (B), we obtain a circuit in the plane which does not intersect any edge of the tree. All vertices of the graph must lie on the same side of this circuit or else our tree T (which fails to intersect the circuit) would not be connected. Without loss of generality we can assume that it is the inside of the circuit that contains no vertex since this can always be arranged by suitably connecting the path ends in (B). This implies that the circuit can be shrunk continuously to a point without touching a vertex in the process. This in turn implies that every edge of our graph crossed by the circuit is crossed an even number of times, which leads us, finally, to the conclusion that the graph edges crossed an odd number of times by an interior to exterior path disjoint from T are path independent for a given tree T: Any two such A-B paths for any tree T must cross exactly the same edges an odd number of times. Thus every tree may be associated uniquely with the set of edges crossed an odd number of times by paths in the plane from interior to exterior which don't intersect it. We now ask, what sets of edges can be crossed an odd number of times by such interior to exterior paths? Referring to a drawing of the graph as in Figure 1, let us call an edge which bounds A an interior edge, an edge which bounds B an exterior edge, and all other edges straight edges. Now at interior to exterior path must cross (an odd number of times) (1) some interior edge, (2) a sequence of an odd number, $(2k+1)$, of successive straight edges, and (3) an exterior edge. If we examine the edges that can lie in trees associated with such a path, we find that these must be trees in a "strip graph" having $n-2k$ vertices, along with two k length tails. That is, if we delete the edges crossed an odd number of times by our path, the edges left form a "strip graph" with two pendant k length tails.

For each interior-exterior path characterized by parameter k, there will thus be $F(n-2k)$ or $f_{2n-4k-2}$ associated trees. They will be the $F(n-2k)$ trees definable on the "strip graph" associated with the path, along with the two attached tails.

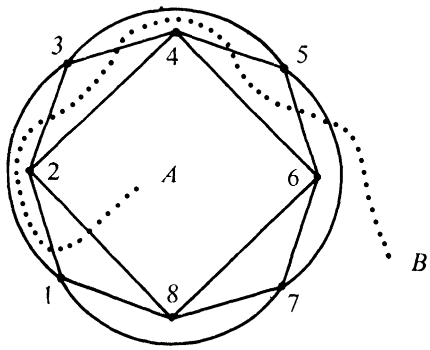

FIG. 4: A $k = 2$ path example for the graph of Fig. 1

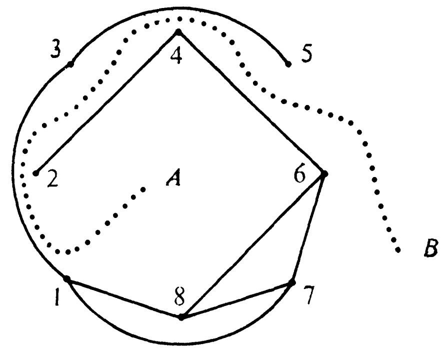

FIG. 5: Result of eliminating edges intersecting path in Fig. 4

In Figure 4, we have an $n = 8$ example. The path A-B is an interior to exterior path in the plane that intersects an interior edge (2–8), five intermediate edges, (1,2), (2,3), (3,4), (4,5), (5,6), and an exterior edge, (5,7), of the given graph. Thus it represents a $k = 2$ example of an interior-exterior path. A tree in our graph which fails to intersect this path must contain the two tails (1,3), (3,5) and (2,4), (4,6), and must have its remaining edges form a tree in the "strip graph" defined by the edges (1,8)(8,7)(7,6) (8,6) (1,7) on the vertices 1, 8, 7, 6, as illustrated in Figure 5.

We now count the number of such paths.

Each interior-exterior path must cross (an odd number of times) some straight edge first in going from inside to outside. There is complete symmetry among the n straight edges so that the number of paths is n times the number which first pass through any particular straight edge. The "oddly crossed" edges of the path are uniquely specified once this edge is chosen and k is fixed. We therefore have $nF(n-2k)$ trees for each value of k, and the number of trees in the entire graph is therefore the sum over all k of this quantity.

We may therefore conclude that, for even n, the desired number of trees is $\sum_{k=0}^{(n/2)-1} nF(n-2k)$ or, by the evaluation of F already discussed, $X_n = n(f_2 + f_6 + \cdots + f_{2n-2})$.

LEMMA 3. *For odd n, $X_n = n(1 + f_4 + \cdots + f_{2n-6} + f_{2n-2})$.*

Proof. For odd n, one can reason as for even n, except that the graph in question is non-planar. Thus one cannot simply draw it in the plane and refer to an interior region (A) and an exterior region (B). However, we can draw the graph in the plane with one crossing, as illustrated in Fig. 2, and reason as above with respect to this drawing. Again, one can associate trees with interior to exterior paths, but must allow the paths to intersect any tree edge an even number of times; one can characterize such paths and count the number of trees associated with each path. Though it is not obvious, the resulting countings work out just as in the even case; interior-

exterior paths may be characterized by a parameter k and trees that intersect these an even number of times must be trees on a "strip graph" of $n-2k$ vertices, along with two k length tails. One can again verify that there are $F(n-2k)$ trees associated with each parameter k path. One finally obtains the equation

$$X_n = n \sum_{k=0}^{(n-1)/2} F(n-2k).$$

Here $F(1)$ counts the number of trees for a given $k=(n-1)/2$ path that are all tail. Thus we have $F(1)=1$, and for odd n,

$$X_n = n(1+f_4+f_8+\cdots+f_{2n-6}+f_{2n-2}).$$

THEOREM. $X_n = nf_n^2$.

Proof. The expression in Lemma 2 can be reduced to $n(f_n)^2$ by the following inductive argument. We note that $f_2 = 1 = f_2^2$, $f_2+f_6 = 1+8 = 9 = f_4^2$, $f_2+f_6+f_{10} = 1+8+55 = 64 = f_6^2$. Suppose that for n even, we have $f_2+f_6+\cdots+f_{2(n-2)-2} = f_2+f_6+\cdots+f_{2n-6} = f_{n-2}^2$. We need to show that $f_2+f_6+\cdots+f_{2n-6}+f_{2n-2} = f_n^2$. By the Fibonacci formula $f_{n+m} = f_{n-1}f_m+f_nf_{m+1}$ (see e.g.[2]), we have $f_{2n} = f_{n-1}f_n+f_nf_{n+1} = f_n(f_{n-1}+f_{n+1}) = (f_{n+1}-f_{n-1})(f_{n+1}+f_{n-1}) = f_{n+1}^2-f_{n-1}^2$. Thus we have $f_{2n-2} = f_n^2-f_{n-2}^2$ which gives the desired formula. The odd n argument can proceed similarly and is left to the reader.

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References

1. J. Moon in "A Seminar on Graph Theory" ed. by F. Harary, Holt, New York, 1968.
2. N. Varobyov, Fibonacci Numbers, translated and adapted by N. Whaland Jr. and Olga Titelbaum, Heath, Boston 1963.

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A GENERALIZATION OF A THEOREM OF GAUSS ON SUMS INVOLVING [x]

BRUCE C. BERNDT

The proof of the law of quadratic reciprocity in most texts on elementary number theory combines Gauss's lemma with the relation

$$\sum_{r=1}^{\frac{1}{2}(q-1)} [pr/q] + \sum_{r=1}^{\frac{1}{2}(p-1)} [qr/p] = \tfrac{1}{4}(p-1)(q-1), \tag{1}$$

where p and q are relatively prime, positive integers. (See, for example, [7], [**9**], or [**11**].) Formula (1) is generally attributed to G. Eisenstein, and, in fact, the geometric proof of (1) given in almost all texts on elementary number theory is due to Eisenstein [**4**]. Shortly before publishing his proof, Eisenstein had listed a set of five exercises in number theory [**3**], one of which is a generalization of (1). Solutions of all five exercises were given by both M. A. Stern [**12**] and P. Tardy [**14**].

However, (1) is a special case of a theorem of C. F. Gauss [**6**] proven several years earlier, and, in fact, Gauss gives (1) as a particular example of his theorem. The objective of this paper is to give two short, analytic proofs of a generalization of the aforementioned theorem of Gauss. Our result may be thought of as a type of reciprocity formula. As corollaries, we obtain several other results in the literature, and some new results as well, on sums involving $[x]$.

We shall find it easier to work with $((x))$ rather than $[x]$, where

$$((x)) = \begin{cases} x - [x] - \tfrac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA. *Let $a > 0$ and b be real numbers, and let n be a positive integer. Put $c = b - [b]$ and $d = an + c$. Then,*

$$\text{(2)} \quad I = I(a,b,n) \equiv \int_0^n ((ax + b))\,dx = \tfrac{1}{2}n(d + c - 1) + \tfrac{1}{2}[d]([d] - 2d + 1)/a.$$

Proof. Put $x = ny$ and write

$$\begin{aligned} I/n &= \int_0^1 ((any + c))dy \\ &= \int_0^{(1-c)/an} (any + c - \tfrac{1}{2})dy + \sum_{r=1}^{[d]-1} \int_{(r-c)/an}^{(r+1-c)/an} (any + c - r - \tfrac{1}{2})dy \\ &\qquad + \int_{([d]-c)/an}^{1} (any + c - [d] - \tfrac{1}{2})dy \\ &= \int_0^1 (any + c - \tfrac{1}{2})dy - \sum_{r=1}^{[d]-1} r/an - [d](1 - ([d] - c)/an) \\ &= \tfrac{1}{2}an + c - \tfrac{1}{2} - \tfrac{1}{2}[d]([d] - 1)/an - [d](d - [d])/an. \end{aligned}$$

A slight rearrangement of the above yields (2).

RECIPROCITY THEOREM: *Let a, b, c, d, n, and $I(a,b,n)$ be as above. For brevity, put $g(x) = ((ax + b))$. Then,*

$$\text{(3)} \quad \tfrac{1}{2}\{g(0+0) + g(n-0)\} + \sum_{r=1}^{n-1} ((ar + b)) + \sum_{r=1}^{[d]} \left(\left(\frac{r - c}{a}\right)\right) = I(a,b,n).$$

Observe that

$$g(0+0) = \begin{cases} ((b)), & \textit{if } b \textit{ is not an integer}, \\ -\frac{1}{2}, & \textit{otherwise}, \end{cases} \tag{4}$$

and

$$g(n-0) = \begin{cases} ((d)), & \textit{if } k \textit{ is not an integer}, \\ \frac{1}{2}, & \textit{otherwise}. \end{cases} \tag{5}$$

First proof: Let ε, $\delta > 0$ be chosen small enough so that $((x))$ and $((ax+b))$ have no discontinuities on $(0,\varepsilon]$ and $[n-\delta, n)$. Suppose that the common discontinuities (if any) of $((x))$ and $((ax+b))$ on $(\varepsilon, n-\delta)$ are at $x = x_1, \cdots, x_m$. Let $0 < \varepsilon_1, \cdots, \varepsilon_m < \min(\frac{1}{2}, 1/(2a))$ be chosen so that $I_j = [x_j - \varepsilon_j,\ x_j + \varepsilon_j] \subseteq [\varepsilon, n-\delta]$, $1 \leqq j \leqq m$. Let S be the complement in $[\varepsilon, n-\delta]$ of $\bigcup_{j=1}^m I_j$. At each discontinuity in S of $((x))$ or $((ax+b))$ there is a "jump" of -1. Hence,

$$\begin{aligned} &\int_S ((x))d((ax+b)) + \int_S ((ax+b))d((x)) \\ &= a\int_S ((x))dx - \sum_{r=[b]+1}^{[an+b]} \left(\left(\frac{r-b}{a}\right)\right) + \int_S ((ax+b))dx - \sum_{r=1}^{n-1} ((ar+b)). \end{aligned} \tag{6}$$

Observe that we have added to the two sums on the right side of (6) the terms arising from $x_1, \cdots, x_m$. However, recall that $x_1, \cdots, x_m$ are *common* discontinuities of $((x))$ and $((ax+b))$. Thus, x_j and $ax_j + b$ are both integers, $1 \leqq j \leqq m$. Hence

$$\left(\left(\frac{(ax_j+b)-b}{a}\right)\right) = ((ax_j+b)) = 0,\ 1 \leqq j \leqq m,$$

and so each of the terms that we have added to the right side of (6) is equal to zero. On the left side of (6), apply the integration by parts formula for Riemann-Stieltjes integrals to each of the integrals on the intervals

$$[\varepsilon, x_1 - \varepsilon_1],\ [x_1 + \varepsilon_1, x_2 - \varepsilon_2], \cdots, [x_{m-1} + \varepsilon_{m-1}, x_m - \varepsilon_m],\ [x_m + \varepsilon_m,\ n-\delta].$$

Then, letting $\varepsilon_1, \cdots, \varepsilon_m$ tend to zero, we find that after cancellation, the left side of (6) has the limit $((n-\delta))((a(n-\delta)+b)) - ((\varepsilon))((a\varepsilon+b))$. Hence, upon letting $\varepsilon_1, \cdots, \varepsilon_m$ tend to zero, we find that (6) becomes

$$\begin{aligned} &a\int_\varepsilon^{n-\delta} ((x))dx - \sum_{r=[b]+1}^{[an+b]} \left(\left(\frac{r-b}{a}\right)\right) + \int_\varepsilon^{n-\delta} ((ax+b))dr - \sum_{r=1}^{n-1} ((ar+b)) \\ &= ((n-\delta))((a(n-\delta)+b)) - ((\varepsilon))((a\varepsilon+b)). \end{aligned} \tag{7}$$

In the first sum on the left side of (7), let $r = j + [b]$. Now let both ε and δ tend to 0 in (7). Using (2), (4), (5) and the fact that

$$\int_0^n ((x))dx = 0,$$

we find that (7) becomes

$$-\sum_{j=1}^{[d]}\left(\left(\frac{j-c}{a}\right)\right)+I(a,b,n)-\sum_{r=1}^{n-1}((ar+b))$$
$$=\tfrac{1}{2}\{g(n-0)+g(0+0)\},$$

which is equivalent to (3).

Second Proof: Our second proof is based on the classical Poisson summation formula [2, p. 15]. If f is of bounded variation on $[A,B]$,

$$\text{(8)}\qquad \tfrac{1}{2}\sum_{r=A}^{B}{}'\{f(r+0)+f(r-0)\} = \int_A^B f(x)dx + 2\sum_{m=1}^{\infty}\int_A^B f(x)\cos(2\pi mx)dx,$$

where the dash $'$ on the summation sign indicates that for the terms corresponding to $n = A$ and to $n = B$, only $f(A+0)$ and $f(B-0)$, respectively, are counted. Let $A = 0$, $B = n$, and $f(x) = g(x) = ((ax+b))$. Using (2), we find that (8) becomes

$$\text{(9)}\qquad \tfrac{1}{2}\{g(0+0)+g(n-0)\} + \sum_{r=1}^{n-1}((ar+b)) = I(a,b,n) + 2\sum_{m=1}^{\infty}J(a,b,m,n),$$

where

$$J(a,b,m,n) = \int_0^n ((ax+b))\cos(2\pi mx)dx.$$

To evaluate $J(a,b,m,n)$, put $x = ny$ and write

$$\begin{aligned}
J(a,b,m,n) &= n\int_0^1 ((any+c))\cos(2\pi mny)\,dy\\
&= n\int_0^1 (any+c-\tfrac{1}{2})\cos(2\pi mny)dy - n\sum_{r=1}^{[d]-1} r\int_{(r-c)/an}^{(r+1-c)/an}\cos(2\pi mny)dy\\
&\qquad - n[d]\int_{([d]-c)/an}^{1}\cos(2\pi mny)dy\\
&= -n\sum_{j=1}^{[d]-1}\int_{(j-c)/an}^{([d]-c)/an}\cos(2\pi mny)dy + \frac{[d]}{2\pi m}\sin(2\pi m([d]-c)/a)\\
&= \frac{1}{2\pi m}\sum_{j=1}^{[d]}\sin(2\pi m(j-c)/a).
\end{aligned}$$

Hence,

$$\text{(10)}\qquad 2\sum_{m=1}^{\infty}J(a,b,m,n) = \sum_{j=1}^{[d]}\sum_{m=1}^{\infty}\frac{\sin(2\pi m(j-c)/a)}{\pi m} = -\sum_{j=1}^{[d]}\left(\left(\frac{j-c}{a}\right)\right),$$

where we used the well-known Fourier series

$$((x)) = -\sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{\pi m}.$$

Upon substituting (10) into (9), we arrive at (3), and the second proof is complete.

In some of the corollaries below, the hypotheses in the cited references are a bit more restrictive than ours.

COROLLARY 1. *For $a > 0$ and $d = an$,*

$$(11) \quad \sum_{r=1}^{n} ((ar)) + \sum_{r=1}^{[d]} ((r/a)) = \tfrac{1}{2}n(d-1) + \tfrac{1}{2}[d]([d] - 2d + 1)/a + \tfrac{1}{2}(d - [d]).$$

Proof. The above is an easy consequence of the reciprocity theorem and the lemma if b is an integer.

COROLLARY 2. *Let ρ be the number of values of r, $1 \leqq r \leqq n$, such that ar is an integer. Then,*

$$(12) \qquad \sum_{r=1}^{n} [ar] + \sum_{r=1}^{[an]} [r/a] = n[an] + \rho.$$

Proof. If k is an integer such that $1 \leqq k \leqq n$ and $ak = j$ is an integer, then $j/a = k$ is an integer such that $1 \leqq j \leqq [an]$, and conversely. Hence,

$$\begin{aligned}(13) \quad & \sum_{r=1}^{n} ((ar)) + \sum_{r=1}^{[d]} ((r/a)) \\ &= \sum_{r=1}^{n} (ar - [ar] - \tfrac{1}{2}) + \sum_{r=1}^{[d]} (r/a - [r/a] - \tfrac{1}{2}) + \rho \\ &= \tfrac{1}{2}an(n+1) - \tfrac{1}{2}n + \tfrac{1}{2}[d]([d]+1)/a - \tfrac{1}{2}[d] - \sum_{r=1}^{n} [ar] - \sum_{r=1}^{[an]} [r/a] + \rho.\end{aligned}$$

On comparing (13) with (11), we arrive at (12) forthwith.

In the case $\rho = 0$, Corollary 2 was first proved by Gauss [**6**].

COROLLARY 3. *Let $e, f, k, l, m, p, q, (p-e)/m$ and $(q-f)/l$ be positive integers with $ke \leqq m$ and $kf \leqq l$. Then,*

$$(14) \qquad \sum_{r=1}^{k(q-f)/l} [plr/qm] + \sum_{r=1}^{k(p-e)/m} [qmr/pl] = \frac{k^2(q-f)(p-e)}{lm} + \rho,$$

where ρ is the number of values of r, $1 \leqq r \leqq k(q-f)/l$, such that plr/qm is an integer.

Proof. In Corollary 2, put $n = k(q-f)/l$ and $a = pl/qm$. An elementary calculation shows that $an \geqq k(p-e)/m$ if and only if $qe \geqq pf$, which we may assume

without loss of generality by symmetry. On the other hand, it is readily seen that $an - 1 < k(p-e)/m$ since $ke \leqq m$. Thus, $[an] = k(p-e)/m$. Equation (14) is now immediate from Corollary 2.

If we set $e = f = k = 1$ and $l = m$ in (14), we arrive at the following result.

COROLLARY 4. *Let m, p, q, $(p-1)/m$ and $(q-1)/m$ be positive integers, and let ρ be as in Corollary 3. Then,*

$$(15) \qquad \sum_{r=1}^{(q-1)/m} [pr/q] + \sum_{r=1}^{(p-1)/m} [qr/p] = \frac{(q-1)(p-1)}{m^2} + \rho.$$

For $\rho = 0$, Corollaries 3 and 4 have a long history. Stern [12] was the first to point out that the results could be deduced from Gauss's theorem [6], and (14) was, indeed, proved by Stern (for $\rho = 0$). Equation (15) (for $\rho = 0$) is one of the aforementioned five exercises set forth by Eisenstein [3]. If we let $m = 2$ in (15), we obtain (1). If in Corollary 3 we put $l = m$ and $e = f = 1$, we obtain a result of J. J. Sylvester (for $\rho = 0$) [13]. See also the later paper of J. Hacks [8].

The next corollary is another special case of Corollary 2.

COROLLARY 5. *Let $\lambda > 0$ and let p and q be positive integers. Let ρ be the number of values of r, $1 \leqq r \leqq [\lambda q]$, such that pr/q is an integer. Then,*

$$\sum_{r=1}^{[\lambda q]} [pr/q] + \sum_{r=1}^{[\lambda p]} [qr/p] = [\lambda p][\lambda q] + \rho.$$

Proof. In Corollary 2, put $n = [\lambda q]$ and $a = p/q$. Assume that $q[\lambda p] \leqq p[\lambda q]$ which we may do without loss of generality because of symmetry. It is easy to see that $[an] \leqq [\lambda p]$. On the other hand, suppose that $[an] \leqq [\lambda p] - 1$. Then if $\{x\}$ denotes the fractional part of x,

$$p\lambda - \frac{p}{q}\{\lambda q\} - 1 < \left[p\lambda - \frac{p}{q}\{\lambda q\}\right] = \left[\frac{p}{q}[\lambda q]\right] \leqq \lambda p - \{\lambda p\} - 1.$$

Thus, $q\{\lambda p\} < p\{\lambda q\}$, or $p[\lambda q] < q[\lambda p]$, which is contrary to our assumption $q[\lambda p] \leqq p[\lambda q]$. Hence, $[an] = [\lambda p]$. Corollary 5 is now immediate from Corollary 2.

In the case $\rho = 0$, Corollary 5 was first proved by Sylvester [13]. Stern [12] also gave a proof of Sylvester's result. In full generality, Corollary 5 was first established by Hacks [8].

COROLLARY 6. *Let an be an integer. Then,*

$$\sum_{r=1}^{n} ((ar+b)) + \sum_{r=1}^{an} \left(\left(\frac{r-c}{a}\right)\right) = 0.$$

Proof. Corollary 6 follows readily from the reciprocity theorem with the use of (4), (5), and the Lemma. Note that by the Lemma, $I(a,b,n) = 0$. But this is also

easily seen geometrically without the Lemma from considerations of periodicity and symmetry.

Corollary 6 takes another interesting shape if we let $a = p/n$ and $b = q/n$, where p is a positive integer and $0 < q < n$. Then, we get

$$\sum_{r=1}^{n} \left(\left(\frac{pr+q}{n}\right)\right) + \sum_{r=1}^{p} \left(\left(\frac{nr-q}{p}\right)\right) = 0. \tag{16}$$

If q is an integer,

$$\sum_{r=1}^{n} \left(\left(\frac{pr+q}{n}\right)\right)$$

has been evaluated in closed form by J. M. Gandhi and K. S. Williams [**5**], and in this case (16) would follow trivially from their result.

COROLLARY 7. *Let a and b be positive integers and x an arbitrary real number. Then,*

$$\sum_{r=0}^{b-1} \left(\left(\frac{ar}{b} + ax\right)\right) = \sum_{r=0}^{a-1} \left(\left(\frac{br}{a} + bx\right)\right).$$

Proof. In the reciprocity theorem, replace a by a/b and b by ax. Let $n = b$. By symmetry considerations, $I = 0$. We then arrive at

$$\sum_{r=0}^{b-1} \left(\left(\frac{ar}{b} + ax\right)\right) + \sum_{r=1}^{a} \left(\left(\frac{br - (ax - [ax])b}{a}\right)\right) = 0. \tag{17}$$

Letting $r = j - [ax]$, we find that

$$\begin{aligned}\sum_{r=1}^{a} \left(\left(\frac{br - (ax - [ax])b}{a}\right)\right) &= \sum_{j=1+[ax]}^{a+[ax]} \left(\left(\frac{bj - abx}{a}\right)\right) \\ = \sum_{j=1}^{a} \left(\left(\frac{bj}{a} - bx\right)\right) &= -\sum_{r=0}^{a-1} \left(\left(\frac{br}{a} + bx\right)\right),\end{aligned}$$

where we have used the fact that $((x))$ has period 1 and lastly let $j = a - r$ and used the oddness of $((x))$. Putting the above in (17), we have completed the proof of Corollary 7.

In a straightforward fashion, we may derive from Corollary 7 the following result stated as a problem by E. Cesaro [**1**, Question 45, p. 560]:

$$\sum_{r=0}^{b-1} \left[\frac{ar}{b} + ax\right] = \sum_{r=0}^{a-1} \left[\frac{br}{a} + bx\right].$$

For further references on sums involving $[x]$, see the *Encyclopédie* [**10**] edited by J. Molk.

Professor Wolfgang Jurkat has pointed out to the author that the introduction of ε, δ, $\varepsilon_1, \cdots, \varepsilon_m$ in the first proof of the reciprocity theorem is not necessary if one uses Lebesgue-Stieltjes integrals rather than Riemann-Stieltjes integrals. However, in order to keep the exposition at a more elementary level, the author has decided still to use Riemann-Stieltjes integrals.

References

1. E. Cesaro, Questions proposées par M. E. Cesaro, Nouv. Ann. Math., (3) 4 (1885) 556–565.

2. H. Davenport, Multiplicative number theory, Markham Publ., Chicago, 1967.

3. G. Eisenstein, Aufgaben und Lehrsätze, J. Reine Angew. Math., 27 (1844) 281–284.

4. ———, Geometrischer Beweis des Fundamentaltheorems für die quadratischen Reste, J. Reine Angew. Math., 28 (1844) 246–248.

5. J. M. Gandhi and K. S. Williams, On certain sums of fractional parts, Archiv der Math., 25 (1974) 41–44.

6. C. F. Gauss, Theormatis Arithmetici, Werke, vol. 2, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1876, 3–8.

7. E. Grosswald, The theory of numbers, Macmillan, New York, 1966.

8. J. Hacks, Über Summen von grössten Ganzen, Acta Math., 10 (1887) 1–52.

9. G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, fourth ed., Clarendon Press, Oxford, 1960.

10. J. Molk, editor, Encyclopédie des Sciences Mathématiques, Tome 1, Théorie des Nombres, Gauthier-Villars, Paris, 1906.

11. I. Niven and H. S. Zuckerman, An introduction to the theory of numbers, third ed., Wiley, New York, 1972.

12. M. A. Stern, Über einige Eigenschaften der Function Ex, J. Reine Angew. Math., 59 (1861) 146–162.

13. J. J. Sylvester, Sur la fonction $E(x)$, C. R. Acad. Sc. Paris, 50 (1860) 732–734.

14. P. Tardy, Sopra alcuni teoremi aritmetici, Ann. Mat. Pura Appl., (2) 3 (1869/70) 331–338.

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A COMBINATORIAL PROPERTY OF q-EULERIAN NUMBERS

L. Carlitz

1. We define polynomials $A_{mk} = A_{mk}(q)$ by means of

$$[x]^m = \sum_{k=1}^{m} A_{mk} \begin{bmatrix} x+k-1 \\ m \end{bmatrix} \qquad (m \geqq 1), \tag{1}$$

where $[x] = (1-q^x)/(1-q)$ and

$$\begin{bmatrix} x \\ m \end{bmatrix} = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Alternatively, if we define the rational function $H_m = H_m(x, q)$ by means of the

symbolic relation

$$(qH+1)^m = xH^m \qquad (m>1), \tag{2}$$

where it is understood that, after expansion of the left member, H^k is replaced by H_k, then we have

$$H_m(x,q) = A_m(x,q)\prod_{s=1}^{m}(x-q^s)^{-1},$$

where

$$A_m(x,q) = \sum_{k=1}^{m} A_{mk}x^{k-1} \qquad (m \geqq 1) \tag{3}$$

and the coefficients A_{mk} are the same as those occurring in (1). For $q=1$, A_{mk} reduce to the ordinary Eulerian numbers (see for example [**1**], [**4**, Ch. 8]).

It follows from (1) that A_{mk} satisfies the recurrence

$$A_{m+1,k} = [m-k+2]A_{m,k-1} + q^{m-k+1}[k]A_{m,k}. \tag{4}$$

Also it is easy to show that A_{mk} is divisible by $q^{(m-k)(m-k+1)/2}$. If we put

$$A_{mk} = q^{(m-k)(m-k+1)/2}A^*_{mk}, \tag{5}$$

then (4) becomes

$$A^*_{m+1,k} = [m-k+2]A^*_{m,k-1} + [k]A^*_{mk}. \tag{6}$$

Moreover,

$$A^*_{m,m-k+1} = A^*_{m,k} \tag{7}$$

and

$$\sum_{k=1}^{m} A_{mk} = [m]! = [m][m-1]\cdots[1]. \tag{8}$$

It follows from (6) that A^*_{mk} is a polynomial, in q of degree $(k-1)(m-k)$ with non-negative, integral coefficients. We may put

$$A_{mk} = \sum_{i=0}^{(k-1)(m-k)} a(m,k,i)q^i. \tag{9}$$

The object of the present note is to get a combinatorial interpretation of the coefficient $a(m,k,i)$. An equivalent result (due to John Riordan) was stated without proof in [**2**]; see also [**5**, p. 44].

2. Let π denote a permutation of $Z_n = \{1,2,\cdots,n\}$ with k *rises* and therefore $n-k+1$ *falls*. We count a conventional rise on the extreme left and a conventional fall on the extreme right. We shall label both rises and falls by the *positions* of their

left hand elements. Let the rises of π have the positions $i_0, i_1, \cdots, i_{k-1}$ and let the falls have the positions $j_1, j_2, \cdots, j_{n-k+1}$. For example, the permutation

$$.2.4_0 3_0 1.6_0 5_0 \tag{10}$$

has rises at positions 0, 1, 4 and falls at 2, 3, 5, 6.

Put

$$i = i_0 + i_1 + \cdots + i_{k-1}, \qquad j = j_1 + j_2 + \cdots + j_{n-k+1}, \tag{11}$$

so that

$$i + j = \tfrac{1}{2}n(n+1). \tag{12}$$

Let $\bar{a}(n,k,i)$ denote the number of permutations π of Z_n with k rises and i as defined in (11). Consider the effect of inserting the element $n+1$ in π. If it is inserted in the rise of position t, the number of rises remains unchanged but i becomes $i+k-t-1$. If, however, it is inserted in the fall of position t, then the number of rises becomes $k+1$ but the number of falls remains unchanged. Moreover, j becomes $j' = j + n - k - t + 2 = \frac{1}{2}(n+1)(n+2) - i - k - t + 1$, by (12). Hence i becomes

$$i' = \tfrac{1}{2}(n+1)(n+2) - j' = i + k + t - 1.$$

It follows that

$$\begin{aligned} \bar{a}(n+1,k,i) &= \sum_{t=0}^{k-1} \bar{a}(n,k,i-k+t+1) + \sum_{t=1}^{n-k+2} \bar{a}(n,k-1,i-k-t+2) \\ &= \sum_{t=0}^{k-1} \bar{a}(n,k,i-t) + \sum_{t=0}^{n-k+1} \bar{a}(n,k-1,i-n+t). \end{aligned} \tag{13}$$

If we put

$$\bar{A}_{n,k} = \sum_i \bar{a}(n,k,i)q^i, \tag{14}$$

it follows from (13) that

$$\begin{aligned} \bar{A}_{n+1,k} &= \sum_i q^i \sum_{t=0}^{k-1} \bar{a}(n,k,i-t) + \sum_i q^i \sum_{t=0}^{n-k+1} \bar{a}(n,k-1,i-n+t) \\ &= \sum_i \bar{a}(n,k,i)q^i \sum_{t=0}^{k-1} q^t + \sum_i \bar{a}(n,k-1,i)q^i \sum_{t=0}^{n-k+1} q^{n-t}. \end{aligned}$$

Thus

$$\bar{A}_{n+1,k} = [k]\bar{A}_{n,k} + q^{k-1}[n-k+2]\bar{A}_{n,k-1}. \tag{15}$$

Since $\bar{A}_{1,1} = A^*_{1\,1} = 1$, comparison of (15) with (6) yields

(16) $$\bar{A}_{n,k} = q^{\frac{1}{2}k(k-1)} A^*_{n,k}.$$

Then, by (5), (7) and (16), $A_{n,n-k+1} = q^{\frac{1}{2}k(k-1)} A^*_{n,n-k+1} = q^{\frac{1}{2}k(k-1)} A^*_{n,k}$, so that

(17) $$A_{n,n-k+1} = \bar{A}_{n,k}.$$

It follows that

(18) $$\bar{a}(n,k,i) = a(n,n-k+1,i).$$

We may now state the following

THEOREM 1. *Let $\bar{a}(n,k,i)$ denote the number of permutations of Z_n with k rises and i as defined by* (11). *Then $\bar{a}(n,k,i)$ satisfies* (18), *where*

(19) $$A_{n,k}(q) = \sum_{i=0}^{(k-1)(n-k)} a(n,k,i)q^i.$$

3. Put

(20) $$\bar{a}(n,i) = \sum_k \bar{a}(n,k,i),$$

so that $\bar{a}(n,i)$ is the number of permutations of Z_n with fixed i. Then, by (18), (19) and (8),

(21) $$\sum_i \bar{a}(n,i)q^i = \sum_k A_{n,k} = [n][n-1]\cdots[1].$$

On the other hand, recall that an *inversion* in a permutation $(a_1, a_2, \cdots, a_n)$ is a pair i, j such that $(i-j)(a_i - a_j) < 0$. If $I(n,i)$ denotes the number of permutations of Z_n with i inversions, it is known that [**3**, p. 94]

(22) $$\sum_i I(n,i)q^i = [n][n-1]\cdots[1].$$

Comparing (22) with (21) we get the following corollary of Theorem 1.

THEOREM 2. *The enumerant $a(n,i)$ is equal to the number of permutations with i inversions.*

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References

1. L. Carlitz, Eulerian numbers and polynomials, Math. Mag. 33 (1959) 247–260.
2. ———, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc., 76 (1954) 332–350.
3. E. Netto, Lehrbuch der Combinatorik, Teubner, Leipzig and Berlin, 1927.
4. J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.
5. R. P. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc., no. 119.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

A PROBLEM ON BIPARTITE GRAPHS

J. H. VAN LINT

The following problem originated as a minimal-connexion problem in computer designing. We now give a formulation of the problem using telephones. Let there be $2n$ persons, each having a number of telephones. A telephone is connected to one of l exchanges, where $l \geqq n$. Each exchange can handle only one call at a time. It is required that, no matter how the $2n$ persons are divided into n disjoint pairs, it is possible for the n pairs to conduct telephone conversations simultaneously. This is of course trivial if $l = n$ and each person has n telephones, one for each of the exchanges. The question is to determine (as a function of n) the minimal number of telephones the person with the most telephones has.

Let us reformulate the problem as a problem on bipartite graphs. Let the bipartite graph G have vertices $P_1, P_2, \cdots, P_{2n}$ and $C_1, C_2, \cdots, C_l$. All edges of G are of type $\{P_i, C_j\}$.

The set $S_{n,l}$ consists of all such bipartite graphs which have the property that, for any partition $\{P_{i_1}, P_{j_1}\}, \{P_{i_2}, P_{j_2}\}, \cdots, \{P_{i_n}, P_{j_n}\}$ of $P_1, P_2, \cdots, P_{2n}$ into n disjoint pairs, there are vertices $C_{k_1}, C_{k_2}, \cdots, C_{k_n}$ such that $\{P_{i_\nu}, C_{k_\nu}\}$ and $\{P_{j_\nu}, C_{k_\nu}\}$ are edges of G for $\nu = 1, 2, \cdots, n$. Let $\phi(G)$ denote the maximum of the valences of the vertices $P_1, P_2, \cdots, P_{2n}$. Then we define

(1) $\lambda(n, l) := \min\{\phi(G) \mid G \in S_{n,l}\}$,
(2) $\lambda(n) := \min\{\lambda(n, l) \mid l \geqq n\}$.

The question stated above is to determine $\lambda(n)$.

Recently (cf. [**2**]) this problem was proposed in the Problem Section of *Nieuw Archief voor Wiskunde*. Several (partial) solutions were submitted. These are given in [**1**]. We briefly state the results:

(3) $\lambda(n, n) \leqq [\frac{2}{3} n] + 1$ (a simple construction but a very difficult proof),
(4) $\lambda(n, n) \leqq [\frac{9}{14} n] + 1$ (a complicated construction involving block designs; possible improvements are indicated),
(5) $\lambda(n) = O(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}})$ $(n \to \infty)$, (not constructive; proof by probabilistic arguments),

(6) $\lim\inf_{n\to\infty} n^{-\frac{1}{2}}\lambda(n) \geqq 2^{\frac{1}{2}}e^{-\frac{1}{2}}$,
(7) $\lambda(n,n) \geqq \frac{1}{2}n$ (this is trivial),
(8) $\lim_{n\to\infty} n^{-1}\lambda(n,n) = \frac{1}{2}$ (again not constructive).

The following questions remain open.

(9) Find a constructive method of estimating $\lambda(n,n)$, which leads to result (8),
(10) Determine whether $\lim_{n\to\infty} n^{-\frac{1}{2}}\lambda(n)$ exists,
(11) If $l(n)$ is defined as the minimal l for which $\lambda(n,l) = \lambda(n)$, then determine $l(n)$. It is known that $\lim_{n\to\infty} n^{-1}l(n) = 1$.

From a practical point of view, it is of interest to find a "solution" to the problem, together with a simple algorithm which determines the exchanges to be used for each possible partition of $P_1, P_2, \cdots, P_{2n}$. The only reasonable solution to the practical problem which has proved feasible has all the edges $\{P_i, C_j\}$ with $i - j \geqq 0$. In this case, the average number of telephones is $(3n+1)/4$, but in the sense of the original formulation this solution is very bad because $n+1$ of the persons all have n telephones. Several variations of the problem are possible and could prove useful for applications.

References

1. D. Bolton, P. Erdös, J. H. van Lint, P. Seymour, J. Spencer, R. Tijdeman, Solution to Problem 350, Nieuw Archief voor Wiskunde, 22 (1974) 94–109.

2. J. H. van Lint, Problem 350, Nieuw Archief voor Wiskunde, 21 (1973) 179.

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CLASSROOM NOTES

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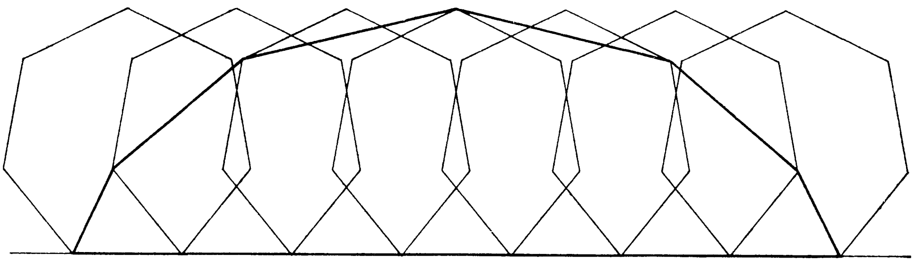
A POLYGONAL ARCH GENERATED BY ROLLING A POLYGON

DUANE W. DETEMPLE

Among the many interesting properties of the cycloid, possibly the two most striking are: (1) the area under a cycloidal arch is three times the area of the generating circle; (2) the length of a cycloidal arch is four times the diameter of the generating circle. These results were obtained respectively by Gilles Persone de

Roberval in 1634 and Sir Christopher Wren in 1658, each employing the newly emerging infinitesimal methods which would later evolve into the calculus (see [1] and [2]).

In this note we show that analogous properties hold for polygonal arches generated by rolling a regular polygon upon a baseline. Of course a vertex will actually describe a roulette of circular arcs, but the desired polygonal arches are obtained if these arcs are replaced by their chords.



THE ARCH GENERATED BY ROLLING A HEPTAGON

The proofs for the theorems below depend only on the elementary geometry and algebra of complex numbers. For polygons with a sufficiently small number of sides, many alternative proofs are possible. Indeed the referee has produced interesting dissection proofs for all of the theorems.

THEOREM 1. *If a regular polygon is rolled along a baseline, the area of the polygonal arch generated by a vertex is three times the area of the regular polygon.*

Proof. We first derive a formula giving the area of an arbitrary polygonal figure. Recall that the area of a triangle with vertices at $(0,0)$, (α_1,β_1), and (α_2,β_2) is given by $\left|\frac{1}{2}(\alpha_1\beta_2-\alpha_2\beta_1)\right|$. In the complex plane this is more compactly expressed as $\left|\frac{1}{2}\operatorname{Im}\bar{a}_1a_2\right|$, where $a_j=\alpha_j+i\beta_j$. It follows that the area of the polygonal figure having successive (positively oriented) vertices at $a_1,a_2,\cdots,a_n$ is given by

$$\tfrac{1}{2}\operatorname{Im}(\bar{a}_1a_2+\bar{a}_2a_3+\cdots+\bar{a}_na_1).$$

We now utilize this formula to compute the area of the regular polygon P_n having vertices on the unit circle at $1,\omega,\omega^2,\cdots,\omega^{n-1}$, where $\omega=\exp(2\pi i/n)$. In fact, since $\omega\bar{\omega}=1$ we find P_n has area $\frac{1}{2}\operatorname{Im}\sum_{j=0}^{n-1}\bar{\omega}^j\omega^{j+1}=\frac{1}{2}\operatorname{Im}n\omega$. Next we generate an arch by rolling P_n downwards along a baseline taken to pass through 1 and $\omega^{n-1}=\bar{\omega}$. The center of P_n translates by $\bar{\omega}-1$ at each step and the vector from the center to the distinguished vertex is successively $1,\omega,\omega^2,\cdots,\omega^{n-1}$. The vertices of the arch so generated are then $a_{j+1}=j(\bar{\omega}-1)+\omega^j$, $j=0,1,\cdots,n-1$. Note that the formula gives $a_{n+1}=a_n$, so that the next arch begins where it should, and in the following expression for the area, in which the sum would naturally have j run from 0 to $n-2$, we may introduce an extra term for $j=n-1$, since $\operatorname{Im}\bar{a}_na_{n+1}=\operatorname{Im}\bar{a}_na_n=0$. The area of the arch is

$$\tfrac{1}{2}\operatorname{Im}\sum_{j=0}^{n-1}[j(\omega-1)+\bar{\omega}^j][(j+1)(\bar{\omega}-1)+\omega^{j+1}]+\tfrac{1}{2}\operatorname{Im}[n(\omega-1)+1]$$

$$=\tfrac{1}{2}\operatorname{Im}\sum_{j=0}^{n-1}[j(\omega-1)\omega^{j+1}+(j+1)\bar{\omega}^j(\bar{\omega}-1)+\omega]+\tfrac{1}{2}\operatorname{Im}n\omega$$

$$=\tfrac{1}{2}\operatorname{Im}(\omega-1)\sum_{j=0}^{n-1}[j\omega^{j+1}-(j+1)\omega^j]+\operatorname{Im}n\omega,$$

where we have used the fact $\operatorname{Im}\bar{w} = -\operatorname{Im}w$. But $\sum_{j=0}^{n-1}\omega^j = (\omega^n-1)/(\omega-1) = 0$, so the above sum can be put into telescopic form. In fact

$$\sum_{j=0}^{n-1}[j\omega^{j+1}-(j+1)\omega^j]=\sum_{j=0}^{n-1}[(j+2)\omega^{j+1}-(j+1)\omega^j]=(n+1)\omega^n-1=n,$$

from which it follows the arch has area $3\cdot\frac{1}{2}\operatorname{Im}n\omega$. □

THEOREM 2. *The $n-2$ obtuse angles of the arch generated by rolling a regular n-gon are all equal, each being $(1-(1/n))\pi$; the two acute angles at the base are each $(\frac{1}{2}-(1/n))\pi$.*

Proof. Again referring to the representation in the complex plane, let $b_j = a_{j+1} - a_j$ denote the jth side of the arch. Since $a_{j+1} = j(\bar{\omega}-1)+\omega^j$ a short computation shows $b_j = (1-\bar{\omega})(\omega^j-1)$.

In particular $b_1 = -|1-\omega|^2 = -2+2\cos(2\pi/n) = -4\sin^2(\pi/n) < 0$, from which the result for the acute angles follows.

Next let $\eta = \exp(i\pi/n)$, so that $\omega = \eta^2$. Then

$$\frac{b_{j+1}}{b_j}=\frac{\eta^{2j+2}-1}{\eta^{2j}-1}=\eta\,\frac{\eta^{j+1}-\bar{\eta}^{j+1}}{\eta^j-\bar{\eta}^j}=\eta\,\frac{\sin((j+1)\pi/n)}{\sin(j\pi/n)}$$

which shows $\arg b_{j+1} = \arg b_j + \pi/n$. □

THEOREM 3. *Let a regular polygon have inscribed and circumscribed circles of radii r and R respectively. Then the length L of the arch generated by that polygon is $L = 4(r+R)$.*

Proof. Again refer to the complex plane representation, for which $r = \cos(\pi/n)$ and $R = 1$. A consequence of the formula displayed in the proof of Theorem 2 is that

$$b_j = \eta^{j-1}b_1\,\frac{\sin(j\pi/n)}{\sin(\pi/n)}.$$

But $b_1 = -4\sin^2(\pi/n)$ and hence

$$L \equiv \sum_{j=1}^{n-1}|b_j| = 4\sin(\pi/n)\sum_{j=1}^{n-1}\sin(j\pi/n).$$

To complete the result we note that

$$\sum_{j=1}^{n-1} \sin(j\pi/n) = \operatorname{Im} \sum_{j=1}^{n-1} e^{\,ij\pi/n} = \operatorname{Im} \frac{1+e^{i\pi/n}}{1-e^{i\pi/n}} = \frac{1+\cos(\pi/n)}{\sin(\pi/n)}. \quad \square$$

References

1. D. J. Struik (ed.), A Source Book in Mathematics, 1200–1800, Harvard University Press, Cambridge, Mass., 1969.

2. E. A. Whitman, Some historical notes on the cycloid, this MONTHLY, 50 (1943) 309–315.

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SELF-COLLINEATIONS OF DESARGUESIAN PROJECTIVE PLANES

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It is well known that every desarguesian projective plane π can be coordinatized by homogenous coordinates taken from a field (which need not be commutative). By a non-degenerate self-collineation α of such a projective plane π is meant a one-to-one mapping of the set of points of π onto itself, say $P(x,y,z) \to P^{\alpha}(x',y',z')$, with the property that the images of collinear points are again collinear. Such collineations are of one of three types:

(i) *projective collineations*, which are linear mappings of the type

$$P(x,y,z) \to P^{\alpha}(a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z)$$

where $\det(a_{ij}) \neq 0$;

(ii) *automorphic collineations*, which are mappings of the type

$$P(x,y,z) \to P^{\alpha}(x^{\sigma}, y^{\sigma}, z^{\sigma}),$$

where σ is an automorphism of the field F of coordinates and where f^{σ} denotes the image of f under the automorphism σ for each $f \in F$; and

(iii) products of the previous two types of collineations, that is mappings of the type

$$P(x,y,z) \to P^{\alpha}(a_{11}x^{\sigma} + a_{12}y^{\sigma} + a_{13}z^{\sigma}, a_{21}x^{\sigma} + a_{22}y^{\sigma} + a_{23}z^{\sigma}, a_{31}x^{\sigma}+a_{32}y^{\sigma}+a_{33}z^{\sigma}),$$

where σ is an automorphism of F as before. This fact was first proved by C. Segré [7], [8] for the particular case when F is the field of complex numbers, though Segré made the additional assumption that the collineation α was a continuous mapping. He called collineations of type (ii) *anticollineations*. More recently, the result has been proved for arbitrary fields of coordinates by several authors, in particular by R. Baer [2], G. Pickert [5], E. Artin [1], B. Segré [6], and D. Pedoe [4].

However, all the proofs which the present author has seen appeal to results extrinsic to the given coordinatization by a system of homogeneous coordinates.

In the present note, a simple proof in which the argument is conducted entirely within the framework of a coordinatizing system of homogeneous coordinates is given. The essence of the proof lies in Lemma II and this is the only item which the author claims to be new.

Before proceeding to the proof, it is worth mentioning that the field of real numbers has no non-identity automorphisms and so the only self-collineations which exist in the real projective plane are of the type which we have called projective collineations.

THEOREM. *Every self-collineation of a desarguesian projective plane may be regarded as the product of a projective collineation and an automorphic collineation.*

Proof. We shall prove our result with the aid of the following two lemmas. (In the case that the coordinatizing field is non-commutative, we shall assume that our coordinate system is that of a left-hand projective plane as defined in chapter V of [3].)

LEMMA I. *There is a unique projective collineation which transforms four given points, no three collinear, into four other given points, no three collinear.*

Proof. Let $\mathbf{a}_i = (a_i, b_i, c_i)$, $i = 1$ to 4, be the given points and let $\mathbf{a}_i^* = (a_i^*, b_i^*, c_i^*)$ be the points into which they are to be transformed. We first show that there exists a unique collineation which maps $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$, and $\mathbf{e}_4 = (1,1,1)$ onto any four assigned points, no three of which are collinear.

Consider the collineation

$$\rho(x'\ y'\ z') = (x\ y\ z)\begin{bmatrix} k_1a_1 & k_1b_1 & k_1c_1 \\ k_2a_2 & k_2b_2 & k_2c_2 \\ k_3a_3 & k_3b_3 & k_3c_3 \end{bmatrix}.$$

This evidently maps $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ onto the points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ respectively provided that no $k_i = 0$. It will map $\mathbf{e}_4$ onto $\mathbf{a}_4$ if we can find non-zero values of the ratios $\rho : k_1 : k_2 : k_3$ such that

$$\begin{aligned} k_1a_1 + k_2a_2 + k_3a_3 &= \rho a_4 \\ k_1b_1 + k_2b_2 + k_3b_3 &= \rho b_4 \\ k_1c_1 + k_2c_2 + k_3c_3 &= \rho c_4 . \end{aligned}$$

However, since $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are not collinear

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

and so these equations are soluble with ρ non-zero. Moreover, none of k_1, k_2, k_3 can be zero since the point $\mathbf{a}_4$ is not collinear with any two of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

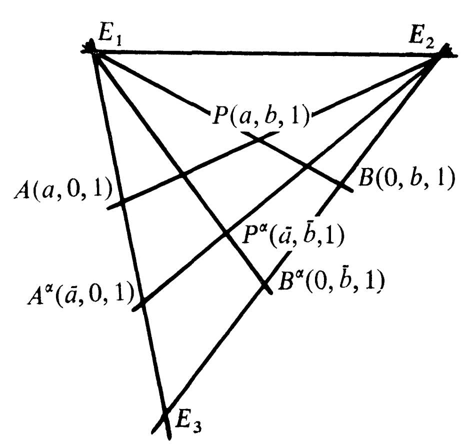
Let the matrix obtained be

$$A = \begin{bmatrix} k_1a_1 & k_1b_1 & k_1c_1 \\ k_2a_2 & k_2b_2 & k_2c_2 \\ k_3a_3 & k_3b_3 & k_3c_3 \end{bmatrix}$$

and let A^* be the matrix of the collineation which maps $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ onto the points $\mathbf{a}_1^*$, $\mathbf{a}_2^*$, $\mathbf{a}_3^*$, $\mathbf{a}_4^*$ respectively. Then the collineation $\rho \mathbf{x}' = \mathbf{x}A^{-1}A^*$ is a collineation mapping $\mathbf{a}_i$ onto $\mathbf{a}_i^*$.

If this is not unique, let $\rho\mathbf{x}' = \mathbf{x}B$ be another collineation having the same effect. Since $\mathbf{e}_iA = \rho\mathbf{a}_i$, $i = 1$ to 4, we should have $\mathbf{e}_iAB = \rho\mathbf{a}_i^*$. Thus, both the matrices A^* and AB define collineations which map the $\mathbf{e}_i$ onto the $\mathbf{a}_i$. The preceding argument shows that, in that case, A^* and AB differ only by a scalar multiplier. Consequently, $A^{-1}A^*$ and B likewise differ only by a scalar multiplier and define the same collineation.

LEMMA II. *Any collineation α of a desarguesian projective plane which leaves fixed the four points $E_1(1, 0, 0)$, $E_2(0, 1, 0)$, $E_3(0, 0, 1)$ and $E_4(1, 1, 1)$ is necessarily an automorphic collineation.*



Proof. We note first that the collineation α maps each point $B(0, b, 1)$ of the line E_2E_3 onto another point $B^\alpha(0, \bar{b}, 1)$ of the same line. It maps each point $A(a, 0, 1)$ of the line E_3E_1 onto another point $A^\alpha(\bar{a}, 0, 1)$ of the same line.

Hence, since α is a collineation, the point $P(a, b, 1)$ which lies at the intersection of the lines E_1B and E_2A is mapped onto the point $P^\alpha(\bar{a}, \bar{b}, 1)$ which lies at the intersection of the lines E_1B^α and E_2A^α. We see that generally the point

$$P(x, y, 1) \xrightarrow{\alpha} P^\alpha(f(x), g(y), 1),$$

where $f(x)$ is independent of y and $g(y)$ is independent of x.

Since α fixes the line $y = x$ joining the points E_3 and E_4, we see that whenever $y = x$ we also have $g(y) = f(x)$. That is, $g(x) = f(x)$ for all $x \in K$, the field of coordinates. Therefore $g(x) \equiv f(x)$ and so $P(x, y, 1) \xrightarrow{\alpha} P^\alpha(f(x), f(y), 1)$. But since the point $E_4(1, 1, 1)$ is left fixed by α, we must have $f(1) = 1$. Therefore, we may write $P(x, y, 1) \xrightarrow{\alpha} P^\alpha(f(x), f(y), f(1))$.

In an exactly similar way we could show that there is a function h such that $P(1, y, z) \xrightarrow{\alpha} P^\alpha(h(1), h(y), h(z))$, where $h(1) = 1$.

But then, $P(1, y, 1) \to P^\alpha(f(1), f(y), f(1)) \equiv P^\alpha(1, f(y), 1)$ and

$$P(1, y, 1) \to P^\alpha(h(1), h(y), h(1) \equiv P^\alpha(1, h(y), 1).$$

Therefore, $h(y) \equiv f(y)$.

We now find it convenient to write f as a mapping. Summarizing our results so far, we have $P(a, b, 1) \to P^\alpha(a^f, b^f, 1)$ and $P(1, b, c) \to P^\alpha(1, b^f, c^f)$. In particular, $P(a, 1, 1) \to P^\alpha(a^f, 1, 1)$. Also, $P(1, a^{-1}, a^{-1}) \to P^\alpha(1, (a^{-1})^f, (a^{-1})^f)$. But these are the same point, hence $\underline{(a^{-1})^f = (a^f)^{-1}}$. Again, $P(a, ab, 1) \to P^\alpha(a^f, (ab)^f, 1)$. Also, $P(1, b, a^{-1}) \to P^\alpha(1, b^f, (a^{-1})^f)$. These too are the same point, so $\underline{(ab)^f = a^f b^f}$. Using these results, we have

$$P(a, b, c) \equiv P(1, a^{-1}b, a^{-1}c) \to P^\alpha(1, a^{-1})^f b^f, (a^{-1})^f c^f) \equiv P^\alpha(1, (a^f)^{-1} b^f, (a^f)^{-1} c^f).$$

That is, $\underline{P(a, b, c) \to P^\alpha(a^f, b^f, c^f)}$.

It now follows that $P(a, b, a + b) \to P^\alpha(a^f, b^f, (a + b)^f)$. But the line $z = x + y$ joining the points $(0, 1, 1)$ and $(1, 0, 1)$ is left fixed by α and so we must have $\underline{(a + b)^f = a^f + b^f}$. Since we have already shown that $(ab)^f = a^f b^f$ and since f is a one-to-one mapping of the coordinatizing field K onto itself, we deduce that f must be an automorphism of K, as required. This completes the proof.

We can now prove our main theorem. Let α be an arbitrary collineation of the desarguesian projective plane π and suppose that α maps the points $E_1(1, 0, 0)$, $E_2(0, 1, 0)$, $E_3(0, 0, 1)$, $E_4(1, 1, 1)$ onto the points $A_1(a_1, b_1, c_1)$, $A_2(a_2, b_2, c_2)$ $A_3(a_3, b_3, c_3)$, $A_4(a_4, b_4, c_4)$ respectively.

By Lemma I above, there exists a unique projective collineation β which maps E_1, E_2, E_3, E_4 onto A_1, A_2, A_3, A_4, respectively. Hence the collineation $\gamma = \alpha\beta^{-1}$ leaves the four points E_1, E_2, E_3, E_4 fixed. By Lemma II, γ is an automorphic collin-

eation and so $\alpha = \gamma\beta$ is the product of a projective collineation and an automorphic collineation, as required.

If γ has equations $\rho(x', y', z') = (x^\sigma, y^\sigma, z^\sigma)$, where σ is an automorphism of the coordinatizing field, and β has equations $\rho(x', y', z') = (x, y, z)A$ then α has equations $\rho(x', y', z') = (x^\sigma, y^\sigma, z^\sigma)A$. Thus, any collineation of π has equations of this form.

References

1. E. Artin, Geometric Algebra, Interscience, London and New York, 1957.
2. R. Baer, Linear Algebra and Projective Geometry, Academic Press, New York, 1952.
3. W. V. D. Hodge and D. Pedoe, Methods of Algebraic Geometry, vol. 1, Cambridge University Press, New York, 1953.
4. D. Pedoe, An Introduction to Projective Geometry, Pergamon, London and New York, 1963.
5. G. Pickert, Projective Ebenen, Springer, Berlin, 1955.
6. B. Segré, Lectures on Modern Geometry, Cremonese, Rome, 1961.
7. C. Segré, Un nuovo campo di ricerche geometriche, Atti. Accad. Sci. Torino, 25 (1889–90) 276–301.
8. ———, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Math. Ann. 40 (1892) 413–467.

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LIMIT OF THE COMPOSITE OF TWO FUNCTIONS

P. Ramankutty and M. K. Vamanamurthy

Almost any book on Calculus contains the theorem regarding the continuity of the composite of two functions, but only very few introduce this question in the context of limits; those who do introduce the topic usually leave it incompletely discussed [**3**, **4**]. Nor do books on Introductory Analysis come as exceptions to this [**5**, p. 156]. Some even have it wrong! [**1**, p. 231; **2**, p. 124.] The purpose of this note is to reach a complete settlement on this question.

Theorem. *If* $\lim_{x\to a} f(x) = A$ and $\lim_{x\to A} g(x) = B$, *then either* $\lim_{x\to a} g(f(x)) = B$ *or* $\lim_{x\to a} g(f(x)) = g(A)$ *or* $\lim_{x\to a} g(f(x))$ *does not exist.*

Proof. Let $\varepsilon > 0$. Then there are $\gamma > 0$ and $\delta > 0$ such that $|g(x) - B| < \varepsilon$ whenever $0 < |x - A| < \gamma$, and $|f(x) - A| < \gamma$ whenever $0 < |x - a| < \delta$.

(i) Suppose there is a neighborhood N of a such that $f(x) \neq A$ for any $x \in N\backslash\{a\}$. Then choosing δ also such that $(a - \delta, a + \delta) \subset N$, we have $0 < |f(x) - A| < \gamma$ whenever $0 < |x - a| < \delta$. This shows: $\lim_{x\to a} g(f(x)) = B$.

(ii) If the assumption in (i) does not hold, then for each neighborhood N of a we have $f(x) = A$ for some $x \in N\backslash\{a\}$ and at such an x, $g(f(x)) = g(A)$. Consequently, if $\lim_{x\to a} g(f(x))$ exists, this limit must be $g(A)$.

The theorem is now completely proved.

Remark: Each of the alternatives mentioned in the theorem can indeed occur as shown by the following examples in which

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

1. $f(x) = x,\ a = 0,$

2. $f(x) = [x],\ a = \frac{1}{2},$

3. $f(x) = \begin{cases} x & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \quad a = 0.$

References

1. Bell, Blum, Lewis and Rosenblatt, Modern University Calculus, Holden-Day, San Francisco, 1966.
2. L. Bers, Calculus, Vol. I. Holt, Rinehart & Winston, New York, 1969.
3. R. C. Fisher, and A. D. Ziebur, Calculus and Analytical Geometry, Prentice-Hall, New York, 1965.
4. E. E. Moise, Calculus, 2nd ed., Addison-Wesley, Reading, Mass., 1972.
5. D. A. Sprecher, Elements of Real Analysis, Academic Press, New York, 1970.

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MATHEMATICAL EDUCATION

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COMPUTING AND MATHEMATICS DEPARTMENTS — THE YEARS AHEAD

RONALD HARROP

Introduction. We are concerned in this paper with some aspects of the impact that the growth of computing is making and could make within the next five or ten years on Mathematics Departments in Universities and Colleges. Detailed reports, such as [1,2,3,4], have been prepared by national or international committees which deal with matters such as the relative administrative positions of computing science and mathematics in universities, the appropriate content for 'Computing Science' courses offered by Mathematics Departments, and the application of computer techniques within mathematics teaching. Many individuals, alone or in small groups, are also actively concerned with such topics, as can be seen by even a casual glance at the papers on Mathematical Education in the 1972 volume of this MONTHLY

References

1. Recommendations on the Undergraduate Program for Work in Computing, May 1964, A report of the CUPM of the Mathematical Association of America.

2. Curriculum 68. Recommendations for Academic Programs in Computer Science. A report of the ACM Curriculum Committee on Computer Science. Communications of the Association for Computing Machinery 11 (1968) 151–196.

3. Report of CUPM Conference on the Uses of Computers in Teaching Undergraduate Mathematics, Jan. 1971, A report of the CUPM of the Mathematical Association of America.

4. Report of the Committee on the Undergraduate Program in Mathematics, January 1972, this MONTHLY, 79 (1972) 796–771.

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PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E.P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before April 30, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 1073* [1953, 417]. *Proposed by G. W. Walker*

A polygonal spiral $A_1A_2A_3\cdots$ of unit segments winds counterclockwise and is constructed in the following manner: Point A_1 is at the origin, point A_2 is at $(1,0)$, $\measuredangle A_{n-1}A_nA_{n+1} = 2\pi/n$ for all $n \geqq 2$. Is there a point lying within the interior of each $\measuredangle A_{n-1}A_nA_{n+1}$? If so, what are its coordinates?

E 1445.* [1960, 1028]. *Proposed by M. S. Klamkin*

A number n is defined as *almost perfect* if $\sum_{d|n} d = 2n \pm 1$. Are there any other almost perfect numbers besides numbers of the form $n = 2^m$? (Cf. R. P. Jerrard and Nicholas Temperly, *Almost perfect numbers*, Math. Mag., 46 (1973) 84–87.)

E 2510. *Proposed by Saul Singer, Brooklyn College*

If n is a natural number, let

$$Q(n) = \prod_{k=1}^{n-1} k^{2k-n-1}.$$

(1) Show that $Q(n)$ is an integer whenever n is prime.

(2)* For which composite n, if any, is $Q(n)$ an integer?

E 2511. *Proposed by Morris Olitsky, Indianapolis, Indiana*

We observe that

$$\tfrac{1}{3} = 0.333333\cdots = 3[(0.1) + (0.1)^2 + (0.1)^3 + \cdots]$$

and that

$$\tfrac{1}{7} = 0.142857\cdots = 7[(0.02) + (0.02)^2 + (0.02)^3 + \cdots].$$

Are there any other positive integers x for which

$$\frac{1}{x} = x\left[\sum_{j=1}^{\infty} (m10^{-n})^j\right]$$

for suitable integers m, n?

E 2512. *Proposed by E. A. Herman, Brandeis University*

Let T_1 and T_2 be two triangles with circumcircles C_1 and C_2 respectively. Show that if T_1 meets T_2, then some vertex of T_1 lies in (or on) C_2 or vice versa. Generalize.

E 2513. *Proposed by Neal Felsinger, Branford, Connecticut*

Let P be a simple (non-self-intersecting) planar polygon. If A is a point in the plane, and if E is an edge of P, then E is *viewable from A* if for every point x of E, the line segment joining A to x contains no point of P other than x. (I.e., one's view of E from A is not "blocked" by any part of P.)

(a) Let A and P be arbitrary. Must some edge of P be viewable from A? Examine the cases of A exterior to P and interior to P separately.

(b) Find sufficient conditions on A in order that some edge of P be viewable from A.

E 2514. *Proposed by G. A. Tsintsifas, Thessaloniki, Greece*

Let P be a convex polygon and let K be the polygon whose vertices are the midpoints of the sides of P. A polygon M is formed by dividing the sides of P (cyclically directed) in a fixed ratio $p:q$ where $p + q = 1$. Show that

$$|M| = (p-q)^2|P| + 4pq|K|,$$

where $|M|$ denotes the area of M, etc.

E 2515. *Proposed by C. L. Mallows, Bell Telephone Laboratories, Murray Hill, New Jersey*

A careless file clerk has documents $D_1, \cdots, D_d$ that should go respectively into files $F_1, \cdots, F_d$; instead he places them independently, at random, into a total of f files ($f \geqq d \geqq 1$) so that each of the f^d possible arrangements is equally likely. Show that the event that some non-empty subset S of the files $F_1, \cdots, F_d$ can be made to have the correct contents by redistributing within S the union of their contents, has probability d/f.

SOLUTIONS OF ELEMENTARY PROBLEMS

A.P.-Free Permutations

E 2440 [1973, 1058]. *Proposed by R. C. Entringer, University of New Mexico, and D. E. Jackson, Los Alamos Scientific Laboratories*

Does every permutation of the integers $0, 1, \cdots, n$ contain an arithmetic progression of at least three terms?

I. *Solution by Tom Odda, Bell Telephone Laboratories, Murray Hill, N.J.* The answer is no, for suppose that $A_m = (a_0, \cdots, a_{m-1})$ is a permutation of $\{0, \cdots, m-1\}$ containing no 3-term A.P. Let $A_{2m} = (2a_0 + 1, \cdots, 2a_{m-1} + 1, 2a_0, \cdots, 2a_{m-1})$. Then A_{2m} is a permutation of $\{0, \cdots, 2m-1\}$ and since in any 3-term A.P. the first and third term have the same parity, any such A.P. contained in A_{2m} would have to be entirely in the first half of A_{2m} or entirely in the second half, either of which is impossible by the choice of A_m. Starting with $A_1 = (0)$, this construction yields permutations A_N where $N = 2^t$ from which a suitable permutation of $\{0, \cdots, n\}$ can be obtained by restriction. (Note that given A_m, we could also have taken

$$A_{2m} = (2a_0, \cdots, 2a_{m-1}, 2a_0 + 1, \cdots, 2a_{m-1} + 1). \text{ — Ed.)}$$

It is obvious that any permutation of the set of *all* nonnegative integers N contains a 3-term A.P. (in fact, increasing). It has been shown (R. L. Graham, unpublished) that there exists a permutation of N which contains no 5-term A.P. The case of 4 is still open. If we are allowed to write N as $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$ then it is known that a 3-term A.P. still must always occur (J. H. Folkman, unpublished) but it is now possible to prevent any 4-term A.P. from occurring.

More specific information on the references mentioned above can be obtained from R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey 07974.

II. *Solution by R. C. Lyndon, University of Michigan.* The answer is no. This follows from the fact that there exists a linear ordering $\prec$ of the nonnegative integers with the property that $a \prec b \prec c$ implies that $b - a \neq c - b$; i.e., a, b, c do not form an arithmetic progression.

To prove this, for every nonnegative integer a let $(a_0, a_1, \cdots)$ be the sequence of coefficients $a_i = 0$ or 1 in its dyadic expansion

$$a = \sum_{i=0}^{\infty} a_i 2^i.$$

Define $a \prec b$ to mean that $(a_0, a_1, \cdots)$ comes before $(b_0, b_1, \cdots)$ in the lexicographic order; that is, for some $k \geqq 0$, $a_0 = b_0, \cdots, a_{k-1} = b_{k-1}$, but $a_k = 0$, $b_k = 1$. Then $b - a = 2^k + u$ where $2^{k+1} \mid u$, so that 2^k is the highest power of 2 that divides $b - a$. Similarly, if $b \prec c$ and if l is the first place where the b_i and c_i differ, then 2^l is the highest power of 2 that divides $c - b$. Since $b_k = 1$ while $b_l = 0$, it follows that $k \neq l$ and so $b - a \neq c - b$.

(Although the ordering defined above is technically a "simple" ordering, it is rather complicated; for example, given any a, b with $a \prec b$, there are infinitely many distinct c with $a \prec c \prec b$. We note that this ordering is the same as what would be obtained by repeated application of the second method of Odda. — Ed.)

III. *Solution by H. E. Thomas, Jr., University of Michigan.* There are at least 2^n permutations of $\{0, \cdots, n\}$ that do not contain an arithmetic progression of three or more terms. By varying the arbitrary choices in the following algorithm, one obtains 2^n such permutations.

1. Set the modulus M to 2, and let the original file be $(0, 1, \cdots, n)$; consider it as a single subfile.

2. If no subfile has more than one entry, stop. If not, go to Step 3.

3. Partition each subfile with more than one entry into two subfiles whose entries are determined by their residue class mod M. (Which subfile is on the right and which is on the left is arbitrary.)

4. Double the modulus and return to Step 2.

Note that after the completion of Step 2, the elements in any subfile fall into only two residue classes mod M, so that the partitioning in Step 3 is always well-defined. Furthermore, at the completion of Step 3, the common difference of any three-term arithmetic progression made from the entries in the file is divisible by the current value of M, since every such progression must be wholly in some subfile. (This is shown as in Solution I. — Ed.) These two statements are true no matter how many times one has looped through Statements 2 through 4.

The construction terminates when all entries are distinct mod M; since each subfile has but a single element, no subfile can contain a three-term arithmetic progression and hence the resulting permutation of $\{0, 1, \cdots, n\}$ is free of three-term progressions.

To show that the algorithm generates 2^n progression-free permutations depending on the choices made in Step 3, we show that precisely n partitions are made. Since each partition does not disturb the entries in the other subfiles, the 2^n permutations

thus generated are all distinct. The first time through Step 3 ($M = 2$), there is made one partition for a total of 2 subfiles. The second time through ($M = 4$), there are made two partitions for a total of $2^2 = 4$ subfiles. Continuing on, the kth time through ($M = 2^k$), there are made $\frac{1}{2}M = 2^{k-1}$ partitions for a total of $M = 2^k$ subfiles. Suppose that $2^k < n + 1 \leqq 2^{k+1}$ so that the construction terminates after $k + 1$ passes through Step 3. After the penultimate pass through Step 3, each of the $M = 2^k$ subfiles contains either one or two elements. Say p of them contain a single element and q of them contain two elements. Since $p + q = 2^k$ and $p + 2q = n + 1$, we see that $q = n + 1 - 2^k$ so that on the final pass through Step 3, only $n + 1 - 2^k$ new partitions are made. Thus the total number of partitions made is

$$1 + 2 + \cdots + 2^{k-1} + n + 1 - 2^k = n.$$

The same result can be obtained more simply by convincing oneself that rearrangement does not alter the number of partitions necessary to separate the file into singleton subfiles. Since there are $n + 1$ entries, it is obvious that n partitions are necessary to separate them.

With a little care, the same result can be derived for any set of $n + 1$ distinct integers. Note however that the construction does not yield in general all permutations without progressions. For example, if $n = 3$, there are actually 10 such permutations, whereas the algorithm generates only $2^3 = 8$: the permutations (1, 0, 3, 2) and (2, 3, 0, 1) are not constructed. We note that if $n + 1$ is a power of 2, then one of the constructions is essentially bit-reversal; for example, if $n = 7$, then bit-reversal on (0, 1, 2, 3, 4, 5, 6, 7) gives (0, 4, 2, 6, 1, 5, 3, 7). (This is the same as the second method of Odda. — Ed.)

IV. *Comment by G. J. Simmons, Sandia Laboratories, Albuquerque, New Mexico.* Let $P(n)$ denote the number of permutations on the n elements $\{0, 1, \cdots, n - 1\}$ which contain no three-term arithmetic progressions. We have, with a prodigious expenditure of computer time, calculated $P(n)$ for $n \leqq 20$. The computing time for the case $n = 20$ alone was almost 40 minutes!

n	$P(n)$	n	$P(n)$
1	1	11	2460
2	2	12	6128
3	4	13	12840
4	10	14	29380
5	20	15	73904
6	48	16	212728
7	104	17	368016
8	282	18	659296
9	496	19	1371056
10	1066	20	2937136

It is tempting to assume that every admissible permutation of $\{0,1,\cdots,n-1\}$ can be extended to an admissible permutation of $\{0,1,\cdots,n\}$ by inserting n in the "right place"; this is not true, as is seen by examining (2, 4, 3, 5, 0, 1). Thus if we construct the rooted tree of admissible permutations by extension with n, certain branches terminate.

Also solved by S. Baskaran (India), J. Binz (Switzerland), Bro. Alfred Brousseau, D. Ž. Djoković, Gertrude Ehrlich, T. E. Elsner, D. W. Ehrbach (England), D. B. Erickson & C. V. Heuer, Michael Goldberg, David Grinstein, Kent Harris & Lucian Wernick, Howard Hiller, M. D. Humphries, Ralph Jones, O. P. Lossers (Netherlands), Carolyn MacDonald, Ivan Niven, Robert Patenaude, G. J. Simmons, M. C. Tews, E. H. Umberger, and the proposer.

Editor's comment. It would be interesting to investigate the asymptotic behavior of $P(n)$. Several conjectures are suggested by the table of Simmons; for example, is $P(n+1)/P(n) \sim 2$?

Although the number $P(n)$ of admissible permutations of $\{0,1,\cdots,n-1\}$ is not easily determined, the number of linear orderings $\prec$ of $Z^+ = \{0,1,\cdots,n,\ldots\}$ with the property that $a \prec b \prec c$ implies that a,b,c are not in arithmetic progression (as in Solution II) is easily determined: it is the power of the continuum. Every such ordering can be thought of as a subset of ordered pairs of Z^+; since there are $(\aleph_0)^2 = \aleph_0$ such ordered pairs, there are no more than $\mathfrak{c} = 2^{\aleph_0}$ such orderings. But let $x = (x_0, x_1, \ldots)$ be a sequence of zeros and ones. Define the *x-lexicographic order* $\prec_x$ on Z^+ as follows: As in Solution II, if $a \in Z^+$, we associate with a the (unique) sequence $(a_0, a_1 \ldots)$ of zeros and ones defined by

$$a = \sum_{i=0}^{\infty} a_i 2^i.$$

Define $a \prec_x b$ if for some $k \geqq 0$, $a_0 = b_0, \ldots, a_{k-1} = b_{k-1}$ and $a_k < b_k$ if $x_k = 0$ or $a_k > b_k$ if $x_k = 1$. (The ordering of Solution II is seen to be $\prec_x$ for $x = (0,0,\ldots)$.) Another way of looking at this ordering is to note that the ordering $\prec$ of Solution II extends to all sequences of zeros and ones and not only those generated by $a \in Z^+$, which must terminate in infinitely many zeros. If we define $a \oplus x$ by $(a_0, a_1, \ldots) \oplus (x_0, x_1, \ldots) = (a_0 + x_0 \pmod 2, a_1 + x_1 \pmod 2, \ldots)$, then we have that $a \prec_x b$ if and only if $a \oplus x \prec b \oplus x$. It is clear that $\prec_x$ is a distinct linear ordering for each x, and that if $a \prec_x b \prec_x c$, then a, b, c cannot be in arithmetic progression. Since there are precisely $\mathfrak{c} = 2^{\aleph_0}$ such x, it follows that there are precisely $\mathfrak{c}$ such linear orderings.

Unique Cube Roots Modulo m

E 2446 [1973, 1139]. *Proposed by H. D. Ruderman, Hunter College Campus School*

Characterize those moduli m for which both $x^3 \equiv 1 \pmod m$ implies $x \equiv 1 \pmod m$ and $x^3 \equiv 0 \pmod m$ implies $x \equiv 0 \pmod m$; show that these are precisely, the moduli for which $x^3 \equiv y^3 \pmod m$ implies $x \equiv y \pmod m$ for all x, y.

Solution by the Bennett College Team. We prove a more general result.

THEOREM. *Let $k > 1$ be a natural number. For any natural number m, the following three properties are equivalent:*

(a) $x^k \equiv y^k \pmod m$ *implies* $x \equiv y \pmod m$;

(b) $x^k \equiv 0 \pmod m$ *implies* $x \equiv 0 \pmod m$ *and* $x^k \equiv 1 \pmod m$ *implies* $x \equiv 1 \pmod m$;

(c) m *is a squarefree number with the property that every prime divisor* p *of* m *satisfies* $(p-1,k) = 1$.

Proof. Obviously (a) implies (b), so suppose that (b) holds. If $x^k \equiv 0 \pmod m$ implies $x \equiv 0 \pmod m$, then clearly m is squarefree (and conversely). But if m is squarefree, and the congruence $x^k \equiv 1 \pmod m$ has the unique solution $x \equiv 1 \pmod m$, then it is known that every prime divisor p of m must satisfy $(p-1,k) = 1$ (J. Alonso, *Number of solutions of the congruence* $x^m \equiv r \pmod n$, Math. Mag., 46 (1973) 215–217).

Suppose then that (c) holds, and that $x^k \equiv y^k \pmod m$ for some x, y. Then $x^k \equiv y^k \pmod p$ for every prime divisor p of m and it follows (*ibid.*) that $x \equiv y \pmod p$ and thus $x \equiv y \pmod m$.

We remark that if k is even, then the only modulus m for which any (hence all) of the above conditions holds is $m = 2$.

Also solved by Arnold Adelberg, Anders Bager (Denmark), S. Baskaran (India), Sarah Christiansen, Charles Church & Theresa Vaughan, D. L. Costa, R. A. Gibbs, M. G. Greening (Australia), G. A. Heuer (Germany), Carl Hurd, L. Kuipers, J. R. Kuttler, L. E. Mattics, Gary McDonald & Merry McDonald, Wanda Mourant, L. A. Parson, Robert Patenaude, Bart Rice, F. T. Rush, SCSC Problem Solving Group, Kenneth Schilling, Southern University Primer for Research Group, Temple University Problem Solving Group, Charles Wexler, and the proposer.

Editor's comment. Heuer refers to the result that the mapping $x \to x^k$ in a group of order n is one-to-one if and only if $(k, n) = 1$, which appears as Theorem 2 of W. R. Utz, *Square roots in groups*, this MONTHLY 60 (1953), 185–186; see also E 1794 [1966, 892]. He obtains essentially the same characterization as the Bennett Team by considering the multiplicative group of numbers relatively prime to m, which is of order $\varphi(m)$, where φ is Euler's totient function. His conditions are that m must be squarefree and $(k, \varphi(m)) = 1$; this is seen to be equivalent to condition (c) of the published proof since if $m = p_1p_2\ldots p_n$, then $\varphi(m) = (p_1 - 1)(p_2 - 1)\ldots(p_n - 1)$.

Bounds for k-Satisfactory Sequences

E 2447 [1973, 1139]. *Proposed by E. T. H. Wang, Wilfrid Laurier University*

A *k-satisfactory sequence* is a k-tuple $S = (a_1, a_2, \cdots, a_k)$ of natural numbers with $a_1 \leqq a_2 \leqq \cdots \leqq a_k$ such that $\Sigma a_i = \Pi a_i$. Let $v(S)$ denote this common value. Show that $v(S) \leqq 2k$ with equality if and only if $S = (1, \cdots, 1, 2, k)$, and investigate the problem of finding a lower bound for $v(S)$. (Cf. E 2262 [1971, 1021].)

I. *Solution by Carl Hurd, Altoona Campus of Pennsylvania State University.* The problem is false in the case $k = 1$, so we assume that $k \geqq 2$. We note first that if m, n are positive integers, then since $mn - m - n + 1 = (m-1)(n-1) \geqq 0$ it follows that

$$mn + 1 \geqq m + n \tag{1}$$

with equality if and only if $m = 1$ or $n = 1$.

THEOREM 1. *If $S = (a_1, \cdots, a_k)$ is k-satisfactory, then $v(S) \leqq 2k$ with equality if and only if $S = (1, \cdots, 1, 2, k)$.*

Proof. Write $b_i = a_i - 1$. Then

$$
\begin{aligned}
k + \sum_{i=1}^{k} b_i = \sum_{i=1}^{k} a_i = v(S) = \prod_{i=1}^{k} a_i &= \prod_{i=1}^{k} (1 + b_i) \\
&= 1 + \sum_{i=1}^{k} b_i + b_k \sum_{i=1}^{k-1} b_i + \cdots \\
&\geqq 1 + \sum_{i=1}^{k} b_i + b_k \sum_{i=1}^{k-1} b_i
\end{aligned}
\tag{2}
$$

from which it follows that

$$k \geqq 1 + b_k \sum_{i=1}^{k-1} b_i. \tag{3}$$

Since S is k-satisfactory, we know that $k \geqq 2$ and $a_k \geqq a_{k-1} \geqq 2$, so that $b_k \geqq b_{k-1} \geqq 1$ and we can apply (1) to infer that

$$k \geqq 1 + b_k \sum_{i=1}^{k-1} b_i \geqq \sum_{i=1}^{k} b_i. \tag{4}$$

Hence

$$v(S) = \sum_{i=1}^{k} a_i = k + \sum_{i=1}^{k} b_i \leqq 2k. \tag{5}$$

If $S = (1, \cdots, 1, 2, k)$, then certainly $v(S) = 2k$ so we must show that if $v(S) = 2k$, then necessarily $S = (1, \cdots, 1, 2, k)$. Now $v(S) = 2k$ implies that there is equality in (5), (4), and (3) and hence the additional terms on the right-hand side of (2) must be all zero. Among these are all possible products of three distinct b_i, so all but two of the b_i are zero; only b_{k-1} and b_k are nonzero. Now (4) becomes

$$k = 1 + b_k b_{k-1} = b_k + b_{k-1},$$

and by (1), this can hold if and only if $b_k = 1$ or $b_{k-1} = 1$. But $1 \leqq b_{k-1} \leqq b_k$, so in either circumstance $b_{k-1} = 1$. Thus $a_{k-1} = 2$ from which it follows that $a_k = k$.

Before obtaining a lower bound for $v(S)$, we need a lemma.

LEMMA. *Suppose that $b_1, \cdots, b_r$ are nonnegative integers and that $\prod_{i=1}^{r}(1 + b_i) > 2^t$. Then $\sum_{i=1}^{r} b_i > t$.*

Proof. We can assume without loss of generality that all b_i are positive, for any b_i which are zero can be removed from the product and later replaced in the sum without affecting either. We use induction on t. If $t = 0$, the proposition is obvious

so suppose it is true for all nonnegative integers $t < w$ and assume that $\prod_{i=1}^{r}(1+b_i) > 2^w$. If $r = 1$, there is no problem, so assume that $r \geqq 2$. There is a nonnegative integer u such that $2^u \leqq b_r < 2^{u+1}$. Now

$$\prod_{i=1}^{r-1}(1+b_i) = \frac{\prod_{i=1}^{r}(1+b_i)}{1+b_r} > \frac{2^w}{2^{u+1}} = 2^{w-u-1},$$

so, by the induction hypothesis, $\sum_{i=1}^{r-1} b_i > w-u-1$ from which it follows that $\sum_{i=1}^{r} b_i > w-u-1+b_r \geqq w-u-1+2^u \geqq w-u-1+u+1 = w$. [A proof can also be given by using the Mean Value Theorem on the function $f(x) = \log(1+x)$ on the interval $[1, b_i]$.—Ed.]

THEOREM 2. *Suppose that $S = (a_1, \cdots, a_k)$ is k-satisfactory. Let s be the least positive integer such that $2^s - s \geqq k$. Then $v(S) \geqq k+s$.*

Proof. Again let $b_i = a_i - 1$. Then $k > 2^{s-1} - s + 1 \geqq 2^{s-2}$. Since $\prod_{i=1}^{k}(1+b_i) = v(S) = k + \sum_{i=1}^{k} b_i \geqq k > 2^{s-2}$, we can apply the Lemma to infer that $\sum_{i=1}^{k} b_i > s-2$. Thus $v(S) = k + \sum_{i=1}^{k} b_i > k+s-2$; but $k > 2^{s-1} - s+1$, so that $k \geqq 2^{s-1} - s + 2$, and $v(S) > k+s-2 \geqq 2^{s-1}$. Therefore $\prod_{i=1}^{k}(1+b_i) = v(S) > 2^{s-1}$ and we can apply the Lemma again to infer that $v(S) > k+s-1$, so that $v(S) \geqq k+s$.

In the case that $k = 2^s - s$ for some s, this lower bound is attained by the sequence consisting of $k-s$ ones and s twos.

II. *Comment by D. M. Bloom, Brooklyn College.* As for the lower bound, clearly $v(S) = \sum a_i \geqq \sum 1 = k$; thus the inequality $v(S) \geqq ck$ holds when $c = 1$. It is false for any larger value of c as shown by the counterexample $S = (1, \cdots, 1, s+2, (k+s)/(s+1))$, where s is chosen so large that $(s+2)/(s+1) < c$ and then $k \equiv 1 \pmod{s+1}$ is chosen so large that

$$\frac{k+s}{k} \cdot \frac{s+2}{s+1} < c.$$

Also solved by D. M. Bloom, Kenneth Schilling, Temple University Problem Solving Group, and P. H. Young.

Editor's comment. Only Bloom and Hurd proved correct lower bound results. The Temple Group stated without proof a lower bound similar to that given by Hurd.

A Matrix and its Matrix of Reciprocals Both Positive Semi-definite

E 2448 [1973, 1139]. *Proposed by Gérard Letac, Université de Clermont, France*

Find all positive semi-definite Hermitian matrices $A = (a_{ij})$ with the property that the matrix of reciprocals $(1/a_{ij})$ is also positive semi-definite.

Solution by John Hearon, National Institutes of Health. Let $H = (h_{ij})$ be a positive semi-definite Hermitian matrix. Certainly h_{ij} must be non-zero for all i, j

before we can even form its matrix of reciprocals G. The required characterization is that G is positive semi-definite if and only if H is of rank one, i.e., if and only if $H = \mathbf{aa}^*$ where $\mathbf{a}$ is a column vector and $\mathbf{a}^*$ is its conjugate transpose.

Suppose first that both G and H are positive semi-definite, so that $H = X^*X$ for some matrix X; let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be the columns of X. Then $h_{ij} = \mathbf{x}_i^* \mathbf{x}_j$ for all i, j and the requirement that the determinant of every 2×2 principal minor be nonnegative reads

$$(1) \qquad \|\mathbf{x}_i\|^2 \|\mathbf{x}_j\|^2 \geqq |\mathbf{x}_i^*\mathbf{x}_j|^2.$$

The same requirement for G reads

$$(2) \qquad \|\mathbf{x}_i\|^{-2} \|\mathbf{x}_j\|^{-2} \geqq |\mathbf{x}_i^*\mathbf{x}_j|^{-2}.$$

But (1) and (2) are inconsistent unless there is equality in both, which means that for each i, j the vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ are dependent since this is the necessary and sufficient condition for equality in the Schwartz inequality. Thus H is of rank one and can be written $H = \mathbf{aa}^*$, where $\mathbf{a}$ spans the range of H.

Conversely, suppose that $H = \mathbf{aa}^*$ where of course no component of $\mathbf{a}$ is zero. Letting G denote the matrix of reciprocals, and $\mathbf{v}$ any column vector, we see that

$$\mathbf{v}^*G\mathbf{v} = \sum_i \sum_j \frac{\bar{v}_i v_j}{a_i \bar{a}_j} = \left(\sum_i \frac{\bar{v}_i}{a_i}\right)\left(\sum_j \frac{v_j}{\bar{a}_j}\right) = \left|\sum_i \frac{\bar{v}_i}{a_i}\right|^2 \geqq 0.$$

Also solved by T. S. Bolis, Bradley Dickinson, P. W. Epasinghe (Ceylon), G. A. Heuer (Germany), Marvin Marcus, R. S. Sacher, William Watkins, and the proposer.

Polynomial Quotients

E 2450 [1974, 84]. *Proposed by S. R. Conrad, B. N. Cardozo High School, Bayside, New York*

Prove that $x^{4n} + x^{3n} + x^{2n} + x^n + 1$ is divisible by $x^4 + x^3 + x^2 + x + 1$ whenever n is a positive integer which is not a multiple of 5.

I. *Solution by Graham Lord, Temple University.* Let ζ be any one of the four primitive fifth roots of unity and suppose that $5 \nmid n$. Then

$$\zeta^{4n} + \zeta^{3n} + \zeta^{2n} + \zeta^n + 1 = \frac{\zeta^{5n} - 1}{\zeta^n - 1} = 0.$$

(Note that if n were a multiple of 5, the preceding sum would equal 5.) Fix ω as any one of these primitive fifth roots; the other three are ω^2, ω^3, and ω^4 and hence a factor of $x^{4n} + x^{3n} + x^{2n} + x^n + 1$ is $(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)$ which is equal to $x^4 + x^3 + x^2 + x + 1$.

II. *Generalization by Norman Bauman, Nanuet, N.Y., and D. M. Bloom, Brooklyn College. (Editor's composite.)* More generally, $x^{(m-1)n} + \cdots + x^{2n} + x^n + 1$

is divisible by $x^{m-1} + \cdots + x^2 + x + 1$ whenever m and n are relatively prime positive integers. We have

$$\frac{x^{(m-1)n} + \cdots + x^{2n} + x^n + 1}{x^{m-1} + \cdots + x^2 + x + 1} = \frac{\dfrac{x^{mn}-1}{x^n - 1}}{\dfrac{x^m - 1}{x-1}} = \frac{(x^{mn}-1)(x-1)}{(x^m-1)(x^n-1)}.$$

The roots of $(x^m - 1)(x^n - 1)$ are 1 (twice) and the non-unit mth and nth roots of 1; since $(m,n) = 1$, these are all distinct. But $(x^{mn}-1)(x-1)$ has roots 1 (twice), and the non-unit mnth roots of 1, which include all $m + n - 2$ of the distinct non-unit mth and nth roots of 1. Hence divisibility is established.

III. *Comment by Jesse Deutsch, Brooklyn, N.Y.*, Problem E 2450 resembles problem # 22, p. 4, in S. Straszewicz, *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, Pergamon Press, 1965, which is to show that $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by $x^4 + x^3 + x^2 + x + 1$. Three solutions are given.

IV. *Comment by Kenneth Rosen, Massachusetts Institute of Technology.* A solution of E 2450 can be found in Bryant Tuckerman, *Factorization of* $x^{2n} + x^n + 1$ *using cyclotomic polynomials*, Math. Mag., 42 (1969) 41–42.

Also solved by W. B. Adams, Peter Ash, Richard Bagby, Anders Bager (Denmark), S. Baskaran (India), Bennett College Team, K. A. Beres, David Bienenfeld (Israel), T. S. Bolis, Robert Breusch (New Zealand), Bro. Alfred Brousseau, B. Carlat, C. S. K. Chetty (India), David Cochener, S. H. Cox, Jr. (Puerto Rico), Kay Dundas, E. S. Eby, H. M. Edgar, M. A. Fechter, David Foley, Stanley Fox, Owen Fraser, G. J. Galloway, R. A. Gibbs, Robert Gilmer (Australia), D. J. Gittinger, G. K, Goff, Donald Goldberg & Susan Moy, Michael Goldberg, J. F. Golightly, W. J. Gorman, III, Sylvan Green, M. G. Greening (Australia), Emil Grosswald, Eino Halminen (Finland), K. J. Heuvers, Myron Hlynka, Se June Hong, Karel Horák (Czechoslovakia), Carl Hurd, Yasuhiko Ikeda, Geoffrey Kandall, I. G. Kastanas (Greece), H. Kasube, M. S. Klamkin, C. F. Koch, Ben-Zion Kurtaran (Israel), Albert Leisinger, H. S. Lieberman, P. W. Lindstrom, Carolyn MacDonald, L. F. Meyers, Wanda Mourant, M. R. Murty & V. K. Murty, J. L. Nanda, R. B. Nelson, F. D. Parker, Aron Pinker, J. H. Pitt, Radford College Problems Group, H. J. Ricardo, F. T. Rush, St. Olaf College Students, W. M. Sanders, F. W. Saunders, SCSC Problem Solving Group, B. L. R. Shawyer, Harry Sherman, Michael Shimshoni (Israel), Joseph Silverman, Paul Smith, Southern University Primer for Research Group, Art Steger, A. H. Stein, Oto Strauch (Czechoslovakia), N. L. Swanson, Temple University Problem Solving Group, S. J. Tillman, Guy Torchinelli, C. W. Trigg, W. P. Turgeon, E. T. H. Wang, Charles Wexler, R. D. Whittekin, W. G. Wild, Ken Yocom, David Zeitlin, Aleksandras Zujus, and the proposer.

The Iterated Sine

E 2451 [1974, 84]. *Proposed by F. W. Hartmann, Villanova University*

Let $a_0 = 1$, $a_1 = \sin 1$, $a_2 = \sin a_1, \cdots, a_{n+1} = \sin a_n$. Show that the sequence $\{na_n^2\}$ has a limit and find that limit. What can be said for arbitrary a_0?

Solution by the Temple University Problem Solving Group. Suppose first that $0 < a_0 < \pi$. Then obviously $a_n > 0$ for all n, and if we let $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x > 0$ for $0 < x < \pi$; since $f(0) = 0$, it follows that $a_n - a_{n+1} = a_n - \sin a_n = f(a_n) > 0$. Hence the sequence $\{a_n\}$ is strictly monotonically decreasing and bounded below by 0, so if $L = \lim a_n$, then $L = \sin L$, implying that $L = 0$. The same can be said for any a_0 which is congruent $(\mathrm{mod}\, 2\pi)$ to some $a \in (0, \pi)$: $a_1 > a_2 > \cdots > 0$ and $a_n \to 0$. Similarly, if a_0 is congruent $(\mathrm{mod}\, 2\pi)$ to some $a \in (-\pi, 0)$, then $a_1 < a_2 < \cdots < 0$ and $a_n \to 0$. (If a_0 is an integral multiple of π, then $a_1 = a_2 = \cdots = 0$.)

Consider then for any $a_0 \neq k\pi$ the limit

$$\lim_{n\to\infty} (a_{n+1}^{-2} - a_n^{-2}) = \lim_{n\to\infty} (\csc^2 a_n - a_n^{-2}) = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$$

which is equal to $\frac{1}{3}$ by four applications of L'Hospital's rule. (Or note that $\sin^2 x = (x - \frac{1}{6}x^3 + - \cdots)^2 = x^2 - \frac{1}{3}x^4 \pm \cdots$.—Ed.)

If $y_n = a_n^{-2} - a_{n-1}^{-2}$, then the sequence $\{y_n\}$ has $\frac{1}{3}$ as its limit, hence the sequence $\{\bar{y}_n\}$ of Cesàro means also has $\frac{1}{3}$ as its limit. I.e.,

$$\frac{1}{3} = \lim_{n\to\infty} \bar{y}_n = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (a_k^{-2} - a_{k-1}^{-2})$$

$$= \lim_{n\to\infty} \frac{1}{n}(a_n^{-2} - a_0^{-2}) = \lim_{n\to\infty} \frac{1}{n a_n^2}.$$

It follows that $na_n^2 \to 3$ for any $a_0 \neq k\pi$, whereas if $a_0 = k\pi$, then na_n^2 is the null sequence and therefore $na_n^2 \to 0$.

Also solved by T. S. Bolis, D. Borwein & B. L. R. Shawyer, Robert Breusch (New Zealand), L. B. Deziel, Jr. & O. G. Ruehr, Leon Gerber, I. N. Katz, M. S. Klamkin, O. P. Lossers (Netherlands), L. E. Mattics, L. F. Meyers, Ram Murty & V. K. Murty, D. F. Neu, H. J. Ricardo, St. Olaf College Students, Harry Sherman, Wolfe Snow, Oto Strauch (Czechoslovakia), Ken Yocom, and the proposer.

Editor's comment: The Murtys, Meyers, and Ricardo note that this problem is essentially Problem 173 (Section I) of G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. I, Dover, New York, 1945. The problem can be found on p. 31 and its solution on p. 187.

Deziel & Ruehr, Klamkin, Lossers, and Ricardo observe that the iterated sine is extensively investigated in N. G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, Amsterdam, 1958, pp. 153 ff.

Ricardo also comments that Problem 283 in the *Pi Mu Epsilon Journal* [1973, 481] is to show that $\sum a_n$ diverges: this is an immediate consequence of our problem since $a_n \sim \sqrt{(3/n)}$ and $\sum n^{-1/2}$ diverges.

Even though the problem has been solved in the literature, your editors felt that the Temple Group's solution was so elegant that it deserved publication.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before April 30, 1975.

6006. *Proposed by Frank Uhlig, University of Würzburg, Germany*

Let A_i be a finite family of complex square matrices that have no eigenvalues in common. Let p_i be a family of real polynomials and define $B_i = p_i(A_i)$ for each i. If each A_i is similar to a real matrix, prove that there is a real polynomial p such that $p(A_i) = B_i$ for every i.

6007. *Proposed by Rollin Sandberg, California State University at Fullerton.*

Let f be a non-decreasing, continuous function from $[0, a]$ onto $[0, b]$ such that f' vanishes almost everywhere. Determine the length of this arc.

6008. *Proposed by P. B. Gilkey, University of California, Berkeley*

For $\xi = (x_1, x_2, x_3) \in \mathbb{R}^3$, set $A = A(\xi) = x_1A_1 + x_2A_2 + x_3A_3$ where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix},$$

and let $\Gamma = \Gamma(\xi)$ be any positively oriented closed curve enclosing the eigenvalues of $A(\xi)$. Show that the integral

$$I(B) = \int_{\mathbb{R}^3} \oint_{\Gamma} \operatorname{tr}\{(\lambda - A)^{-1}[B(\lambda - A)^{-1}]^3\} \lambda \exp[-\lambda^2]\, d\lambda d\xi$$

vanishes for every 4×4 matrix B.

6009. *Proposed by J. A. Goldstein, Tulane University*

Let X be a finite dimensional topological vector space whose topology is given by a metric d. Let T be a surjective isometry on X such that $T0 = 0$. If d is invariant, i.e., if $d(p, q) = d(p - q, 0)$ for all $p, q \in X$, so that X is a Fréchet space, then T is necessarily linear [Z. Charzyński, Studia Math. 13 (1953), 94–121]. Must T still be linear if the assumption that d is invariant is dropped? What if $\dim X = 1$?

6010. *Proposed by L. Carlitz, Duke University*

Coefficients $c_{m,n}^{(k)}$ are defined by means of

$$(1 + x)^m (1 - x)^n = \sum_{k=0}^{m+n} c_{m,n}^{(k)} x^k \qquad (m \geqq 0, n \geqq 0).$$

Show that

$$\sum_{k=0}^{m+n} (c_{m,n}^{(k)})^2 = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

6011. *Proposed by M. Slater, The University, Bristol, England*

According to a question in the 1973 William Lowell Putnam competition, B–1, this MONTHLY, 81 (1974) 1090, the group Z of integers has the following property X:

For any n, suppose A is a list of $(2n+1)$ terms in Z, such that on removal of any one term, the remainder can be divided into two batches of n terms having equal sums. Then all the terms of A are equal.

Determine exactly what abelian groups G have property X.

SOLUTIONS OF ADVANCED PROBLEMS

Least Common Multiple of Consecutive Terms in a Sequence

5413 [1966, 783]. *Proposed by J. L. Selfridge, Pennsylvania State University, and Paul Erdös*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Denote by $A_n^{(k)}$ the least common multiple of $a_n, a_{n+1}, \cdots, a_{n+k-1}$. Prove that the number of indices n for which $A_n^{(k)} < x$ is less than $cx^{1/k}$.

Comment by Paul Erdös, University of Calgary. E. Szemerédi and I disproved this. The result probably fails for every $k \geqq 3$, but this we cannot show. We show that it fails for $k = 6$. Denote by $f(x, k)$ the number of indices n for which $A_n^{(k)} < x$. We define our sequence as follows: Consider the integers $4m+1$, $4m+2$, $4m+3$, $4m+4$, $m = 1, 2, 3, \cdots$. Form the six products $(4m+i)(4m+j)$, $1 \leqq i < j \leqq 4$; these are our a's. The six a's $(4m+i)(4m+j)$, $1 \leqq i < j \leqq 4$ are clearly consecutive and their least common multiple is $(4m+1)(4m+2)(4m+3)\cdot(4m+4) < x$ for $m < \frac{1}{4}x^{1/4} - 4$. Thus $f(x, 4) > \frac{1}{4}x^{1/4} - 4 > cx^{1/6}$, which completes the proof.

This proof works also for $k = 5$, but not for $k = 4$ and $k = 3$. For $k = 3$ it is easy to see that $f(x, 3) < cx \log x$.

For $k = 2$ it is easy to see that

$$\overline{\lim}\, f(x,2)/x^{\frac{1}{2}} = \sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k+1}. \tag{1}$$

To prove (1) observe that $[a, a+t] \geqq a(a+t)/t$. Thus if $A_n^{(2)} \leqq x$ and $n > \sqrt{kx}$ then $\overline{a_{n+1} \cdot a_n} \geqq k+1$, or there are at most $(\sqrt{k+1} - \sqrt{k})x^{\frac{1}{2}}/(k+1)$ choices for n if $\sqrt{kx} < a_n < \sqrt{(k+1)x}$ and (1) follows easily.

Perhaps

$$\underline{\lim}\, f(x,2)/\sqrt{x} \leqq 1, \tag{2}$$

but we have not yet proved (2).

It would be interesting to determine the least upper bound for α for which

$$\overline{\lim}\ f(x,k)/x^{\alpha} = \infty.$$

The Conjugates of Algebraic Integers

5931 [1973, 949]. *Proposed by C. J. Smyth, University of Turku, Finland*

Let γ be a PV-number (an algebraic integer with $|\gamma| > 1$, and conjugates $\gamma = \gamma_1, \gamma_2, \cdots, \gamma_n$, with $|\gamma_i| < 1$ $(i = 2, \cdots, n)$). Show that for $i \neq j$, $|\gamma_i| = |\gamma_j|$ implies $\gamma_i = \bar{\gamma}_j$.

Solution by the proposer. Consider

$$f(x) = \prod_{1 \leq i \leq j \leq n} (x - \gamma_i \gamma_j).$$

Now $f(x) \in \mathbb{Z}[x]$, the product being a symmetric function of the γ_i. Assume $|\gamma_i| = |\gamma_j|$ but $\gamma_i \neq \bar{\gamma}_j$ for some $i \neq j$. Then both $(x - \gamma_i \bar{\gamma}_i)$ and $(x - \gamma_j \bar{\gamma}_j)$ divide $f(x)$, so $(x - \phi)^2$ divides $f(x)$, where $\phi = \gamma_i \bar{\gamma}_i = \gamma_j \bar{\gamma}_j$. But $0 < \phi < 1$, so ϕ has a conjugate ϕ' over the rationals, with $|\phi'| > 1$. Hence $(x - \phi')^2$ divides $f(x)$. But the only zeros of $f(x)$ of absolute value greater than 1 are $\gamma\gamma_i$ $(i = 1, 2, \cdots, n)$, and these are all distinct and occur with multiplicity one. So $(x - \phi')^2$ cannot divide $f(x)$, and we have a contradiction.

Also solved by D. Ž. Djoković.

Equivalence Relation in the Symmetric Group

5932 [1973, 949]. *Proposed by Richard Stanley, University of California at Berkeley*

Call two permutations σ and ρ in the symmetric group S_n equivalent (denoted $\sigma \sim \rho$) if every cycle c in the disjoint cycle decomposition of σ is some power (depending on c) of a cycle in the disjoint cycle decomposition of ρ. It is easily seen that $\sim$ is an equivalence relation. Let $E(n)$ denote the number of equivalence classes. Show that

$$\sum_{n=0}^{\infty} E(n)x^n/n! = \exp\left(\sum_{i=1}^{\infty} x^i/i\phi(i)\right),$$

where ϕ is the Euler totient function.

Solution by Albert Nijenhuis, University of Pennsylvania. Let T be any subset of $\{1, 2, \cdots, n+1\}$ which contains the last element, and let ρ be any permutation on $\{1, \cdots, n+1\} \backslash T$. Any cyclic ordering of T yields a cycle C; and a permutation $\sigma \in S_{n+1}$ is obtained by combining ρ and C; i.e., $\sigma = \rho C$. This construction produces each $\sigma \in S_{n+1}$ precisely once as T, ρ and the cyclic ordering of T are allowed to vary. Elements $\sigma = \rho C$, $\sigma' = \rho' C' \in S_{n+1}$ are equivalent if and only if (i) $T = T'$,

(ii) $\rho \sim \rho'$, (iii) $C' = C^m$, where m and $|T|$ are coprime. Hence,

$$E(n+1) = \sum_{k=0}^{n} \binom{n}{k} E(n-k) \frac{k!}{\phi(k+1)}, \qquad E(0) = 1.$$

Let $f(x) = \sum_{n=0}^{\infty} E(n)x^n/n!$; then this implies

$$f'(x) = f(x) \sum_{k=0}^{\infty} \frac{x^k}{\phi(k+1)}; \quad f(0) = 1.$$

Solution of this ordinary differential equation gives the result.

Also solved by L. Carlitz, L. E. Clarke (England), M. G. Greening (Australia), A. A. Jagers (Netherlands), Brian Peterson, Allen Stenger, and the proposer.

Editor's Note: Letting $E(n, k)$ denote the number of equivalence classes of permutations as defined but with exactly k cycles, Carlitz shows that

$$\sum_{n,k=0}^{\infty} E(n,k) \frac{x^n}{n!} t^k = \exp\left(t \sum_{n=1}^{\infty} \frac{x^n}{n\phi(n)}\right).$$

Infinite Complete Subgraph of a Random Graph

5933 [1973, 949]. *Proposed by P. L. Renz, Wellesley College*

Let $\mathscr{G}$ be a family of random graphs constructed on a fixed countably infinite vertex set V and having the property that the probability of $\{v, v'\}$ being an edge of a graph G of $\mathscr{G}$ is p, with $0 \leqq p \leqq 1$, for each distinct pair of vertices v and v' in V. What is the probability that a graph G in $\mathscr{G}$ contains an infinite complete subgraph?

Solution by L. E. Clarke, University of East Anglia, England. The required probability is clearly 0 if $p = 0$ and 1 if $p = 1$, so let us assume that $0 < p < 1$. It is possible to construct a probability space $(\Omega, \mathscr{F}, P)$, and to define on it independent random variables

$$X_{ij} \,(= X_{ji}) \qquad (i,j \in V;\, i \neq j),$$

each of which takes the values 1, 0 with respective probabilities p, $1 - p$. A suitable probability model is obtained by making correspond to each point ω of Ω the graph $G(\omega)$ with vertex set V defined by $\{i,j\}$ is an edge of $G(\omega)$ if and only if $X_{ij}(\omega) = 1$.

Now let S be a any non-empty finite subset of V containing, say, n vertices, and let A_S'be the set of those ω for which the vertices of S are the vertices of a maximal complete subgraph of $G(\omega)$. Then $\omega \in A_S$ if and only if any two vertices of S are joined by an edge of $G(\omega)$, but to each vertex j of $V - S$ there corresponds at least one vertex of S not joined to it by an edge of $G(\omega)$. Therefore

$$P(A_S) = p^{n(n-1)/2} \prod_{j \in V-S} (1 - p^n) = 0,$$

the second factor vanishing since $V - S$ is countably infinite. Therefore the probability that G has a finite maximal complete subgraph is

$$P(\bigcup_S A_S) \leqq \sum_S P(A_S) = 0$$

(since the class of non-empty finite subsets S of V is countable). Since any graph has at least one maximal complete subgraph (this follows readily from Zorn's lemma), it follows that the probability that G has an infinite maximal complete subgraph is 1.

Also solved by O. P. Lossers (Netherlands), M. S. Martin, J. H. Pitt, Joel Spencer, and the proposer.

A Constrained Equality

5935 [1973, 1067]. *Proposed by K. Selucký, Brno, Czechoslovakia*

Suppose

$$\frac{1}{x_1} + \frac{1}{s - x_2} + \frac{1}{s - x_3} + \frac{1}{x_4} = \frac{1}{s - x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{s - x_4},$$

where $s > x_1 \geqq x_2 \geqq x_3 \geqq x_4 \geqq \frac{1}{2}s$. Prove $x_1 = x_2$ and $x_3 = x_4$.

I. *Solution by R. O. Davies, The University of Leicester, England.* Rewriting the equation as

$$\frac{2x_2 - s}{(s - x_2)x_2} + \frac{2x_3 - s}{(s - x_3)x_3} = \frac{2x_1 - s}{(s - x_1)x_1} + \frac{2x_4 - s}{(s - x_4)x_4},$$

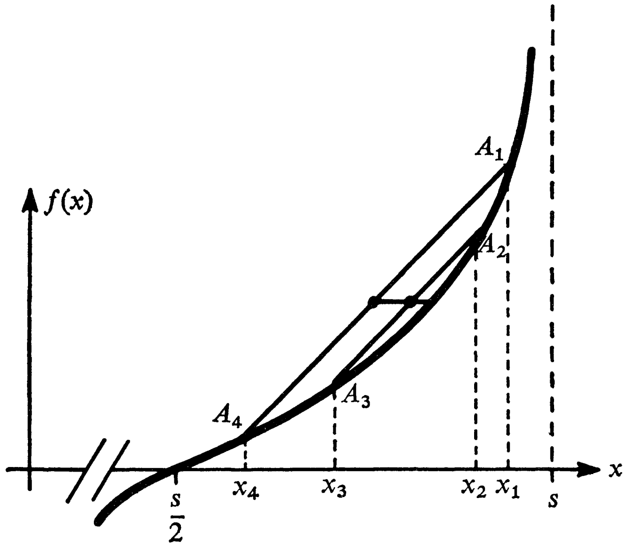
and observing that $(2x - s)/(s - x) \cdot x$ is strictly increasing (from 0 to ∞) in the range $s/2 < x < s$, we see that counterexamples are easily found. It is more difficult to find integer ones. I decided to try putting $x_2 = x_3$ and $x_4 = (3s/4) - t$, $x_1 = (3s/4) + t$ and by making judicious use of the known solution of $x^2 + y^2 = z^2$. I was able to discover the following:

$$s = 104,\ x_1 = 91,\quad x_2 = x_3 = 84,\ x_4 = 65.$$

II. *Solution by David Hertzig, University of Miami.* The change of variables $x_i = \frac{1}{2}(1 + \tan(\frac{1}{2}\theta_i))s$ converts the problem to:

Suppose $\tan\theta_1 - \tan\theta_2 = \tan\theta_3 - \tan\theta_4$, where $\frac{1}{2}\pi > \theta_1 \geqq \theta_2 \geqq \theta_3 \geqq \theta_4 \geqq 0$. Prove $\theta_1 = \theta_2$ and $\theta_3 = \theta_4$. Further, letting $y_i = \tan\theta_i$ gives $y_1 - y_2 = y_3 - y_4$, where $y_1 \geqq y_2 \geqq y_3 \geqq y_4 \geqq 0$. In this form, it is manifestly false that necessarily $y_1 = y_2$ and $y_3 = y_4$. For example, take $y_1 = 12/5$, $y_2 = 15/8$, $y_3 = 3/4$, $y_4 = 9/40$ corresponding to $x_1 = \frac{5}{6}s$, $x_2 = \frac{4}{5}s$, $x_3 = \frac{2}{3}s$, $x_4 = \frac{5}{9}s$.

III. *Solution by W. J. Gilbert, University of Waterloo, Ontario.* The problem as stated is clearly false. Perhaps there is some information missing. (If we impose the condition $x_1 + x_4 \geqq x_2 + x_3$, then it will follow, from the concavity, that $x_1 = x_2$ and $x_3 = x_4$.)



Let $f(x) = 1/(s-x) - 1/x$, so that the equality can be written as $f(x_2) + f(x_3) = f(x_1) + f(x_4)$. Let A_i be the point $(x_i, f(x_i))$. Then the equality holds when the midpoint of A_1A_4 is on the same level as the midpoint of A_2A_3. The graph of $f(x)$ is concave upwards when $s/2 < x < s$ because $f''(x) = 2/(s-x)^3 - 2/x^3 > 0$ if $0 < s - x < x$. Hence, for any choices of s, x_4, x_3 and x_2, such that $s > x_2 \geqq x_3 > x_4 \geqq s/2$ the equality can be solved for x_1 to give a counterexample. For example, if $s = 1$, $x_4 = \frac{1}{2}$, $x_3 = x_2 = \frac{2}{3}$, then $x_1 = (1 + \sqrt{13})/6$.

Also solved by Bennett College Team, J. C. Binz (Switzerland), James Boone & Mary King, Robert Breusch (New Zealand), L. E. Clarke (England), George Crofts, E. de Jonge (Netherlands), G. J. Ford, D. P. Giesy, F. Gobel (Netherlands), M. G. Greening (Australia), A. C. Hindmarsh, R. A. Jacobson, P. T. Joslin, Woon-Chung Lam & Radha G. Nath, O. P. Lossers (Netherlands), D. E. Mackenzie (Australia), L. E. Mattics, S. C. Otermat, J. P. Robertson, Saint Olaf College Students, J. W. Shaw, Jr., E. D. Shirley, Michael Skalsky, and Southern University Primer for Research Group. There were seven alleged proofs of the stated result.

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.

Contemporary Mathematics. By Bruce E. Meserve, Max A. Sobel. Prentice-Hall, Englewood Cliffs, N. J., 1972. x + 550 pp. $12.95. (Telegraphic Review, October 1972.)

This is an excellent text for a one-or two-semester mathematics course designed to prepare teachers for grades kindergarten through 8. I used it for a one-semester 4-credit course on content and methods of elementary school mathematics, a course which satisfies the Missouri state requirements for a methods course. Methods are taught most effectively, I believe, in the context of the subject matter.

The contents include recreational mathematics, sets, logic, structure of the number system, high-school algebra and geometry, coordinate geometry, statistics, and probability. I covered sets and logic, then skipped to non-metric and metric geometry, concluding with number structure. I did not want to slight geometry by leaving it to the end of the course, because I feel this subject is particularly difficult for self-study. Other subjects are perhaps more amenable to the exposition in the teacher's manual of an elementary text.

Some people criticize texts such as this for duplicating junior-high school mathematics. It is true that much of the content is in the junior-high curriculum, but the treatment here is much more sophisticated than in junior-high texts. Most of my students thought the level was appropriate.

The text avoids the Scylla of triviality and the Charybdis of formalism. The exposition is clear. The problems are unusually well-chosen, with enough routine exercises for students to master concepts and a few starred difficult problems to challenge the most able. A major virtue are the pedagogical exercises in every section. Here is an example on modular arithmetic:

> This exercise may be used to provide a laboratory type experience. Consider a soldier facing in a given direction. He is then given various commands, such as "right face," "left face," "about face," and "as you were." The last command tells him to retain whatever position he may be in at the time.

> A few specific examples of his movements should clarify matters. If he is facing to the north and is given the command "about face," then in his new position he is facing to the south. If we let R represent the command "right face," A represent the command "about face," and L represent the command "left face," then R followed by A is equivalent to the single command L....
> (a) Make a [multiplication] table summarizing all possible movements....

One student taught this exercise to the others. For each section of the book, students chose a pedagogical exercise, presenting it to the class with the overhead projector. They improved markedly in poise.

Many of the pedagogical exercises serve the important function of familiarizing future teachers with texts and other media used in schools. Unfortunately there is no bibliography, and the professor must comb the text carefully to get references. I put on reserve in the library a number of elementary texts. Other useful resources included Cuisinaire rods, a large set of blocks, a balance, and a kit for making three-dimensional models. I also asked the students to participate in and report on an elementary mathematics class over spring vacation.

It is a tragedy that many elementary teachers dislike mathematics and transmit this attitude to students. A well-planned college course with such an appealing text as *Contemporary Mathematics* can change attitudes and impart self-confidence to future teachers.

ELIZABETH BERMAN, Rockhurst College

Real Analysis and Probability. By Robert B. Ash. Academic Press, New York, 1972. xv + 476 pp. $16.50. (Telegraphic Review, August-September 1972.)

I used this book as the text for our standard first year graduate level course in real analysis. The principal auxiliary texts were Hewitt and Stromberg, *Real and Abstract Analysis* (Springer Verlag, 1965) and Rudin, *Real and Complex Analysis* (2nd edition, McGraw-Hill, 1974). In many ways, this book is comparable with Rudin's book. Both books treat measure and integration theory as the means to other ends. They both cover the basic material as quickly and efficiently as possible. Both leave the more technical aspects of the theory to more encyclopedic books like Hewitt and Stromberg. Both give elegant but terse proofs, sometimes at the expense of needed motivation. And overall, both are excellent books. The development in Rudin is slanted towards functional analysis and complex analysis, while the development in Ash is slanted towards probability theory. I chose Ash as the text partly because I felt his bias for probability would balance my bias for functional analysis.

Here is how I used this text. Fall quarter was devoted to Chapters 1 and 2 on Measure and Integration Theory. Topological considerations are avoided but Lebesgue-Stieltjes measures in R^n are treated. Winter term was spent on Chapter 3: Functional Analysis. Extra time was devoted to applications to Fourier series. Section 3.5 is an excellent concise introduction to topological vector spaces; very little

extra effort was needed to get the Krein-Milman theorem. Chapter 4 is aptly titled "The Interplay between Measure Theory and Topology." The Daniell integral is quickly developed and several versions of the Riesz representation theorem are obtained. Part of Spring quarter dealt with this material. The remainder of the quarter was devoted to a brief introduction to probability: Chapter 5 in Ash plus the Kolmogorov Strong Law of Large Numbers in Section 7.2.

Chapters 1 to 4 and the Appendix on Topology are available in a separate book entitled *Measure, Integration, and Functional Analysis.* I assigned the larger text because it cost very little more and because every mathematician should own at least half of a probability book. The book is actually designed for a graduate course in modern probability; Chapters 6–8 are titled "Conditional Probability and Expectation," "Strong Laws of Large Numbers and Martingale Theory," and "The Central Limit Theorem." The first third of such a course would probably cover Chapters 1 and 2 as well, and so the first term of these two courses could be combined into one.

A few specific comments are in order. Several topics that I regard as important were relegated to elaborate exercises in Ash. Among them are the change of variable formula involving Jacobians (Ex. 2.3.6), the theory of complex measures (Ex. 2.2.6, 2.4.10 and 2.4.11), and the conjugate spaces of the various L^p spaces (Ex. 3.3.11). The Riesz representation theorem is obtained for compact spaces, but not for locally compact spaces. Section 2.3 concerning absolutely continuous functions and functions of bounded variation is nicely organized but just too terse. There seems to be a gap in the proof of Theorem 2.3.9; one needs to know that if f is of bounded variation and $f(x) = 0$, except for a countable set, then $f'(x) = 0$ a.e. (see Rudin's 8.19). A nice feature is that complete answers are given (at the back) for over one-third of the exercises, including the more elaborate ones. An answer book is available with answers to the remaining problems.

As happens with Rudin's book, the better students liked Ash and the weaker students found it difficult going. I found Ash a satisfactory alternative to Rudin's fine book and would not hesitate to use it again.

KENNETH A. ROSS, University of Oregon

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S*, P**, L**, *Russian for the Mathematician.* S.H. Gould. Springer-Verlag, 1972, xi + 211 pp, $8.80 (P). A splendid self-study "crash course" textbook, consisting of five chapters of preparation for 70 pages of reading selections. The selections are from analytic geometry and calculus, elementary algebra and analysis, and more advanced topics, and include English titles and summaries (but not word-for-word translations). The preliminary chapters cover the Cyrillic alphabet, pronunciation (important even for the reader who merely wishes to learn to read), endings and mathematical vocabulary, systematically developed. All of it is clearly explained and motivated. PJC

HISTORY, L*, *The Einstein Decade (1905-1915).* Cornelius Lanczos. Acad Pr, 1974, xiv + 230 pp, $12.50. An interesting and valuable book written by one of Einstein's collaborators. In the first half of the book, the author discusses Einstein the person, Einstein the philosopher, Einstein the physicist. In the second half he discusses briefly each of Einstein's papers written during this period. An informative tribute to Einstein by a friend and colleague. Highly recommended. SG

COMBINATORICS, L*, *Combinatorial Identities.* Henry W. Gould. (1239 College Ave., Morgantown, W. Va., 26506) 1972, $10. Correction of Jan., 1973, TR. Published by author.

LINEAR ALGEBRA, T(16-18: 2), P, *Generalized Inverses: Theory and Applications.* Adi Ben-Israel, Thomas N.E. Greville. Wiley, 1974, xi + 395 pp, $22.50. The authors classify and characterize generalized inverses for square or rectangular matrices according to the four basic equations of R. Penrose. From this unified viewpoint, this book surveys various types of generalized inverses including operators between Hilbert spaces. Some computing methods and applications to linear programming and to electrical networks are included. The chapter on the relation between generalized inverses and minimum-norm least-squares solutions inspires readers to suspect (or expect) some other areas to which the inverses apply. Indeed, applications to statistics, control and system theory are cited in the suggested further reading. The over 450 exercises are primarily intended to impart information rather than to test understanding. I-CH

ALGEBRA, P*, *Topics in Algebraic and Analytic Geometry.* Phillip Griffiths, John Adams. Princeton U Pr, 1974, vi + 219 pp, $5 (P). Although classified as algebra, this book also has a lot of topology. It's essentially concerned with differences between algebraic varieties, topological manifolds, and analytic manifolds, mostly over the complex numbers. An excellent book, reasonably priced. Readable for anyone with some background in sheaves and complex manifolds. PJM

ALGEBRA, T*(15-17: 2), S, L. *Basic Algebra I.* Nathan Jacobson. Freeman, 1974, xvi + 472 pp, $13.95. Not that basic, this book includes real mathematics. It is a new viewpoint of the really basic structures of modern algebra. A quick study of the integers à la Herstein sets the level at about that of the classic "Topics in Algebra." But the succeeding chapters--monoids and groups, modules over a PID, Galois theory, polynomial equations, multilinear algebra and classical groups (geometry), Lie and Jordan and Boolean algebras reflect the passage of time and a serious rethinking of what is basic in algebra. Lots of examples, exercises, and good writing should make this book worthy of consideration for a really substantial year course. JAS

ALGEBRA, T(16-18), S, L. *Groups, Rings, Modules.* Maurice Auslander, David A. Buchsbaum. Har-Row, 1974, x + 470 pp, $26. A second course in abstract algebra, categorically phrased. Groups and monoids are used as motivation for categories, and the emphasis then passes to modules and rings. No homological algebra, but just about everything else. Part I could be a very condensed one semester algebra course (the authors consider it leisurely). Lots of exercises, good index, but no bibliography. PJM

FINITE MATHEMATICS, T*(13-14: 1). *Applied Finite Mathematics.* Howard Anton, Bernard Kolman. Acad Pr, 1974, xv + 475 pp, $10.95. Fine pedagogic effort. Problems and examples well above average. Clear writing, attractive format. Matrices, linear programming including non-standard problems, probability and statistics with Bayes formula and hypothesis testing, Markov chains, game theory, computers. LH

CALCULUS, T**(13-14: 3), L. *Calculus.* Lynn Loomis. A-W, 1974, xvi + 1024 pp, $15.50. The order and selection of topics is traditional for a three term text; the treatment is intuitive at first--"at about the level of Cauchy" the author claims. Rigor is slowly increased as the reader's sophistication increases. A persistent theme: good approximations. Huge problem sets with an excellent variety of problems. Covers through triple integrals with a final chapter containing rigorous overview. An excellent text. TAV

CALCULUS, T(13-14: 3). *Analytic Geometry and the Calculus, Third Edition.* A.W. Goodman. Macmillan, 1974, xxii + 1031 pp, $14.95. The author has deleted the chapter on linear algebra from the second edition (TR, October 1969) and reorganized the chapter on multiple integrals, but a great deal of the text remains unchanged. LLK

CALCULUS, T(13: 1, 2). *Calculus with Applications to Social and Life Sciences.* William L. Hart. Page Ficklin, 1973, xiv + 498 pp, $11.95. Differentiation and integration, exponential, logarithmic and trigonometric functions, solid analytic geometry, partial differentiation, continuous probability distributions, infinite series, double integrals, least square methods. Applications throughout to social and biological (and physical) sciences. Theoretical level is right for its intended audience. First fifty pages are "Precalculus Review." DFA

CALCULUS, T(13: 1). *Calculus with Applications: A Brief Course.* William L. Hart. Page Ficklin, 1973, xii + 387 pp, $11.25. The first two-thirds of Hart's *Calculus with Applications to Social and Life Sciences* (see above), which does not contain the chapters on the calculus of trigonometric functions, infinite series, double integrals and least square methods. DFA

REAL ANALYSIS, T*(15: 1, 2). *Mathematical Analysis, Second Edition.* Tom M. Apostol. A-W, 1974, xvii + 492 pp, $14.95. An updating of the first edition. Gone are line integrals, surface integrals and vector analysis; included are two chapters on Lebesgue integration. Careful logical independence of several chapters allows great flexibility in choice of topics. An impressive book. TAV

GEOMETRY, S(15-18), P, L. *Arrangements and Spreads.* Branko Grünbaum. CBMS Reg. Conf. in Math., No. 10. AMS, 1972, 114 pp, $4.50 (P). Neglected aspects of elementary geometry in two dimensions, presented in an understandable and attractive format. *Arrangements* of lines are the structures determined by finite families of straight lines in the real projective plane; *spreads* of curves may be considered as continuous analogues of arrangements of curves. Also treated are arrangements of pseudolines and curves, and topological planes. Replete with conjectures and research problems. PJC

TOPOLOGY, S(16-17), P. *Fixed Point Theorems.* D.R. Smart. Tracts in Math., No. 66. Cambridge U Pr, 1974, viii + 93 pp, $8.95. Starting with Brouwer's fixed point theorem, the author explores generalizations and extensions: Schauder's theorem, the fixed point property for topological spaces, fixed points for families of mappings, and fixed points for many-valued mappings. Also included are applications to differential equations, game theory and functional analysis. The final chapter deals with numerical invariants used in fixed point theory (e.g., the Lefschetz number). No exercises, except completing proofs; a short index and a good bibliography. 10¢ a page seems a little steep, but it is a good book. PJM

TOPOLOGY, T(16-17: 1), S, L. *Differential Topology.* Victor Guillemin, Alan Pollack. P-H, 1974, xvi + 222 pp, $14.95. A good subtitle for this book would be "without algebraic topology." The authors do discuss cohomology, but from a classical, integration-on-manifolds viewpoint. References are rated G, PG, R, X in order of increasing difficulty. Plentiful exercises. An excellent book at an (almost) reasonable price. PJM

TOPOLOGY, P. *The Atiyah-Singer Theorem and Elementary Number Theory.* F. Hirzebruch, D. Zagier. Publish or Perish, 1974, xii + 262 pp, $10. Signature theorems which are consequences of the Atiyah-Singer index theorem are specialized to obtain well known number theoretic results which involve Dedekind sums. CEC

PROBABILITY, T*(14: 1, 2), L. *Elementary Probability Theory with Stochastic Processes.* Kai Lai Chung. Springer-Verlag, 1974, x + 325 pp, $12. An important book, not only because it signals the entrance of Springer into undergraduate texts, but because it is an extremely well-conceived and executed text. The material chosen would provide the reader enough background to read most post-calculus probability texts, e.g., Feller's classic. Many exercises and examples. A concise, self-contained introduction to Markov chains concludes the work. TAV

PROBABILITY, S(18), P. *Gibbs States on Countable Sets.* Christopher J. Preston. Tracts in Math., No. 68. Cambridge U Pr, 1974, ix + 128 pp, $10.95. Introduction to recent results and techniques in classical lattice statistical mechanics, aimed at mathematicians with no background in physics. Obtains Markov random fields on finite and countable graphs, then removes the graph structure. Valuable, with many notes and references for further reading. DFA

STATISTICS, T(2). *Statistical Analysis: A Decision-Making Approach.* Robert Parsons. Har-Row, 1974, xvi + 836 pp, $15.95. Comprehensive introductory pre-calculus text, designed for business administration and economics students at either the undergraduate or graduate level. Particularly good, though sophisticated, presentation of Bayesian materials, and excellent, though slightly biased, discussions of the Bayesian versus the classical point of view. A shorter version is available (see below). Besides the usual topics, both versions include simple and multiple regression and correlation analysis, and index numbers. RSK

STATISTICS, T(1, 2). *Statistics for Decision Makers.* Robert Parsons. Har-Row, 1974, xiv + 638 pp, $12.95. Shorter version of the author's *Statistical Analysis: A Decision-Making Approach.* Omits chapters on the F-distribution, decision-making under uncertainty, decision tree analysis and time series analysis. RSK

STATISTICS, T*(16-17: 2), L. *Introduction to the Theory of Statistics, Third Edition.* Alexander M. Mood, *et al.* McGraw, 1974, xvi + 564 pp, $12.50. Thorough revision and reorganization of the well-known 1963 second edition by Mood and Graybill. Essentially all material requiring matrices (e.g., general linear model, experimental design models, general multivariate normal distribution) has been eliminated. Bayesian methods have been de-emphasized. What remains is a strong updated introduction to classical statistics. RSK

STATISTICS, T?(13: 1). *Practical Statistics and Probability.* Robert Loveday. Cambridge U Pr, 1974, xiii + 205 pp, $3.50 (P). Primarily descriptive with little actual content, e.g., no mention of the t-distribution. Approximately one-third of the pages are exercises. One wonders about a table that states that a probability less than 2 1/2% is "very significant" while a probability less than 1% is "very significant indeed."RSK

STATISTICS, T(13-14: 1, 2), S, L. *Statistical Analysis for Managerial Decisions, Second Edition.* John C.G. Boot, Edwin B. Cox. McGraw, 1974, xiv + 651 pp, $13.50. This edition has a new chapter on nonparametric statistics, more emphasis on the t-test, and more on hypotheses testing and decision theory. First edition reviewed here in August 1970. FLW

STATISTICS, T(14: 1), S. *Statistics for Biologists, Second Edition.* R.C. Campbell. Cambridge U Pr, 1974, xiv + 385 pp, $5.95 (P); $15.50. For a precalculus course. Covers nonparametric methods, point and interval estimation, ANOVA, regression, and non-normal distributions (Poisson, binomial). The examples and problems are taken from biology, and should appeal to students of this field. Others wishing to learn statistics will fail to grasp the generality of the methods presented. Extensive problem sets and tables. TAV

STATISTICS, P. *Reliability and Biometry: Statistical Analysis of Lifelength.* Ed: Frank Proschan, R.J. Serfling. SIAM, 1974, x + 815 pp, $24.50. Papers from a July, 1973 conference at Florida State U., Tallahassee, whose purpose was to encourage dialog between two historically distinct statistical specialities with an identical objective: the study of lifelength. LAS

COMPUTER SCIENCE, S(14-15). *Understanding the IBM 360 and 370 Computers with Machine Language Programming.* William T. Batten. P-H, 1974, xxiii + 453 pp, $15.50. Instruction sets, communication, loading, use of registers, arithmetic, etc., at an introductory level. Well illustrated. RWN

COMPUTER SCIENCE, T(15-17), S, P. *Implementing Software for Non-Numeric Applications*. W.M. Waite. P-H, 1973, xv + 510 pp, $14.95. The operation and implementation of processors for lists and strings. The fundamental structure emphasized are linked lists. Emphasizes LISP and HELP (a baby LISP) on WISP, an abstract machine. General enough to be useful to other implementations. RWN

COMPUTER SCIENCE, T(13-15: 1, 2), S, P, L. *Business, Computer Systems and Applications*. Alan L. Eliason, Kent D. Kitts. SRA, 1974, xi + 335 pp, $6.95 (P). This text describes the most common business computer applications. Flowcharting and INPUT/OUTPUT are stressed while actual computer programming has been avoided. Focuses on accounting applications. Review questions and exercises conclude each chapter. RB

COMPUTER SCIENCE, T(13: 1), L. *Mathematics for Data Processing*. Frank J. Clark. Reston, 1974, xii + 305 pp, $13. Positional numeration systems, Boolean algebra and networks, and their relationship to computers. Elementary functions (including logarithms), matrices, linear programming, statistics, integrated with FORTRAN language. Exercises. LCL

COMPUTER SCIENCE, S. *The Little LISPer*. Daniel P. Friedman. SRA, 1974, 58 pp, $3.95 (P). A highly unconventional yet effective informal introduction to LISP, a powerful programming language derived from recursive function theory by John McCarthy. This booklet consists entirely of brief questions paired (in a separate column) with equally brief answer-explanations. All "instruction" is done in the reader's mind by a Socratic synthesis of relationships entailed by bits and pieces of previously revealed facts. LAS

COMPUTER SCIENCE, S, P. *Decision Tables*. Michael Montalbano. SRA, 1974, 191 pp, $7.50 (P). A short textbook designed for programmers, system analysts and computer scientists. Discusses logic, flow charts, decision tables and decision-table translators. Exercises with answers. RWN

COMPUTER SCIENCE, T(14-15: 1), L*. *Discrete Computational Structures*. Robert R. Korfhage. Comp. Sci. and Appl. Math. Acad Pr, 1974, xiii + 381 pp, $13.95. A selection of topics from discrete mathematics relevant to computing. Includes graphs, trees, groups, lattices, and Boolean algebras. Also formal languages, propositional calculus, combinatorics and discrete probability. Good applications of and to computing. Not intended to be thorough or rigorous. Exercises. Too few references. RWN

COMPUTER SCIENCE, S(14-15). *Floating-Point Computation*. Pat H. Sterbenz. P-H, 1974, xiv + 316 pp, $15. Number systems, underflow, overflow, double-precision, error analysis, etc. Especially oriented toward users of the IBM 360. Exercises. RWN

COMPUTER SCIENCE, T(14), S, L. *Computers: A Systems Approach*. Ned Chapin. Van N-Rein, 1971, xvi + 686 pp, $14.95. A survey of hardware, software, man-machine communication and system management. Glossary. RWN

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-1: Gesellschaft für Informatik e. V.: 3. Jahrestagung*. Ed: Wilfried Brauer. Springer-Verlag, 1973, xi + 508 pp, $13.20 (P). Proceedings of the third annual meeting in Hamburg, October 1973. JAS

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-2: Gessellschaft für Informatik e. V.: 1. Fachtagung über Automatentheorie und Formale Sprachen*. Ed: Karl-Heinz Böhling, Klaus Indermark. Springer-Verlag, 1973, vii + 322 pp, $10.70 (P). Proceedings of the conference on automata theory and formal languages, Bonn, July 1973. JAS

APPLICATIONS (BEHAVIORAL SCIENCES), S(14-16), P, L**, *Topics in Behavioral Mathematics.* T.L. Saaty. MAA, 1973, iii + 465 pp, $3.50 (P). These notes, as those immediately below, were produced from lectures given at the 1973 Summer Seminar conducted at Williams College by the MAA. Prepared explicitly to import motivation and new ideas to college teachers. In addition to the content (e.g., optimization, scheduling, queues, conflict resolution, epidemics) a reading could well influence an individual's teaching methods and research interests. LCL

APPLICATIONS (BEHAVIORAL SCIENCES), S(14-16), P, L**, *Notes of Lectures on Mathematics in the Behavioral Sciences.* Henry A. Selby. MAA, 1974, 325 pp, $3 (P). Serves to enlighten the entire mathematical community to significant applications of mathematics (other than statistics) to the behavioral sciences. Evidence provided by six prominent researchers in the behavioral sciences. The mathematics here is elementary (matrix algebra, calculus, probability). Should be available to every undergraduate mathematics teacher. LCL

APPLICATIONS (BIOLOGY), T*(17-18: 2), S*, P*, L*, *The Genetic Structure of Populations.* Albert Jacquard. Biomathematics, V. 5. Springer-Verlag, 1974, xviii + 569 pp, $39.40. Translated, expanded and revised in consultation with the author from the 1970 French original. A massive introductory survey, beginning with basic genetic and mathematical facts (two appendices cover matrix algebra and linear difference equations), and proceeding through eleven chapters on, first, non-evolutionary and second, evolutionary theory to a concluding three chapters on human population structure. Many specific case studies illuminate the theory throughout the book. LAS

APPLICATIONS (BUSINESS), T(13-14: 2), *Mathematical Methods for Social and Management Scientists.* T. Marll McDonald. HM, 1974, 553 pp, $11.95. Primarily for business students. Emphasizes mechanics and elementary applications. Good problem sets, clear examples. Finance, statistics, linear systems, linear programming, elementary calculus. *Instructor's Manual* provides answers to problems without answers in the text itself, BASIC programs, and hints for classroom presentation. LH

APPLICATIONS (ECOLOGY), P, L, *The Estimation of Animal Abundance and Related Parameters.* G.A.F. Seber. Hafner, 1973, xii + 506 pp, $39.95. Comprehensive research treatise on methods of census-taking for various wildlife populations, with methods classified according to population studied and nature of sampling information available for the population. Demands from the reader a facility in mathematical statistics. Includes numerical examples based on "real-life" data. Bibliography of 26 pages. PJC

APPLICATIONS (ECONOMICS), S(17-18), P, L*, *The Computation of Economic Equilibria.* Herbert Scarf. Yale U Pr, 1973, x + 249 pp, $11.50. "One of the major triumphs of mathematical economics...has been the proof of the existence of a solution for the neoclassical model of economic equilibrium." Unfortunately, the proof of this existence is wholly nonconstructive, relying on Brouwer-type fixed point theorems. "As a consequence, general equilibrium analysis has remained...far removed from its ultimate purpose as a method for the evaluation of economic policy." Scarf develops in this monograph an efficient computational procedure (using simplicial subdivisions and combinational results similar to Sperner's lemma) for the approximation of the required fixed points. LAS

APPLICATIONS (ECONOMICS), T(17: 1), S, P, *The Theory of Income Distribution.* Harry G. Johnson. Lect. in Econ., No. 3. Gray-Mills, 1973,

292 pp, $17. Rewritten and polished lecture notes--discursive and diagrammatic, not mathematical--from the author's graduate course taught during the past eight years at the University of Chicago. Focuses on investment and growth in the context of full employment. Includes final (and part of Ph.D. qualifying) examinations. LAS

APPLICATIONS (ECONOMICS), T(16-17: 2), S, L*, *Theory of Microeconomics.* Trout Rader. Acad Pr, 1972, xvi + 369 pp, $14.95. A rigorous systematic treatment of production, consumption and trade using mathematical methods of elementary real analysis. Intended as a text for economics graduate students, it would serve as an excellent source for mathematicians wishing to learn basic economic equilibrium theory. LAS

APPLICATIONS (ECONOMICS), S(16-18), P, *Lecture Notes on Economics and Mathematical Systems-90: Posterior and Predictive Densities for Simultaneous Equation Models.* J.-F. Richard. Springer-Verlag, 1973, vii + 226 pp, $7.70 (P). Deals with the "identification problem" when estimating parameters for econometric models. Bayesian approach uses prior stochastic information to get posterior marginal densities or posterior moments. Considers prior densities whose corresponding posterior densities are stable under linear transformations. LH

APPLICATIONS (ECONOMICS), T(16-17: 1), *Econometric Techniques and Problems.* C.E.V. Leser. Griffin's Stat. Mono., No. 20. Hafner Pr, 1974, viii + 144 pp, $4.95 (P). Revision of the author's 1966 monograph (TR, June 1967). Presumes a background in calculus and statistics through regression analysis, and "a good acquaintance with the concepts of economics." Half devoted to the methods of simple and multiple regression and simultaneous equation estimation, half to applications in the areas of production functions, demand analysis, and macro-economic models. RSK

APPLICATIONS (ECONOMICS), T**(13-16: 1), S**, L**, *Introduction to Quantitative Methods in Economics.* D.E. James, C.D. Throsby. Wiley, 1974, xv + 335 pp, $14.95. Very well-written introduction to mathematical economics. Elementary single and multi-variable calculus, matrix algebra presented via economic concepts. Optimization, equilibrium, multipliers, input-output systems, linear programming, regression models. Many examples, diagrams, problems with answers. LH

APPLICATIONS (ECONOMICS), S(16-18), P, *Lecture Notes in Economics and Mathematical Systems-91: General Equilibrium with Price-Making Firms.* Thomas Marschak, Reinhard Selten. Springer-Verlag, 1974, xi + 246 pp, $8.50 (P). Studies static general equilibrium in an economy of noncolluding firms and consumers. Firms treated as players in a noncooperative game able to imagine, with limited information, others' responses to their deviation from current situation. Equilibrium study extended to oligapolistic economy. LH

APPLICATIONS (ENGINEERING), P, *Variational Methods Applied to Problems of Diffusion and Reaction.* W. Strieder, R. Aris. Tracts in Nat. Philo., V. 24. Springer-Verlag, 1973, ix + 109 pp, $13.90. Bounds on the diffusion coefficient of a porous medium, on the rate of diffusion limited reaction, and on effectiveness factors in heterogeneous catalysis. Some analysis of experimental data. DFA

APPLICATIONS (ENGINEERING), P, *Network Theory.* Ed: R. Boite. Gordon, 1972, x + 577 pp, $35. Proceedings of the second NATO Advanced Institute on Network Theory held in Knokke, Belgium in September 1969. JAS

APPLICATIONS (ENGINEERING), P*, L*. *Traffic Science.* Ed: Denos C. Gazis. Wiley, 1974, ix + 293 pp, $19.95. Four coordinated chapters by four different authors review all principal models of traffic problems: flow theories, delay problems at intersections, urban control problems (e.g., synchronization of traffic signals), and traffic generation, distribution and assignment. Mathematical techniques include differential equations, probability theory, mathematical programming, control theory and network theory. LAS

APPLICATIONS (ENGINEERING), P. *Computation for Process Engineers.* G.L. Wells, P.M. Robson. Halsted Pr, 1973, xi + 192 pp, $12.75. Introduction to computational terminology, numerical methods, optimization, simulation and the use of computer programs in the chemical industry. RWN

APPLICATIONS (ENGINEERING), T(16-17: 1), P. *Integrated Theory of Finite Element Methods.* John Robinson. Wiley, 1973, xxi + 428 pp, $29.50. A physical approach using characteristic matrices for stress and strain elements, especially isoparametric elements. Applications to curved beams, cracks, and large coupled systems of structures. RWN

APPLICATIONS (ENGINEERING), P. *Mechanics Today, Volume 1, 1972.* Ed: S. Nemat-Nasser. Pergamon Pr, 1974, xix + 384 pp, $30. A collection of seven surveys of current research, inaugurating a new series published on behalf of the American Academy of Mechanics. LAS

APPLICATIONS (ENGINEERING), T(18: 1), P. *Theory and Application of the z-Transform Method.* E.I. Jury. Krieger, 1973, xiii + 330 pp, $16.50. For engineers and systems theorists, with enough rigor for applied mathematicians. Solution of linear difference equations, stability for linear discrete systems, application of the convolution z-transform to and periodic modes of oscillation for nonlinear discrete systems, approximation techniques for continuous systems. Problems. 300 citations. Reprint of 1964 edition. DFA

APPLICATIONS (ENGINEERING), T(16-17: 2). *Computer Simulation of Dynamic Systems.* Ralph J. Kochenburger. P-H, 1972, xii + 530 pp, $16.50. Prerequisites: some programming, circuits, Laplace transforms. Introduces fundamentals of simulating time-dependent systems using analog, digital and hybrid computers. Includes simulation of linear and nonlinear operations, function generation, random disturbances, and systems with distributed parameters. RWN

APPLICATIONS (ENGINEERING), S(16), P. *Matrix Computer Methods of Vibration Analysis.* D.J. Hatter. Halsted Pr, 1973, 206 pp, $15.75. Introduction to computational matrix algebra, with application to problems arising with various kinds of vibrating systems. Flow charts throughout, with corresponding Fortran programs in appendices. Of use to practicing engineers and programmers. DFA

APPLICATIONS (ENGINEERING), S(17), P. *Lectures in Mathematical Models of Turbulence.* B.E. Launder, D.B. Spalding. Acad Pr, 1972, v + 169 pp, $2,50. Description of available models for simulating turbulence transport processes in convective flows. From lectures at Imperial College, London, 1971. Reproduces projected slides, gives commentary, provides 80 citations. Discoursive, with no calculations or exercises. DFA

APPLICATIONS (ENGINEERING), S(14-15), L. *The Geometry of Rotations About A Point: A Practical Treatise.* Milton Felstein, R 1, Box 126, Albany, Ohio 45710. 1974, 69 pp, $7.25 (P). Practical applications of 3x3 matrices for solving rotation problems in solid geometry of importance to

engineers (e.g., sequences of rotations, change of coordinates, etc.). A cookbook approach which is unnecessary given the present emphasis on linear algebra in the undergraduate curriculum; thus the principal value of this presentation may be as a source of problems for linear algebra instructors. LCL

APPLICATIONS (ENGINEERING), T(15-17: 1), S, P, L. *The Fast Fourier Transform.* E. Oran Brigham. P-H, 1974, xiii + 252 pp, $18.95. The FFT, an efficient algorithm (due to Cooley and Tukey) for approximating the Fourier transform, is developed intuitively in terms of Fourier transforms and discrete Fourier transforms. Properties, applications and extensions. Includes Algol and Fortran codes. Exercises. RWN

APPLICATIONS (ENGINEERING), T(17), P. *The Finite Element Method: Fundamentals and Applications.* Douglas H. Norrie, Gerard de Vries. Acad Pr, 1973, xiii + 322 pp, $23. Variational and residual trial function methods. Ritz finite element method (classical and Hilbert space). Application of the finite element methods to problems in solid and structural mechanics and fluid flow and generally to those involving the Laplace, Helmholtz, wave, and diffusion equations. Many examples and references. A few exercises. DFA

APPLICATIONS (ENGINEERING), P. *Heat and Concentration Waves: Analysis and Applications.* G. Alan Turner. Acad Pr, 1972, xxviii + 233 pp, $16.50. Treats time-varying temperatures of concentrations as waves, thereby attempting to unify analyses of various physical problems. Allows the medium to be moving, and lets stationary material affect the wave. Theoretical, but studies problems of measurement and computational processes. DFA

APPLICATIONS (HISTORY), T?(13), S, P, L*. *An Introduction to Quantitative Methods for Historians.* Roderick Floud. Princeton U Pr, 1973, ix + 220 pp, $7.50. Simple techniques of descriptive and analytical statistics, to the level of simple linear regression. No previous mathematical knowledge or maturity assumed. Careful use of data sets taken from historical literature as examples; no exercises. Discussions of "imperfect" data, how and when to use the computer. Useful bibliography. PJC

APPLICATIONS (INFORMATION THEORY), P. *Second International Symposium on Information Theory.* B.N. Petrov, F. Csáki. Akademiai Kiado, 1973, 451 pp, $20. Papers on information theory, coding theory, statistical methods, communications systems from a September, 1971 conference at Tsahkadsor, Armenia. LAS

APPLICATIONS (INFORMATION THEORY), S(15), L. *Information Theory.* John F. Young. Wiley, 1971, viii + 168 pp, $8.95. Elementary, discursive introduction to the subject, written by a cybernetician. Intended audience includes engineers, mathematicians, physicists, psychologists. Extensive bibliography directs the reader to more specialised works. DFA

APPLICATIONS (INFORMATION THEORY), S(15-18), P, L. *The Measurement of Verbal Information in Psychology and Education.* Klaus Weltner. Springer-Verlag, 1973, xiii + 185 pp, $22.40. Extends information theory concept of a bit to subjective information in order to model man as a receiver with some internal state. Develops and justifies empirical measuring techniques for bits of subjective information, retaining notion of value as novelty. Application to teaching: bits will index complexity. LH

APPLICATIONS (LAW), P, L. *Law and Logic: A Critical Account of Legal Argument.* Joseph Horovitz. Lib. of Exact Philo., V. 8. Springer-Verlag,

1972, xv + 213 pp, $19.90. Philosophical investigation of the ideal nature of legal argument. The author proceeds by analyzing the views of contemporary writers on the subject and concludes that "legal argument, insofar as it is rational, must be considered formal in nature, i.e., formalizable at least in principle." PJC

APPLICATIONS (LINGUISTICS), S(14-16), L. *Set Theory and Syntactic Description.* William S. Cooper. Mouton, 1974, 52 pp, $3.90 (P). Reprint of a 1964 monograph gently sketching the elements of set theory (with certain additional operations) as a basis for syntactic description. Illustrated with detailed grammars for several sublanguages of spoken English. LAS

APPLICATIONS (LINGUISTICS), P. *Éléments de linguistique mathématique.* A.V. Gladkij, I.A. Mel'čuk. Transl: J. Cohen, D. Hérault. Mono. mathematique, T. 4. Dunod, 1972, x + 178 pp, (P). Expository description of the fundamentals of mathematical linguistics, translated from the Russian. Formal grammars, properties of generative grammars, generative grammars and natural languages, other models of language. Interesting style has theorems informally worked into the exposition, with marginal notes referring the reader to precise formulations in the appendix. References are given for theorems not proved in their entirety. Examples taken largely from Russian, here transliterated. No formal prerequisites, but demands "in actuality, a certain culture in the two disciplines." PJC

APPLICATIONS (LINGUISTICS), S, P*, L. *Linguistics and Information Science.* Karen Sparck Jones, Martin Kay. Acad Pr, 1973, xii + 244 pp, $14.50. This comprehensive survey, commissioned by the *Fédération Internationale de Documentation*, examines the current state-of-the-art, with a view toward directing future research into the linguistic aspects of automatic document analysis, description, and retrieval. LCL

APPLICATIONS (LINGUISTICS), P. *Understanding Natural Language.* Terry Winograd. Acad Pr, 1972, vii + 191 pp, $8.95. Description of the implementation of a systemic grammar in a computer program for understanding English. Includes overview of literature in artificial intelligence on natural language processing, semantics, and theorem-proving. Develops program capable of parsing, interpreting, and acting on information from an English sentence. Program itself (not given) was written in LISP for a PDP-10 time-sharing system with 80K core and graphic display. This book originally appeared as *Cognitive Psychology* (1972), V. 3, No. 1.PJC

APPLICATIONS (MATERIALS SCIENCE), P. *An Introduction to Computer Simulation in Applied Sciences.* Ed: Farid F. Abraham, William A. Tiller. Plenum Pr, 1972, xiv + 219 pp, $14.50. Set of lectures on six topics growing out of a course designed "to increase the awareness of students in the materials area of computer simulation techniques and potentialities." Topics include the rationale for computer simulation in this area, and applications to incompressible fluid flows, heat diffusion, vapor deposition, computational theoretical chemistry, and weather. Gives theoretical developments as well as computational algorithms and computer program listings. RSK

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Assistant Professor D. H. Anderson, Southern Methodist University, has been promoted to Associate Professor.

Dr. C. D. Gallant, St. Francis Xavier University, has been promoted to Associate Professor and appointed Chairman of the Mathematics Department.

Assistant Professor R. B. Levow, Florida Atlantic University, has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics.

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN THE MATHEMATICAL SCIENCES

In its annual brochure on Fellowship and Research Opportunities in the Mathematical Sciences, the Division of Mathematical Sciences of the National Research Council calls attention to a number of fellowships and other kinds of support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1974–75. Copies of this brochure are available from:

Division of Mathematical Sciences
National Research Council
2101 Constitution Avenue, N. W.
Washington, D. C. 20418

FIRST NATIONAL CONVENTION OF TYC FACULTY HELD

The First National Convention for two-year college mathematics educators, organized by the Editors of the Mathematics Associations of Two-Year Colleges (MATYC) Journal, was held on April 25 and 26, 1974, at the Essex House in New York City.

For a list of the topics of discussions and guest speakers, see the February 1974 issue of this MONTHLY, pp. 206–207, or *The MATYC Journal*, Vol. 8, No. 2, p. 4.

The MATYC Journal has been published for the past eight years, by two-year college educators for two-year college educators, with Professors Frank J. Avenoso and George M. Miller as Editors-in-Chief, and is available for subscription ($5/year) from *The MATYC Journal*, Nassau Community College, Department of Mathematics and Computer Science, Stewart Avenue, Garden City, N. Y. 11530.

SECOND ANNUAL CONFERENCE OF THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

The second annual conference of the O.A.M.E. will be held on May 23–25, 1975, at the University of Western Ontario, London, Ontario. There will be presentations, workshops, and educational displays for elementary and secondary schools, faculties of education, universities, and community colleges. For further information, please write to the conference co-chairmen: Tom Griffiths, North Lambton S. S., Forest, Ontario, Canada N0N–1J0, and Brock Rachar, City Math Centre, c/o Oak Park School, London, Ontario, Canada N6H–3Y3.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE OHIO SECTION

The Ohio Section of the MAA held its annual Spring Meeting at Muskingum College, New Concord, May 3 and 4, 1974. Two hundred-four people attended the meeting. Chairman Frederick Leetch presided; James Murtha was Program Chairman.

The following invited addresses were presented:

A loop switching problem, by H. O. Pollak, Bell Telephone Laboratories, President-Elect of the MAA.

Non-trivial applications of graph theory, by M. D. Plummer, Vanderbilt University.

On finite Moore graphs, by S. E. Payne, Miami University.

Graphs and geometries, by D. K. Ray-Chaudhuri, The Ohio State University.

The following contributed papers were presented:

Polynomial division using matrices, by L. D. Rodabaugh, University of Akron.

Some remarks on the diophantine equation $Y^2 + k = X^2$, by Joseph Blass, Bowling Green State University.

A new course for linear algebra: computer oriented through flowcharted algorithms, by Robert Ducharme, Baldwin-Wallace College.

On a lattice point property, by Necdet Akin, Ohio University.

Computer graphics, by Thomas Hern and Clifford Long, Bowling Green State University.

Planar graphs associated with cardiac muscle, by Cloyd Payne, University of Toledo.

Signed digraphs and the environmental sciences, by Deborah Young, Miami University.

Computation of dual unit balls, by Terry Morrison, Kent State University.

The approximation of some transcendental functions by algebraic means, by R. L. Wilson, Ohio Wesleyan University.

The generalized Steiner problem, by Glenn Clark, Mount Union College.

An ancient problem in graph theory, by David Kullman, Miami University.

Cones with multiplicative structure, by C. E. Cleaver, Kent State University.

Some mathematical aspects of Piaget's developmental psychology, by W. D. Markel, Defiance College.

The program also included the following swap sessions: Mathematics Education, led by Harold Brockman, Capital University; Computer Graphics, led by Thomas Hern and Clifford Long, Bowling Green State University; Department Chairmen Get-Together, led by Luke Zaccaro, Youngstown State University; Pre-calculus Individualized Instruction Survey, led by Marilyn Studer, Kent State University; and Applications in Calculus, led by R. L. Wilson, Ohio Wesleyan University.

Also on the agenda were the annual business meeting of the Section, a meeting of the Executive Board of the Section, and meetings of the *ad hoc* committees: Committee on Teacher Training and Certification, Committee on Curriculum, and Committee on Cooperation between Colleges and Universities.

The officers for 1974–75 are: Louis Green, Case-Western Reserve University, Chairman; Richard Laatsch, Miami University, Chairman-Elect; R. H. Rolwing, University of Cincinnati, Secretary-Treasurer; Gus Mavrigian, Youngstown State University, Secretary-Treasurer-Elect; Richard Little, Kent State University, Canton Campus, Program Chairman; R. S. Varga, Kent State University, and Marion Wetzel, Denison University, Program Committee.

R. H. ROLWING, *Secretary-Treasurer*

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CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.

FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.

ILLINOIS, Rockford College, Rockford, May 9–10, 1975.

INDIANA

IOWA, Iowa State University, Ames, April 18–19, 1975.

KANSAS

KENTUCKY, Murray State University, Murray, April 11–12, 1975.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 1975.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MISSOURI, Missouri Western State College, St. Joseph, April 18–19 1975.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.

NEW JERSEY

NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.

NORTHEASTERN

NORTHERN CALIFORNIA, Menlo College, Menlo Park, February 8, 1975.

OHIO

OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.

PACIFIC NORTHWEST

PHILADELPHIA

ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.

SEAWAY

SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1975.

SOUTHWESTERN, Glendale Community College, Glendale, Arizona, April 11–12, 1975.

TEXAS, Angelo State University, San Angelo, April 11–12, 1975.

WISCONSIN, University of Wisconsin-Superior, April or May 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC

ASSOCIATION FOR WOMEN IN MATHEMATICS, Washington, D. C., January 24, 1975.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

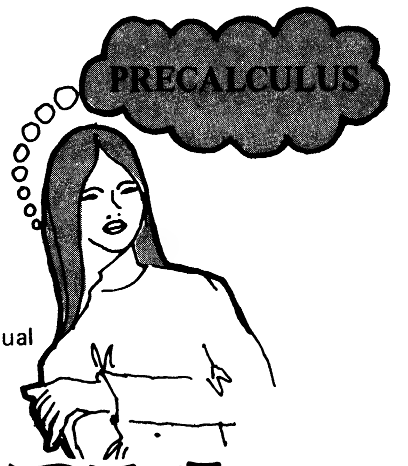
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Denver, Colorado, April 23–26, 1975.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30-May 2, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS



ALGEBRA AND TRIGONOMETRY, Third Edition
ALGEBRA, TRIGONOMETRY, AND ANALYTIC GEOMETRY, Second Edition
both by **Paul K. Rees,** Louisiana State University; **Fred W. Sparks,** Texas Technological University; and **Charles Sparks Rees,** University of New Orleans
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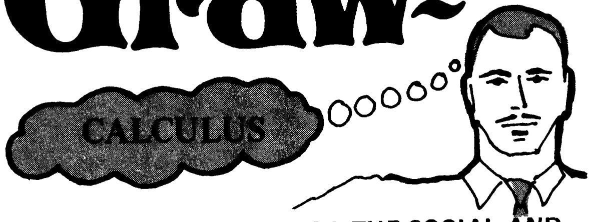
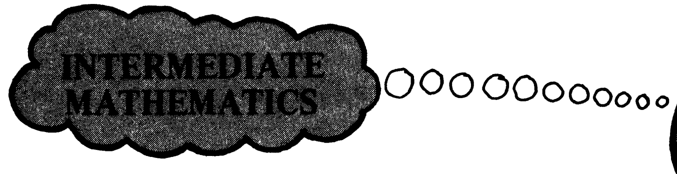
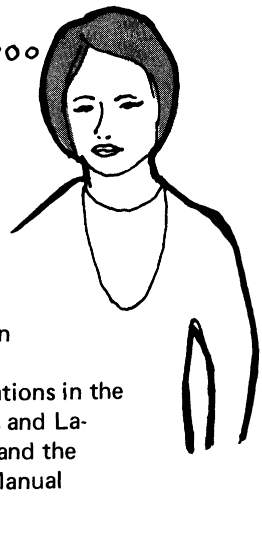


PRACTICAL CALCULUS FOR THE SOCIAL AND MANAGERIAL SCIENCES
Laurence D. Hoffmann, Claremont Men's College
Emphasizes an intuitive understanding and provides firm working knowledge of introductory calculus in one term. Stresses practical applications. 1975, 384 pages, $9.95; Solutions Manual





INTRODUCTION TO ABSTRACT ALGEBRA
Louis Shapiro, Howard University
Includes more historical perspective and examples than most other texts. Survey chapters on Wedderburn theorems, Group Representations, and Galois Theory introduce students to more advanced topics. 1975, 448 pages, $10.95

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Prices subject to change.

THE CARUS MATHEMATICAL MONOGRAPHS

The Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields.

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13. *A Primer of Real Functions* (Second edition), by Ralph P. Boas, Jr.
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16. *Dedekind Sums,* by Hans Rademacher and Emil Grosswald.
17. *The Schwarz Function and its Applications,* by Philip J. Davis.

One copy of each Carus Monograph may be purchased by individual members of the Association for $5.00 each; additional copies and copies for nonmembers are priced at $10.00 each.

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 82 NUMBER 2

CODEN: AMMYAE

CONTENTS

(Continued on inside cover)

FEBRUARY 1975

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 16 months, Math. Notes 19 months, Research Problems 11 months, Classroom Notes 22 months, Math. Education 20 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to Alex Rosenberg, Department of Mathematics, Cornell University, Ithaca, N.Y. 14853; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to Leonard Gillman, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

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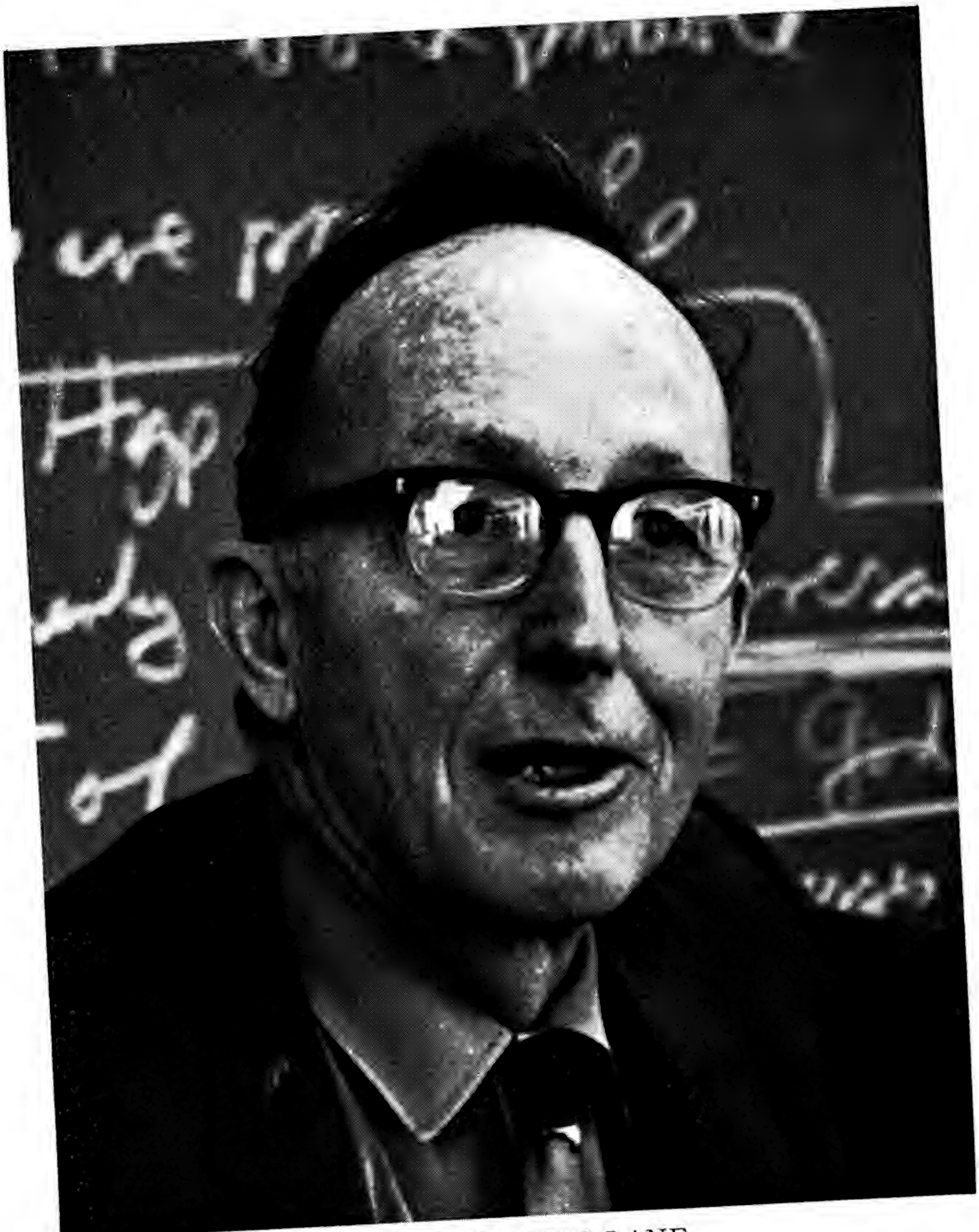
AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR SAUNDERS MAC LANE

As you know, our Distinguished Service Award recognizes service to Mathematics other than in the creation of new mathematics. This year's award is made to Saunders Mac Lane, an outstanding counterexample to the folk metatheorem that research mathematicians of the highest standing either won't or can't do anything besides creating new mathematics. His contributions to research have been recognized by his membership in the National Academy of Sciences (of which he is vice-president) and his presidency (just ended) of the American Mathematical Society. That he is also a member of the National Science Board, a position that one does not get to just by proving good theorems, is an indication of his many other contributions. I shall mention only some highlights.

Saunders Mac Lane was born in Norwich, Connecticut, in 1909; he received a Ph.B. from Yale in 1930, an M.A. from Chicago in 1931, and a Ph.D. from Göttingen in 1934. He was a Sterling Fellow at Yale in 1933–34, a Peirce Instructor at Harvard in 1934–36, and taught at Cornell and Chicago before joining the Harvard faculty in 1938 as an assistant professor. In 1947 he left Harvard as a full professor to go to Chicago, where he has taught ever since. In 1944–45 he was head of the "Applied Mathematics Group — Columbia," a research activity of the Office of Scientific Research and Development, which worked on problems of air-borne fire control. His work there, both administrative and technical, earned him the admiration of his colleagues.

Mac Lane's service just to the Mathematical Association of America is sufficiently impressive. Among other things, he was a Governor (1943–45), a Vice-President (1948–49), and President for 1951–52. His presidency made an enduring impression on those who were active in the Association at that time. He saw the office less as an honor than as an opportunity, attacked the problems of collegiate mathematics with characteristic imagination and energy, and set the Association on the active course that it has followed ever since. Here are some of the activities that began under Mac Lane's presidency: the first steps were taken toward forming CUPM, and in particular toward improving the training of school teachers in mathematics; the Hedrick Lectures were invented; what is now the Employment Register was begun (by the MAA alone); the MAA and AMS began to cooperate in producing a *Combined Membership List.*

Mac Lane has not only been influential in the cause of better education in mathematics, he has set a good personal example. I know of no more enthusiastic teacher or skillful expositor. He received our Chauvenet Prize in 1941. In 1973 he gave a stimulating talk to the high school students at the Awards Ceremony of the U.S. Mathematical Olympiad. I recall that about 1931, when I was an undergraduate, I was warned that the real stuff was what was being done at Göttingen; Mac Lane was one



SAUNDERS MAC LANE

of the first to return to this country with the new ideas, and was one of their most vigorous proponents. The *Survey of Modern Algebra* by Garrett Birkhoff and Saunders Mac Lane introduced these ideas to several generations of students, and its later editions have continued the influence of the first. It is a measure of how far we have progressed since 1941 that the *Survey* was said at the time by some mathematicians to be too far over the heads of American students; in fact, it was apparently never reviewed in the MONTHLY, presumably because it was thought to be too far above the undergraduate level. More recently, Mac Lane's impassioned advocacy of the power of category theory has brought that subject into everyday use in a remarkably short time.

As President of the American Mathematical Society, Mac Lane has given that Society the dynamic leadership that one would have anticipated, initiating new activities and vigorously pursuing old ones. Many members of the Society especially valued the openness of his administration and his new lines of communication with the membership. He handled difficult problems with exemplary skill which many of us have enjoyed observing at first hand. His presidency of the Society, like his presidency of the Association, will be remembered with admiration.

I could go on like this for some time, but no list of official activities can do full justice to the person we honor today, a man of truly outstanding personal qualities, always willing to consider suggestions, a patient and fair arbiter, and above all, a stout advocate of the proper role of Mathematics in our nation's affairs. As long as our professional organizations can find leaders like Saunders Mac Lane, they will be able to carry out effectively the high aims of their founders.

R. P. BOAS, JR.

AWARD OF THE 1975 CHAUVENET PRIZE TO PROFESSORS MARTIN D. DAVIS AND REUBEN HERSH

The Board of Governors of the Mathematical Association of America voted to award the 1975 Chauvenet Prize to Professors Martin D. Davis and Reuben Hersh for their paper "Hilbert's 10th Problem," *Scientific American*, 229 No. 5, (November, 1973) 84–91.

A certificate and monetary award in the amount of two hundred fifty dollars was presented to Professor Davis and to Professor Hersh at the Annual Business Meeting of the Association on January 26, 1975, in Washington, D. C.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English, which comes within the range of profitable reading for members of the Association. The purpose of the Prize is to stimulate the writing of expository and survey articles. The 1975 Prize, awarded for a paper

published in the three-year period 1971–73, is the twenty-third award of the Chauvenet Prize since its institution by the Association in 1925. For the list of names of previous winners, see this MONTHLY, 71 (1964), p. 589, 72 (1965), pp. 2–3, 74 (1967), p. 3., 75 (1968), pp. 3–4, 77 (1970), pp. 117–118, 78 (1971), pp. 112–113, 79 (1972), pp. 112–113, 80 (1973), p. 120, and 81 (1974), pp. 113–114.

Professor Davis was born on March 8, 1928, in New York City. He received his B. S. degree at the City College of New York in 1948, his M. A. in 1949, and Ph.D. in 1950, both at Princeton University. He began his academic career in 1950 as a Research Instructor in Mathematics at the University of Illinois; in 1952 he joined the School of Mathematics at the Institute for Advanced Study in Princeton. From 1954 on, he advanced rapidly through the ranks at various institutions beginning with the University of California at Davis and ending with his current position, held as of 1965, as Professor of Mathematics at the Courant Institute of Mathematical Sciences.

Professor Davis' research interests are in mathematical logic, and, in particular, recursive functions, Diophantine decision problems and mechanical theorem-proving. His many contributions to these areas of mathematics, which have received a great deal of attention, are contained in his nearly thirty publications including several books and the paper *Hilbert's Tenth Problem is Unsolvable*, which appeared in this MONTHLY 80 (1973) 233–269, for which he was awarded a 1974 Lester R. Ford Award by the Association and a LeRoy P. Steele Prize by the American Mathematical Society.

Professor Hersh was born in 1927 in New York City. He received his B. A. degree in English Literature at Harvard University in 1946, and with the aid of the G. I. Bill of Rights, earned a certificate in machine-shop practice at Machine and Metal Trades High School in Manhattan in 1952. He received his Ph. D. in mathematics at New York University under Professor P. D. Lax in 1962. He was an assistant professor at Fairleigh Dickinson University in 1962, an instructor at Stanford University from 1962 to 1964, and has been at the University of New Mexico since that time, advancing to full professor in 1970.

He was a participant in the Battelle Mathematics Physics Rencontres in Seattle in 1968, a visiting member of the Courant Institute of Mathematical Sciences in 1970–71, and directed the Rocky Mountain Mathematics Consortium's 1972 Symposium in Edmonton on "Stochastic Differential Equations."

Professor Hersh's mathematical research has centered on partial differential equations, or "random evolutions." The article for which he is being awarded the Chauvenet Prize is the fifth he has written for the *Scientific American*, with various co-authors, on recent discoveries in pure mathematics.

In their acceptances, Professor Davis and Professor Hersh expressed deep gratitude for the honor of having been awarded the 1975 Chauvenet Prize.

HENRY L. ALDER

On the occasion of his retirement
as Secretary of the Mathematical Association of America

On January 27, 1975, at the conclusion of the 58th Annual Meeting of the Mathematical Association of America, Henry L. Alder will retire from the office of Secretary, a post he has held for fifteen years. In deep appreciation of his long and dedicated service, we record here a few reminiscences from the Alder era and some comments on the Alder influence on the Association.

In its almost 60 years, the MAA has had 35 presidents, 51 vice-presidents, 5 treasurers, and only 4 secretaries. Each of these officers has influenced the development of the MAA, but the opportunities as well as the energies have been varied. A president is in an obvious position to direct the course of the Association, and historians tend to view the development of an organization in units of presidential terms. However, a president is allotted but two years of "burn." His influence on the Association, however strong, is a momentary thrust, a mid-course correction. An MAA secretary, by contrast, serves at least five years and all but one have served more than ten. A secretary's influence on the course of the Association is therefore a steady gravitational force. A wise president uses it. A strong president can oppose it. But, a president who attempts to oppose rather than use this pervasive gravitational force quickly spends his fuel and then must watch helplessly while the Association slowly bends toward its original course.

Like any gravitational force, a secretary's influence is directed. The focal point the secretary's "goal" for the organization — may be definable only empirically and after the fact, but it exists, nevertheless. It is clear to anyone who has been associated with Henry Alder that his goal has been a stronger, more vigorous MAA dedicated to elevating the teaching of mathematics in our colleges and universities to its highest plane. It is also clear to anyone who has felt it that his gravitational force has been irresistible.

Born in Duisburg, Germany, in 1922, Henry Alder moved with his family to Zürich, Switzerland in 1933. After graduating from the Kantonschule in Zürich in 1940, he studied chemistry for a semester at the *Eidgenössische Technische Hochschule.* When his family moved to the United States in 1941 he enrolled immediately in the University of California at Berkeley, where he received his A.B. degree in 1942 and, after service in the U.S. Air Force, his Ph.D. in mathematics in 1947. After receiving his Ph.D., he was appointed to the faculty at Berkeley. In 1948 he joined the faculty of California at Davis where he rose through the ranks, becoming Professor of Mathematics in 1965. Joining the MAA in 1950, he quickly became actively involved in the affairs of the Northern California Section, and in 1957 became the first president of Mu Alpha Theta. In 1965, Mu Alpha Theta awarded him its International Distinguished Service Award. Also in 1957 he became vice-chairman of the Board of Governors

of the *Pacific Journal of Mathematics*, and he has been active in the leadership of that journal to this day.

By the late 1950's it was clear that the MAA was becoming too large and its activities too numerous and varied to be administered by a single secretary-treasurer. Accordingly, in 1959, the By-laws of the Association were amended to separate the offices of Secretary and Treasurer, and a search committee was appointed to nominate a candidate for the office of Secretary. Happily, the committee discovered Henry Alder at a time when he was willing to consider such a challenge. Then-Secretary-Treasurer Harry M. Gehman and Alder vigorously attacked the job of separating duties and responsibilities that had been married since the formation of the MAA. It was a difficult task. Gehman writes, "... (Henry) approached his new job with his characteristic vigor and enthusiasm. There were many problems connected with the division of my previous job as Secretary-Treasurer, but Henry and I solved all of these problems and remained good friends in the process. I was delighted that such a good choice had been made by the Committee..."

The team of Alder and Gehman came to personify orderliness and precision. It is hard to imagine another pair better able to guide the MAA through the breathtaking '60's. A list of the Committees of the MAA that were created during this period or that burst into frenzies of activity at that time — more than likely encouraged and financed by the NSF—inspires awe in the mind of anyone familiar with the former ways of the Association and terror in the heart of any administrator: The Committee on the Undergraduate Program in Mathematics (CUPM), The Committee on Visiting Lecturers, The Committee on Educational Media (CEM), The Committee on Secondary School Lecturers, The Committee on Institutes, The Committee on High School Contests, The Committee on Advisement and Personnel. The list is not complete, but each of these Committees was busy during the '60's preparing proposals, submitting them to foundations, receiving grants and conducting projects large and small. Two of them, CUPM and CEM, were alone responsible for *tripling* MAA expenditures during some of those years. The demands for coordination, review, supervision and fiscal management of grant activities, on top of all of the normal MAA activities, were unrelenting. Without efficient and untiring administration, the Association would not have been able to contain the energy that had been released within it.

Fortunately, Alder and Gehman provided more than enough gravitational force to hold the MAA together. One must have experienced that force fully to appreciate it. Officers, committee chairmen, project directors, and at least one Executive Director, owe Henry Alder large debts of gratitude for hundreds of occasions when pointed reminders, friendly advice, carefully worded inquiries or leading compliments averted costly and embarrassing errors or oversights. Using an incredible memory and an inspired filing system, Henry seemed able to store every detail of MAA activities and every word uttered by an officer or committee chairman in the line of duty and to recall any bit of information instantly on demand. His bulging and worn briefcase,

augmented by a battered cardboard box or two, seemingly contained all records of the MAA at least a decade into the past for ready reference at a meeting of the Executive and Finance Committees or the Board of Governors.

The Alder pen has produced detailed agendas and minutes of at least 70 meetings of the Executive and Finance Committees and nearly 30 meetings of the Board of Governors. It would not be useful to try to count the pages of these documents — before the number could be reported, there would be 100 more — but it has been estimated that he leaves at least 5000 pages to guide future generations.

Henry Alder is the embodiment of the cliché that mathematics trains the mind to approach life through logic and order. He elevated the art of advocacy to a state of Euclidean precision. Those who have worked most closely with him agree that every working day of his fifteen years as Secretary, Henry must have produced at least five tightly structured and ordered reasons why we should or shouldn't do something, leaving a legacy of 18,000 carefully numbered reasons for or against. Who has not received a letter from him beginning, "I can easily think of a dozen reasons why we should (shouldn't) . . . but I will mention only seven." It is impossible to resist a smile in the face of such logical overkill, but more often than not, when the debate has run its course, all that remains of the opposition is the smile. Part of the genius of the Alder technique is that the tight smile of defeat soon becomes the wide grin of friendship — one is not beaten by Henry's assault, one is converted.

A recent letter from Henry contained a definitive example of the Alder avalanche. On page three of a letter containing no less than seven precise examples from recent MAA affairs, listed in careful succession in support of a point in contention, he writes, "If it would help you, I would be very glad to write you a long and detailed letter to prove this point." Case dismissed!

Henry's colleagues in the administration of the MAA marvel at the occasional glimpses of his active professional life apart from the Association. He teaches a full schedule at Davis, and reports indicate that his lectures are meticulously prepared and that he is always available for consultation by his students. Somehow he has found time to co-author several textbooks, including one of the most widely used texts in elementary statistics, to be active on faculty committees in his department and his university, and to be a leader in the administration of the *Pacific Journal of Mathematics*. He has traveled widely and continues to publish papers on number theory, his first mathematical love. One of his papers, "Partition Identities—from Euler to the Present," this MONTHLY, 76 (1969) 733–746, won him a Lester R. Ford Award in 1970. It was this wide interest and enthusiasm that caused him to request replacement as Secretary of the Association at the end of his present term, his third. There are no signs of fatigue, no flagging of enthusiasm. His participation in planning for the future of the Association has increased, if anything, in vigor. We confidently anticipate that Henry Alder will be a part of the Association's future. We know certainly that he is a giant part of the past and present. We do not say "Good-bye." We say "Thank you . . . , and, Henry, do you think . . . ?"

A. B. WILLCOX

LOCALIZATION IN TOPOLOGY

PETER HILTON

1. Introduction. Recently, the theory of localization (at a prime or set of primes), which had proved such a powerful tool in commutative ring theory, has found application in modern topology. The theory was first introduced into topology by Sullivan [**16**], who also effectively adopted the technique of completion from algebra; however, it may be said that localization techniques were already implicit in Zabrodsky's method of mixing homotopy types [**18**]. Subsequently, localization theory has been exploited by many topologists, e.g. [**1, 5, 10, 11, 15**]. The author, Mislin, and Roitberg [**6**] have used the method extensively in studying *non-cancellation phenomena*; for example, one finds that there are four smooth closed manifolds X_1, X_2, X_3, X_4, all of different homotopy types, of which X_1 is the exceptional Lie group G_2, and which are such that

$$X_1 \times S^3 \simeq X_2 \times S^3 \simeq X_3 \times S^3 \simeq X_4 \times S^3, \tag{1.1}$$

meaning that, on taking the topological product with the sphere S^3, we get homotopically equivalent manifolds. We mention this example in order to illustrate a key point about several of the applications of localization theory to topology, namely, that the theory leads to results which make no mention of localization.

A comprehensive treatment of a process more general than the localization we will discuss, but executed in the semi-simplicial category, is given in [**1**]. A general account of localization in commutative ring theory was given by Cohn in this journal [**2**]. Here we will be content to sketch the foundations of the theory as it applies to topology. In Sections 2 and 3, we give a reasonably comprehensive and self-contained account of the localization theory of abelian groups. This may be regarded as the prototype for more sophisticated localization theories, but it is not merely as a paradigm that it is presented; the results of the localization theory of abelian groups are of direct importance for the development of a localization theory in topology. In Section 4, we outline that theory as it applies to connected, simply-connected polyhedra. We emphasize that we apply localization in *homotopy theory*, that is, our constructions are carried out "up to homotopy" and the properties claimed for the localization functor in topology are valid "up to homotopy." We show, in particular, in Section 4, how the localization of a polyhedron X is constructed in a manner which imitates, step-by-step, the process of building up the polyhedron X by successive attachment of cells. We also describe, in this section, how localization theory leads to the discovery of new *Hopf manifolds*, that is, manifolds admitting a continuous multiplication with two-sided unity element.

We open Section 5 by drawing attention to the need to enlarge the domain of applicability of localization techniques, even if we were content only to elucidate further properties of the localization of simply-connected polyhedra. However, as

we point out, it would also be well worthwhile to be able to apply localization techniques to certain non-simply-connected spaces since, for example, they have proved so successful in the study of simply-connected Hopf manifolds, and we would wish to be able to apply them to the study of arbitrary connected Hopf manifolds. A category of connected spaces, namely that of *nilpotent* spaces, is described, which turns out to permit localization and which is broad enough to include all connected Hopf spaces. Moreover, one may execute various constructions in this category, without leaving the category, which lead to important theorems in the localization theory of simply-connected spaces; and such theorems remain valid in the larger category. Indeed it would seem that virtually over that whole area of homotopy theory in which it has been customary to assume simple-connectedness (e.g., the Serre theory of the homology of fibre spaces and its numerous applications), one may replace simple-connectedness assumptions by nilpotency assumptions and obtain useful generalizations of the known results, at some cost, of course, in complicating the argument.

The present expository paper is based on joint work with Guido Mislin and Joseph Roitberg. A report on localization theory in homotopy, incorporating the localization of nilpotent groups and spaces, will appear under joint authorship as a monograph [7] in the series *Notas de Matematica*, published by North Holland.

2. Localization of abelian groups. The theory of localization of 1-connected topological spaces bears a resemblance to the simpler theory of localization of abelian groups. However, the resemblance is far closer than that of a mere analogy, in fact, we base the definition of a P-local space on that of a P-local abelian group, and many of the crucial properties of the localization functor in topology are proved by exploiting the corresponding property of the localization functor in abelian group theory.

Let P be a family of primes. Then the *ring of P-local integers*, $\mathbb{Z}_P$, is the subring of the rationals, $\mathbb{Q}$, consisting of those rationals expressible as a/b, where p divides $b \Rightarrow p \notin P$. Notice that the effect of passing from $\mathbb{Z}$ to $\mathbb{Z}_P$ is that arithmetical questions involving the primes in P retain their significance, whereas the same questions posed for the primes in the complementary family P' become banal. Thus, for example, if P consists only of the prime 2, then $\mathbb{Z}_P$ consists of rationals expressible as fractions a/b with *odd* denominators. The question whether a/b is divisible by 2 in $\mathbb{Z}_P$ is a significant question, the answer being affirmative if and only if a is even; on the other hand, every element of $\mathbb{Z}_P$ is divisible by 3, for example, in $\mathbb{Z}_P$. Thus in general we may say that, by passing to $\mathbb{Z}_P$, we are confining attention, or *localizing*, to the primes in P.

Now let A be any abelian group. We define its *P-localization* to be the abelian group

$$A_P = A \otimes \mathbb{Z}_P, \tag{2.1}$$

and the *P-localizing map* is the canonical homomorphism $e\colon A \to A_P$, given by

$$a \mapsto a \otimes 1, \quad a \in A. \tag{2.2}$$

Further, we call an abelian group B *P-local* if the map $p\colon B \to B$, given by $b \mapsto pb$, $b \in B$, is an automorphism, whenever $p \in P'$. This amounts to saying that we have unique division by p in B, $p \in P'$. We then prove the following *universal property* for the P-localizing map:

PROPOSITION 2.1. *A_P is P-local. Moreover, if $\phi\colon A \to B$ is any homomorphism of A to a P-local abelian group B, then there exists a unique homomorphism $\psi\colon A_P \to B$ such that $\psi e = \phi$.*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\phi"'] & A_P \arrow[dl, dashed, "\psi"] \\ B & \end{tikzcd}$$

Proof. $\mathbb{Z}_P$ is clearly P-local. Now $p\colon \mathbb{Z}_P \to \mathbb{Z}_P$ induces $p\colon A \otimes \mathbb{Z}_P \to A \otimes \mathbb{Z}_P$. Thus, if $p \in P'$, $p\colon A_P \to A_P$ is an automorphism, so A_P is P-local.

Given $\phi\colon A \to B$ with B P-local, consider an element $a \otimes \rho$ of A_P, with $\rho \in \mathbb{Z}_P$. Then ρ may be expressed as $\rho = m/n$, where the prime factors of n are all in P'. Since B is P-local, we have unique division by n in B and we may set $\psi(a \otimes \rho) = (1/n)\phi(ma)$. The reader will easily check that ψ, so defined, is a homomorphism such that $\psi e = \phi$. That the equation $\psi e = \phi$ uniquely determines ψ is plain; for, given any element y of A_P, there is a natural number n, whose prime factors are in P', with $ny \in eA$. Then $\psi(ny)$ is determined by the equation $\psi e = \phi$, and $\psi(y)$ is determined as the unique result of dividing $\psi(ny)$ by n in B.

Notice now that the universal property of A_P determines it up to canonical isomorphism — this is a simple exercise in "abstract nonsense." Thus if we are content, as we are, to determine the localization up to canonical isomorphism, we may regard Proposition 2.1 as expressing the defining property of the P-localizing map, and the explicit construction given earlier as providing a proof of existence. Since the construction is so simple, this point of view would be rather sophisticated if our sole interest were in abelian groups. It is the correct point of view, however, when we come to localize topological spaces, as the reader will observe in Section 4.

It is an important observation that the concept of a P-local abelian group coincides precisely with that of a *$\mathbb{Z}_P$-module.* In fact, we have

PROPOSITION 2.2. *An abelian group A is P-local if and only if it admits the structure of a $\mathbb{Z}_P$-module. Moreover, if A admits the structure of a $\mathbb{Z}_P$-module, it does so in only one way; and every homomorphism $\alpha\colon A \to B$ of P-local abelian groups is a $\mathbb{Z}_P$-module homomorphism.*

Proof. If A is P-local, we let $\mathbb{Z}_P$ act on A by

$$\frac{m}{n} \cdot a = \frac{1}{n}(ma), \tag{2.3}$$

where the prime factors of n are all in P', so A has unique division by n. It is plain that (2.3) turns A into a $\mathbb{Z}_P$-module. Conversely, if A has the structure of a $\mathbb{Z}_P$-module, then for $p \in P'$, the element $a' = (1/p) \cdot a$ satisfies $pa' = a$, so that we may divide by p in A; and, since $(1/p) \cdot pa' = a'$, by the module axioms, it follows that division by p is unique.

Our proof shows that (2.3) yields the only way that $\mathbb{Z}_P$ can act on A. Moreover, if A, B are P-local then

$$\alpha\left(\frac{1}{n}a\right) = \frac{1}{n}\alpha(a),$$

since $n\alpha(a/n) = \alpha(a)$, from which it immediately follows that α is a $\mathbb{Z}_P$-module homomorphism.

Notice that Proposition 2.2 allows us to say that a P-local abelian group *is* a $\mathbb{Z}_P$-module. Of course, for an arbitrary unitary ring Λ, we would not be allowed to say that an abelian group is a Λ-module, since, in general, it could be given the structure of a Λ-module in several different ways.

We know from Propositions 2.1, 2.2 that A_P is a $\mathbb{Z}_P$-module; indeed, it is easy to write down the explicit $\mathbb{Z}_P$-module structure. The converse is striking:

PROPOSITION 2.3. *Every $\mathbb{Z}_P$-module is isomorphic to some A_P. In fact, A is P-local if and only if $e: A \to A_P$ is an isomorphism.*

Proof. We only have to prove that if A is P-local, then e is an isomorphism. The reader may like to prove explicitly that $e: A \to A \otimes \mathbb{Z}_P$, as defined above (2.2), is an isomorphism if A is P-local. However, we have available a slick, but slightly sophisticated, proof. For it is easy to see that, if A is P-local, then the identity map $1: A \to A$ has the universal property ascribed to e in Proposition 2.1 (take $\psi = \phi$!). But the universal property determines A_P up to canonical isomorphism, so e is an isomorphism.

Before proceeding further let us make some elementary observations about localization that enable us to make calculations. These observations arise from the remark that localization is achieved by tensoring (2.1) with a *torsion-free* abelian group $\mathbb{Z}_P$. We thus may state without proof

PROPOSITION 2.4. *P-localization is exact and preserves direct sums.*

Let us just explain this statement of exactness. Obviously localization is *functorial*: given any homomorphism of abelian groups $\phi: A \to B$, there is an induced homomorphism $\phi_P: A_P \to B_P$, given by

$$\phi_P(a \otimes \rho) = \phi a \otimes \rho, \quad a \in A, \quad \rho \in \mathbb{Z}_P; \tag{2.4}$$

and thus we have a localizing functor L_P, such that e is a natural transformation from the identity functor to L_P. Given a short exact sequence of abelian groups

$$A' \rightarrowtail^{\mu} A \xrightarrow{\varepsilon} \twoheadrightarrow A''$$

(essentially, A' is a subgroup of A and $A'' = A/A'$; μ is the embedding and ε the projection), the exactness of localization asserts that we have a commutative diagram

$$\begin{tikzcd} A' \arrow[r, tail, "\mu"] \arrow[d, "e"] & A \arrow[r, two heads, "\varepsilon"] \arrow[d, "e"] & A'' \arrow[d, "e"] \\ A'_P \arrow[r, tail, "\mu_P"] & A_P \arrow[r, two heads, "\varepsilon_P"] & A''_P \end{tikzcd}$$

and *the lower sequence is also short exact.*

We use Proposition 2.4 to facilitate our observations.

PROPOSITION 2.5. *Let T be the torsion subgroup of A. Then T_P is the torsion subgroup of A_P.*

Proof. Let $F = A/T$, so that F is torsion-free. Since the tensor product of two torsion-free abelian groups is torsion-free, it follows that F_P is torsion-free. Since $T \rightarrowtail A \twoheadrightarrow F$ is short exact, so is $T_P \rightarrowtail A_P \twoheadrightarrow F_P$. Certainly T_P is a torsion group. Since F_P is torsion-free, T_P is the torsion subgroup of A_P.

PROPOSITION 2.6. *Let A be a finitely generated abelian group. Then $A_P \cong F_P \oplus T_P$, where T_P is the P-localization of the torsion subgroup of A and F_P is a free $\mathbb{Z}_P$-module whose rank is the rank of A.*

Proof. Since A is finitely generated, $A \cong F \oplus T$, where F is free and T is the torsion subgroup of A. Moreover, the rank of F is, by definition, the rank of A. Thus Proposition 2.6 follows from the second part of Proposition 2.4.

Since a torsion abelian group is the direct sum of its p-primary components, we may complete our calculations of the localization of torsion groups by proving

PROPOSITION 2.7. *Let p be a prime and let T be a p-torsion abelian group. Then*

$$T_P = \begin{cases} T & \textit{if } p \in P \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. A p-torsion group is certainly P-local if $p \in P$. Thus the first assertion follows from Proposition 2.3. On the other hand, it is easy to see that if $p \in P'$, then the only homomorphism from T to a P-local group is the zero homomorphism. It is thus clear from the universal property that $T \to 0$ P-localizes!

Propositions 2.6, 2.7 enable us, in particular, to compute the P-localization of any finitely-generated abelian group.

The argument used above may easily be generalized. We close this section with a result which is crucial in establishing the results of Section 4 and which contains such a generalization.

PROPOSITION 2.8. *Let A be a P'-torsion abelian group and let B be P-local. Then* $\operatorname{Hom}(A,B) = 0$ *and* $\operatorname{Ext}(A,B) = 0$.

Proof. Let $\phi\colon A \to B$ and let $a \in A$. Then there exists n whose prime factors are all in P' with $na = 0$. Thus $n\phi(a) = 0$; but we have unique division by n in B, so $\phi(a) = 0$ and so $\phi = 0$.

Now we may interpret $\operatorname{Ext}(A,B)$ as follows. Let $B \overset{\mu}{\rightarrowtail} I \overset{\varepsilon}{\twoheadrightarrow} J$ be any short exact sequence with I divisible. Then ε induces ε_*: $\operatorname{Hom}(A,I) \to \operatorname{Hom}(A,J)$ and $\operatorname{Ext}(A,B)$ is the cokernel of ε_*. We proceed by choosing I judiciously; namely, since B is a $\mathbb{Z}_P$-module, we may choose for I an injective $\mathbb{Z}_P$-module. J is then also a $\mathbb{Z}_P$-module and, since $\mathbb{Z}_P$ is torsion-free, it is easy to see that I is divisible, so the choice is legitimate. However, with this choice, $\operatorname{Hom}(A,I) = 0$, $\operatorname{Hom}(A,J) = 0$, as already proved, so $\operatorname{Ext}(A,B) = 0$.

Notice that the conclusion $\operatorname{Ext}(A,B) = 0$ may be expressed by saying that any short exact sequence

$$B \rightarrowtail E \twoheadrightarrow A$$

of abelian groups, with A P'-torsion and B P-local, is equivalent to the direct sum

$$B \rightarrowtail A \oplus B \twoheadrightarrow A.$$

We close this section with a characterization of P-localization which is valuable in the arguments in Section 4. Given abelian groups A, B and given $\phi\colon A \to B$, we say that ϕ is *P-injective* if $\ker\phi$ is P'-torsion; *P-surjective* if coker ϕ is P'-torsion; and *P-bijective* if it is both P-injective and P-surjective. We then prove

PROPOSITION 2.9. *The homomorphism $\phi\colon A \to B$ P-localizes A if and only if B is P-local and ϕ is P-bijective.*

Proof. We first show that $e\colon A \to A_P$ is P-bijective (certainly, A_P is P-local!). Now the quotient C of $\mathbb{Z}_P$ by $\mathbb{Z}$ is easily seen to be P'-torsion, and it is then a standard result of homological algebra that

$$\ker e = \operatorname{Tor}(A,C), \quad \operatorname{coker}\, e = A \otimes C.$$

Moreover, since C is P'-torsion, so are $A \otimes C$, $\operatorname{Tor}(A,C)$, thus establishing the fact that e is P-bijective.

Now suppose that B is P-local and $\phi\colon A \to B$ is P-bijective. Let $e\colon A \to A_P$ P-localize A. By the universal property (Proposition 2.1) there exists a (unique) homomorphism $\psi\colon A_P \to B$ with $\psi e = \phi$. We want to prove that ψ is an isomorphism. Thus our argument is complete when we have established the following two propositions.

PROPOSITION 2.10. *Suppose given the commutative diagram of abelian groups*

$$\begin{tikzcd} A \arrow[rr, "\theta"] \arrow[dr, "\phi"'] & & A' \arrow[dl, "\psi"] \\ & B & \end{tikzcd} \qquad \psi\theta = \phi.$$

Then (i) *if ϕ is P-surjective, so is ψ,* (ii) *if ϕ is P-injective and θ is P-surjective, then ψ is P-injective.*

Proof. (i) Let $b \in B$. Since ϕ is P-surjective, there exists n, whose prime factors are in P', with $nb \in \operatorname{im}\phi$. But then $nb \in \operatorname{im}\psi$, so ψ is P-surjective.

(ii) Let $a' \in \ker\psi$. Since θ is P-surjective, there exists n, whose prime factors are in P', with $na' \in \operatorname{im}\theta$, say $na' = \theta a$. Then $\phi a = \psi\theta a = n\psi a' = 0$. Since ϕ is P-injective, there exists m, whose prime factors are in P', with $ma = 0$. Then $mna' = 0$, showing that $\ker\psi$ is P'-torsion and ψ is P-injective.

PROPOSITION 2.11. *Let $\psi: C \to B$ be a homomorphism with B, C P-local. Then* (i) *if ψ is P-injective, ψ is injective,* (ii) *if ψ is P-surjective, ψ is surjective.*

Proof. (i) Let $c \in \ker\psi$. Then $nc = 0$ for some n whose prime factors are in P'. But C is P-local, so we may divide by n, obtaining $c = 0$. Thus $\ker\psi = (0)$ and ψ is injective.

(ii) Let $b \in B$. Then $nb \in \operatorname{im}\psi$, say $nb = \psi c$, for some n whose prime factors are in P'. Since C is P-local, $c = nc_1$, for some (unique) $c_1 \in C$, so that $n\psi c_1 = nb$. Since B is P-local, we may divide by n, obtaining $\psi c_1 = b$ and thus proving that ψ is surjective.

Reverting to the proof of Proposition 2.9, the reader will observe (recalling that e is P-bijective) that Proposition 2.10 shows that ψ: $A_P \to B$ is P-bijective; and then Proposition 2.11 shows that ψ is, in fact, an isomorphism.

NOTATIONAL REMARK: If P consists of a single prime p, we write A_p for A_P. If P is the empty set of primes, we write A_0 instead of A_ϕ, and call A_0 the *rationalization* of A; however, we shall retain the notation $\mathbb{Q}$ rather than $\mathbb{Z}_0$.

3. The family of primes. The results of Section 2 may be broadly described as showing how information about an abelian group A may be used to obtain information about its localizations A_P. In this section we are chiefly concerned with going in the opposite direction. The results of this section are all relevant to the study of localization in topology.

PROPOSITION 3.1. *Let C be an abelian group. Then*

$$C = 0 \Leftrightarrow \forall \text{ primes } p,\ C_p = 0.$$

Proof. Let T be the torsion subgroup of C, with $F = C/T$. Then the projection $C \twoheadrightarrow F$ induces $C_p \twoheadrightarrow F_p$. If we assume $C_p = 0$, we infer $F_p = 0$. But $F_p = F \otimes \mathbb{Z}_p$,

and the tensor product of two torsion free abelian groups can only be zero if one of them is zero. Thus $F = 0$ and $C = T$ is a torsion group. Then C_p is just the p-primary component of C, by Proposition 2.7, so that $C = 0$ if every $C_p = 0$.

COROLLARY 3.2. *Let ϕ: $A \to B$ be a homomorphism of abelian groups. Then ϕ is injective (surjective) $\Leftrightarrow \forall$ primes p, ϕ_p is injective (surjective).*

Proof. This follows immediately from Proposition 3.1 and the exactness of localization. For the kernel (cokernel) of ϕ localizes, by exactness, to the kernel (cokernel) of ϕ_p.

Corollary 3.2 should not be read as implying that, given two abelian groups A, B such that $A_p \cong B_p$ for all p, then $A \cong B$. For in the corollary it is essential that there be a *global* homomorphism ϕ which localizes to an isomorphism at each p. Indeed, in general, we cannot infer $A \cong B$ from $A_p \cong B_p$, for all p. Before giving a counterexample, however, we point out that the inference *is* valid if A, B are finitely generated.

PROPOSITION 3.3. *Let A, B be finitely generated abelian groups such that $A_p \cong B_p$ for all p. Then $A \cong B$.*

Proof. Since A, B are finitely generated, we may write $A = F \oplus T$, $B = F' \oplus T'$, where F, F' are free and T, T' are torsion. Since $A_p \cong B_p$, we have $A_0 \cong B_0$ or $F_0 \cong F'_0$. Thus, F, F' have the same rank so that $F \cong F'$. Now

$$A_p = F_p \oplus T(p),\ B_p = F'_p \oplus T'(p),$$

where $T(p)$, $T'(p)$ are the p-primary components of T, T'. Moreover, F_p, F'_p are torsion-free, so that $T(p)$, $T'(p)$ are the torsion subgroups of A_p, B_p respectively. Thus, since $A_p \cong B_p$, we infer that $T(p) \cong T'(p)$, for all primes p, so that $T \cong T'$ and hence $A \cong B$.

Counterexample 3.4. This example is due to G. Mislin and will be presented in detail* in [7]. Let $A = \Pi_p \mathbb{Z}/p$. Then it is easy to see that T, the torsion subgroup of A, is $\oplus_p \mathbb{Z}/p$. It may be shown that T is not a direct summand in A. Now T_p is the torsion subgroup of A_p. Since $T_p = \mathbb{Z}/p$, it follows from a general theorem on abelian groups that T_p *is* a direct summand in A_p. Consequently, A and $B = T \oplus A/T$ furnish an example of two non-isomorphic abelian groups such that $A_p \cong B_p$, for all primes p.

It will be one of the most interesting features of the theory of topological localization that the analog of Proposition 3.3 is false! If, as is reasonable, we regard the condition of compactness for polyhedra as corresponding to the condition of finite generation for abelian groups — and we explain in the next section (Proposition 4.7 and the preceding remark) just how reasonable that is — then we find homotopically distinct polyhedra whose localizations, at all primes, are equiv-

* *Added in proof*: A far simpler example is given in [7; preamble to Theorem 3.14]!

alent. This is indeed a triumph, not a weakness, of the topological theory, since it enables us to construct interesting new polyhedra (even manifolds), and to introduce a potent new concept into homotopy theory.

To return to abelian group theory, we now discuss two results of direct use in topology. The importance of these results for our purposes leads us to refer to them as theorems. We abbreviate "finitely generated" to "fg."

THEOREM 3.5. *Let A be an* fg *abelian group. Then A is the pull-back of its localizations A_p over its rationalization A_0.*

Before proving this theorem, we explain what it is asserting. Given A, we have localizations $e_1\colon A \to A_{p_1}$, $e_2\colon A \to A_{p_2}, \cdots$, where we have enumerated the primes as $p_1, p_2, \cdots$, and we may rationalize each A_{p_i} by $r_i\colon A_{p_i} \to A_0$. Now, given any element $a \in A$, we obtain elements $a_i = e_i(a) \in A_{p_i}$, and it is obvious that $r_1(a_1) = r_2(a_2) = \cdots$. The theorem asserts the converse if A is fg. That is, it asserts that if we take elements $a_1, a_2, \cdots, a_i \in A_{p_i}$, such that $r_1(a_1) = r_2(a_2) = \cdots$, then there exists a unique element $a \in A$ such that $e_i(a) = a_i$, $i = 1, 2, \cdots$.

Proof of theorem. It is standard that tensor products (and hence localization) commute with direct sums (see Proposition 2.4) and that pull-backs commute with direct products. Since direct sums coincide with direct products over a *finite* indexing set, it follows that, if we have any finite collection $A_1, A_2, \cdots, A_k$ of abelian groups for which the assertion of the theorem holds, then it holds for $A = \oplus_{i=1}^{k} A_i$. Thus to prove the assertion for A fg, it suffices to prove it in the two cases

$$\text{(i)}\quad A = \mathbb{Z}/p^k, \qquad \text{(ii)}\quad A = \mathbb{Z}.$$

The first case being quite trivial, we concentrate on the second. We are given elements $a_i \in \mathbb{Z}_{p_i}$, which are identical elements of $\mathbb{Q}$. Express a_i as a fraction in its lowest terms, thus $a_i = m_i/n_i$, with $n_i \geqq 1$. Then $p_i \nmid n_i$ and

$$m_1 = m_2 = \cdots = m, \text{ say; } n_1 = n_2 = \cdots = n, \text{ say.}$$

However, since $p_i \nmid n$, for all i, we must have $n = 1$, so that we have the (unique, of course) element $m \in \mathbb{Z}$, which p_i-localizes to a_i for all i.

This theorem does *not* extend to arbitrary abelian groups, as the example $A = \oplus_p \mathbb{Z}/p$ shows.

For our next theorem, we take an fg abelian group A and let $B = \Pi_p A_p$. There is then a canonical map $\varepsilon\colon A \to B$, given by the localizing maps $e_i\colon A \to A_{p_i}$.

THEOREM 3.6. *Let A be an* fg *abelian group. Then the square*

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ A_0 & \xrightarrow{\varepsilon_0} & B_0 \end{array} \tag{3.1}$$

is bicartesian, where r is the rationalizing map, and $B = \Pi_p A_p$.

Proof. We first recall that the assertion that (3.1) is bicartesian amounts to claiming two properties: (a) every element x of B_0 is expressible as $x = rb + \varepsilon_0 y$, $b \in B$, $y \in A_0$; and (b) if $rb = \varepsilon_0 y$, $b \in B$, $y \in A_0$, then there exists a unique element $a \in A$ with $\varepsilon a = b$, $ra = y$. We then remark that precisely the same considerations as at the outset of the proof of Theorem 3.5 enable us to say that it is sufficient to consider the cases

$$\text{(i)} \quad A = \mathbb{Z}/p^k, \qquad \text{(ii)} \quad A = \mathbb{Z}.$$

Again the case $A = \mathbb{Z}/p^k$ is trivial, since then $A = B$, $\varepsilon =$ identity, $A_0 = B_0 = (0)$. Thus we have only to consider the case (ii), $A = \mathbb{Z}$. In this case, ε and ε_0 are injective, and standard arguments of abelian group theory then show that (3.1) is bicartesian if and only if the induced map coker $\varepsilon \to$ coker ε_0 is an isomorphism; this, in turn, holds if and only if coker ε is torsion-free and divisible, since, by Proposition 2.4, coker $\varepsilon_0 =$ coker $\varepsilon \otimes \mathbb{Q}$. We use ε to embed $\mathbb{Z}$ in $\Pi_p \mathbb{Z}_p$ and write C for the quotient. Thus we have to prove that

$$C = (\prod_p \mathbb{Z}_p)/\mathbb{Z}$$

is torsion-free and divisible. Note that $\mathbb{Z}$ is embedded in $\Pi_p \mathbb{Z}_p$ *diagonally*; that is, we identify $n \in \mathbb{Z}$ with the element $\{n\}$ of $\Pi_p \mathbb{Z}_p$, whose component in $\mathbb{Z}_p$ is n for every p.

Proof that C is torsion-free: Enumerate the primes as $p_1, p_2, \cdots$ and write $\Pi_i \mathbb{Z}_{p_i}$ for $\Pi_p \mathbb{Z}_p$. Let $[a_i] \in C$ be the element (coset) containing $\{a_i\} \in \Pi_i \mathbb{Z}_{p_i}$, $a_i \in \mathbb{Z}_{p_i}$. Write a_i as m_i/n_i in its lowest terms with $n_i \geqq 1$, $p_i \nmid n_i$. Suppose that $n[a_i] = 0$, $n \geqq 1$, so that $n\{a_i\} \in \mathbb{Z}$. Thus there exists $m \in \mathbb{Z}$ such that

$$m_1/n_1 = m_2/n_2 = \cdots = m/n.$$

As in the proof of Theorem 3.5, we infer that $n_1 = n_2 = \cdots = 1$, and $m_1 = m_2 = \cdots = d$, say. Thus $m/n = d$, so that $\{a_i\} = \{d\}$ and $[a_i] = 0$.

Proof that C is divisible: It is plainly sufficient to prove that every element of C is divisible by p, where p is an arbitrary prime. For convenience of notation, set $p = p_1$. Then if $[a_i] \in C$ and $a_i = m_i/n_i$ as above, we have $p_1 \nmid n_1$, so that there exist integers a, b with $ap_1 + bn_1 = 1$. Then $[a_i] = [a_i']$ where $a_i' = a_i - m_1 b$, and

$$a_1' = \frac{m_1}{n_1} - m_1 b = \frac{am_1 p_1}{n_1}.$$

Thus $[a_i] = [a_i'] = p_1[a_i'']$, where $a_1'' = am_1/n_1$, $a_i'' = a_i'/p_1$, $i \neq 1$, so that $[a_i]$ is divisible by $p = p_1$.

Our final proposition shows how we may reduce questions involving localization to a study of a *finite* number of primes. We again suppose the primes enumerated as

$p_1, p_2, \cdots$ and write P_l for the complement of the first l primes. Let $e_l\colon A \to A_{P_l}$ $r_l\colon A_{P_l} \to A_0$ be the localizing and rationalizing maps.

PROPOSITION 3.7. (i) *Let A be an* fg *abelian group. Then there exists m such that $r_l\colon A_{P_l} \to A_0$ is injective if $l \geqq m$.* (ii) *Let $\phi\colon B \to A_0$ be a homomorphism of the* fg *abelian group B into the rationalization of the* fg *abelian group A. Then there exists n such that ϕ lifts uniquely to $\psi\colon B \to A_{P_l}$ (i.e., $r_l\psi = \phi$) if $l \geqq n$.*

Proof. (i) Choose m so that P_m excludes all the primes p such that A has p-torsion. It is then plain from Propositions 2.6 and 2.7 that r_l is injective if $l \geqq m$.

(ii) First choose m as in (i). Now let $(b_1, b_2, \cdots, b_k)$ generate B. We can express each ϕb_i as

$$\phi b_i = a_i \otimes \frac{1}{d_i}, \quad i = 1, 2, \cdots, k, \quad a_i \in A, \quad d_i \in \mathbb{Z}, \quad d_i \neq 0.$$

Choose $n \geqq m$ so that P_n excludes all the prime factors of $d_1 d_2 \cdots d_k$. Then we may regard $a_i \otimes 1/d_i$ as an element of A_{P_l} if $l \geqq n$ and, since r_l is then injective, the rule

$$\psi b_i = a_i \otimes 1/d_i, \qquad i = 1, 2, \cdots, k$$

determines the unique homomorphism $\psi\colon B \to A_{P_l}$ such that $r_l\psi = \phi$.

4. Localization of 1-connected polyhedra. We first prescribe the category in which we shall work. We shall be considering 1-connected (i.e., connected and simply-connected) polyhedra X, and we will suppose each such space X furnished with a base-point which will be a vertex of X. We shall further be considering base-point-preserving continuous functions, or *maps*, of such polyhedra and we will identify such maps $X \to Y$ if they are connected by a base-point-preserving homotopy. We sum this up by saying that we work in the *homotopy category* $\mathbf{H}_1$ *of based* 1-*connected polyhedra.*

Given such a polyhedron X, we say that X is *P-local* if its homotopy groups $\pi_n X$ are P-local for all $n \geqq 1$. Of course, since X is 1-connected, $\pi_1 X$ is trivial, but we prefer to include $\pi_1 X$ in the definition in view of later generalizations to less restrictive classes of polyhedra (see Section 5). We now make the basic definition (compare Proposition 2.1). Recall that we are always in the category $\mathbf{H}_1$.

DEFINITION 4.1. The map $e\colon X \to X_P$ *P-localizes* X if X_P is P-local and if, for any map $f\colon X \to Y$, where Y is P-local, there exists $g\colon X_P \to Y$, unique (up to homotopy) with $ge = f$ in $\mathbf{H}_1$.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & X_P \arrow[dl, dashed, "g"] \\ Y & \end{tikzcd}$$

It is trivial to prove that, if e exists, then it is unique up to canonical equivalence (just as for abelian groups).

Now any map $f: X \to Y$ in $\mathbf{H}_1$ induces homomorphisms of homotopy groups

$$\pi_n f: \pi_n X \to \pi_n Y, \qquad n \geqq 1, \tag{4.1}$$

and homomorphisms of homology groups (with integer coefficients)

$$H_n f: H_n X \to H_n Y, \quad n \geqq 1. \tag{4.2}$$

Then the two basic theorems of the theory are the following:

THEOREM 4.1. *Every X in $\mathbf{H}_1$ may be P-localized.*

THEOREM 4.2. *The following statements about $f: X \to Y$ in $\mathbf{H}_1$ are equivalent:*
(i) *f P-localizes X;*
(ii) *$\pi_n f$ P-localizes $\pi_n X$, $\quad n \geqq 1$;*
(iii) *$H_n f$ P-localizes $H_n X$, $\quad n \geqq 1$.*

In fact, the proofs of Theorems 4.1 and 4.2 proceed simultaneously. We do not go into details, but merely draw the readers attention to certain important features of the argument.

The first step is to prove the equivalence of (ii) and (iii) in Theorem 4.2. In the light of Proposition 2.9, it suffices to prove the following two assertions.

PROPOSITION 4.3. *Let $f: X \to Y$ in $\mathbf{H}_1$. Then $\pi_n f$ is P-bijective for all $n \geqq 1$ if and only if $H_n f$ is P-bijective for all $n \geqq 1$.*

PROPOSITION 4.4. *Let Y be in $\mathbf{H}_1$. Then $\pi_n Y$ is P-local for all $n \geqq 1$ if and only if $H_n Y$ is P-local for all $n \geqq 1$.*

Proposition 4.3 is proved by standard arguments of homotopy theory [**12**]. Proposition 4.4 is reduced to standard arguments by the following crucial observation.

PROPOSITION 4.5. *Let Y be a connected space. Then $H_n Y$ is P-local for all $n \geqq 1$ if and only if $H_n(Y; \mathbb{Z}/p) = 0$ for all primes $p \in P'$, and all $n \geqq 1$.*

This proposition is really purely algebraic, since one has only to invoke the universal coefficient theorem in homology [**9**] and the easily proved fact that the abelian group A is P-local if and only if

$$A \otimes \mathbb{Z}/p = 0, \qquad \operatorname{Tor}(A, \mathbb{Z}/p) = 0$$

for all primes $p \in P'$.

The second step in the proof of Theorems 4.1 and 4.2 is to show that (iii) implies (i); here we use a standard obstruction argument. We then turn our attention to Theorem 4.1, proving, in fact, that to any X in $\mathbf{H}_1$ we may always find $f: X \to Y$

satisfying (iii) of Theorem 4.2. We argue by induction on the dimension of X (a final, easy limiting argument then handles the case when X is infinite-dimensional). Since X is 1-connected it is sufficient, for starting the induction, to suppose that $X = \bigcup S^2$, a union of 2-spheres with a single common (base) point. In fact, if we can find a localization S^2_P of S^2, then we may localize this space X by $\bigcup S^2_P$, so that we only have to construct $e\colon S^2 \to S^2_P$ satisfying (iii). It is plain that S^2_P must be a 1-connected polyhedra whose only non-vanishing homology group is $\mathbb{Z}_P$ in dimension 2; and then e is the map inducing $\mathbb{Z} \to \mathbb{Z}_P$ in homology. Since we can always construct a space $M(G,n)$ — called a Moore space — having only one non-vanishing homology group, namely G, in dimension n, when $n \geqq 2$, the induction is easily started. Indeed we can localize $\bigcup S^n$ as $\bigcup S^n_P$, for any $n \geqq 2$.

Now suppose X is $(n+1)$-dimensional, $n \geqq 2$, and we know how to construct, for any n-dimensional 1-connected polyhedron X', a map $f'\colon X' \to Y'$ satisfying (iii). We regard X as constructed from its n-skeleton X^n by attaching $(n+1)$-cells; let $g\colon \bigcup S^n \to X^n$ be the attaching map. Thus we have a diagram

$$\begin{array}{ccccc} \bigcup S^n & \xrightarrow{g} & X^n & \xrightarrow{j} & X \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f'} & & \\ \bigcup S^n_P & & Y' & & \end{array} \tag{4.3}$$

where e, f' satisfy (iii) and j is the inclusion. It is now easy to prove that there is a map $h\colon \bigcup S^n_P \to Y'$ such that $he = f'g$ in $\mathbf{H}_1$. Use h to attach the cone on $\bigcup S^n_P$ to Y', to form Y. Then we may further find $f: X \to Y$ so that $fj = kf'$ in $\mathbf{H}_1$, where k embeds Y' in Y.

$$\begin{array}{ccccc} \bigcup S^n & \xrightarrow{g} & X^n & \xrightarrow{j} & X \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \bigcup S^n_P & \xrightarrow{h} & Y' & \xrightarrow{k} & Y \end{array} \tag{4.4}$$

From (4.4) and the exact homology sequence for a pair of spaces, together with the exactness of localization of abelian groups (Proposition 2.4), we infer that f satisfies (iii) so that the inductive step is complete.

Finally, we complete the proof of Theorem 4.2 by showing that (i) $\Rightarrow$ (iii). Thus let $f'\colon X \to Y'$ satisfy (i). We have proved that we can construct $f\colon X \to Y$ satisfying (iii). Hence $f\colon X \to Y$ satisfies (i). Since both $f'\colon X \to Y'$, $f\colon X \to Y$ satisfy (i), there exists a homotopy equivalence $u\colon Y \to Y'$ with $uf = f'$ in $\mathbf{H}_1$. Since f satisfies (iii), so does f'.

Let us look again, more closely, at the constructive aspect of our proof of Theorem 4.1. A cell-complex X is built up by successively attaching cells by means of maps of spheres. We see, inductively, from (4.4) that the P-localization X_P of X

is built up by attaching *P-local cells* by means of maps of *P-local spheres*. Here a P-local n-sphere S_P^n is a Moore space $M(\mathbb{Z}_P, n)$, and a P-local $(n+1)$-cell is a cone on a P-local n-sphere. Moreover, as (4.4) again shows, the attaching maps in the construction of X_P are the P-localizations of the attaching maps in the construction of X. This suggests that X_P be regarded not merely as a cell-complex but rather as a *P-local-cell-complex*, whose local-cellular structure imitates the cellular structure of X. This point of view has been adopted, for example, by Slagle [**14**] in his study of the homotopy theory of local-cell-complexes.

We now discuss certain key properties of localization. Recall that we are only concerned with 1-connected polyhedra at this stage.

PROPOSITION 4.6. *Let $f: X \to Y$ in $\mathbf{H}_1$. Then f is a homotopy equivalence if and only if each f_p is a homotopy equivalence.*

We note, however, that there is no result corresponding to Proposition 3.3. It is easy to construct examples of compact 1-connected polyhedra X, Y such that $X_p \simeq Y_p$, for all primes p, but $X \not\simeq Y$. A simple example of this phenomenon is provided by the spaces below, obtained by attaching 7-cells to the 3-sphere S^3 by means of suitable maps $S^6 \to S^3$:

$$X = S^3 \cup_\omega e^7, \quad Y = S^3 \cup_{7\omega} e^7. \tag{4.5}$$

Here $\omega \in \pi_6(S^3)$ is a generator; indeed $\pi_6(S^3) = \mathbb{Z}/12$ and ω designates the element which measures the non-commutativity of quaternion multiplication. First, one shows that if $X_k = S^3 \cup_{k\omega} e^7$, then $X_k \simeq X$ if and only if $k \equiv \pm 1 \bmod 12$, so that $X \not\simeq Y$. Second, it is plainly only necessary to localize at the primes 2 and 3, since ω is annihilated by localizing at any other prime. Now, localizing at 2, ω_2 is of order 4, so that

$$\omega_2 = -7\omega_2;$$

and, localizing at 3, ω_3 is of order 3, so that

$$\omega_3 = 7\omega_3.$$

It is thus clear that $X_2 \simeq Y_2$, $X_3 \simeq Y_3$, proving our assertion.

A second and closely related example is obtained by considering principal S^3-bundles over S^7. Again, these bundles are classified by elements of $\pi_6(S^3)$, and we consider the total spaces E_ω, $E_{7\omega}$ of the bundles classified by ω, 7ω respectively. Then we obtain, as above, the conclusion that $E_\omega \not\simeq E_{7\omega}$ but $E_{\omega p} \simeq E_{7\omega p}$ for all primes p. In fact E_ω, $E_{7\omega}$ have cellular structures

$$E_\omega = S^3 \cup_\omega e^7 \cup e^{10}, \quad E_{7\omega} = S^3 \cup_{7\omega} e^7 \cup e^{10} \tag{4.6}$$

which explains why this phenomenon is related so closely to that of our first example.

Indeed, (4.6) has considerable further interest. For E_ω is, in fact, the symplectic group $\mathrm{Sp}(2)$, whereas $E_{7\omega}$ is not a Lie group, nor even of the homotopy type of a

Lie group. However, it was noted in [**8**] that $E_{7\omega}$ is a Hopf manifold, that is, a manifold admitting a continuous multiplication with two-sided unity element, and Stasheff proved [**15**] that it is even of the homotopy type of a topological group. However, this topological group cannot be a Lie group and must therefore be infinite-dimensional — a remarkable fact, since $E_{7\omega}$ is itself 10-dimensional. The resemblance between $E_\omega = \mathrm{Sp}(2)$ and $E_{7\omega}$ is, indeed, remarkably close; it was proved in [**8**] that

$$E_\omega \times S^3 = E_{7\omega} \times S^3$$

(in the sense of diffeomorphism); and in [**11, 13**] that

$$E_\omega \times E_\omega \simeq E_{7\omega} \times E_{7\omega}.$$

Mislin [**11**] introduced the term *genus* to describe a set of homotopy type $X, Y, \cdots$, such that

$$X_p \simeq Y_p, \text{ for all primes } p.$$

Thus we may say that E_ω and $E_{7\omega}$ belong to the same genus; in fact, it may be proved that they constitute an entire genus. Zabrodsky has recently proved that a genus containing a compact Hopf manifold is always finite; and it is reasonable to conjecture the stronger statement that a genus containing a compact polyhedron is finite.

We deduce immediately from Proposition 3.3 (since compact 1-connected polyhedra have finitely-generated homology and homotopy groups)

PROPOSITION 4.7. *Let X, Y be compact polyhedra in $\mathbf{H}_1$ of the same genus. Then*

$$H_nX \cong H_nY, \quad n \geqq 1, \qquad \pi_nX \cong \pi_nY, \quad n \geqq 1.$$

Indeed, it is very hard to detect the difference between X and Y by means of the usual invariants of algebraic topology. For example, the entire mod p Steenrod algebra, in cohomology, is shared by X and Y. Thus an important task in the development of this theory is to obtain invariants which distinguish homotopy types of the same genus.

We would also like to mention a result, to appear in [7], relating to the genus of a Hopf manifold. We recall that a Hopf manifold M is a closed topological manifold, with base point e, furnished with a continuous multiplication $\mu: M \times M \to M$ such that $\mu(x,e) = \mu(e,x) = x$, for all $x \in M$. Among the Hopf manifolds are the Lie groups; S^7 and $\mathbb{R}P^7$ (real projective 7-space) also admit the structure of Hopf manifolds; and it is easy to see that topological products of Hopf manifolds and retracts of Hopf manifolds are again Hopf manifolds. Recently, and largely by the use of localization techniques, many new Hopf manifolds have been discovered — for example, the manifold $E_{7\omega}$ referred to earlier (4.6).

Let us refer to a *Hopf space* X if we discard from the definition of a Hopf manifold

the requirement that the underlying space be a manifold. It is then plain that if X is a Hopf space, so are all its localizations X_P and, moreover, in such a way that $e\colon X \to X_P$ is an H-map (that is, a homomorphism up to homotopy). The question thus naturally arises whether the converse holds: suppose we are given a space X such that its localizations at each prime, X_p, admit the structure of an H-space — can we deduce that X admits the structure of an H-space?

Now, as explained above, an H-space structure on X_p determines an H-space structure on the rationalization X_0. From Theorem 3.6 we may then readily deduce

THEOREM 4.8. *If X is a compact polyhedron in $\mathbf{H}_1$ and if the H-space structure in each X_p determines the same H-space structure in X_0, then X admits an H-space structure such that each $e_p\colon X \to X_p$ is an H-map.*

This result, however, is often difficult to apply, because the condition that the induced H-space structures in X_0 coincide, can be hard to verify. In [7] a stronger result is proved:

THEOREM 4.9. *If X is a Hopf manifold in $\mathbf{H}_1$ and Y is a compact polyhedron belonging to the genus of X, then Y is a retract of $X \times X$. Thus, in particular, Y admits the structure of a Hopf manifold.*

This theorem, while being rather powerful, is unsatisfactory for the following reason. It certainly shows that Y may be endowed with a continuous multiplication $\mu_Y\colon Y \times Y \to Y$, with two-sided unity, if X may be endowed with a continuous multiplication $\mu_X\colon X \times X \to X$, with two-sided unity. However, it gives no indication of what properties of μ_X might be inherited by μ_Y. For example, μ_X might be (homotopy) associative, and we would like to be able to infer (were it true!) that Y admitted a (homotopy) associative multiplication. Theorem 4.9 would not help here because the multiplication on Y which we infer from the fact that Y is a retract of $X \times X$ would inherit no interesting properties (apart from the existence of a two-sided unity) from a given multiplication μ_X on X.

5. A generalization. A very important, indeed crucial, theorem of localization theory is the following (see [5]).

THEOREM 5.1. *Let W be a compact, connected polyhedron and let X in $\mathbf{H}_1$ be of finite type, that is, the homotopy groups of X are finitely generated. Then $[W, X]$ is the pull-back of the sets $[W, X_p]$ over $[W, X_0]$.*

Here $[W, X]$ is the set of base-point-preserving homotopy classes of based maps $W \to X$; it is thus a set with distinguished element, the class of the constant map. Then the assertion of the theorem is to be understood in the sense of Theorem 3.5, and, indeed, Theorem 3.5 is required in its proof. However, Theorem 5.1 is clearly more subtle, because in Theorem 3.5 we are dealing with abelian groups whereas the sets $[W, X]$, $[W, X_p]$, $[W, X_0]$ have no algebraic structure, in general.

It turns out that to prove this theorem we need to make a study of the function space X^W of maps of W into X; and we would wish to be able to localize this space. However, we meet an immediate difficulty since X^W will not be 1-connected. Even if, as we should, we confine attention to a single component of X^W, the component will fail to be simply-connected. Thus, even to prove a theorem about localizations of spaces in $\mathbf{H}_1$ we need to move outside the category $\mathbf{H}_1$.

We would also wish to move outside $\mathbf{H}_1$ in order to be able to apply localization techniques to a broader class of spaces; in particular, we would like to include all connected Lie groups, indeed all connected Hopf manifolds, and not to be confined to those which are simply-connected.

Fortunately, a very convenient and appropriate category is to hand, the category of *nilpotent* spaces. To explain the notion of a nilpotent space we need some preparation.

Given a group G, we define the *lower central series* of G by the rule

(5.1) $$\Gamma^1 G = G,\ \Gamma^{i+1} G = [G, \Gamma^i G], \qquad i \geqq 1,$$

where $[A, B]$ is the group generated by commutators $a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Then we say that G is *nilpotent of class* $\leqq c$ if $\Gamma^{c+1} = \{1\}$, the trivial group. Thus (nontrivial) abelian groups are nilpotent of class 1.

Now let the group G act on the abelian group A; we define the *lower central G-series* of A by the rule

(5.2) $$\Gamma_G^1 A = A,\ \Gamma_G^{i+1} A = \mathrm{gp}(a - xa),\ a \in \Gamma_G^i A,\ x \in G, \qquad i \geqq 1,$$

where $\mathrm{gp}(a - xa)$ is the group generated by the elements $a - xa$. Then we say that the G-action on A is *nilpotent of class* $\leqq c$ if $\Gamma_G^{c+1} A = \{0\}$. Thus if G acts trivially on A, that is, $xa = a$ for all $x \in G$, $a \in A$, then the G-action is nilpotent of class 1.

If X is a connected space, then the fundamental group $\pi_1 X$ acts on the higher homotopy groups $\pi_n X$, $n \geqq 2$, which are, of course, abelian. We may thus make our basic definition.

DEFINITION 5.1. The connected polyhedron X is *nilpotent* if $\pi_1 X$ is nilpotent and operates nilpotently on the higher homotopy groups $\pi_n X$.

Plainly, 1-connected polyhedra are nilpotent. Thus, if $\mathbf{N}$ is the homotopy category of nilpotent polyhedra, then

$$\mathbf{H}_1 \subseteq \mathbf{N}.$$

Also it is easy to prove that if M is a (connected) Hopf manifold, then $\pi_1 M$ is abelian and acts trivially on the higher homotopy groups of M. Thus $M \in \mathbf{N}$. Further, the category $\mathbf{N}$ takes on great significance from the proposition.

PROPOSITION 5.2. *Let $X \in \mathbf{H}$ and let W be a compact polyhedron. Then each component of the function space $X^W \in \mathbf{H}$.*

(Strictly speaking, X^W is not a polyhedron but, as shown by Milnor, it has the homotopy type of polyhedron.)

It now turns out that Theorems 4.1, 4.2 remain true in the larger category $\mathbf{N}$ — of course, the proofs become more complicated, but the only decisive change in the line of reasoning is in the actual construction of the P-localization X_P, which no longer proceeds cellularly. Further, Theorem 5.1 also generalizes to the case in which X is nilpotent and a key step in the proof is the following.

PROPOSITION 5.3. *Let* $X \in \mathbf{N}$ *and let* W *be a compact polyhedron. Then* (*restricting to components*)

$$X_P^W = (X^W)_P.$$

Notice that this proposition could not even have been formulated without extending the notion of localization to $\mathbf{N}$, since X^W is not, in general, in $\mathbf{H}_1$ even if X is in $\mathbf{H}_1$.

Our description of localization above still contains one important omission. Generalizing the definition given in Section 4, we could say that the nilpotent space X is P-local if all its homotopy groups $\pi_n X$, $n \geqq 1$, are P-local. Now, if $n \geqq 2$, $\pi_n X$ is abelian and we have already explained in Section 2 what it means for an abelian group to be P-local. However, $\pi_1 X$ is only given to be nilpotent, so we must explain what it means for a nilpotent group to be P-local. In fact, we can give a significant meaning to the assertion that an *arbitrary* group is P-local; namely, G is *P-local* if the function

$$f_p \colon x \mapsto x^p, \qquad x \in G,$$

is bijective for all primes p not in P. Notice, first, that this does generalize the definition given in Section 2 for abelian groups and, second, that we can only assert that the function f_p is *bijective* since it is not, in general, a homomorphism.

Now we rested much of the argument of Section 4 — naturally — on the localization theory for abelian groups. Thus we would expect that the arguments used in the case in which we permit nilpotent spaces, instead of confining ourselves to 1-connected spaces, would require, as foundation, a localization theory for nilpotent groups. Indeed, such a theory may be developed, independently of any application of topology (though, of course, motivated by such an application!), and this is done in [3]. In particular, it is proved in [3] that any nilpotent group has a P-localization, and the P-localization is characterized in a manner generalizing Proposition 2.9 in a fairly obvious way.

REMARK. The importance of Theorems 4.1, 4.2 (and their extensions to nilpotent spaces) is evident once it is granted that we want to be able to use localization techniques. Theorem 4.1 says that we *can* localize spaces, and Theorem 4.2 tells us how to recognize the localization when we have it! However, the significance of Theorem 5.1 may be less obvious. In order to explain its importance, let us first make the following observation. Suppose, in the notation of Theorem 5.1, that W

can also be localized (say, W is simply-connected). Then, by the universal property described in Definition 4.1, we may replace the conclusion of the theorem by the equivalent assertion

(5.3) $[W, X]$ is the pull-back of the sets $[W_p, X_p]$ over $[W_0, X_0]$.

Recall, from the explanation of Theorem 3.5, that (5.3) is an *existence* statement together with a *uniqueness* statement; let us extract from (5.3) the uniqueness part of the assertion. This reads:

(5.4) Suppose given two maps $f, g\colon W \to X$ such that $f_p \simeq g_p\colon W_p \to X_p$ for all primes p. Then $f \simeq g$.

It is statement (5.4) that tells us that, when we localize, we are really still doing homotopy theory.

References

1. A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics 304, Springer (1972).

2. P. M. Cohn, Rings of fractions, this Monthly, 78 (1971) 596–616.

3. P. J. Hilton, Localization and cohomology of nilpotent groups, Math. Zeit., 132 (1973) 263–286.

4. ———, Remarks on the localization of nilpotent groups, Comm. Pure and Applied Math., XXVI (1973) 703–713.

5. P. J. Hilton, G. Mislin and J. Roitberg, Homotopical localization, Proc. Lond. Math. Soc. 3, XXVI (1973) 693–706.

6. P. J. Hilton, G. Mislin and J. Roitberg, H-spaces of rank 2 and non-cancellation phenomena, Inv. Math., 16 (1972) 325–334.

7. P. J. Hilton, G. Mislin and J. Roitberg, Localization theory for nilpotent groups and spaces, Notas de Matematica, North Holland (to appear).

8. P. J. Hilton and J. Roitberg, On principal S^3-bundles over spheres, Ann. Math., 98 (1969) 91–107.

9. P. J. Hilton and S. Wylie, Homology theory, Cambridge University Press, (paperback edition 1967).

10. M. Mimura, G. Nishida and H. Toda, Localization of CW-complexes and its applications, J. Math. Soc. Japan, 23 (1971) 593–624.

11. G. Mislin, The genus of an H-space, Lecture Notes in Mathematics 249, Springer (1971) 75–83.

12. J.-P. Serre, Groupes d'homotopie et classes de groupes abéliens, Ann. Math., 58 (1953) 258–294.

13. A. Sieradski, Square roots up to homotopy type, Amer. J. Math., 94 (1972) 73–81.

14. J. Slagle, Combinatorial theory of local CW-complexes, Ph.D. Thesis, University of Washington 1973.

15. J. Stasheff, Manifolds of the homotopy type of (non-Lie) groups, Bull. A. M. S., 75 (1969) 998–1000.

16. D. Sullivan, Geometric topology, part I: Localization, periodicity and Galois symmetry, MIT, June 1970, (mimeographed notes).

17. J. H. C. Whitehead, Combinatorial homotopy I, Bull. A. M. S., 55 (1949) 213–245.

18. A. Zabrodsky, Homotopy associativity and finite CW-complexes, Topology, 9 (1970) 121–128.

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SIMPLE POINTS OF AN AFFINE ALGEBRAIC VARIETY

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1. Introduction and notation. During the past one hundred years, the study of algebraic geometry has provided a primary impetus for the development of what we refer to as modern algebra. The correspondences between radical ideals and algebraic varieties, prime ideals and irreducible varieties, and maximal ideals and points, in the classical setting of David Hilbert, provide interesting illustrations of the interplay between geometric notions and ring-theoretic concepts. Of course the ideals referred to in these correspondences are polynomial ideals, but these are by no means the only ideals whose study can provide us with information of a geometric nature.

In this expository article, after recalling one definition of an affine algebraic variety, we shall present two distinct definitions of a simple point of an affine algebraic variety. The first definition is classical, and says in effect that a point of an irreducible affine algebraic variety is simple if the tangent space at the point in question is not too large. We refer to this notion as that of **classical simplicity**. The second definition was made by Oscar Zariski roughly thirty years ago. It says that a point of an irreducible affine algebraic variety is simple when a related local ring is regular. We shall refer to this notion as **algebraic simplicity**, since it has its roots in the ideal theory which forms a basic part of modern algebra.

In [6], Oscar Zariski proved that in a rather general setting the definitions of algebraic simplicity and classical simplicity coincide. We present here a different proof of that equivalence, one whose general approach follows that which appeared in [1]. We present elementary examples to illustrate some of the concepts, as we proceed. For the sake of readability, we assume that fields are of characteristic zero, but the proofs given are valid in general if the ground field is perfect. We conclude the article with a brief presentation of an example of Zariski. The example illustrates the requirement that the ground field be a perfect field if the two formulations of simplicity are to be equivalent.

Throughout this article, we let k denote the arbitrary **ground field** in which the coefficients of all polynomials will lie. We consider polynomials in n variables over k, and for convenience we write $k[X]$ for $k[X_1, \cdots, X_n]$. We will evaluate polynomials at points in K^n, where K is an algebraically closed field containing k and with infinite transcendence degree over k. With these hypotheses, the field K is called a **universal domain** for the ground field k. We refer to K^n as **affine space** and we call K the **coordinate field**. As we have remarked, we usually assume that k and K are of characteristic zero, although nearly all results hold if k is an arbitrary perfect field.

The polynomial ideals in $k[X]$ play a critical role in defining geometric objects in the affine space K^n. Before 1900, David Hilbert had shown that any ideal in a polynomial ring over a field can be generated by a finite number of polynomials. This convenient property was later found to hold for the ideals in a large class of

rings. Such rings are called **Noetherian rings** after Emmy Noether, who extended Hilbert's work by studying arbitrary commutative rings with this property.

DEFINITION 1.1. Let I be a polynomial ideal in $k[X]$. The set of simultaneous zeros in K^n of all polynomials in I is called an **affine algebraic variety**. We denote this variety by $V(I)$, or simply by V if only one variety is under consideration.

If V is a variety in K^n defined over k, then the set of all polynomials which vanish at all points of V is a polynomial ideal which we denote by $I(V)$ and refer to as **the ideal of** V. Although all polynomial ideals in $k[X]$ define a variety in K^n, it is not true that all such ideals are ideals of some affine algebraic variety. Those polynomial ideals which are the ideals of some variety can be characterized as radical ideals. We recall the definition of a radical ideal in an arbitrary commutative ring:

DEFINITION 1.2. Let I be an ideal in a commutative ring R. If, for any $f \in R$, the existence of a power f^m of f in I implies that $f \in I$, then the ideal I is called a **radical ideal**. If I is any ideal of a commutative ring R, then the **radical of** I is defined.

$$\text{rad}(I) = \{f \in R : f^m \in I \text{ for some } m > 0\}.$$

If $V(I)$ is the variety of a polynomial ideal I, then it can be shown that the ideal of $V(I)$ is rad (I). One version of the Hilbert *Nullstellensatz* asserts that there is a one-to-one order-inverting correspondence between radical polynomial ideals in $k[X]$ and the affine algebraic varieties in K^n. In the event that $I(V)$ is a prime ideal, the variety V is called an **irreducible variety**. Such a variety is characterized by the property that it cannot be written as a union of varieties, except in a trivial way. For the remainder of this article, we assume that V is an irreducible variety, and thus that the ideal of V is prime. We denote this polynomial ideal by $P(V)$.

The residue class ring $k[X]/P(V)$ will be denoted by R, and the n-tuple of canonical images in R of the indeterminates $X_1, \cdots, X_n$ will be denoted by (x). We reserve the symbol (v) for a point of V which has a special property:

DEFINITION 1.3. A point (v) of V is a **generic point** for V if the set of all polynomials in $k[X]$ which vanish at (v) is the prime ideal $P(V)$.

The **dimension of** V is equal to the transcendence degree over k of the field $k(v)$ generated by the elements in the n-tuple (v), where (v) is generic for V. The field $k(v)$ can also be replaced by the field $k(x)$, in defining the dimension of V. Since K is assumed to be a universal domain for k, it can be shown that any irreducible variety V has a generic point (v) in K^n. In the event that K were not a universal domain for k, a given irreducible variety V might not possess a generic point, and in this case the n-tuple (x) could play a role quite analogous to that of a generic point.

Example 1.4. Consider the prime polynomial ideal in $\mathbb{Q}[X_1, X_2]$ generated by the irreducible polynomial $g(X_1, X_2) = X_1^3 - X_2^2$. Let V be the variety of this ideal in $\mathbb{C}^2$. Since π is a transcendental number, it can be shown that (π^2, π^3) is a point of V which is generic for V.

If $f_1, \cdots, f_m$ are polynomials in $k[X]$ which generate $P(V)$, then the partial derivatives $\partial f_i / \partial X_j$ form an $m \times n$ matrix of polynomials. If (w) is any point of V, we can evaluate the entries of this matrix at (w), and the entries of the resulting matrix are denoted $\partial f_i / \partial X_j (w)$. The matrix $(\partial f_i / \partial X_j (w))$ is an $m \times n$ matrix with entries in the coordinate field K, and it will be referred to as the **Jacobian matrix of V at the point** (w). The symbol

$$\rho \left(\frac{\partial f_i}{\partial X_j}(w) \right)$$

will denote the rank of this matrix.

2. The classical formulation of a simple point. Let V be an arbitrary r-dimensional irreducible variety in K^n over a ground field k, and assume that the prime ideal $P(V)$ is generated by the set of polynomials $\{f_1, \cdots, f_m\}$.

DEFINITION 2.1. A point (w) of V is **classically-simple** for V if the Jacobian matrix at (w), with entries $\partial f_i / \partial X_j (w)$, is of rank $n - r$.

It is not difficult to show that this definition is independent of the choice of a generating set for $P(V)$, as it must be if the definition is to have real meaning. A point of V which is not classically-simple will be called a **classical singularity** of V.

The Jacobian matrix of V at (w) can be used to associate to (w) a geometrical object which we shall call the tangent linear variety to V at (w):

DEFINITION 2.2. If $(w) = (w_1, \cdots, w_n) \in K^n$ is a point of V, consider the equations

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(w) \cdot (X_j - w_j) = 0$$

defined by the entries of the Jacobian matrix, where $\{f_1, \cdots, f_m\}$ is any finite set of generators for $P(V)$. The solutions in K^n of this linear system of polynomial equations over K form a variety in K^n which we call the **tangent K-linear variety to V at the point** (w). We denote this variety by $L_K(w)$.

We remark that Definition 2.2 could also be formulated with the field K replaced by the field $k(w)$ which is generated over k by the coordinates of the point (w). Of course $k(w)$ need not be an algebraically-closed field, and it is somewhat more natural here to have the tangent variety lie in the space K^n. However, we shall have occasion to consider the tangent $k(w)$-linear variety which we denote by $L_{k(w)}(w)$. We remark also that these definitions are easily shown independent of the choice of generators for $P(V)$.

If the origin of the coordinate system for K^n is translated to the point (w), then the tangent linear variety $L_K(w)$ of V at (w) becomes a K-vector space. This vector space has dimension over K which we call the linear dimension of $L_K(w)$:

DEFINITION 2.3. The **linear dimension** of $L_K(w)$ is the dimension over K of the

solution subspace in K^n of the system of homogeneous equations:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(w) \cdot X_j = 0.$$

We denote this vector space by $L^0_K(w)$.

The proof of the next result is immediate from standard results of elementary linear algebra.

PROPOSITION 2.4. *The linear dimension of* $L^0_K(w)$ *is equal to*

$$n-\rho\left(\frac{\partial f_i}{\partial X_j}(w)\right).$$

It is also not difficult to show that the dimension of $L_K(w)$ as a K-variety coincides with its linear dimension:

PROPOSITION 2.5. *For any point* (w) *of* V, *the equality*

$$\dim L_K(w) = n - \rho\left(\frac{\partial f_i}{\partial X_j}(w)\right) \text{ holds.}$$

We omit the proof of this result, but we refer the reader to [**5**, p. 16]. In the course of the proof, one can easily show that $L_K(w)$ is an irreducible variety over K. In light of the last two propositions, one can interpret the dimension of $L_K(w)$ as being either a linear dimension or as the dimension of this K-linear variety as a variety.

The propositions provide geometric meaning to the quantity, $n-\rho\ (\partial f_i/\partial X_j(w))$, which is used in determining whether or not (w) is a classically-simple point of V. For, by Definition 2.1, the point (w) is classically-simple for V if and only if $n-\rho(\partial f_i/\partial X_j(w)) = r$, where r is the dimension of V. It follows that (w) is classically-simple for V if and only if the tangent K-linear variety $L_K(w)$ has dimension equal to that of the variety V over the ground field k.

It can be shown that if k is of characteristic zero, or more generally, if k is a perfect field, the inequality

$$n-\rho\left(\frac{\partial f_i}{\partial X_j}(w)\right) \geqq r$$

always holds. Thus (w) is classically simple for V exactly when the tangent linear variety $L_K(w)$ is "as small as possible." We postpone the proof of this result until section 5, where it will appear as a corollary to the exactness of a certain sequence.

We conclude the discussion of classical simplicity by considering some examples. Let $f(X_1,X_2) = X_1 - X_2^2$ be a polynomial defined over the rational numbers, with solution space in $\mathbb{C}^2$. Then $\partial f/\partial X_1\ (0,0) = 1$ and $\partial f/\partial X_2\ (0,0) = 0$. The tangent linear variety to this variety at the origin is then the line $X_1 = 0$ in $\mathbb{C}^2$. Since, in this

setting, $n = 2$ and $r = 1$, it follows that $(0,0)$ is a classically-simple point for the variety defined by f.

Consider next the polynomial $g\ (X_1, X_2) = X_1^3 - X_2^2$. Then both partial derivatives of g are zero, when they are evaluated at the origin. The rank of the Jacobian matrix of g at the origin is zero rather than one, so the origin is not a classically-simple point for the variety in $\mathbb{C}^2$ defined by g. In this case, the attached linear variety at the origin is the entire complex plane, which is too large.

The preceding examples suggest that there is some similarity between the concept of a singularity which we have presented here, and the one which is familiar to students of elementary calculus. Indeed, the curve defined by the equation $X_1 - X_2^2 = 0$ over the reals is differentiable as a function of one variable, X_2. The curve defined by $X_1^3 - X_2^2 = 0$ has a cusp at the origin when considered over the reals, and the related function of one variable $h(X_2) = X_2^{2/3}$ has no derivative at the origin when considered over $\mathbb{R}$.

We wish to emphasize that this type of argument fails in general: Polynomials whose curves have a classical singularity at some point on the variety in $\mathbb{C}^2$ may not have a singular behavior when the graph is viewed in $\mathbb{R}^2$. We cite the example defined by the equation $X_1^3 - X_2^4 = 0$. Thus we cannot generally restrict our attention from $\mathbb{C}^2$ to $\mathbb{R}^2$.

3. The algebraic formulation of a simple point. The algebraic study of simple points uses local rings and a process called localization.

DEFINITION 3.1. Let R be a commutative ring with identity. R is called a **local ring** if it possesses a unique maximal ideal.

DEFINITION 3.2. Let R be a commutative ring with identity, and let S be a multiplicatively closed subset of R. Consider pairs (r, s) with $r \in R$ and $s \in S$. We define a relation

$$(r, s) \sim (r', s')$$

between such pairs, by the condition that there exists an $s_0 \in S$ such that

$$s_0(s'r - r's) = 0.$$

It is easily verified that this is an equivalence relation, and we denote the equivalence class containing (r, s) by r/s. The set of all equivalence classes is denoted R_S. By a direct computation, one can show that the usual addition and multiplication of fractions are well-defined and thus that R_S is a ring. We call R_S the **localization of R at the set S.**

DEFINITION 3.3. Let R be a commutative ring with identity and let P be a prime ideal in R. Then $S = R - P$ is a multiplicatively closed set, and we consider R_S in this case. The ring R_S is a local ring with maximal ideal $R_S \cdot P$. We sometimes denote this ring R_P and refer to it as the **localization of R at P,** although this notation and terminology are inconsistent with that given in the preceding definition.

For a comprehensive discussion of local rings and the process of localization, we refer to Nagata's book [**4**, Chapter 1]. We turn now to the applications of localization in the local study of irreducible affine algebraic varieties.

If $(w) \in K^n$ is any point, we let $P((w))$ denote the prime ideal of all polynomials in $k[X]$ which vanish at (w). When (w) is a point of an irreducible variety V, it is easy to show that $P((w)) \supset P(V)$, and the containment is proper unless (w) is a generic point for V. We let Q denote the image of $P((w))$ in the residue class ring $R = k[X]/P(V)$, and we note that Q is a prime ideal in the Noetherian integral domain R. The localization of R at the ideal Q is a Noetherian local integral domain which we denote by R_Q. We denote the unique maximal ideal of R_Q by the letter M.

In general, the point (w) may not be an algebraic variety, but it is generic for some variety W, namely the set of all zeros of the polynomial ideal $P((w))$. We denote the dimension of W by s and we recall that, by definition, $s = \text{tr deg}\,(k(w): k)$.

We refer to the length of the longest possible chain of prime ideals in a ring as the **Krull dimension** of the ring. Since there is a one-to-one, order-inverting correspondence between prime polynomial ideals which contain $P(V)$ and the irreducible subvarieties of V, the Krull dimension of R_Q can be seen to coincide with the length of the longest chain of irreducible subvarieties of V which contain W. The following result, proved essentially by Wolfgang Krull in a less geometric setting, is then not surprising:

THEOREM 3.4. *The Krull dimension of the local ring R_Q is $r-s$, where $r = \dim V$ and $s = \dim W$.*

(The proof follows easily from [**7**, Volume II, pp. 192–194].)

There is another dimension which is often associated with a local ring: the number of elements in any minimal generating set for the maximal ideal M. By a well-known result, often referred to as Nakayama's Lemma, this number equals the vector space dimension of M/M^2 over the field R_Q/M, and thus it is well-defined. We denote this dimension by m_0 and we refer to it as the **vector space dimension associated with the local ring**.

The notion of a regular local ring was first defined by Wolfgang Krull in 1938 [**3**, p. 207]. There are various equivalent formulations of this definition, and we present here one which was provided by I. S. Cohen [**2**, p. 86]:

DEFINITION 3.5. A local ring is **regular** if its Krull dimension is equal to the associated vector space dimension m_0.

In his study of chains of prime ideals in commutative Noetherian rings, Wolfgang Krull had shown that in any local ring, the Krull dimension is bounded above by m_0, the vector space dimension associated with the ring. Thus a local ring was seen by Cohen to be regular precisely when its associated vector space dimension is as small as possible.

It was Oscar Zariski who pioneered the use of local rings in the study of points on an irreducible affine algebraic variety. His idea was to define a point of a variety to be simple when the associated local ring is regular.

DEFINITION 3.6. Let (w) be a point of an irreducible affine algebraic variety V, and let R_Q denote the local ring of (w) on V. Then (w) is an **algebraically-simple** point of V whenever R_Q is a regular local ring.

As a corollary to our definitions and Theorem 3.4, we obtain the following result:

COROLLARY 3.7. *If r is the dimension of V, and s is the dimension of the variety W which is defined by the polynomial ideal $P((w))$, then (w) is algebraically-simple for V if and only if $r - s = m_0$.*

Although Definition 3.6 has its basis in notions of modern algebra, Corollary 3.7 shows that this definition does have a geometric interpretation. The quantity $r - s$ is a difference in sizes of two varieties while m_0 is a vector space dimension. We refer the reader to [**6**, pp. 10–12] for a more detailed study of the geometric interpretation of the algebraic notions introduced in this section.

4. Exact sequences and derivations. In this section, we develop briefly some results which we shall use in the next section to assist in proving the equivalence of the two formulations of simplicity under reasonable hypotheses. We assume some familiarity with short exact sequences of modules, and with the notion of a split exact sequence. It is well known that short exact sequences of vector spaces over a field can always be split. However, it will be useful to define a stronger notion when considering a ring with a k-algebra structure:

DEFINITION 4.1. Let R be a k-algebra where k is a field, and let I be an ideal of R. Consider the split short exact sequence of k-vector spaces

$$0 \to I \xrightarrow{j} R \xrightarrow{\pi} R/I \to 0,$$

where j and π are the canonical homomorphisms. Let $h: R/I \to R$ be a k-vector space homomorphism which splits the sequence. If h is actually a k-algebra homomorphism, we shall say that $h: R/I \to R$ **splits the sequence as a k-algebra homomorphism.** In this case, it can be shown that $R = I + h(R/I)$ where $h(R/I) \cap I = 0$ and $h(R/I)$ is a subalgebra. We summarize these circumstances by writing

$$R = I \dotplus h(R/I).$$

We observe that in Definition 4.1, the residue k-algebra R/I always contains a copy of k, since I can contain no units of R and thus $k \cap I = 0$.

We can now state and prove a result which will be a basic step in the proof of the equivalence of the two notions of simple points when the ground field is of characteristic zero (or a perfect field of characteristic p). The theorem as stated is a

special case of the Cohen structure theorems [7, Volume II, pp. 304–313], but we provide a direct proof which is part of mathematical folklore.

THEOREM 4.2. *Let k be a field of characteristic zero. Let R be a local k-algebra with a maximal ideal M for which $M^2 = 0$. Assume further that the residue class field R/M is finitely generated over k. Then there exists a k-algebra homomorphism $h\colon R/M \to R$ such that $R = M + h(R/M)$.*

Proof. Since R/M is finitely generated over k, we may write

$$R/M = k(\bar{Y}_1, \cdots, \bar{Y}_t, \bar{y}),$$

where the $\bar{Y}_i$ are transcendental over k and are algebraically independent, and where $\bar{y}$ is separably algebraic over the field $k(\bar{Y}_1, \cdots, \bar{Y}_t)$. We are using here the fact that a separably algebraic, finitely generated extension of any field can always be generated by a single element. The representation for R/M is a standard part of classical field theory.

Let $Y_1, \cdots, Y_t$ be any elements in R such that $\pi(Y_i) = \bar{Y}_i$, where $\pi\colon R \to R/M$ is the canonical surjection. Let f be a nonzero polynomial in $k[X_1, \cdots, X_t]$ where the X_i are variables over k. If $f(Y_1, \cdots, Y_t)$ is in M then $\pi \circ f(Y_1, \cdots, Y_t) = 0$. But since π is a k-algebra homomorphism, we observe that $f(\bar{Y}_1, \cdots, \bar{Y}_t)$ is also zero. Since $\bar{Y}_1, \cdots, \bar{Y}_t$ are algebraically independent, this event is an impossibility and thus $f(Y_1, \cdots, Y_t)$ is not an element of M. Since R is a local ring, $f(Y_1, \cdots, Y_t)$ is then a unit of R. We have thus shown that all nonzero elements of $k[Y_1, \cdots, Y_t]$ are units of R, and so

$$k(Y_1, \cdots, Y_t) \subseteq R.$$

Let $y_0 \in R$ be such that $\pi(y_0) = \bar{y}$. Let $\bar{g} \in k(\bar{Y}_1, \cdots, \bar{Y}_t)\ [X]$ be a minimal polynomial of $\bar{y}$ over the field $k(\bar{Y}_1, \cdots, \bar{Y}_t)$. Let g be any polynomial in

$$k(Y_1, \cdots, Y_t)\ [X]$$

which maps to $\bar{g}$ when π is applied to the coefficients of g. Then $\pi \circ g(y_0) = \bar{g}(\bar{y}) = 0$ by definition of $\bar{g}$.

We now look for an element $m \in M$ for which $g(y_0 + m) = 0$. Since by hypothesis $M^2 = 0$, for any positive integer n we have

$$(y_0 + m)^n = y_0^n + m\,n\,y_0^{n-1}.$$

We immediately conclude from this fact that $g(y_0 + m) = g(y_0) + m\,g'(y_0)$ where g' denotes the usual derivative of g. Since $\bar{y}$ is separably algebraic over

$$k(\bar{Y}_1, \cdots, \bar{Y}_t),$$

we know that $\pi(g'(y_0)) = \bar{g}'(\bar{y}) \neq 0$. Thus $g'(y_0)$ is not in M and so is a unit in R. We can then solve the equation $g(y_0 + m) = g(y_0) + m\,g'(y_0) = 0$ for $m \in M$, and we obtain $m = -g(y_0)/g'(y_0)$. We remark that $g(y_0) \in M$ because $\pi(g(y_0)) = 0 \in R/M$,

by the definitions of g and $\bar{g}$. If we let $y = y_0 - g(y_0)/g'(y_0)$ then we have shown that $\pi(y) = \pi(y_0) = \bar{y}$, and that $g(y) = 0$.

Consider the ring $k(Y_1, \cdots, Y_t)[y]$ in R. This ring is an integral domain and a finite dimensional vector space over the field $k(Y_1, \cdots, Y_t)$, and it can thus easily be shown to be a field isomorphic to R/M. We denote it by $k(Y_1, \cdots, Y_t, y)$.

We define $h: R/M \to R$ by setting $h(\bar{Y}_i) = Y_i$ and $h(\bar{y}) = y$, and by then extending h to all of R/M as a k-algebra homomorphism. A direct computation shows that h is a well-defined k-algebra homomorphism, and that $\pi \circ h: R/M \to R/M$ is the identity map. By Definition 4.1, we conclude that $R = M \dotplus h(R/M)$. This completes the proof of Theorem 4.2.

It is useful in studying classically-simple points of a variety to have available the notion of a derivation. Derivations are maps of rings into bimodules which generalize partial derivatives of polynomials, and so the theory of derivations is useful in examining the Jacobian matrix at a point on an irreducible variety.

Let R be a commutative ring with identity and let N be an R-bimodule. We recall that an R-**bimodule** is both a left and right R-module with the property that

$$(r_1 n)r_2 = r_1(nr_2)$$

for all r_1 and $r_2 \in R$ and $n \in N$.

DEFINITION 4.3. Let R be a commutative ring with identity and let N be an R-bimodule. A mapping $D: R \to N$ is called a **derivation from R to N** if it is additive and if, for every r_1 and $r_2 \in R$, the map D satisfies:

$$D(r_1 r_2) = r_1 D(r_2) + D(r_1) r_2 \text{ (the derivation property).}$$

DEFINITION 4.4. If in addition R' is a subring of R for which $D(R') = 0$, then D is called an R'-**derivation** of R to M. We denote by $\mathrm{Der}_{R'}(R, N)$ the set of all R'-derivations from R to N.

In the preceding definition, if N is replaced by a field L containing R, and if R contains a copy of a field k, then we recall that $\mathrm{Der}_k(R, L)$ is a vector space over the field L. Here, $l \cdot D$ for $l \in L$ and $D \in \mathrm{Der}_k(R, L)$ is defined by the equation $(l \cdot D)(r) = l \cdot D(r)$ for each $r \in R$.

REMARK 4.5. It is not difficult to show that derivations generalize the usual partial derivatives. For example, if $R = k[u_1, \cdots, u_m]$ is finitely generated as a ring over the field k, consider $D \in \mathrm{Der}_k(R, L)$. It can be shown that for $h(u_1, \cdots, u_m) \in R$, the derivation D can be written

$$D(h(u_1, \cdots, u_m)) = \sum_{i=1}^{m} \frac{\partial h}{\partial X_i}(u_1, \cdots, u_m) D(u_i),$$

where $\partial h / \partial X_i$ represents the ordinary partial derivative of the polynomial

$$h(X_1, \cdots, X_m)$$

in m variables over k.

If R is a local ring and M is its unique maximal ideal, there is a relationship between certain derivations with domain R and vector space homomorphisms with domain M/M^2. We make this relationship explicit in the following result which is fundamental in our development:

THEOREM 4.6. *Let k be a field of characteristic zero, and let R be a local k-algebra with unique maximal ideal M. Let $0 \to M \xrightarrow{j} R \xrightarrow{\pi} R/M \to 0$ denote the canonical short exact sequence of k-vector spaces. Assume further that N is an R/M vector space.*

(a) *The sequence*

$0 \to \mathrm{Der}_k(R/M, N) \xrightarrow{\pi^*} \mathrm{Der}_k(R,N) \xrightarrow{j^*} \mathrm{Hom}_{R/M}(M/M^2, N)$ *is exact, where j^* and π^* will be defined below.*

(b) *If R/M is finitely generated as a field over k, then the following sequence is exact:*

$$0 \to \mathrm{Der}_k(R/M, N) \xrightarrow{\pi^*} \mathrm{Der}_k(R,N) \xrightarrow{j^*} \mathrm{Hom}_{R/M}(M/M^2, N) \to 0.$$

Proof. (a) We remark first that

$$\mathrm{Der}_k(R/M, N), \ \mathrm{Der}_k(R,N), \text{ and } \mathrm{Hom}_{R/M}(M/M^2, N)$$

are all R/M-vector spaces in the obvious way. For since N is an R/M-vector space, derivations and homomorphisms with images in N may be multiplied by elements of R/M and a vector space structure results.

If $D \in \mathrm{Der}_k(R/M, N)$, we define $\pi^*(D) = D \circ \pi \colon R \to N$. It is easily verified that π^* is indeed an R/M homomorphism.

We next define the map j^*. Let $D' \colon R \to N$ be a k-derivation. We define $j^*(D') = j_{D'} \in \mathrm{Hom}_{R/M}(M/M^2, N)$ as follows: $j_{D'}(\bar{m}) = D'(m)$ where $m \in M$ is a representative of $\bar{m} \in M/M^2$. Since N is an R/M-module, $MN = 0$ and it follows by the derivation rule for D' that $D'(M^2) = 0$. Thus $j_{D'}$ is a well-defined map. It is routine to verify that $j_{D'}$ is R/M-linear. Direct computations also show that $j_{D'}$ is additive, that $j_{\bar{r}D'} = \bar{r} j_{D'}$ where $\bar{r} \in R/M$, and that

$$j_{D'_1 + D'_2} = j_{D'_1} + j_{D'_2}.$$

Thus D' does induce an R/M-homomorphism from M/M^2 to N, and the map j^*: $\mathrm{Der}_k(R,N) \to \mathrm{Hom}_{R/M}(M/M^2, N)$ is an R/M-homomorphism.

We claim that the sequence

$$\mathrm{Der}_k(R/M, N) \xrightarrow{\pi^*} \mathrm{Der}_k(R,N) \xrightarrow{j^*} \mathrm{Hom}_{R/M}(M/M^2, N)$$

is exact at $\mathrm{Der}_k(R,N)$. Suppose $j^*(D') = 0$ where $D' \in \mathrm{Der}_k(R,N)$. Then $D'(M)$

$= 0 \in N$, and so D' induces a derivation $D\colon R/M \to N$. The image of D under π^* is D', so that we have shown that $D' \in \text{Kernel}\ (j^*)$ implies that $D' \in \text{Image}\ (\pi^*)$. Thus the image of π^* contains the kernel of j^*.

On the other hand, if $D' \in \text{Der}_k(R, N)$ is in the image of π^*, then $D' = D \circ \pi$ where $D \in \text{Der}_k(R/M, N)$. We need to show that $j^*(D \circ \pi) = 0$. But

$$j^*(D \circ \pi) = j_{D\circ\pi} \text{ and } j_{D\circ\pi}(\bar{m}) = D \circ \pi(m) = D(0) = 0$$

since $\pi(M) = 0$ in R/M. Altogether, we have shown that Image $(\pi^*) =$ Kernel (j^*).

We finally show that π^* is injective and thus that the sequence

$$0 \to \text{Der}_k(R/M, N) \xrightarrow{\pi^*} \text{Der}_k(R, N) \xrightarrow{j^*} \text{Hom}_{R/M}(M/M^2, N)$$

is exact. Suppose $\pi^*(D) = 0$ where $D \in \text{Der}_k(R/M, N)$. Then $D \circ \pi = 0$ and since π is a surjective map from R to R/M, it follows that D is identically zero on R/M. Thus $\pi^*(D) = 0$ implies $D = 0$ and so π^* is injective.

(b) We first show that we can replace M by M/M^2 and thus reduce the proof to the case where $M^2 = 0$. The sequence $0 \to M/M^2 \to R/M^2 \to R/M \to 0$ is an exact sequence of k-vector spaces. The ring R/M^2 is a local k-algebra, with maximal ideal M/M^2, and it satisfies the hypotheses of Theorem 4.6(b), with the added property that its maximal ideal has square equal to zero. To reduce to the case where $M^2 = 0$, we observe that $\text{Der}_k(R, N) \cong \text{Der}_k(R/M^2, N)$. For, $MN = 0$ because N is an R/M-module, and by the derivation property, $D(M^2) = 0$ for every $D \in \text{Der}_k(R, N)$. It is thus evident that $\text{Der}_k(R, N)$ and $\text{Der}_k(R/M^2, N)$ may be identified, and this identification provides the R/M-isomorphism from $\text{Der}_k(R, N)$ to $\text{Der}_k(R/M^2, N)$. The exactness of the sequence

$$0 \to \text{Der}_k(R/M, N) \to \text{Der}_k(R/M^2, N) \to \text{Hom}_{R/M}(M/M^2, N) \to 0$$

is thus sufficient to show the exactness of the sequence in (b). Since in the proof of (b), the ideal M can be replaced by M/M^2 in R/M, we assume without loss of generality that $M^2 = 0$.

Under the hypotheses of (b), it remains to show that j^* is a surjection. By Theorem 4.2, we can assume that $R = M \dotplus h(R/M)$ where $h\colon R/M \to R$ is a k-algebra homomorphism for which $\pi \circ h$ is the identity map of R/M.

Suppose that $g\colon (M = M/M^2) \to N$ is an R/M-homomorphism. We define $D_g\colon R \to N$ by $D_g(m + h(\bar{r})) = g(\bar{m})$, and it is easily verified that D_g is a k-derivation.

Finally, $j^*(D_g) = j_{Dg} = g$ and so j^* is surjective. The verification of this fact is straightforward from the definitions of j^* and D_g. We conclude that

$$0 \to \text{Der}_k(R/M, N) \xrightarrow{\pi^*} \text{Der}_k(R, N) \xrightarrow{j^*} \text{Hom}_{R/M}(M/M^2, N) \to 0$$

is an exact sequence of R/M-vector spaces.

5. Equivalence of the formulations of simplicity. The results of the preceding section have immediate application to the local study of algebraic varieties. We

revert to the notation which was introduced at the outset of this article and also to that explained after Definition 3.3.

LEMMA 5.1. *If* (w) *is a point of* V *and* W *is the subvariety of* V *for which* (w) *is generic, then*

$$0 \to M \xrightarrow{j} R_Q \xrightarrow{\pi} k(w) \to 0$$

is an exact sequence of k-vector spaces where the homomorphisms are canonical and will be defined below.

Proof. We recall that if (v) is a generic point for V then $R \cong k[v]$. Thus $R_Q \cong \{f(v)/g(v)\colon f(v)$ and $g(v)$ are in $k[v]$ and $g(w) \neq 0\}$. Now since (w) is a point of V, it is clear that $P((w)) \supseteq P(V)$, and it follows directly that the canonical map from $k[v]$ to $k[w]$ is well-defined. Thus the canonical map from R_Q to $k(w)$ which is obtained by replacing (v) with (w) is also well-defined and is evidently a surjection. We let π denote this map. The kernel of π is

$$\left\{\frac{f(v)}{g(v)} : \frac{f(v)}{g(v)} \in R_Q \text{ and } f(w) = 0\right\},$$

which is easily seen to be the ideal M.

COROLLARY 5.2. *With the usual hypotheses and notation, we assume that the ground field* k *is of characteristic zero. Let* N *be an* R_Q/M*-vector space. Then there is a short exact sequence of* R_Q/M*-vector spaces*:

$$0 \to \mathrm{Der}_k(R_Q/M, N) \to \mathrm{Der}_k(R_Q, N) \to \mathrm{Hom}_{R_Q/M}(M/M^2, N) \to 0.$$

Proof. This result follows from Lemma 5.1 and Theorem 4.6.

COROLLARY 5.3. *In the setting of Corollary* 5.2, *there is a short exact sequence of* $k(w)$*-vector spaces*:

$$0 \to \mathrm{Der}_k(k(w), k(w)) \to \mathrm{Der}_k(R_Q, k(w)) \to \mathrm{Hom}_{k(w)}(M/M^2, k(w)) \to 0.$$

Proof. In Corollary 5.2, we replace N by $k(w)$ and we use the fact that

$$R_Q/M \cong k(w).$$

It is Corollary 5.3 which will enable us to study the relationship between algebraically-simple points and classically-simple points of a variety. We first undertake a study of the three terms which compose this sequence.

REMARK 5.4. If k has characteristic zero, the dimension of $\mathrm{Der}_k(k(w), k(w))$ over $k(w)$ is s, where $s = \mathrm{tr\,deg}(k(w)\colon k)$ is the dimension of the variety W for which (w) is a generic point. A similar result holds more generally if k has arbitrary characteristic, provided that $k(w)$ is separably generated over k [7, Volume I, p. 127]. This result is guaranteed, in particular, if k is a perfect field.

LEMMA 5.5. *The dimension of* $\mathrm{Hom}_{k(w)}(M/M^2, k(w))$ *over* $k(w)$ *is equal to the dimension of* M/M^2 *over* $R_Q/M \cong k(w)$. *Thus, both dimensions are equal to the number of elements in a minimal generating set for* M *over* R_Q.

Proof. The space $\mathrm{Hom}_{k(w)}(M/M^2, k(w))$ is the dual space of the $k(w)$-vector space M/M^2, and by elementary linear algebra both spaces have the same finite dimension. The last statement of the lemma now follows by Nakayama's Lemma [4, pp. 12–13].

It remains to study the middle term of the exact sequence of Corollary 5.3, and we begin that study with a lemma:

LEMMA 5.6. *Let* (v) *be a generic point for* V *and let* R *denote the coordinate ring* $k[X]/P(V)$. *Then*

$$\mathrm{Der}_k(R_Q, k(w)) \cong \mathrm{Der}_k(R, k(w)) \cong \mathrm{Der}_k(k[v], k(w)),$$

where the isomorphisms are as $k(w)$-*vector spaces.*

Proof. Since $R \cong k[v]$, we need only show that

$$\mathrm{Der}_k(R_Q, k(w)) \cong \mathrm{Der}_k(R, k(w)).$$

It is easily seen that any derivation D in $\mathrm{Der}_k(R, k(w))$ can be extended uniquely to a derivation D' in $\mathrm{Der}_k(R_Q, k(w))$. If $f(v)/g(v) \in R_Q$, then let

$$D'\left(\frac{f(v)}{g(v)}\right) = \frac{g(w)D(f(v)) - f(w)\,D(g(v))}{g(w)^2}.$$

We emphasize that such an extension works only because $g(w) \neq 0$ by definition of R_Q, and an analogous extension to the entire field of fractions of R could not be performed. In the present case, we can identify $\mathrm{Der}_k(R, k(w))$ and $\mathrm{Der}_k(R_Q, k(w))$.

We can relate the $k(w)$-vector space $\mathrm{Der}_k(k[v], k(w))$ to our brief study of classical simplicity. To do so, we consider the tangent K-linear variety to V at w which we called $L_K(w)$. If in Definition 2.2 we replace the field K by its subfield $k(w)$, we can define a tangent $k(w)$-linear variety $L_{k(w)}(w)$ which lies in $k(w)^n$. This $k(w)$-linear variety can be related to the $k(w)$-vector space of derivations being considered.

PROPOSITION 5.7. *Let* $L^0_{k(w)}(w)$ *denote the vector space which is associated with* $L_{k(w)}(w)$. *Then*

$$\dim_{k(w)} \mathrm{Der}_k(k[v], k(w)) = \dim_{k(w)} L^0_{k(w)}(w),$$

where the dimensions are as $k(w)$-*vector spaces.*

Proof. The present proof follows closely that given in [5, pp. 20–21] in a slightly altered setting.

Let $\{f_1, \cdots, f_m\}$ be a generating set for the prime ideal $P(V)$, and identify $k[v]$ with $k[X]/P(V)$. If $D \in \mathrm{Der}_k(k[v], k(w))$, then since $f_i(v) = 0$ we have $D(f_i(v)) = 0$.

If $h(v) \in k[v]$, then

$$D(h(v)) = \sum_{j=1}^{n} \frac{\partial h}{\partial X_j}(w)D(v_j)$$

so that

$$D(f_i(v)) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(w)D(v_j) = 0.$$

Thus $(D(v_1), \cdots, D(v_n))$ is a point of $L^0_{k(w)}(w)$. The map given by

$$D \to (D(v_1), \cdots, D(v_n))$$

is then a map from $\mathrm{Der}_k(k[v], k(w))$ to $L^0_{k(w)}(w)$, and it is injective since a derivation from $k[v]$ to $k(w)$ is uniquely specified by giving its values on the set of k-algebra generators $\{v_1, \cdots, v_n\}$ for $k[v]$. Thus

$$\dim_{k(w)} \mathrm{Der}_k(k[v], k(w)) \leqq \dim_{k(w)} L^0_{k(w)}(w).$$

Conversely, let $(y) = (y_1, \cdots, y_n)$ be an element of $L^0_{k(w)}(w)$ and thus a solution in $k(w)^n$ of the system of $k(w)$-linear equations:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(w) \cdot X_j = 0.$$

Then the mapping D_y from $k[X]$ to $k(w)$ given by

$$D_y(f(X)) = \sum_{j=1}^{n} y_j \frac{\partial f}{\partial X_j}(w)$$

is a k-linear map. Since $(y) \in L^0_{k(w)}(w)$, the polynomials $\{f_1, \cdots, f_m\}$ are in the kernel of D_y. Also, the equality

$$D_y(f(X)g(X)) = f(w)D_y(g(X)) + g(w)D_y(f(X))$$

is easily seen to hold for all f and g in $k[X]$. We have thus shown that D_y is in $\mathrm{Der}_k(k[X], k(w))$ and that $D_y(P(V)) = 0$. Thus D_y induces in a canonical way an element of $\mathrm{Der}_k(k[X]/P(V), k(w)) = \mathrm{Der}_k(k[v], k(w))$, and we denote this element by D'_y. The map $(y) \to D'_y$ is a $k(w)$-linear map from $L^0_{k(w)}(w)$ to $\mathrm{Der}_k(k[v], k(w))$, and is injective. For if $D'_y = 0$, then $D'_y(v_j) = y_j$ for all $j = 1, \cdots, n$, so that $(y) = 0$. Thus

$$\dim_{k(w)} \mathrm{Der}_k(k[v], k(w)) = \dim_{k(w)} L^0_{k(w)}(w).$$

LEMMA 5.8. *The respective linear dimensions of the two tangent linear varieties which we have considered are equal*:

$$\dim_{k(w)} L^0_{k(w)}(w) = \dim_K L^0_K(w).$$

Proof. Both linear varieties are defined by the same system of homogeneous linear equations: $\sum_{j=1}^{n} \partial f_i/\partial X_j(w) \cdot X_j = 0$, for $i = 1, \cdots, m$. By Proposition 2.4,

it is evident that both linear dimensions are equal to

$$n - \rho\left(\frac{\partial f_i}{\partial X_j}(w)\right).$$

We now summarize the results of this section which relate to the definition of a classically-simple point of V.

PROPOSITION 5.9. *Let k be a ground field of characteristic zero, and let (v) be generic for V. Let (w) be a point of V where V is an r-dimensional irreducible variety in affine n-space K^n. The following conditions are equivalent:*

(a) *The point (w) is a classically-simple point of V.*

(b) *The rank of the Jacobian matrix $(\partial f_i/\partial X_j(w))$ is $n-r$. (It assumes its maximal value at the point (w).)*

(c) *The linear dimension over K of the vector space $L_K^0(w)$ associated with the tangent linear variety $L_K(w)$ at (w) is r. (It assumes its minimal value at the point (w).)*

(d) *The linear dimension over $k(w)$ of the vector space $L_{k(w)}^0(w)$ associated with the tangent $k(w)$-linear variety $L_{k(w)}(w)$ at (w) is r.*

(e) *The dimension over $k(w)$ of* $\mathrm{Der}_k(k[v], k(w))$ *is r.*

(f) *The dimension over R_Q/M of* $\mathrm{Der}_k(R_Q, R_Q/M)$ *is r.*

We now prove the classical theorem of Zariski which shows the equivalence of the two formulations of simplicity.

THEOREM 5.10. *Let k be a ground field of characteristic zero for an irreducible variety $V \subseteq K^n$. Then a point (w) of V is algebraically-simple for V if and only if it is classically-simple for V.*

Proof. Let (v) be generic for V and let V have dimension r. Assume that the variety W for which (w) is generic has dimension s. Consider the short exact sequence of Corollary 5.3:

$$0 \to \mathrm{Der}_k(k(w), k(w)) \to \mathrm{Der}_k(R_Q, k(w)) \to \mathrm{Hom}_{k(w)}(M/M^2, k(w)) \to 0.$$

By Lemma 5.6, we may substitute $\mathrm{Der}_k(k[v], k(w))$ for the middle term of this sequence. We consider the $k(w)$-vector space dimensions of the terms of this sequence.

By Remark 5.4, the dimension of $\mathrm{Der}_k(k(w), k(w))$ over $k(w)$ is s. By Lemma 5.5, the dimension of $\mathrm{Hom}_{k(w)}(M/M^2, k(w))$ is m_0, the number of elements in a minimal generating set for M over the ring R_Q.

By Theorem 3.4 and the fact that the Krull dimension of the local ring is bounded above by m_0, we know that $m_0 \geqq r - s$.

By Definitions 3.5 and 3.6, we can conclude that the $k(w)$ -dimension of

$$\mathrm{Hom}_{k(w)}(M/M^2, k(w))$$

is $r - s$ if and only if (w) is algebraically-simple for V.

By a dimension count, $\dim_{k(w)} \mathrm{Der}_k(k[v], k(w)) = s + m_0 \geqq r$, and thus this dimension equals r if and only if the point (w) is algebraically-simple for V. By Proposition 5.9 (parts (a) and (e)), the point (w) is classically-simple for V if and only if it is algebraically-simple for V.

REMARK 5.11. In reference to the remarks at the end of section 2, we note that the dimension of $\mathrm{Der}_k(k[v], k(w))$ is always $\geqq r$. Thus the dimension of $L_K^0(w)$, or equivalently of $L_K(w)$, is always greater than or equal to r by Proposition 5.7 and Lemma 5.8. A point is then classically-simple for V exactly when its tangent linear variety is "as small as possible."

6. Generalizations and a counterexample. In several instances in this article, we have assumed that the ground field k has characteristic zero. At some points a remark was made that a result holds if the ground field k is perfect, i.e., if $k = k^p$ where p is the characteristic of k. In fact, the chief results of this article were shown to hold, provided that k is perfect, by Oscar Zariski [6]. The proofs which we have used here are also applicable in that more general case, and are considerably simpler than Zariski's original proofs.

By 1947, Zariski had shown that it is always true that classically-simple points are algebraically-simple. The converse of this statement, however, is not true if the ground field k is nonperfect. We conclude the article with an example of Zariski which demonstrates that the requirement that k be a perfect ground field is essential in Theorem 5.10.

Example. [**6**, p. 2] Let k be an arbitrary nonperfect field of characteristic $p \neq 0$. Let b be an element of k such that $b^{1/p}$ is not in k. Let K denote any universal domain for k. Consider the curve V in K^2 defined by the polynomial equation:

$$f(X_1, X_2) = X_1^p + X_2^p - b = 0.$$

Then it can be directly shown that all points of V are algebraically-simple, but no points of V are classically-simple.

References

1. H. Cartan and C. Chevalley, Géométrie Algébrique, École Normale Supérieure, Paris, 1965.

2. I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc., 59 (1964) 54–106.

3. W. Krull, Dimensiontheorie in Stellenringen, J. Reine Angew. Math., 179 (1938) 204–226.

4. M. Nagata, Local Rings, Wiley, New York, 1962.

5. G. Seligman, Algebraic Groups, Yale University, New Haven, 1964.

6. O. Zariski, The concept of a simple point of an abstract algebraic variety, Trans. Amer. Math. Soc., 62 (1947) 1–52.

7. O. Zariski and P. Samuel, Commutative Algebra, Volumes I and II, Van Nostrand, Princeton, 1958.

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INTEGRAL DOGGEREL

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Inspired by Tom Lehrer's "The Derivative Song"

You take the interval from a to b
And divide it in n pieces arbitrarily;
Find the maximum of f in each little bit
And multiply the length of that bit by it.
Now add up the products — that's the upper sum —
But don't stop there because there's more to come.

Find the minimum of f in each little bit
And multiply the length of that bit by it.
Then add up the products as you did before;
That's the lower sum, but there still is more.

Now you send that n to infinity
So the lengths tend to zero simultaneously.
Then the upper sums get smaller but they're bounded below
By all of the lower sums (it's easy to show)
While the lower sums get bigger and keep closing in;
And if the gap between gets so very thin
That there's only room for one quantity,
Then that's the integral of f from a to b.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

Reply to Query 8. This Query asked for expository articles dealing with mathematical models in the social sciences. The following have been suggested:

Debreu, PNAS, 40 (1954) 588–592.

T. C. Koopman, *Three Essays on the State of Economic Science.*

E. Batschelet, *Introduction to Mathematics for Life Scientists,* Springer Verlag, New York, 1973.

J. M. Tanur, et al. Eds., *Statistics: A Guide to the Unknown,* Holden-Day, San Francisco, 1972.

Casstevens and Porterfield, Behavioral Science, 13 (1968) 234–237.

Casstevens, The American Political Science Review, 62 (1968) 205–207.

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

DEGREE OF MAPS AND PERIODIC MAPS ON S^1

J. E. CONNETT

In 1933 Borsuk proved that if $f: S^n \to S^n$ is a map of the n-sphere to itself such that $f \circ T = T \circ f$, where $T: S^n \to S^n$ is the antipodal map, then f has odd degree. Milnor [2] noted (and generalized) a simple converse to this result, namely that if $f: S^n \to S^n$ is a map of odd degree, then there exists $x \in S^n$ such that $f(T(x)) = T(f(x))$. This note shows how it is possible to generalize both of the above theorems when $n = 1$:

THEOREM. *Let $T: S^1 \to S^1$ be an orientation-preserving map of period m, and $f: S^1 \to S^1$ a map. Then*

(a) *If $f(T(x)) = T^n(f(x))$ for all $x \in S^1$, $\deg(f)$ is congruent to n* modulo m.

(b) *Conversely, if f has degree n, there exists $x \in S^1$ such that $f(T(x)) = T^n(f(x))$.*

The proof makes use of Brouwer's theorem [1] that orientation-preserving periodic maps on S^1 are conjugate to rotations, i.e., if T_0 is periodic on S^1, there exists a homeomorphism $g: S^1 \to S^1$ and a rotation T_1 of S^1 such that $T_1 = g^{-1} \circ T_0 \circ g$. Using this fact one easily shows that it suffices to prove (a) and (b) only for maps T that actually are periodic rotations of S^1.

Proof of (a). Let $p: R^1 \to S^1$ denote the usual covering map, $p(t) = e^{2\pi i t}$. By [3], p. 55, there exists a map $F: R^1 \to R^1$ such that $p \circ F = f \circ p: R^1 \to S^1$. Note that $F(t+1) = F(t) + \deg(f)$, for all $t \in R^1$. If T is a rotation of S^1 or period m, then for some k relatively prime to m, the map $\theta^1: R^1 \to R^1$ defined by $\theta^1(t) = t + (k/m)$ is such that $p \circ \theta^1 = T \circ p$. If $\theta^n(t) = t + (nk/m)$ denotes the nth iterate of θ^1, then $p \circ \theta^n = T^n \circ p$. It follows that $p \circ F \circ \theta^1 = f \circ p \circ \theta^1 = f \circ T \circ p$, and $p \circ \theta^n \circ F = T^n \circ p \circ F = T^n \circ p \circ F = T^n \circ f \circ p$. Thus from the hypothesis in (a), $p \circ F \circ \theta^1 = p \circ \theta^n \circ F$. Therefore, there exists an integer j such that $F \circ \theta^1(t) = \theta^n \circ F(t) + j$ for each $t \in R^1$, i.e.,

$$F\left(t + \frac{k}{m}\right) = F(t) + \frac{nk}{m} + j.$$

This yields

$$F\left(t + \frac{2k}{m}\right) = F\left(t + \frac{k}{m} + \frac{k}{m}\right) = F\left(t + \frac{k}{m}\right) + \frac{nk}{m} + j = F(t) + \frac{2nk}{m} + 2j,$$

and repeating this process m times gives

$$F(t+k) = F(t) + nk + mj,$$

which is equivalent to

$$F(t) + k \cdot \deg(f) = F(t) + nk + mj.$$

Since k is relatively prime to m, this last equation implies that $\deg(f) \equiv n(\bmod m)$, and (a) is proved.

Proof of (b). Using the above notation, and letting $n = \deg(f)$, it will be shown that $F(t + (nk/m)) = F(t) + (nk/m)$ for some $t \in R^1$. Suppose there is no such t. Then either $F(t + (k/m)) < F(t) + nk/m$ for all $t \in R^1$, or the opposite inequality is true. Assuming the former case, one obtains

$$F\left(t + \frac{2k}{m}\right) = F\left(t + \frac{k}{m} + \frac{k}{m}\right) < F\left(t + \frac{k}{m}\right) + \frac{nk}{m} < F(t) + \frac{2nk}{m}.$$

Continuation of this process eventually gives the inequality $F(t+k) < F(t) + nk$, which is false since $F(t+1) = F(t) + n$. Similarly the inequality $F(t + (k/m)) > F(t) + (nk/m)$ leads to $F(t+k) > F(t) + nk$, again a contradiction. Therefore there exists $t_0 \in R^1$ such that

$$F\left(t_0 + \frac{k}{m}\right) = F(t_0) + \frac{nk}{m},$$

i.e., $F \circ \theta^1(t_0) = \theta^n \circ F(t_0)$. Applying the map p to both sides produces the equation

$$f \circ T(p(t_0)) = T^n \circ f\,(p(t_0)),$$

which proves (b).

An immediate corollary of (b) is the following:

COROLLARY 1. *If $f\colon S^1 \to S^1$ is an orientation-preserving homeomorphism, and $T\colon S^1 \to S^1$ is an orientation-preserving periodic map, then there exists $x \in S^1$ such that $f \circ T(x) = T \circ f(x)$.*

From this one may deduce:

COROLLARY 2. *If G is any group acting freely on S^1, then the elements of G of finite order are central in G.*

Proof (cf. [2]): If G acts freely on S^1, then each element of G is an orientation-preserving homeomorphism. Let $f \in G$ and let $T \in G$, where T has finite order. By Corollary 1, there exists $x \in S^1$ such that $T \circ f(x) = f \circ T(x)$. Therefore $Tf = fT$, which proves Corollary 2.

This last corollary shows that (b) of the theorem cannot be generalized to higher-dimensional spheres, since, e.g., there are infinitely many finite nonabelian groups acting freely on S^3.

References

1. L. E. J. Brouwer, Über die periodischen Transformationen der Kugel, Math. Annalen, 80 (1919) 39–41.

2. J. Milnor, Groups which act on S^n without fixed points, Amer. J. Math., 79 (1957) 623–630 MR 19 p. 761.

3. E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966. MR 35 # 1007.

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A LOWER BOUND FOR $\pi(n)$

ROBERT E. DRESSLER

Introduction. Let $\pi(n)$ stand for the number of primes not exceeding the positive integer n. It was Erdös, I believe, who gave (in a very simple way) the following non-trivial lower bound for $\pi(n)$ [2, pg. 84].

$$\pi(n) \geqq \frac{\log n}{2 \log 2} \qquad (n \geqq 1).$$

This result is not nearly so strong as the Prime Number Theorem, or even Chebycheff's Theorem. However, its worth lies in the brevity and elegance of the proof and also in the fact that it holds for all positive integers.

Our purpose is to improve the above inequality by essentially doubling the lower bound without sacrificing any of the brevity or simplicity in the proof.

The result. In this section we obtain crude upper and lower bounds for $s(n)$, the number of positive square-free integers not exceeding the positive integer n. These bounds yield an inequality involving n and $\pi(n)$ which leads to

THEOREM A. *There is a constant C such that*

$$\pi(n) \geqq \frac{\log n}{\log 2} + C \qquad (n \geqq 1).$$

Proof. For any prime p, the number of multiples of p^2 which do not exceed n is at most n/p^2. Thus, $s(n) \geqq (1 - \sum_p 1/p^2)n = C_1 n$, where $C_1 > 0$ because

$$\sum_p \frac{1}{p^2} < \sum_{n \geqq 2} \frac{1}{n^2} < \int_1^\infty \frac{1}{x^2} dx = 1.$$

Also it is immediate that $s(n) \leqq 2^{\pi(n)}$. Hence $C_1 n \leqq 2^{\pi(n)}$ and so $\pi(n) \geqq (\log n/\log 2) + C$, where $C = (\log C_1/\log 2)$.

A slightly better result. The proof of Theorem A requires nothing in the way of prerequisite machinery. We shall finish by giving an improvement of Theorem A which is based upon a recently obtained result on partitions.

It has been proven [1] (using no theorems on $\pi(n)$) that every integer greater than 45 can be written as the sum of distinct primes greater than 7. We thus see that $n - 45 \leqq 2^{\pi(n)-4} - 1$ and so by transposing and taking logarithms we have

THEOREM B. $\pi(n) \geqq \dfrac{\log(n-44)}{\log 2} + 4 \qquad (n > 44).$

References

1. R. E. Dressler, Sums of distinct primes, Norsk Matematisk Tidsskrift, 21 (1973) 31–32.
2. W. J. LeVeque, Topics in Number Theory, Vol. I, Addison-Wesley, Reading, Mass., 1956.

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SOME UNIQUENESS THEOREMS FOR ANALYTIC FUNCTIONS

DAVID BURDICK AND F. D. LESLEY

1. Introduction. Among the joys of analytic function theory are the uniqueness theorems:

Suppose f is a function which is analytic in a domain D containing the origin. Then $f \equiv 0$ in D if the derivatives $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \cdots$, or if there exists a sequence $\{z_n\}_{n=0}^{\infty}$ converging to $z = 0$ for which $f(z_n) = 0$.

These are apparently generalizations of a property of polynomials; namely that a polynomial of degree N is determined by $N + 1$ pieces of information, which may be either the values of f at $N + 1$ points or the values of the first $N + 1$ derivatives (including the 0th) at a single point. Motivated by the metatheorem that analytic functions are very long polynomials, one might consider the mixed case where we have a sequence $\{z_n\}_{n=0}^{\infty}$ converging to $z = 0$ for which $f^{(n)}(z_n) = 0$, $n = 0, 1, 2, \cdots$. Certainly, a polynomial of degree N is determined by $N + 1$ such pieces of information. However, the metatheorem fails in general. In fact, a beautiful theorem of Polya (see [3], p. 32) implies that for $f(z) = 1/(1 + z^2)$, any point x on the real axis has the property that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ for which $z_n \to x$ as $n \to \infty$ and $f^{(n)}(z_n) = 0$.

Holding our metatheorems dearly, we conclude that meromorphic functions are not enough like long polynomials and we restrict ourselves to more well-behaved functions. Alternatively, we may consider the rate of convergence of the sequence $\{z_n\}$. These questions have been investigated rather thoroughly by a number of authors (see references in [1] and [3]) and the purpose of this note is to give a very simple proof of three well known theorems. The proof is motivated by a proof of the uniqueness theorem for polynomials, wherein one solves for the coefficients in terms of the values $f^{(n)}(z_n)$. In extending this to analytic functions one is led to infinite matrices. However, the absolute convergence of power series and the idea

behind Olga Taussky's simple proof [2] that diagonally dominant finite matrices are non-singular are enough for our purposes.

2. The Basic Lemma. The "niceness" of an analytic function may be determined by the size of the coefficients of the Taylor series at the origin. This is our point of attack.

LEMMA. *Suppose f is analytic in a disk U centered at $z = 0$, $\{z_n\}_{n=0}^{\infty} \subset U$, and $f^{(n)}(z_n) = 0 \quad n = 0,1,2,\cdots$. Suppose also that $g(n)$ is a positive function for which*

$$\lim_{n\to\infty} \frac{|f^{(n)}(0)|}{g(n)} = 0,$$

(i.e., $f^{(n)}(0) = o(g(n))$), *and that there exists N such that for all $m > N$,*

$$\sum_{n=1}^{\infty} \frac{g(m+n)}{g(m)} \frac{|z_m|^n}{n!} < 1.$$

Then $f \equiv 0$ in D.

Proof. Writing the Maclaurin series for the mth derivative $f^{(m)}$ we have

$$0 = \sum_{n=0}^{\infty} \frac{f^{(m+n)}(0)}{n!} z_m^n.$$

Subtracting the first term from both sides and using the absolute convergence of the series,

$$|f^{(m)}(0)| \leqq \sum_{n=1}^{\infty} |f^{(m+n)}(0)| \frac{|z_m|^n}{n!}$$

or

$$\frac{|f^{(m)}(0)|}{g(m)} \leqq \sum_{n=1}^{\infty} \frac{|f^{(m+n)}(0)|}{g(m+n)} \frac{g(m+n)}{g(m)} \frac{|z_m|^n}{n!}.$$

Now choose some $m > N$ such that $|f^{(m)}(0)|/g(m) \geqq |f^{(m+n)}(0)|/g(m+n)$, $n = 0,1,2,\cdots$. This is the crucial step and may be done as $f^{(n)}(0) = o(g(n))$ as $n \to \infty$. Then we have

$$\frac{|f^{(m)}(0)|}{g(m)} \leqq \frac{|f^{(m)}(0)|}{g(m)} \sum_{n=1}^{\infty} \frac{g(m+n)}{g(m)} \frac{|z_m|^n}{n!} < \frac{|f^{(m)}(0)|}{g(m)}.$$

This is a contradiction which can only be avoided if $f^{(n)}(0) = 0$ for $n \geqq m$. But then f is a polynomial and we use the result for polynomials to conclude that $f \equiv 0$.

3. Applications of the lemma. The first result shows that our metatheorem is true for functions of exponential type.

THEOREM 1. (Takenaka, see [3], p. 44). *Suppose that f is entire and of exponential type τ. Suppose also that $\{z_n\}_{n=0}^{\infty}$ is a sequence for which $f^{(n)}(z_n) = 0$ and for some $k > \tau$, $|z_n| < (\log 2)/k$, $n = 0, 1, 2, \cdots$. Then $f \equiv 0$.*

Proof. Since f is of exponential type τ, for any $k > \tau$, we have $f^{(n)}(0) = o(k^n)$. Thus we let $g(n) = k^n$ and we see that, for all m,

$$\sum_{n=1}^{\infty} \frac{g(m+n)}{g(m)} \frac{|z_m|^n}{n!} = \sum_{n=1}^{\infty} \frac{k^{m+n}}{k^m} \frac{|z_m|^n}{n!} = e^{kz_m} - 1$$

which is less than 1 for $|kz_m| < \log 2$, or $|z_m| < (\log 2)/k$.

We now leap to the other extreme, to functions which are analytic in a neighborhood of the origin. Here the sufficient condition for uniqueness concerns the rate of convergence of the $\{z_n\}$.

THEOREM 2 (Kakeya, see [3], p. 43). *Suppose that f is analytic in $D = \{z: |z| < R\}$ and that $\{z_n\}_{n=0}^{\infty}$ is a sequence in D for which $f^{(n)}(z_n) = 0$* and $\overline{\lim}_{n\to\infty} n|z_n| < R \log 2$. *Then $f \equiv 0$ in D.*

Proof. We first assume that $R > 1$. Then $f^{(n)}(0) = o(n!)$ and letting $g(n) = n!$, we consider

$$\sum_{n=1}^{\infty} \frac{(m+n)!}{m!n!} |z_m|^n = \frac{1}{(1-|z_m|)^{m+1}} - 1.$$

We next observe that

$$-(m+1)\log(1-|z_m|) = m|z_m| + m|z_m| \sum_{n=1}^{\infty} \frac{|z_m|^n}{n+1} - \log(1-|z_m|)$$

$$= m|z_m| + o(1) \qquad \text{as } m \to \infty,$$

since $m|z_m|$ is bounded and both the series and $\log(1-|z_m|)$ converge to 0 as $m \to \infty$. Thus $\overline{\lim}_{m\to\infty} m|z_m| < \log 2$ implies that for m sufficiently large $(1-|z_m|)^{-(m+1)} < 2$ and we may conclude that $f \equiv 0$.

The result for $R < 1$ follows by applying the above argument to $h(z) = f(R_1 z)$ for suitable $R_1 < R$.

Finally we prove a theorem where a combination of conditions on the function and the sequence is used. Theorems of this kind were proved by Ålander and Gontcharoff.

THEOREM 3. *Suppose that f is entire of order ρ. Suppose also that $\{z_n\}_{n=0}^{\infty}$ is a sequence for which $f^{(n)}(z_n) = 0$ and that for some constants $k > 0$, and $\rho' > \rho$, $\overline{\lim}_{n\to\infty} n^{1-1/\rho'}|z_n| < k$. Then $f \equiv 0$.*

Proof. Since f is of order ρ,

$$\overline{\lim_{n\to\infty}} \frac{-n\log n}{\log|f^{(n)}(0)/n!|} = \rho < \rho'$$

so that for any ρ_0 with $\rho < \rho_0 < \rho'$ we have

$$\frac{-n\log n}{\log|f^{(n)}(0)/n!|} < \rho_0$$

for n sufficiently large. This means that $f^{(n)}(0) < n!/n^{n/\rho_0}$ and since $\rho' > \rho_0$, $f^{(n)}(0) = o(n!/n^{n/\rho'})$.

Letting $g(n) = n!/n^{n/\rho'}$, we consider

$$\sum_{n=1}^{\infty} \frac{(m+n)!}{(m+n)^{(m+n)/\rho'}} \frac{|m^{m/\rho'}|}{m!} \frac{|z_m|^n}{n!} < \sum_{n=1}^{\infty} \frac{(m+n)!}{m!n!} \left| \frac{z_m}{m^{1/\rho'}} \right|^n .$$

According to the argument of Theorem 2, this is less than one for all m sufficiently large if $\overline{\lim}_{m\to\infty} m|z_m|/m^{1/\rho'} < \log 2$. Now let $h(z) = f(Rz)$, for $R = k/\log 2$. Then h has the same order as f, but the zeros of h are at $\zeta_m = z_m/R$. Thus

$$\overline{\lim_{m\to\infty}}\, m^{1-1/\rho'}|\zeta_m| = \overline{\lim_{m\to\infty}}\, m^{1-1/\rho'} \frac{|z_m|\log 2}{k} < \log 2 .$$

Therefore $h \equiv 0$ and consequently $f \equiv 0$.

References

1. R. P. Boas, Entire Functions, Academic Press, New York, 1954.
2. O. Taussky, A Recurring Theorem on Determinants, this MONTHLY, 56 (1949) 672–676.
3. J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

IN A COMPACT SEMIGROUP, ARE TOPOLOGICALLY SIMPLE SUBSEMIGROUPS ALSO SIMPLE?

H. L. CHOW

An algebraic semigroup S which is also a Hausdorff space is termed a topological semigroup if the multiplication is (jointly) continuous. A subsemigroup of S is called simple if it has no proper ideals. Now we define a subsemigroup A of S

to be *topologically simple* if $A \subset \bar{I}$ for any ideal I of A, where the bar denotes closure. Evidently a simple semigroup is topologically simple.

CONJECTURE. *If S is compact, any topologically simple subsemigroup of S is simple.*

Note that, in general, a topologically simple semigroup need not be simple, as is easily seen from the following example.

Example: (Dobbins [3, Example 3.1]). Let $S = \{(x, y) \in R^2 : x > 0,\ y \geqq 0\}$ with the usual topology and multiplication defined by

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2,\ x_1 y_2 + y_1)$$

for $(x_1, y_1), (x_2, y_2) \in S$. Clearly its minimal ideal is the set $\{(x, y) \in S\colon y > 0\}$. Hence, S is topologically simple but not simple.

A result related to the conjecture is the theorem below, which implies that a *closed* topologically simple semigroup in a compact semigroup must be simple.

THEOREM (Chow [2, Lemma 4]). *Suppose A is a subsemigroup of a compact semigroup S. Then A is topologically simple if and only if $\bar{A}$ is simple.*

We remark that the conjecture has been disproved for the separately continuous case; see ([1], IV.7.1) or Example 2 of [2].

References

1. J. F. Berglund and K. H. Hofmann, Compact semitopological semigroups and weakly almost periodic functions, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

2. H. L. Chow, On supports of semigroups of measures, Proc. Edinburgh Math. Soc., (2) 19 (1974) 31–33.

3. J. G. Dobbins, On the kernel of a locally compact semigroup, Duke Math. J., 39 (1972) 327–331.

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CLASSROOM NOTES

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COMPUTATION OF THE MATRIX EXPONENTIAL

EDWARD P. FULMER

Let A be an $n \times n$ matrix of constants, then the fundamental matrix of solutions of the system of linear differential equations $X'(t) = AX(t)$ is given by the matrix

exponential [1, p. 62 ff]

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{t^k A^k}{k!}.$$

The purpose of this note is to present a pedagogically simple method of constructing e^{At}.

In undergraduate courses on differential equations, the student usually learns to construct e^{At}, in the case when A is similar to a diagonal matrix, by using the similarity transformation $X(t) = PY(t)$, where P is chosen so that $P^{-1}AP$ is diagonal. Of course, to use this method, one must be able to find n linearly independent eigenvectors of A. An alternative method of constructing e^{At} is Kirchner's formula [2], which works whether A is similar to a diagonal matrix or not. The author's experience in teaching Kirchner's formula is that students find it conceptually complicated and awkward to apply.

The method of constructing e^{At} presented here is pedagogically simpler than either method mentioned above. This method is based on the following theorem, which is due to Ziebur [3].

THEOREM. *Each entry of the matrix e^{At} satisfies the n-th order linear differential equation $c(D)y = 0$, where $c(x) = \det(xI - A)$ is the characteristic polynomial of A and $D = d/dt$.*

This result follows immediately from the Cayley-Hamilton theorem and the fact that $d^k/dt^k(e^{At}) = A^k e^{At}$, for $k = 1, 2, \cdots$. In addition, it follows that

$$e^{At}\big|_{t=0} = I \text{ and } \frac{d^k}{dt^k}(e^{At})_{t=0} = A^k, \quad \text{for } k = 1, 2, \cdots.$$

Therefore, e^{At} is the unique solution of the nth order initial value problem

$$c(D)G(t) = 0, \ G(0) = I, \ G'(0) = A, \cdots, G^{(n-1)}(0) = A^{n-1}.$$

To illustrate the method, suppose A has n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, the general solution of the differential equation $c(D)G(t) = 0$ is

$$G(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \cdots + C_n e^{\lambda_n t},$$

where the C_k are $n \times n$ matrices of constants. The n initial conditions yield a system of n linear equations for the n unknown matrices C_k:

$$\begin{aligned}
I &= C_1 + C_2 + \cdots + C_n \\
A &= \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_n C_n \\
A^2 &= \lambda_1^2 C_1 + \lambda_2^2 C_2 + \cdots + \lambda_n^2 C_n \\
&\cdots \\
A^{n-1} &= \lambda_1^{n-1} C_1 + \lambda_2^{n-1} C_2 + \cdots + \lambda_n^{n-1} C_n.
\end{aligned}$$

If one solves this system of equations, one obtains the C_k as polynomials of at most degree $n-1$ in A. The coefficients in these polynomials are just the entries in the rows of the inverse of the Vandermonde matrix, which is the coefficient matrix of the system of equation for the C_k.

The method is essentially the same when A has repeated eigenvalues. Suppose A has k distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ with multiplicities $s_1, s_2, \cdots, s_k$. Then the general solution of $c(D)G(t) = 0$ is

$$(C_{11} + tC_{12} + \cdots + t^{s_1-1}C_{1s_1})e^{\lambda_1 t} + \cdots + (C_{k1} + tC_{k2} + \cdots + t^{s_k-1}C_{ks_k})e^{\lambda_k t}.$$

The n initial conditions again yield a system of n linear equations for the n matrices C_{ij}, but in this case, the coefficient matrix for the system is the confluent Vandermonde matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \lambda_1 & 1 & \cdots & 0 & \cdots & \lambda_k & 1 & \cdots & 0 \\ \lambda_1^2 & 2\lambda_1 & \cdots & 0 & \cdots & \lambda_k^2 & 2\lambda_k & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & \cdots & \dfrac{(n-1)!}{(n-s_1)!}\lambda_1^{n-s_1} & \cdots, & \lambda_k^{n-1} & (n-1)\lambda_k^{n-2} & \cdots & \dfrac{(n-1)!}{(n-s_t)!}\lambda_k^{n-s_t} \end{bmatrix}$$

The matrices C_{ij} are again obtained as polynomials of at most degree $n-1$ in A, and the coefficients in these polynomials are the entries in the rows of the inverse of the confluent Vandermonde matrix.

For example, compute e^{At} for the matrix

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 6 & 3 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

The eigenvalues of A are 1, 1, and 2; hence, the confluent Vandermonde matrix is

$$V_c = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

Thus, $e^{At} = (C_1 + tC_2)e^t + C_3 e^{2t}$, where $C_i = \sum_{j=1}^3 v_{ij}A^{j-1}$ and the v_{ij} are the entries of

$$V_c^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

The primary advantage of the method presented here is its pedagogical simplicity. It is not a practical method of computing e^{At} when A is a large matrix, since the method requires the computation of the first $n-1$ powers of A. This alone requires $n^4 - 2n^3$ multiplications, in general. One should note, however, that Kirchner's formula shares this disadvantage and that in computing n eigenvectors of A, one would do an equivalent amount of work.

References

1. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

2. R. B. Kirchner, An explicit formula for e^{At}, this MONTHLY, 74 (1967) 1200–1204.

3. A. D. Ziebur, On determining the structure of A by analyzing e^{At}, SIAM Review, no. 1, 12 (1970) 98–102.

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EXPLICIT FORMULAS FOR SOLUTIONS OF THE SECOND-ORDER MATRIX DIFFERENTIAL EQUATION $Y'' = AY$

TOM M. APOSTOL

Let A be a given $n \times n$ matrix and let $Y(t)$ denote an $n \times m$ matrix function of a real variable t. It is well known that the first-order matrix differential equation $Y'(t) = AY(t)$, with prescribed initial value $Y(0)$, has the unique solution $Y(t) = e^{tA}Y(0)$ on the interval $(-\infty, +\infty)$. (For example, see [1], p. 200.)

This note discusses the second-order matrix equation

$$Y''(t) = AY(t), \tag{1}$$

with prescribed initial values $Y(0)$ and $Y'(0)$. It is easy to show that (1) has a unique solution on $(-\infty, +\infty)$ given by

$$Y(t) = C(t)Y(0) + S(t)Y'(0), \tag{2}$$

where the matrix functions $C(t)$ and $S(t)$ are given by the everywhere-convergent power series

$$C(t) = \sum_{k=0}^{\infty} \frac{t^{2k}A^k}{(2k)!}, \qquad S(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}A^k}{(2k+1)!}. \tag{3}$$

The solution (2) can be obtained by trying a solution of the form $Y(t) = \sum_{k=0}^{\infty} C_k t^k$ with undetermined matrix coefficients, or by noting that (1) is equivalent to a first-order system of the form

$$Z'(t) = BZ(t), \tag{4}$$

where $Z(t)$ is a $2n \times m$ matrix with block form

with prescribed initial values $Y(0)$, $Y'(0), \cdots, Y^{(r-1)}(0)$. The unique solution on $(-\infty, +\infty)$ is given by the following extension of (2):

$$Y(t) = \sum_{j=1}^{r} C_j(t) Y^{(j-1)}(0),$$

where each matrix function $C_j(t)$ is expressible in the form

$$C_j(t) = \sum_{k=1}^{n} y_{k,j}(t) P_{k-1}(A), \qquad (j = 1, 2, \cdots, r)$$

generalizing (7). The scalar functions $y_{k,j}$ are determined recursively by r triangular systems of rth order linear differential equations with constant coefficients, exactly like those in (8) and (9), except that rth derivatives appear on the left and, for the first equation in each system, all the initial conditions are 0 except $y_{1,j}^{(j-1)}(0) = 1$.

References

1. T. M. Apostol, Calculus, Vol. 2, 2nd ed. Xerox, Waltham, Mass., 1969.
2. E. J. Putzer, Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients, this MONTHLY, 73 (1966) 2–7.

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

A PROPOSAL FOR A PROFESSIONAL PROGRAM IN MATHEMATICS

T. I. SEIDMAN

To an overwhelming extent a student majoring in mathematics in the United States receives training predicated on the assumption that he or she is a potential academic mathematician. The standard track assumes a progression through graduate education culminating in a doctorate and followed by a career of teaching and the creation of new mathematics. Such an academic emphasis is, of course, entirely understandable: this is the track which most university faculty members best understand and which, perhaps, they are most likely to encourage for their students — after all, it is probably the track which they themselves followed.

This assumption is made for students following programs in 'applied' as well as in 'pure' mathematics; the term 'applied mathematics' often refers primarily not to the direct application of mathematics but rather to the theoretical study of certain areas of mathematics which have historically been found useful in applications. In the context of the traditional areas of applied mathematics (i.e., applications to the physical sciences) the successful mathematicians in industry are typically either individuals who, starting with a background of training in another field (e.g., engineering), found in time that their principal activity was the application of mathematics, or individuals with an academic mathematical education who have managed to adapt themselves to a quite different orientation. For the areas of operations analysis and systems theory the unsuitability of much available mathematical training is even more marked. In statistics the situation is only somewhat better. For rare cases is there a standard track designed to provide appropriate training for the non-academic mathematician. Certainly there *should* be a track designed for those who intend employment in the direct application of mathematical concepts and techniques to the formulation and resolution of problems as they arise in the non-academic world.

The view presented here of the non-academic mathematician is that of a 'professional' — providing services to a client or employer. Whereas the academic mathematician is free to choose problems on the bases of interest and the likelihood of obtaining satisfactory, rigorously provable results, it is the task of the 'mathematician as professional' to be able to contribute 'something useful' toward such problems as may be presented. Even when the mathematical backgrounds required would not be dissimilar, the orientation appropriate to such an approach to problems is radically different from that appropriate to academic research. The distinction of orientations might be compared with that between the outlooks of the engineer vis-a-vis the physicist. An article **[2]** by M. Klamkin of the Ford Research Center is worth noting in this context. Of particular interest also is the report [3] of the MAA Committee on Corporate Members.

At this point one must distinguish between the mathematician able to work professionally on problems in, e.g., economics and a mathematically oriented economist. For work on such a problem, the mathematician is likely to be employed as part of a group, including others trained more specifically in economics. Thus, it is possible to rely considerably on others for orientation to the problem (picking this up 'on the job'), which permits a rather less deep background in economics. On the other hand, a reasonable background is certainly desirable at least to the extent that familiarity with the relevant concepts and terminology will markedly facilitate the initiation of a project and will avoid losing time.

The professional mathematician might typically be introduced to an ongoing problem in a specialized context, for example, to make a statistical analysis or in connection with use of a computer for data reduction or solution of equations arising in an established model. It is often at this point that a mathematician can suggest and then profitably participate in a reformulation of larger aspects of the problem. It is not

unlikely that the value of such reformulation may be indicated, not by any specifically mathematical aspect of the problem, but by bringing to bear on it the modes of thinking to which mathematical training is conducive — concern for precision, conceptual clarity and the explication of underlying structure.

A proposal to create a program of training for the 'professional mathematician' raises a number of fundamental questions:

(1) Is this properly the concern of the department of mathematics?
(2) What new demands would be imposed on the mathematics faculty by such a program?
(3) Is such a program advantageous, in that one might expect good employment prospects for its graduates?
(4) How do the capabilities required of a successful 'professional' mathematician differ from those required of a successful 'academic' mathematician?
(5) Is it feasible to impart such capabilities (to a group of appropriately selected individuals) by a course of instruction?
(6) To what extent might existing courses be adapted to such a program? What new courses would have to be developed?

Although it would seem unquestionable that such a program would fill an existing gap, it is less certain that responsibility for its creation lies with the mathematics department. If the program were to be as successful as one might hope, it might ultimately require an administrative structure of its own, as for example, chemical engineering is now departmentally independent of chemistry. For the initiation of curricula of this nature, however, the responsibility must lie with the existing department of mathematics as part of an adequate consideration of the ways in which mathematics can and should enter into the education of 'non-majors' and into the formulation and solution of problems in the non-academic context.

The technical sophistication of much of modern industrial activity, the development of mathematical concepts and techniques (e.g., graph theory, linear programming, convex processes) applicable to types of problems not heretofore amenable to mathematical treatment, and especially the advent of the computer, have greatly increased the range of situations in which direct application of mathematics is likely to be feasible and useful. Everyone is aware that the computer has made feasible the quantitative solution of larger, more complicated problems within the traditional areas of applied mathematics and that there has been an incredible incursion of the computer into such areas as industrial production and design (e.g., numerically controlled machine tools and attendant design techniques) and accounting (payrolls, billing, inventory control, etc.). Less widely known is the extensive mathematization of the biological, behavioral and policy sciences. The construction and analysis of mathematical models are of rapidly increasing importance in a wide variety of fields, for example in cytology, neuroanatomy, epidemiology, cognition theory, linguistics, demography, library science, cryptanalysis, military strategy and economic forecasting. The demand for people competent in the range of mathematical techniques known as 'O.R.' or 'operations analysis' is an indication of this mathematization in the policy sciences.

As has been suggested above, the primary distinction between the roles of academic and professional mathematician is one of orientation of attitude. To initiate a program for professional training in mathematics, a *sine qua non* would be an adequate number of faculty members sympathetic to the problem-oriented approach to mathematics and able to orient their teaching in a way consistent with it. Beyond this, much depends on the desired breadth of the program.

If it is anticipated that students are to be prepared for employment involving applications of kinds already familiar in engineering and the physical sciences, then the required mathematical background is that of traditional applied mathematics: calculus, linear algebra, ordinary and partial differential equations, numerical analysis, etc. Apart from orientation, this would involve only areas represented in almost every moderately good department of mathematics.

On the other hand, much of the new application of mathematics — as noted above and, even in the case of the physical sciences, in the consideration of complex systems — involves background in a number of additional areas: computer science, graph theory and combinatorics, statistics, optimization and programming, etc. If students are to be prepared for employment involving policy and management applications, then it becomes necessary to have some additional faculty capability directed toward these areas.

It should be emphasized that it is important to have a faculty devoted not only to teaching but also to direct involvement, through research and consultation, in the new directions of mathematical application. Since applied mathematics is presently quite active, with new applications and new ideas coming rapidly, it is only such a faculty which can train students who will then have both the knowledge of techniques needed for immediate employability and the fundamental background to enable them to grow to meet their future challenges.

The increased use of mathematics in direct application implies both a pressure on mathematics departments for appropriate 'service courses' and employment opportunity for mathematically trained people. In a recent article **[1]** E. Bareiss observed that many of those employed in industry who are really mathematicians have 'non-mathematical' titles: systems engineer, communications scientist, etc. Thus, there are presumably a large number of positions which, although not recognized explicitly as mathematical, provide opportunities for individuals with suitable mathematical training.

It is difficult to estimate the employment prospects associated with any educational program: these depend greatly on the level (bachelor's, master's, doctoral), on the general economic situation, on the selection of particular students, on the general reputation of the school as well as of the program. For a new program the uncertainties are even greater. In this case, precisely because employers have had some difficulty (cf. **[2]**) in fitting academically trained mathematicians into problem-oriented positions, the graduates of such a program should have a competitive advantage. Further, it may reasonably be hoped that the existence of such a program as is here proposed

may suggest to potential employers the feasibility of greater participation, beginning at the point of problem formulation, by problem-oriented mathematicians. Such a hope becomes even more reasonable if the faculty involved attempts a moderate amount of proselytization.

The requirements for success in academic mathematics are well known; for professional mathematics less so. Summarizing briefly, it is important for the professional mathematician

(a) to be able to provide quickly a rough estimate of the difficulty of the problem,
(b) to interact comfortably with non-mathematicians in eliciting significant auxiliary aspects of a problem, in 'modeling', in developing more precise formulations,
(c) to be able to bring to bear on a problem a wide variety of tools while keeping in mind their limitations (these may be techniques already known but it is most important to know what to look for and where to find it and to be able to learn a technique adequately while employing it).
(d) to keep in mind the 'real' problem and purpose for which the immediate technical problem must be resolved, (i.e., to maintain a teleological orientation to the problem) being able to accept an incomplete solution to the mathematical problem if it is *good enough* for the problem at hand, and
(e) to be able to explain, in an understandable way, the meaning of the 'solution' obtained.

It is worth repeating that, as is implied by (b) and (e) above, it is indispensable for the professional mathematician to be able to communicate with non-mathematicians. In general, this will make a reasonable background in the area of application highly desirable.

Very often an informed common sense will be adequate to handle a problem and 'quick and dirty' methods may be more appropriate than use of the deep and powerful tools of modern mathematics. Often some special property of a particular situation may permit the use of a method which could not be used more generally. On the other hand, the concepts and results of modern mathematics may lead to a deep insight into the underlying structure of a problem, thereby indicating which are the truly significant elements, suggesting appropriate modes of analysis, and facilitating, for the mathematician, the learning of relevant techniques which may not be already familiar.

It is probably impossible to teach 'common sense' though it may be possible to select for it. What *can* be done is to convert 'common sense' to 'informed common sense', to inculcate a teleological mode of approach to problems, to provide some experience with relatively unstructured or imprecisely formulated problems, to provide some experience in interacting with non-mathematicians, to provide a broad background of potentially relevant techniques and of the conceptual framework (both in mathematics and in the potential area of application) which make possible an informed selection of the 'appropriate tool for the job'.

In proposing a specific course of instruction for a program of this sort we observe that much of the 'standard' material would remain essential—calculus, linear algebra, differential equations, as well as such presently optional material as statistics, numeri-

cal analysis, and computer programming. Additional material in optimization and linear programming, elementary functional analysis, data structures and information processing, non-linear analysis, graph theory and combinatorial analysis should also be available. Throughout, the emphasis should be on (a) the conceptual structure of the subject and (b) the 'practical' considerations involved in performing the calculations involved. Proofs, except insofar as they illuminate the significance of the results, may be largely omitted or replaced by heuristic or 'plausibility' arguments. (It might well be argued that such emphases are often equally appropriate for the student who will become an academic mathematician. These courses, so taught, might well be considered common to both the academic and professional tracks.) On the other hand, conceptual accuracy, clarity and precision of statement remain exceedingly important; in professional contexts it is often precisely the ability to abstract (i.e., to isolate significant aspects of a situation) and to formulate a problem with clarity and precision which makes a mathematician valuable. Of particular interest to the potential professional mathematician would be an orientation course discussing, with examples, the approach to mathematics outlined above and a "modeling" course discussing the use and construction of mathematical models and estimation, though these might be combined.

It is worth noting that a recent (1972) CUPM report [**4**] recommends the inclusion of more applied material in the general mathematical curriculum — in particular, the availability of "one or two courses... which ... treat realistic problems and which emphasize model building" — as well as the establishment of a "concentration in applied mathematics." The present proposal agrees entirely with the thrust of that report — subject to the interpretation that the recommended "concentration" be a program explicitly intended to provide training for individuals whose expectation is not primarily the study of mathematics (however potentially useful the specific topic might be) but, rather, the direct application of the techniques and methodology of mathematics to substantive real problems.

References

1. E. Bareiss, The College Preparation for a Mathematician in Industry, this MONTHLY, 79 (1972) 972–984.

2. M. Klamkin, The Ideal Role of an Industrial Mathematician and its Educational Implications, this MONTHLY, 78 (1971) 53–76.

3. R. E. Gaskell and M. S. Klamkin, The Industrial Mathematician Views His Profession, MAA, Washington, D.C., 1973, this MONTHLY, 81(1974) 699–716.

4. CUPM, Mathematical Association of America, Applied Mathematics in the Undergraduate Curriculum, January, 1972.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before May 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2495 [1974, 902]. **Correction.** *Jointly proposed by M. S. Klamkin, University of Waterloo, and L. A. Shepp, Bell Telephone Laboratories.*

E2516. *Proposed by Morris Newman and Charles Johnson, National Bureau of Standards*

Two matrices A and B are *permutation equivalent* if B can be obtained from A by first permuting the rows of A and then permuting the columns of the resulting matrix.

Call an $n \times n$ matrix of zeros and ones a *$k-k$ matrix* if there are precisely k ones in each row and each column. Show that if $n \leqq 5$, then every $k-k$ matrix is permutation equivalent to its transpose, but that this is no longer true if $n \geqq 6$.

E2517. *Proposed by Alex G. Ferrer, Ensenada, Mexico*

Let P be a point interior to the triangle ABC, and let r_1, r_2, r_3 be the distances of P from the sides of the triangle. If p denotes the perimeter of the pedal triangle, show that

$$\Sigma(r_1 + r_2)\cos\tfrac{1}{2}C \leqq p.$$

When does equality occur?

E 2518. *Proposed by Derek A. Zave, Sperry UNIVAC, Roseville, Minnesota*

Let F be the set of real polynomials f with nonnegative coefficients for which $f(1) = 1$. Let $0 < x_0 < 1$ and $0 < \alpha \leqq 1$ be fixed; compute

$$m(x_0;\alpha) = \inf\{f'(1): f \in F \text{ and } f(x_0) \leqq \alpha\}.$$

E 2519. *Proposed by H. L. Montgomery, University of Michigan*

Let P be a complex polynomial of degree n with $P(1) = 0$ and $P(0) = 1$. Show that

$$\max\{|P(z)|: |z| \leqq 1\} \geqq 1 + \frac{1}{3n}.$$

E 2520. *Proposed by G. B. Huff, University of Georgia*

The non-trivial sequence $a_0, a_1, \cdots$ satisfies the following recursion formula:

$$a_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{2k}^2 (n-2k)!\, a_k^2 .$$

Find a_n.

E 2521*. *Proposed by John A. Cross, Snow College*

An instructor has a file of p questions of equal diagnostic value in testing students on a certain topic. He gives q-question tests repeatedly ($q < p$). How many test forms can he compose if any n-size subset, $1 \leqq n < q$, of the p questions may appear on at most two tests, and no subset of size $m > n$ may appear on more than one test? Determine an algorithm for composing the set of possible tests, for any allowable p, q, n.

SOLUTIONS OF ELEMENTARY PROBLEMS

Cutting Corners is not so Easy

E 2452* [1974, 84]. *Proposed by Joel Anderson, California Institute of Technology*

Starting with an arbitrary convex polygon P_1, a sequence of polygons is generated by successively "chopping off corners"; thus if P_i is a k-gon, then P_{i+1} is a $(k+1)$-gon. At the jth step let d_j be the altitude of the cut-off triangle, measured from the cut-off vertex. Prove or disprove: The series Σd_j converges.

Editor's comment. Three contributions were received, one conjecturing convergence (but offering no proof), and the other two claiming to have found divergent examples, both of which were faulty. The problem remains unsolved and a solution is still solicited.

The Linear Dependence of Certain Trigonometric Values

E 2453 [1974, 84]. *Proposed by H. G. Niederreiter, Southern Illinois University*

Determine all rational numbers r for which 1, $\cos 2\pi r$ and $\sin 2\pi r$ are linearly dependent over the rationals.

I. *Solution by Leonard Carlitz, Duke University.* Clearly we need consider only those r which satisfy $0 \leqq r < 1$. Suppose that

$$(*) \qquad a + b\cos 2\pi r + c\sin 2\pi r = 0$$

with a, b, c, r rational. From this it follows that

$$(b^2 + c^2)\cos^2 2\pi r + 2ab\cos 2\pi r + a^2 - c^2 = 0,$$

so that $\cos 2\pi r$ is either rational or a quadratic irrationality; this is also true of $\sin 2\pi r$. Hence (using (*)) the complex number ζ defined by

$$\zeta = \cos 2\pi r + i\sin 2\pi r$$

is, on the one hand, a primitive nth root of unity for some $n \geqq 1$, and is, on the other hand, either rational, a quadratic irrationality, or a biquadratic irrationality. Thus $\phi(n) = 1$, 2, or 4, where ϕ denotes Euler's totient function. We examine each case in turn.

If $\phi(n) = 1$, then $n = 1$ or $n = 2$; if $n = 1$, then $\zeta = 1$ and $r = 0$, whereas if $n = 2$, then $\zeta = -1$ and $r = \frac{1}{2}$.

If $\phi(n) = 2$, then $n = 4$, 3, or 6, giving the possible values for r of $\frac{1}{4}, \frac{3}{4}; \frac{1}{3}, \frac{2}{3}; \frac{1}{6}, \frac{5}{6}$ respectively.

If $\phi(n) = 4$, then $n = 8$, 12, 5, or 10. However, $n = 5$, 10 are not allowed since, for example, $2\sin(2\pi/5) = [\frac{1}{2}(5 + \sqrt{5})]^{1/2}$, which is not a quadratic irrationality. The values $n=8$, 12 give the values for r of $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}; \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$ respectively.

It is easily verified that the values of r thus found do indeed satisfy (*). We can summarize the result by saying that (*) is satisfied if and only if r is of the form $r = m/24$ where $(m, 24) \neq 1$.

II. *Solution by Karl Heuer and G. A. Heuer, Mathematisches Institut der Universität zu Köln.* If $\sin 2\pi r$, $\cos 2\pi r$, or $\tan 2\pi r$ is rational, then r is a solution. Suppose that none of these quantities is rational, but that

$$a\sin 2\pi r + b\cos 2\pi r = c$$

with a, b, c rational (and necessarily all nonzero). Then

$$(a^2 + b^2)\sin^2 2\pi r - 2ac\sin 2\pi r + (c^2 - b^2) = 0;$$

that is, $\sin 2\pi r$ is the root of a quadratic with nonzero linear term in which the sum of the coefficients is $(a-c)^2$—the square of a rational number. K. W. Wegner (*Equations with trigonometric values as roots*, this MONTHLY 66 (1959), 52–53) has given the quadratics having as a root the (irrational) sine of a rational number of degrees (i.e., a rational multiple of 2π), and essentially the only ones with nonzero linear term are $4x^2 \pm 2x - 1 = 0$ and $x^2 \pm 2x - 4 = 0$. Of these, only $4x^2 - 2x - 1 = 0$ has the sum of its coefficients the square of a rational number. Hence we can assume that $a - c = \pm 1$ while $ac = 1$. This requires $c^2 \pm c - 1 = 0$, which has no rational

solutions. Thus the only values of r for which 1, $\cos 2\pi r$, and $\sin 2\pi r$ are linearly dependent over the rationals are those for which either $\sin 2\pi r$ or $\cos 2\pi r$, or $\tan 2\pi r$ is rational. It is well known that the only solutions to this are the "obvious" ones, that is, those r for which either $\sin 2\pi r$ or $\cos 2\pi r$ is 0, $\pm\frac{1}{2}$, or ± 1 or for which $\tan 2\pi r = 0$ or ± 1. See J. M. H. Olmsted, *Rational values of trigonometric functions*, this MONTHLY 52 (1945), 507–508.

Also solved by Robert Breusch (New Zealand), Robert Gilmer (Australia), L. E. Mattics, and the proposer.

Editor's comment. It is curious that the only way that 1, $\cos 2\pi r$, and $\sin 2\pi r$ can be linearly dependent over the rationals is for some two of them to be dependent.

A Combinatorial Identity

E 2454 [1974, 85]. *Proposed by Doug Hensley, Graduate Student, University of Minnesota*

Suppose that $m \leqq n$ are positive integers. Show that

$$\sum \prod_{i=1}^{m} a_i^{-1} = m! \sum \prod_{i=1}^{m} b_i^{-1},$$

where the summation on the left-hand side is taken over all ordered partitions of n into m positive integers, and where the summation on the right-hand side is over all m-element subsets of $\{1, 2, \cdots, n\}$ which contain n.

I. *Solution by A. Hindmarsh, Lawrence Livermore Laboratory.* Let A_{mn} denote the sum on the left above, and let B_{mn} denote the sum on the right. The function $f(x, y) = (1-x)^{-y}$, when expanded about $x = y = 0$ in a two-dimensional Taylor series, will yield the desired identity $A_{mn} = m! B_{mn}$.

Let us define $A_{mn} = B_{mn} = 0$ if $m > n$ or if either $m = 0$ or $n = 0$ (but not both), and $A_{00} = B_{00} = 1$. Writing

$$f(x, y) = (1 - x)^{-y} = \exp\{y(-\log(1-x))\}$$
$$= \sum_{m=0}^{\infty} A_m(x) y^m$$

we must have

$$A_m(x) = \frac{(-\log(1 - x))^m}{m!}$$

so that

$$A_m(x) = \frac{1}{m!}\left\{\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right\}^m = \frac{1}{m!}\sum_{n=0}^{\infty} A_{mn} x^n.$$

Expanding in the reverse order using the Binomial Theorem, we obtain

$$f(x, y) = (1 - x)^{-y} = \sum_{n=0}^{\infty} B_n(y) x^n,$$

where

$$B_n(y) = \binom{-y}{n}(-1)^n = \left(\frac{y}{1}+1\right)\left(\frac{y}{2}+1\right)\cdots\left(\frac{y}{n-1}+1\right)\left(\frac{y}{n}\right) = \sum_{m=0}^{\infty} B_{mn}y^m.$$

Equating the two series for f, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_{mn}}{m!}x^n y^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}y^m x^n,$$

from which the result follows.

II. *Solution by Kenneth Schilling, University of California, Davis.* First, let $a_1, a_2, \cdots, a_m$ be arbitrary positive integers, and let S_m denote the set of permutations of $\{1,2,\cdots,m\}$. We show that

$$\prod_{i=1}^{m} a_i^{-1} = \sum_{\sigma\in S_m}\prod_{i=1}^{m}(a_{1\sigma}+\cdots+a_{i\sigma})^{-1}. \tag{1}$$

The proof is by induction on m. Equation (1) is clear if $m = 1$, so assume true for $m-1$. Note that the right-hand side of (1) can be written

$$(a_1+\cdots+a_m)^{-1}\sum_{\sigma\in S_m}\prod_{i=1}^{m-1}(a_{1\sigma}+\cdots+a_{i\sigma})^{-1}. \tag{2}$$

Now letting S_m^j denote the subset of those permutations in S_m for which $m\sigma = j$, we see that (2) becomes

$$(a_1+\cdots+a_m)^{-1}\sum_{j=1}^{m}\sum_{\sigma\in S_m^j}\prod_{i=1}^{m-1}(a_{1\sigma}+\cdots+a_{i\sigma})^{-1} \tag{3}$$

and upon applying the induction assumption (since S_m^j is essentially S_{m-1}), we note that (3) becomes

$$\begin{aligned}(a_1+\cdots+a_m)^{-1}&\sum_{j=1}^{m} a_1^{-1}\cdots a_{j-1}^{-1}a_{j+1}^{-1}\cdots a_m^{-1}\\ &= (a_1+\cdots+a_m)^{-1}\prod_{i=1}^{m}a_i^{-1}\sum_{j=1}^{m}a_j = \prod_{i=1}^{m}a_i^{-1}.\end{aligned}$$

Thus (1) is established.

Now take (1) and sum both sides over the ordered partitions $a_1+\cdots+a_m = n$ of n into m positive integers:

$$\sum_{a_1+\cdots+a_m=n}\prod_{i=1}^{m}a_i^{-1} = \sum_{a_1+\cdots+a_m=n}\sum_{\sigma\in S_m}\prod_{i=1}^{m}(a_{1\sigma}+\cdots+a_{i\sigma})^{-1}.$$

Next invert the order of summation on the right-hand side

$$\sum_{a_1+\cdots+a_m=n}\prod_{i=1}^{m}a_i^{-1} = \sum_{\sigma\in S_m}\sum_{a_1+\cdots+a_m=n}\prod_{i=1}^{m}(a_{1\sigma}+\cdots+a_{i\sigma})^{-1}. \tag{4}$$

Now fix $\sigma \in S_m$. For every partition $a_1 + \cdots + a_m = n$ of n into m positive integers, there corresponds an m-element subset of $\{1, 2, \cdots, n\}$ which contains n, namely $\{a_{1\sigma}, a_{1\sigma} + a_{2\sigma}, \cdots, a_{1\sigma} + a_{2\sigma} + \cdots + a_{m\sigma}\}$; moreover, this correspondence is one-to-one for if $b_1, \cdots, b_m$ is such a subset, then write $b_1 < b_2 < \cdots < b_m = n$ and let $a_{1\sigma} = b_1$, $a_{2\sigma} = b_2 - b_1, \cdots, a_{m\sigma} = b_m - b_{m-1}$. Since σ is known, the partition $a_1 + \cdots + a_m = n$ is completely determined. Thus the right-hand side of (4) is simply

$$\sum_{\sigma \in S_m} \left\{ \sum \prod_{i=1}^{m} b_i^{-1} \right\},$$

where the inner sum is over all m-element subsets $\{b_1, \cdots, b_m\}$ of $\{1, 2, \cdots, n\}$ which contain n. But this is independent of σ, and since there are $m!$ permutations $\sigma \in S_m$, the result follows.

Also solved by Leonard Carlitz, M. G. Greening (Australia), Myron Hlynka, Carl Hurd, Graham Lord, Stephen Spindler, Temple University Problem Solving Group, and the proposer.

Fermat Numbers, a Result of Legendre, and Two Identities

E2455 [1974, 85]. *Proposed by S. Audinarayana Moorthy, University of Bombay, India*

If n is a natural number, let $g(n)$ denote the number of 1's in the representation of n in the base 2, and let $h(n)$ denote the highest power of 2 which divides $n!$. It is a known result due to Legendre that $g(n) + h(n) = n$.

(1) Let $F_n = 2^{2^n} + 1$ denote the nth Fermat number. Show that

$$\sum_{n=1}^{\infty} 2^{-n} g(n) = 2 \sum_{n=0}^{\infty} F_n^{-1}.$$

(2) Let $h(0) = 0$. Show that

$$\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{n=0}^{\infty} x^{h(n)}$$

Solution by Graham Lord, Temple University, and by O. P. Lossers, Technological University, Eindhoven, the Netherlands (independently). Define

$$\text{(A)} \qquad \phi(a, x) = \prod_{n=0}^{\infty} (1 + a x^{2^n}).$$

From the uniqueness of the representation of a positive integer n as the sum of distinct powers of 2, it follows that

$$\text{(B)} \qquad \phi(a, x) = \sum_{n=0}^{\infty} a^{g(n)} x^n.$$

Computing the partial derivative ϕ_1 from (B) and evaluating at $a = 1$ and $x = \frac{1}{2}$,

we have

$$\text{(C)} \qquad \phi_1(1,\tfrac{1}{2}) = \sum_{n=1}^{\infty} 2^{-n} g(n).$$

Computing ϕ_1 from (A), using logarithmic differentiation, we see that

$$\frac{\phi_1(a,x)}{\phi(a,x)} = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1+ax^{2^n}}$$

and since from (B) we have $\phi(1,\frac{1}{2}) = (1-\frac{1}{2})^{-1} = 2$, it follows that

$$\text{(D)} \qquad \phi_1(1,\tfrac{1}{2}) = 2\sum_{n=0}^{\infty} \frac{1}{2^{2^n}+1} = 2\sum_{n=0}^{\infty} F_n^{-1}.$$

Equating (C) and (D), we obtain (1) of the problem.

There is a misprint in part (2) of the problem; the left-hand side should read

$$\prod_{n=0}^{\infty} (1+x^{2^n-1}),$$

rather than

$$\prod_{n=0}^{\infty} (1+x^{2^n})$$

which is seen to be $\phi(1,x) = \sum_{n=0}^{\infty} x^n$. To prove (2), simply set $a = x^{-1}$ in (A) to get

$$\phi(x^{-1},x) = \prod_{n=0}^{\infty} (1+x^{2^n-1})$$

and also in (B) to get

$$\phi(x^{-1},x) = \sum_{n=0}^{\infty} x^{n-g(n)} = \sum_{n=0}^{\infty} x^{h(n)}$$

by the result of Legendre.

Also solved by the proposer. Part (1) was solved by Bro. Albert Brousseau, and he and Emil Grosswald pointed out that part (2) as printed was incorrect.

A Dirichlet-Like Product

E2457 [1974, 169]. *Proposed by Don Redmond, University of Illinois*

Let $\tau(n)$ denote the number of divisors of the natural number n and let $\theta(n)$ denote the number of decompositions of n into two relatively prime factors. Suppose that f is a multiplicative function and that $g(n) = \sum_{d|n} f(d)$. Show that

$$\sum_{d^2|n} f(d)\tau\left(\frac{n}{d^2}\right) = \sum_{d^2|n} g(d)\theta\left(\frac{n}{d^2}\right).$$

I. *Solution by M. Ram Murty and Kumar Murty, Carleton University.* Suppose that f and g are any arithmetical functions, not necessarily multiplicative. We define the Dirichlet-like product $*_2$ as follows: $h = f *_2 g$ if

$$h(n) = \sum_{d^2|n} f(d)g\left(\frac{n}{d^2}\right).$$

Unlike the usual Dirichlet product (which we denote by $*$) this product is neither commutative nor associative, but it does satisfy the following identity:

$$f *_2 (g *_2 h) = (f * g) *_2 h, \tag{1}$$

as we now show. For any n, the right-hand side of (1), evaluated at n, is

$$\sum_{d^2|n} \sum_{e|d} f(e)g\left(\frac{d}{e}\right) h\left(\frac{n}{d^2}\right). \tag{2}$$

Inverting the order of summation in (2), we get

$$\sum_{e^2|n} \sum_{d} f(e)g\left(\frac{d}{e}\right) h\left(\frac{n}{d^2}\right), \tag{3}$$

the inner sum being taken over all d which are multiples of e and which satisfy $d^2 \mid n$. Writing every such d as $d = te$, we see that (3) becomes

$$\sum_{e^2|n} \sum_{t^2|n/e^2} f(e)g(t)h\left(\frac{n}{t^2e^2}\right),$$

which is the left-hand side of (1) evaluated at n.

With the notation we have established, the problem is to show that

$$f *_2 \tau = (f * 1) *_2 \theta,$$

where $1(n) = 1$ for all n. But by (1),

$$(f * 1) *_2 \theta = f *_2 (1 *_2 \theta)$$

and it is a known result, due to Liouville, that $1 *_2 \theta = \tau$, thus proving the result. (See L. E. Dickson, *History of the Theory of Numbers*, Vol. I, Chelsea, New York, 1952, p. 285, line 11 for these and other identities, together with references to Liouville's original papers.) We note that the condition that f be multiplicative is superfluous.

To obtain a generalization of this result, let k be a fixed natural number and define $*_k$ in the obvious fashion. The analog of (1) for general k is still valid. If $\tau_k(n)$ is the number of k-tuples of natural numbers $(a_1, \cdots, a_k)$ such that $a_1 a_2 \cdots a_k = n$, and if $\theta_k(n)$ is the number of such k-tuples that also satisfy $\mathrm{GCD}(a_1, \cdots, a_k) = 1$, then it is true that $1 *_k \theta_k = \tau_k$, implying that for every arithmetical function f, $f *_k \tau_k = g *_k \theta_k$ whenever $g = f * 1$. (This result holds even in the trivial case $k = 1$, for $\tau_1 = 1$ and $\theta_1 = \delta$, where δ is the identity for the Dirichlet product: $\delta(1) = 1$ and $\delta(n) = 0$ if $n > 1$.—Ed.)

II. *Solution by Manny Yothers, Lower Stillwater College.* If f is an arithmetical function, we define its generating function F by

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

(See G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Third Edition, Oxford, 1954, Chap. XVII.) It is well known that if $h = f * g$, then $H(s) = F(s)G(s)$; similarly it is easily established that if $h = f *_2 g$, then $H(s) = F(2s)G(s)$ from which equation (1) of the previous solution follows immediately.

Generating functions can be used to establish the result $1 *_2 \theta = \tau$ of Liouville also. Evidently $\theta(n) = 2^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime divisors of n, so that the generating function for θ is

$$\frac{\zeta^2(s)}{\zeta(2s)},$$

where ζ is Riemann's zeta-function (*ibid.*, Theorem 301, p. 255). The generating function for 1 is by definition $\zeta(s)$, so that the generating function for $1 *_2 \theta$ is

$$\zeta(2s)\,\frac{\zeta^2(s)}{\zeta(2s)} = \zeta^2(s),$$

which is the generating function for τ (*ibid.*, Theorem 299, p. 255).

Analogous results hold for $*_k$, since if $h = f *_k g$, then $H(s) = F(ks)G(s)$.

III. *Solution by M. V. Subbarao and V. V. S. Sastri, University of Alberta.* The problem is the particular case $k = 2$ of a more general result:

$$\sum_{d^k \mid n} f(d)\,\tau\left(\frac{n}{d^k}\right) = \sum_{d^k \mid n} g(d)\psi_k\left(\frac{n}{d^k}\right), \tag{1}$$

where $\psi_k(n)$ is the number of kth-power-free divisors of n; this is because $\psi_2 = \theta$. (Note that (1) holds even for $k = 1$, since in this case $\psi_1 = 1$, and (1) follows from the well-known result $1 * 1 = \tau$.—Ed.)

Let f be an arbitrary arithmetical function (not necessarily multiplicative) and let $g = f * 1$. Define $f_k(n)$ to be $f(n^{1/k})$ if n is a kth power and 0 otherwise, and define $g_k(n)$ similarly. Let $q_k(n)$ be 0 or 1 according as n has or does not have a kth power as a factor. Since $\tau = 1 * 1$, the left-hand side of (1) is the value of the following function at n

$$f *_k \tau = f_k * \tau = f_k * (1 * 1) = (f_k * 1) * 1$$

and since $\psi_k = q_k * 1$, the right-hand side of (1) is the value of the following function at n

$$g *_k \psi_k = g_k * \psi_k = g_k * (q_k * 1) = (g_k * q_k) * 1.$$

We see then that (1) is equivalent to the following:

$$(2) \qquad f_k * 1 = g_k * q_k .$$

Let n be arbitrary and write $n = a^k b$, where b is kth-power-free. The right-hand side of (2), when evaluated at n, is

$$\sum_{d|n} g_k(d) q_k\left(\frac{n}{d}\right) = \sum_{d^k|n} g(d) q_k\left(\frac{n}{d^k}\right) = \sum_{d|a} g(d) q_k\left(\frac{a^k b}{d}\right) = g(a).$$

The left-hand side of (2) when evaluated at n is

$$\sum_{d|n} f_k(d) = \sum_{d^k|n} f(d) = \sum_{d|a} f(d) = g(a),$$

since $g = f * 1$. Thus (2) is established and so is (1). We remark that (2) follows from results of R. Vaidyanathaswamy involving the kth convolute. (R. Vaidyanathaswamy, *The theory of multiplicative arithmetical functions*, Trans. Amer. Math. Soc. 33 (1931), 579–662.) He there shows that $f_k * \varepsilon_k = (f * 1)_k = g_k$ (Theorem VI, p. 595) and $\varepsilon_k * q_k = 1$ (Formula (1), p. 600) from which (2) follows immediately.

(Using the generating functions defined in Solution II above, we can prove (2) rather easily: The generating function for $f_k * 1 = f *_k 1$ is $F(ks)\zeta(s)$, whereas the generating function for $g_k * q_k = g *_k q_k = (f * 1) *_k q_k$ is

$$[F(ks)\zeta(ks)]\left[\frac{\zeta(s)}{\zeta(ks)}\right] = F(ks)\zeta(s)$$

by Theorem 303 (p. 255) of Hardy and Wright.—Ed.)

Also solved by S. J. Benkoski, Gene Gale, M. G. Greening (Australia), Emil Grosswald, Graham Lord, L. E. Mattics, Kenneth Schilling, Temple University Problem Solving Group, C. S. Venkataraman & R. Sivaramakrishnan (India), L. J. Warren, Ken Yocom, and the proposer.

Editor's comment. The result of Liouville, namely that $1 *_2 \theta = \tau$, was proved by most solvers in the following fashion: Since all functions involved are multiplicative, it suffices to prove that

$$(1) \qquad \sum_{d^2|p^n} \theta\left(\frac{p^n}{d^2}\right) = \tau(p^n)$$

for every prime power p^n. The right-hand side of (1) is obviously $n + 1$, whereas the left-hand side is

$$\sum_{k=0}^{m} \theta(p^{n-2k}) = 1 + 2m = n + 1 \qquad \text{if } n = 2m \text{ is even}$$

$$\sum_{k=0}^{m} \theta(p^{n-2k}) = 2(m + 1) = n + 1 \qquad \text{if } n = 2m + 1 \text{ is odd,}$$

since $\theta(1) = 1$ and $\theta(p^i) = 2$ if $i \geqq 1$.

The careful reader will note that the key formula in the original problem is $1 *_2 \theta = \tau$; Solution I presents the generalization $1 *_k \theta_k = \tau_k$, of which the original problem is the special case $k = 2$

since $\theta = \theta_2$ and $\tau = \tau_2$. Solution III presents essentially the generalization $1*_k\psi_k = \tau$; Venkataraman and Sivaramakrishnan present a complementary generalization of the form $1*_k \theta = \alpha_k$, where $\alpha_k(n)$ is defined as follows: write $n = a^k b$ where b is kth-power-free, and let q be the product of the distinct prime divisors of b. Then $\alpha_k(n) = \tau(m^2 q)$. More simply, we note that α_k is multiplicative and if $n = qk + r$ with $0 \leqq r < k$, then $\alpha_k(p^n) = 2q + 1$ if $r = 0$ and $\alpha_k(p^n) = 2q + 2$ if $r \neq 0$.

Approximating Pi with Series-Parallel Circuits

E 2459* [1974, 170]. *Proposed by A. A. Mullin, U.S. Army Research Office, Arlington, Virginia*

One is given an unlimited number of perfect one ohm resistors with which to construct a resistance of π ohms to within an accuracy of 10^{-6} ohms. Only series-parallel circuits are allowed. What is the minimum number of resistors necessary?

I. *Amalgam of essentially identical solutions by J. P. Lambert, Las Vegas, Nevada, and L. E. Mattics, University of South Alabama.* The solution will be given in two parts; the first is a general theorem concerning which rational numbers can be realized with two-terminal series-parallel circuits using precisely n unit resistances, and the second part applies this theory to a particular rational approximation to π.

For this solution, *circuit* will mean *two-terminal series-parallel circuit of unit resistors*, and a rational number is *n-constructible* if it can be realized as a circuit using precisely n unit resistors. We recall that such circuits are defined recursively by the following rules:

1. A single unit resistor is a circuit.
2. If C_1 and C_2 are circuits, then so is the configuration of C_1 and C_2 connected in series or in parallel.
3. Every circuit is defined by a finite number of applications of Rules 1 and 2.

We now present the main theoretical tool for our investigation.

THEOREM. *Define*

$$M_n = \max\{\max(a,b)\colon a/b \text{ is } n\text{-constructible and } (a,b) = 1\}$$

$$S_n = \max\{a + b\colon a/b \text{ is } n\text{-constructible and } (a,b) = 1\}.$$

Then $M_n = f_{n+1}$ and $S_n = f_{n+2}$ where f_n is the n-th Fibonacci number: $f_1 = f_2 = 1$ and $f_{k+2} = f_k + f_{k+1}$ for integral $k \geqq 1$.

Proof. Recall that if two circuits C_1 and C_2 have resistances R_1 and R_2 respectively, then connecting them in series gives a circuit with resistance R_s given by $R_s = R_1 + R_2$, and connecting them in parallel gives a circuit with resistance R_p given by $1/R_p = 1/R_1 + 1/R_2$. Observe that interchanging the words "series" and "parallel" in the description of the circuit changes its resistance from a/b to b/a, so that a/b is n-constructible if and only if b/a is n-constructible and therefore

$$M_n = \max\{b : a/b \text{ is } n\text{-constructible and } (a,b) = 1\}$$
$$= \max\{a : a/b \text{ is } n\text{-constructible and } (a,b) = 1\}.$$

We assume that when we talk about a circuit of resistance p/q, then always $(p, q) = 1$.

PART 1. We show that $M_n \leqq f_{n+1}$ and $S_n \leqq f_{n+2}$. The proof is by induction. Clearly $M_1 = 1$ and $S_1 = 2$ since the only 1-constructible number is $1/1$. Now suppose that $M_k \leqq f_{k+1}$ and $S_k \leqq f_{k+2}$ for all k, $1 \leqq k < n$, and let a/b be n-constructible. Let C be an n-circuit of resistance a/b. By definition of circuit, C consists of two subcircuits C_1 and C_2 connected in either series or parallel, and by our earlier observation we can assume without loss of generality that the connection is series. Let the resistance of C_1 be r/s and that of C_2 be u/v; then r/s is k-constructible and u/v is $(n-k)$-constructible for some k, $1 \leqq k \leqq n-1$. Now

$$\frac{a}{b} = \frac{r}{s} + \frac{u}{v} = \frac{rv + su}{sv},$$

where the right-hand side need not be in lowest terms. [One conjectures that if $(rv + su)/sv$ is not in lowest terms, then a/b can be constructed with a different circuit using fewer resistors.—Ed.] Then

$$\begin{aligned} a + b &\leqq rv + su + sv = rv + s(u + v) \\ &\leqq rM_{n-k} + sS_{n-k} \leqq rf_{n-k+1} + sf_{n-k+2} \\ &= rf_{n-k+1} + sf_{n-k+1} + sf_{n-k} \\ &\leqq S_k f_{n-k+1} + M_k f_{n-k} \leqq f_{k+2} f_{n-k+1} + f_{k+1} f_{n-k} = f_{n+2} \end{aligned}$$

using the known relation

$$f_a f_{b+1} + f_{a-1} f_b = f_{a+b}. \tag{1}$$

Since $S_n = \max\{a + b\}$, it follows that $S_n \leqq f_{n+2}$ completing the induction for S_n.

Now $b \leqq sv \leqq f_{k+1} f_{n-k+1} \leqq f_{n+1}$ by the inductive hypothesis and (1). To complete Part 1, we must show that $a \leqq f_{n+1}$. We can assume without loss of generality that $s \geqq r$; if $r \geqq s$, we can group the terms differently and use essentially the same argument. Consider

$$\begin{aligned} a &\leqq rv + su = r[(v + u) - u] + su \leqq rf_{n-k+2} + u(s - r) \\ &\leqq rf_{n-k+2} + f_{n-k+1}(s-r) = r(f_{n-k+1} + f_{n-k}) + f_{n-k+1}(s - r) \\ &= rf_{n-k} + sf_{n-k+1} = rf_{n-k} + s(f_{n-k} + f_{n-k-1}) \\ &= sf_{n-k-1} + (r + s)f_{n-k} \leqq f_{k+1} f_{n-k-1} + f_{k+2} f_{n-k} = f_{n+1}, \end{aligned}$$

again by (1). Thus $M_n \leqq f_{n+1}$ and the proof of Part 1 is complete.

PART 2. We show that $M_n = f_{n+1}$ and $S_n = f_{n+2}$ by exhibiting an n-circuit of resistance a/b with $\max(a, b) = f_{n+1}$ and $a + b = f_{n+2}$. Starting with a circuit C_1

consisting of a single resistor, form C_2 by connecting C_1 in series with another resistor, and form C_3 by connecting C_2 in parallel with another resistor. Continue, alternately adding another resistor first in series and then in parallel, obtaining circuits as shown in Figure 1.

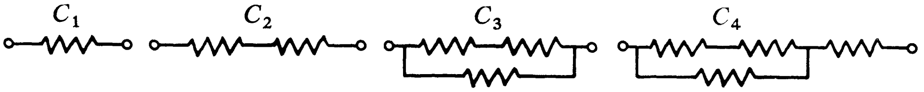

FIG. 1

Certainly C_1 has resistance $1 = 1/1 = f_1/f_2$ and $1 + 1 = 2 = f_3$. Suppose that it has been shown that C_n (n odd) has resistance f_n/f_{n+1}. Then C_{n+1} has resistance $f_n/f_{n+1} + 1 = f_{n+2}/f_{n+1}$ and C_{n+2} has resistance $(1/(f_{n+2}/f_{n+1}) + 1)^{-1} = f_{n+2}/f_{n+3}$, completing the induction. ∎

Studying the convergents to π using the continued fraction expansion of Wallis, we observe that the first convergent a/b which satisfies $|a/b - \pi| < 10^{-6}$ is $a/b = 355/113$. Now 355/113 can be realized by the 15-circuit of Figure 2.

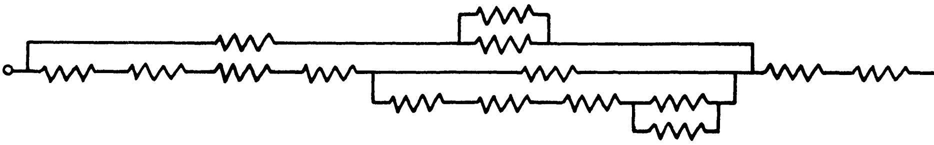

FIG. 2

No other fraction with numerator less than 2000 approximates π to within 10^{-6}, and since $f_{17} = 1597$, at least 16 resistors would be necessary to realize this or any other sufficiently close approximation. If we can show that 355/113 cannot be realized by any k-circuit with $k \leqq 14$, we shall be done.

Suppose, to the contrary, that this can be done with a circuit C. If C consists of two sub-circuits in parallel, then $113/355 = a/b + c/d$, where b and d must be chosen from $\{5, 71, 355\}$. If either denominator were 355 — say $b = 355$ — then $d \geqq 5$ and at least $13 + 4 = 17$ resistors would be necessary, since $M_{12} = 233 < 355$ and $M_3 = 3 < 5$. Hence $113/355 = a/5 + c/71$. But this equation has no solution in positive integers.

It follows that C must consist of two sub-circuits in series, so that $355/113 = a/b + c/d$. Since $M_{10} = 89 < 113$, not both b and d can be 113. Thus we have $355/113 = a + c/113$, where $a = 1, 2,$ or 3.

Case 1. Suppose that $a = 3$, so that $355/113 = 3 + 16/113$. Since $113 > M_{10} = 89$, we see that 16/113 must be 11-constructible. Now 113 is prime so that 16/113 cannot be realized as two sub-circuits in series, implying that $113/16 = p/q + r/s$, where either q or s is 16, but not both can be 16 since $M_6 = 13 < 16$ and this would necessitate at least $7 + 7 = 14$ resistors for 113/16, and hence for 16/113. Say $b = 16$. Then a/b requires at least 7 resistors, so that c/d can use no

more than 4. Then $c/d \leqq 4$, so that $a/b \geqq 113/16 - 4 = 49/16$. Since $S_8 = 55 < 49 + 16$, we see that a/b must require at least 9 resistors, so that c/d can use no more than 2. Using the same argument again, we have $a/b \geqq 113/16 - 2 = 81/16$, and since $S_9 = 89 < 81 + 16$, necessarily a/b must require 10 resistors so that $c/d = 1$. But then $a/b = 97/16$ which needs at least 11 resistors since $M_{10} = 89 < 97$. Thus 16/113 is not 11-constructible so that Case 1 is impossible.

Case 2. Suppose that $a = 2$, so that $355/113 = 2 + 129/113$. Then 129/113 must be realized by two sub-circuits in parallel, since a series connection would reduce to the case just considered. Then $113/129 = p/q + r/s$, and the only possibility using 12 or fewer resistors is $\{b, d\} = \{3, 43\}$ so that we can assume that $113/129 = 2/3 + 9/43$. Since $M_3 = 3$ and $M_8 = 34 < 43$, we see that exactly 3 resistors must be used for 3/2 and 9 must be used for 43/9. The circuit for 43/9 cannot be formed from two sub-circuits in parallel (43 is prime and this would necessitate at least 18 resistors), so that $43/9 = u/v + w/9$; here u/v must use at most 3 resistors, since $w/9$ needs at least 6 resistors ($M_5 = 8$) so that $u/v = 1$, 2, or 3. Now u/v cannot be 3 since $43/9 = 3 + 16/9$ requires at least $3 + 7 = 10$ resistors, and u/v cannot be 2 since $43/9 = 2 + 25/9$ also requires at least $2 + 8 = 10$ resistors. Thus, we can assume that $43/9 = 1 + 34/9$, where 34/9 is 8-constructible. Any series connection of sub-circuits to achieve 34/9 reduces to previous cases, so we must have $9/34 = g/2 + h/17$ (other possibilities for denominators require more than 8 resistors). But this equation has no solution in positive integers and Case 2 cannot occur. (Notice that the situation $u/v = 4$, which is eliminated here because $w/9$ needs at least 6 resistors, is the case which gives the successful realization of 355/113 as a 15-circuit, as given in Figure 2.)

Case 3. Suppose that $a = 1$, so that $355/113 = 1 + 242/113$. Since $M_{12} = 233 < 242$, we see that 242/113 must be 13-constructible. As before, we can show that the only possibility is $113/242 = p/121 + r/2$ (because $M_{10} = 89 < 121$ and $M_2 = 2$). But this equation has no solution in positive integers, and Case 3 is impossible.

We have eliminated all cases and have thus shown that 355/113 cannot be realized with fewer than 15 resistors.

II. *Comment by Michael Goldberg, Washington, D.C.* The determination of a circuit of 355/113 ohms can be done by the division of a rectangle of length 355 and width 113 into squares. If the rectangle is a uniform conducting plate whose resistance (lengthwise) is 355/113, then each square will have unit resistance. If the squares are replaced by one-ohm resistors, then the desired network is obtained. The techniques for obtaining such divisions are described in N. D. Kazarinoff and Roger Weitzenkamp, *Squaring rectangles and squares*, this MONTHLY, 80 (1973), 877–888. See also J. H. Conway, *Mrs. Perkins' Quilt*, Proc. Camb. Phil. Soc. 60 (1964), 363–368, and G. B. Trustrum, Proc. Camb. Phil. Soc. 61 (1965), 7–11. How-

ever, these methods give no assurance that a division into the least number of squares is attained except by exhaustive trials.

One such division into 14 squares is shown in Figure 3; the associated network is shown in Figure 4, where the numbers refer to corresponding square sizes, not to resistance values.

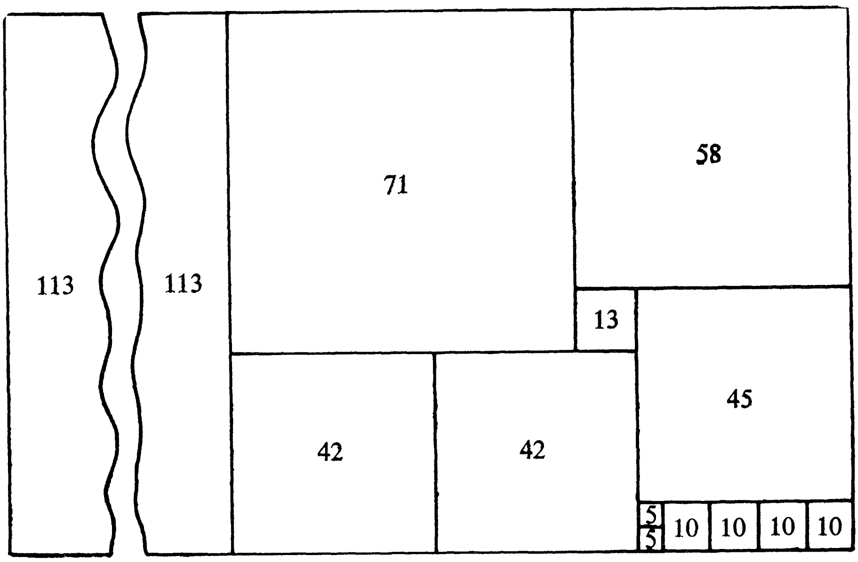

FIG. 3. Division of 113×355 rectangle into 14 squares

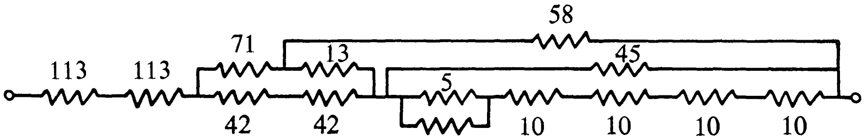

FIG. 4

Editor's comment. D. J. Bordelon and Cornelius Groenewoud (independently) also discovered the 15-resistor circuit of Solution I. Bordelon used the technique of Solution II and offered the following additional references: M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*, Wiley, New York, 1960, p. 28, and I. M. Yaglom, *How to Partition a Square*? (Russian), Nauka, Moscow, 1968, p. 87.

Gene Gale, who found a 29-resistor circuit, pointed out the reference: L. Carlitz and J. Riordan, *The number of labeled two-terminal series-parallel networks*, Duke Math. J. 23 (1956), 435–445. Each of the following found a 26-resistor circuit based on the continued fraction representation of π and recognized that it might not be minimal: L. L. Baggerly & Robert Patenaude, R. B. Eggleton (Israel), Scott Forrest, Eric Rosenthal, F. G. Schmitt, Jr., the Temple University Problem Solving Group, and G. Yuval (Israel). Nine others found solutions of from 18 to 87 resistors, claiming to have established these numbers as minimal.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before May 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6012*. *Proposed by Daniel Shanks, Naval Ship Research and Development*
Prove the following infinite products over certain sets of primes p:

$$\prod_{p=14k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right)=\frac{840}{817},$$

$$\prod_{p=18k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right)=\frac{40}{39}.$$

6013. *Proposed by J. R. Higgins, Cambridgeshire College of Arts and Technology, England*

Let $\{h_r(t)\}_{r=1}^{\infty}$ be the orthonormal Haar functions, defined by

$$h_1(t)\equiv 1,\qquad t\in[0,1]$$

$$h_r(t)=\begin{cases}2^{m/2}\operatorname{sgn}\sin(2^{m+1}\pi t), & t\in\left(\frac{k-1}{2^m},\frac{k}{2^m}\right)\\ 0 \quad \text{elsewhere in } [0,1],\end{cases}$$

where $r=2^m+k$, $m=0,1,\cdots$, and $k=1,\cdots,2^m$.

Let p be any odd positive integer and q any positive integer such that $p<2^q$. Put

$$I(p,q,m,k)=\int_0^{p/2^q}h_r(t)dt.$$

Show that

$$\sum_{m=0}^{q-1}\{I(p,q,m,k)\}^2=\frac{p}{2^q}\left(1-\frac{p}{2^q}\right).$$

The problem has arisen in the theory of the completeness of the Haar functions.

6014. *Proposed by C. H. Kimberling, University of Evansville*
Does there exist an uncountable set of real numbers all of whose closed subsets are countable?

6015. *Proposed by C. W. Anderson, University of California, Berkeley*
Let $n=q_1^{a_1}q_2^{a_2}\cdots q_k^{a_k}$, $k>1$, be the prime decomposition of the integer n, and

define $\text{Ind}(n) = \max\{a_i \mid 1 \leqq i \leqq k\}$. Show that

$$\overline{\text{Ind}(N)} = \lim_{m\to\infty} \frac{1}{m} \sum_{k=2}^{m} \text{Ind}(k)\, 1 + \sum_{n=2}^{\infty} \frac{\mu(n)}{n(n-1)} = 1.705211\cdots$$

where μ is the Moebius function.

6016*. *Proposed by C. J. Moreno, University of Illinois*

Let $D(n)$ be the function defined by $D(n) = \prod p$, where the product runs over those primes p such that $p-1$ divides $2n$. Find an asymptotic formula for the summatory function

$$\sum_{n \leqq x} D(n).$$

By a well-known result, such estimate gives the average size of the denominators of the Bernoulli numbers.

6017. *Proposed by Albert Wilansky, Lehigh University*

In a popular text it is proposed to find a strictly smaller norm for any normed space E by first constructing a strictly larger norm on E'. Show that this construction must fail. The new norm must be equivalent to the old. Give a correct construction.

SOLUTIONS OF ADVANCED PROBLEMS

A Functional Inequality

5934 [1973, 1067]. *Proposed by R. C. Buck, University of Wisconsin.*

What entire functions f obey the inequality

$$|f(z+a)| \leqq |f(z)|\,|f(a)|$$

for all z and for two non-zero choices of a whose ratio $\beta = a_2/a_1$ is irrational?

Solution by L. E. Mattics, University of South Alabama. That $f(z) = Ce^{dz}$ with $|C| \geqq 1$ is such a function is trivial; we shall show that all non-zero entire functions which satisfy the hypotheses are of this form.

If $f(z) \not\equiv 0$ satisfy the hypotheses we have that

$$\left|\frac{f(z+a_1)}{f(z)}\right| \leqq |f(a_1)|$$

for all z such that $f(z) \neq 0$ so that $f(z+a_1)/f(z)$ can be continued to a bounded entire function. Hence $f(z+a_1) = Df(z)$ and likewise $f(z+a_2) = Ef(z)$. Set $F(z) = e^{mz}f(a_1 z)$ where $e^m D = 1$, then $F(z+1) = F(z)$ so that $F(z) = \sum_{-\infty}^{\infty} b_n e^{2\pi niz}$, where the b_n are unique. Now $F(z+\beta) = Ee^{m\beta}F(z)$, so from the theory of Fourier series with real coefficients we can conclude that $Ee^{m\beta}e^{-2\pi ni\beta} = 1$ whenever $b_n \neq 0$. But since β is irrational, $b_n \neq 0$ for at most one n and so $F(z) = Ce^{2\pi niz}$ for some n.

We now have that $f(z) = Ce^{dz}$ and

$$\left|f(z+a_1)\right| = \left|Ce^{dz}\cdot e^{da_1}\right| \leqq \left|f(z)\cdot f(a_1)\right| \text{ only if } \left|C\right| \geqq 1.$$

Also solved by K. F. Andersen, R. Goldstein (England), Shigeru Haruki, Sinan Kunt, O. P. Lossers (Netherlands), and the proposer.

Range of a Holomorphic Function in $|z| < 1$

5936 [1973, 1067]. *Proposed by David Styer, University of Cincinnati*

Is there a function f, bounded and holomorphic in $|z| < 1$, and a polynomial p such that, in $|z| < 1$, f/p assumes every complex value infinitely often with at most one exception?

Solution by O. P. Lossers, Technological University Eindhoven, the Netherlands. The answer is affirmative, as may be illustrated by the following example. Define

$$f(z) = \exp\frac{-1}{1-z} \qquad p(z) = 1-z.$$

The substitution $w = 1/(1-z)$ transforms the unit disk $|z| < 1$ into the half plane $\operatorname{Re} w > \frac{1}{2}$. Obviously in this half plane the transformed function e^{-w} is bounded. The transformed quotient equals the integral function we^{-w}, which, by Picard's theorem, clearly assumes every value $t \neq 0$ an infinity of times. However, for $\operatorname{Re} w \leqq \frac{1}{2}$, $|w| > e^{\frac{1}{2}}|t|$ we have $|we^{-w}| > e^{\frac{1}{2}}|t|e^{-\frac{1}{2}} = |t|$, so that the value t only can be assumed for $|w| \leqq e^{\frac{1}{2}}|t|$ or $\operatorname{Re} w > \frac{1}{2}$. The compactness of the former set yields the conclusion that for every $t \neq 0$ the equation $we^{-w} = t$ has infinitely many solutions satisfying $\operatorname{Re} w > \frac{1}{2}$.

Also solved by P. M. Gauthier, and by the proposer.

On Order-Preserving Automorphisms in a Field

5938 [1973, 1067]. *Proposed by Detlef Laugwitz, Nieder-Ramstadt, Germany*

Let G be an archimedean subfield of the ordered field F, and let A be an order-preserving automorphism of F. Is it true that $AG = G$?

Solution by D. M. Bloom, Brooklyn College. As a counterexample, order the field $R(x)$ (the rational functions with real coefficients) in the usual way (a polynomial is positive if its leading coefficient is), and let $F = Q(\pi, x) \subseteq R(x)$, $G = Q(\pi)$. Since both π and $\pi - (1/x)$ are transcendental over $Q(x)$, a $Q(x)$-automorphism $A: F \to F$ may be defined by $\pi \to \pi - (1/x)$. It is straightforward to verify that positivity of polynomials (elements of $Q[\pi, x]$) is preserved, and hence A is order-preserving; but $AG \neq G$.

Also solved by John H. Smith and by the proposer.

Ideals in Commutative Rings

5940 [1973, 1146]. *Proposed by Donald Minassian, Indianapolis, Indiana*

Let R be a commutative ring with 1 and let I be an ideal of R. On p. 131 of *Introduction to Modern Algebra*, (D. C. Heath, 1963) W. Barnes claims I is noetherian (i.e., as a ring which may lack a unit, and defining the ideal generated by a subset T of a ring as the smallest ideal containing T). Prove, or give a counterexample.

Solution by J. R. Smith, Appalachian State University. We assume that R is to be noetherian. The following is a counter-example. Let Z_4 be the integers modulo 4 and let A be the ideal $\{0 + 4Z, 2 + 4Z\}$. Let R be $Z_4[X]$ and $I = A \cdot R$. Then I is an ideal of R and $I^2 = 0$. Thus any subgroup of I is an ideal of the ring I. Let J_n be the set of elements of I with degree $\leqq n$. Then (J_n) forms an ascending chain of ideals I which does not terminate.

Also solved by W. D. Blair, D. L. Costa, S. H. Cox, Jr. (Puerto Rico), T. E. Elsner, J. A. Dentinger, Jr., Leonard Gillman, Robert Gilmer, H. L. Hiller, Frederick Hoffman, K. E. Hummel. A. A. Jagers (Netherlands), Richard Kerns (Germany), M. L. Laplaza, Jiang Luh, Philip Quartararo, Jr., A. L. Rubin, Art Steger, C. N. Winton, and the proposer.

Note. For papers on the subject of this problem, Gilmer refers us to (1) R. Gilmer, *Integral domains with Noetherian subrings*, Comment. Math. Helv. 45 (1970), 129–134, and (2) Walter Borho, *Die torsionenfreien Ringe mit lauter Noetherschen Unterringen*, Abh. Math. Sem. Univ. Hamburg 38 (1972), 1–7.

Independent Normal Distributions

5942 [1973, 1147]. *Proposed by D. M. Bloom, Brooklyn College*

Let X_1, X_2, X_3 be independent random variables such that $E(X_1) > E(X_2) > E(X_3)$. Assume that the X_i have normal distributions with a common variance. Prove or disprove: if $P(X_1 > X_2) = K$ and $P(X_2 > X_3) = L$, then $P(X_1 > X_3)$ is greater than M where M is defined by

$$\frac{K}{1-K} \cdot \frac{L}{1-L} = \frac{M}{1-M}.$$

Solution by L. E. Clarke, University of East Anglia, England. Let $\phi(x) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}x^2)$ and $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt \ (x \in R)$. Thus ϕ is the probability density function and Φ the distribution function of a normal $(0,1)$ random variable.

Suppose that X_i is normal (μ_i, σ^2) $(i = 1,2,3)$, and so $\mu_1 > \mu_2 > \mu_3$. Then

$$X = \{(X_1 - \mu_1) - (X_2 - \mu_2)\}/(\sigma\sqrt{2})$$

is a normal $(0,1)$ random variable, and so

$$\begin{aligned} K = P(X_1 > X_2) &= P\{X > -(\mu_1 - \mu_2)/(\sigma\sqrt{2})\} \\ &= P(X > -a), \text{ say, } = P(X < a) = \Phi(a). \end{aligned}$$

Similarly, $L = P(X_2 > X_3) = \Phi(b)$, where $b = (\mu_2 - \mu_3)/(\sigma\sqrt{2})$, and

$$P(X_1 > X_3) = \Phi(a + b).$$

Let $f(x) = \Phi(x)/\{1 - \Phi(x)\}$. Then, since $M/(1-M)$ increases for $0 < M < 1$, we have to show that

$$f(a + b) > f(a)f(b) \tag{1}$$

for $a > 0$, $b > 0$.

Let $g(x) = \log f(x)$. Then (1) is equivalent to

$$g(a + b) > g(a) + g(b)$$

or, since $g(0) = 0$, $g(a + b) - g(a) > g(b) - g(0)$. Thus it suffices to show that

$$g'(x + b) - g'(x) > 0 \quad (x > 0),$$

which will hold if g' is increasing on $(0, \infty)$.

Now $\Phi'(x) = \phi(x)$ and $\phi'(x) = -x\phi(x)$. Therefore

$$g'(x) = \phi(x)\{\Phi(x) - \Phi^2(x)\}^{-1}$$

and

$$\begin{aligned} g''(x) &= \phi(x)\{x\Phi^2(x) - x\Phi(x) + 2\phi(x)\Phi(x) - \phi(x)\}\{\Phi(x) - \Phi^2(x)\}^{-2} \\ &= \phi(x)h(x)\{\Phi(x) - \Phi^2(x)\}^{-2}, \text{ say.} \end{aligned}$$

Thus it suffices to show that

$$h(x) > 0 \quad (x > 0). \tag{2}$$

Now $h(0) = 0$, and $h(x) \to 0$ as $x \to \infty$, the second assertion holding since $x\{1 - \Phi(x)\} \sim \phi(x) \to 0$ as $x \to \infty$. Therefore if (2) does not hold, there exists $c \in (0,\infty)$ such that $h(c) \leqq 0$ and $h'(c) = 0$. But $h'(c) = \Phi^2(c) - \Phi(c) + 2\phi^2(c)$, and so

$$\begin{aligned} h(c) &= -2c\phi^2(c) + 2\phi(c)\Phi(c) - \phi(c) \quad \text{(since } h'(c) = 0) \\ &= 2\phi(c)\{\Phi(c) - \tfrac{1}{2} - c\phi(c)\}. \end{aligned}$$

But $\Phi(c) - \frac{1}{2} = \int_0^c \phi(x)dx > c\phi(c)$, since ϕ is decreasing on $[0, c]$, and so $h(c) > 0$, giving a contradiction.

Thus (2) must hold, and the assertion of the problem is established.

Note that the inequality of the problem is, in effect, a sharpening of that of Problem 5555 [1968, 1129].

Also solved by K. Alam & K. Seo, P. W. Epasinghe & J. Sethuraman, Ellen Hertz, N. L. Johnson, O. P. Lossers (Netherlands), M. F. Neuts, G. S. Rogers, and Vincent Sardella.

Richard Olshen has shown with a counterexample that the result of the problem fails if X_i are not independent.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for reviews should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

The Art of Problem Solving. By Stanley Moses. Transworld Publishers, London, 1974.

A new book on the art of problem solving in mathematics is always welcome. Well, almost always. This book is a paperback in a series called the Transworld Student Library, edited by a Reader in mathematics in England's Open University, a series "designed to meet the modern educational needs of the independent learner." The book may well serve a useful purpose in introducing to a wide audience various approaches to the conjecturing and proving of solutions to mathematics problems.

The book is divided into chapters titled Abilities, Strategies, The Inductive Process, Methods of Proof, and Problems and Extensions. The first deals with the background knowledge one must bring to a problem; the second with the plan of attack, namely, the formulation of a mathematical model, an analysis of what knowledge is relevant, the choice of a suitable method, what to do in case of failure and a review when the problem is finally solved. The author's outline of strategy bears a striking resemblance to that in Professor George Polya's *How to Solve It*, and this is acknowledged by the author at the end of Chapter 2, where he compares his method with that of Polya. Much of the language is also from *How to Solve It* and other writings of Polya, such phrases as "problems to find," "problems to prove," "working backwards," and so on.

Chapter 3 deals with plausible reasoning, induction, the use of generalization and specialization, intuition, and analogy. In the preface the author states: "The ideas expressed in chapter 3 are undoubtedly influenced by my study in 1954 of Professor Polya's books on *Mathematics and Plausible Reasoning*. The principles discussed there have over the intervening years become so embedded in my teaching practice that I tend to regard them as traditional." It seems to us that not only the principles have become embedded in the author's teaching practice but also Professor Polya's problems and phraseology. Of the 22 examples in Chapter 3, we recognize 9 of them as being essentially those in Polya's books and a number of examples from other

chapters will also look very familiar to students of Polya's work. The Polya phraseology is also easily recognized in other chapters starting as early as page 28 of Chapter 2. One might also examine pages 31, 39, 44, 46, 48, 52 in the same regard.

The fourth chapter deals with direct and indirect proofs and "proofs" which fail to be proofs at all. In Chapter 5, the author introduces a series of problems and goes on in each case to consider extensions, new problems motivated by the original problem and solved using methods related to the methods used on the original. Some of these are really quite interesting. A collection of 100 problems, answers and hints and a short bibliography follow.

The book seems to have been put together in haste. There are numerous typographical errors and notation suddenly appears with no earlier mention or listing in the "List of Symbols and Notations" at the beginning of the book. Some errors seem more serious: the solutions in some of the examples are incorrect but are so consistently incorrect that one must conclude that the solution may be all right but that it is the solution to a problem different from the one stated. Some of these errors as well as errors in the problems at the end will surely confuse the novice problem solver.

It is frustrating to write on problem solving, because one so easily comes to the view that Polya has said it all. And the present book tends to confirm that view but it is still disappointing to those looking for a new approach or at least for a new set of interesting problems. Even in chapters 4 and 5, where the author does have some worth-while additions to the subject, one still has a feeling of *déjà vu*.

DONALD J. ALBERS, Menlo College, and
G. L. ALEXANDERSON, University of Santa Clara

Convex Functions. By A. Wayne Roberts, Dale E. Varberg. Academic Press, New York, 1973. xx + 300 pp. \$19.50. (Telegraphic Review, April 1974.)

Convex Functions by Roberts and Varberg is a welcome addition to the literature. The book would be excellent for a one semester course at the advanced undergraduate or first-year graduate level. It also serves as a valuable reference for scholars in mathematics, mathematical economics, and operations research. A strong background in advanced calculus and linear algebra is adequate preparation for this text, although some measure theory is required in spots.

Each chapter presents an introduction to a particular aspect of convex function theory and its applications. The authors begin by establishing the usual properties and characterizations of convex functions on the line, using these in subsequent chapters to obtain results for R^n and normed linear spaces. A brief treatment of game theory, linear and convex programming, and inequalities follows. The authors conclude with a survey of generalizations of convex functions presented in an historical perspective.

With the exception of this final section, only the most important concepts and results are presented in the body of the text, and excessive notation is avoided. For example, the authors do not consider algorithms for solving convex programming problems, nor do they examine the relationship between gauges, support functions, and polar convex sets. This restraint on the part of the authors insures the success of their effort; the material will not overwhelm the reader. Hence, the nonspecialist seeking an overview of the subject will find *Convex Functions* more useful than more inclusive texts.

The book is filled with problems, some of a perfunctory nature. Others involve the construction of interesting counterexamples: a convex function whose first derivative fails to exist on a dense set, a discontinuous additive function, etc. Many serve to introduce the reader to the literature. In contrast to the main body of the text, the problems allow the reader a less superficial view of the theory; in addition, the authors here also address the specialist.

By regarding the epigraph of a convex function as a convex set, one can obtain many results for convex functions, such as the Hahn-Banach Theorem, as special cases of theorems for convex sets. Roberts and Varberg have rejected this approach, favoring instead techniques of analysis. They do not malign geometry, but they often avoid obvious pictorial arguments. It is difficult to evaluate the wisdom of their decision in light of the background of the prospective audience. Evidently, they anticipate readers devoid of geometric intuition who equate rigor with an appropriate juggling of coefficients. Perhaps they are correct.

I enjoyed reading *Convex Functions* immensely because of its informality; the book is written in a loose, casual style. The authors are clearly committed to establishing a personal rapport with the reader rather than convincing him of their omniscience.

GERALD BEER, California State University, Los Angeles

Introduction to Probability Theory. By Paul G. Hoel, Sidney C. Port, and Charles J. Stone. Houghton Mifflin, Boston, 1971. xi + 258 pp. $12.95. (Telegraphic Review, November, 1971.)

Introduction to Statistical Theory. By Paul G. Hoel, Sidney C. Port, and Charles J. Stone. Houghton Mifflin, Boston, 1971. x + 237 pp. $13.50. (Telegraphic Review, November, 1971.)

Introduction to Stochastic Processes. By Paul G. Hoel, Sidney C. Port, and Charles J. Stone. Houghton Mifflin, Boston, 1972. x + 203 pp. $12.95. (Telegraphic Review, June–July, 1972.)

Probability and Stochastic Processes: With a View Toward Applications. By Leo Breiman. Houghton Mifflin, Boston, 1969. xii + 324 pp. $13.95. (Telegraphic Review, October 1969.)

Statistics: With a View Toward Applications. By Leo Breiman. Houghton Mifflin, Boston, 1973. xiv + 399 pp. \$15.95. (Telegraphic Review, August-September, 1973.)

Logical meaning and psychological meaning are concepts which psychologists sometimes distinguish in learning theory. A presentation (i.e. course, text, paper, etc.) is logically meaningful if it has coherent meaning for experts in the area. It is presumably well organized and is correct according to the precepts of its discipline. It may even be considered ingenious. Yet it may fall short in its ability to communicate effectively, and thus fail to be psychologically meaningful to its intended audience. Of course both kinds of meaning may be present to varying degrees. Of two mathematics texts, one may be more logically meaningful than the second in that it is more mathematically precise and rigorous. But the second text may provide more motivation and be more stimulating in its exposition and examples. In this review of the texts by Hoel, Port, and Stone, and by Breiman, I will explore the ways in which the necessary compromise between logical and psychological meaning is achieved.

For convenience, I refer to the three texts by Hoel, Port and Stone, in the order in which they appear, as: HPS, v.I; HPS, v.II; and HPS, v.III. I refer similarly to Breiman, v. I and Breiman, v. II.

As the titles suggest, the two series are written with different purposes and thus have distinct flavors. The HPS series presents and emphasizes the unifying theory in each subject area. Presentations are systematic and proofs are provided for nearly all results. While the Breiman texts present much of the same theory, their emphasis is on model building, intuition, and applications. Indeed, Breiman states clearly that his texts are written for students in the applied areas. Both series assume knowledge of calculus through multiple integration, and they eventually require some knowledge of elementary matrix algebra.

I have used the three HPS volumes in three undergraduate courses attended by bright sophomores, juniors, and seniors. The classes included majors from mathematics, biology, physics, and, in one case, Russian. The twelve week courses in probability and statistics were separated by an intensive winter term course in stochastic processes which met ten hours per week for five weeks. The probability course covered nearly all of the first seven chapters of HPS, v. I. The winter term course treated the last chapter of the first volume, the first four chapters of the third volume, and a few supplemental topics. The statistics course covered most of HPS, v. II and a few additional topics, including sufficiency. We did none of the nonparametric statistics as presented in the last chapter of the text. I supplemented HPS, v. II with some mimeograph notes by Professor Jack Kiefer. In all three courses, many problems were assigned, some from the texts and others which I made up or borrowed from Professor Kiefer. I have not had classroom experience with the Breiman volumes.

Both text series are logically meaningful by any reasonable standard. From a mathematical perspective, the HPS series excels in this respect. Its three volumes

are well organized, mathematically correct, and concise perhaps to a fault. The Breiman texts are considerably less formal in their presentation. In Breiman's words, he "never gives a proof unless the proof helps one substantially to understand the workings of the theorem."

While both series are also psychologically meaningful for their intended audiences, here Breiman gets the nod. He tries to help the reader understand what definition one *ought* to give even before he gives it. When a theorem is not proved, Breiman often spends a few moments explaining why one should feel that it *ought* to be true. He personally interacts with the reader. His volumes are filled with interesting and often nontrivial physical examples and exercises designed to illustrate or amplify the basic material. In comparison, the examples and exercises of HPS are usually mathematically-derived rather than being cast in a realistic physical setting. Often the motivation or illustration comes after the presentation of theory rather than before it.

Probability theory is axiomatically presented by HPS in a set-theoretic framework, using σ-fields of sets and probability measures. Breiman's less axiomatic treatment contains no mention of fields of sets, but does emphasize the frequency interpretation of probability and the construction of physical models. The first four chapters of HPS deal exclusively with discrete probability, while the next three present the continuous case. Breiman handles the two cases simultaneously. In this instance I find the HPS approach more psychologically meaningful. HPS devotes the second chapter to combinatorics, to the extent that one might never again wish to see playing cards, balls, and boxes. By omitting combinatorics, Breiman arrives more quickly at the study of stochastic processes. HPS carefully derives distributions of sums and quotients of random variables, but only much later does the series give real insight into their significance. Such topics as convolution, Jacobian transformation of densities, order statistics, and generating functions are concisely treated in HPS, while none of these appear in Breiman. HPS proves the Central Limit Theorem, modulo a few missing details in the proof that distribution functions depend continuously on their characteristic functions. Breiman's discussion of the C.L.T. is largely heuristic, but he presents it in greater generality than does HPS. In some instances, Breiman is careless with mathematical detail. In his discussion of the C.L.T., he repeatedly refers to "small" components of a sum, while never giving precision to this notion. At an earlier point (p. 16), Breiman fails to require that a density function be continuous when he makes an assertion requiring continuity.

The presentation of statistics in HPS, v. II is considerably less complete but somewhat more modern than the traditional coverage in Breiman, v. II. Both texts mention loss, risk, and crossing risk functions. HPS includes optional but somewhat scant sections on Bayesian statistics at the end of three chapters, while Breiman ignores the subject completely. HPS also describes the theoretical importance of randomized procedures, another topic lacking in Breiman. While HPS devotes the last chapter to nonparametric theory, Breiman devotes three chapters to nonparametric theory

and methods. His coverage of the various classical tests in both parametric and nonparametric settings is exhaustive. To his credit, Breiman emphasizes that normality is not necessarily the norm. In general, Breiman, v. II is more carefully written than the first volume. It is a very adequate reference for classical statistics. Both texts fail to carry the modern axiomatic approach as far as they might: neither interprets classical hypothesis testing in light of risk defined with 0–1 loss, nor does either discuss the importance of sufficient statistics from even an intuitive perspective.

The topical coverages of stochastic processes in the two series are nearly identical, but once again the presentations are markedly different. HPS carefully treats Markov chains in the discrete case for two chapters, while Breiman blends the discrete and continuous cases. Breiman's definitions are sometimes unnumbered and stated imprecisely: consider his definition of a stable system (p. 168). His proofs are sometimes not easily distinguishable from his heuristic arguments nor are the ends of some proofs easily discerned (e.g. page 178). Results and proofs are always carefully stated in HPS, v. III. Occasionally, heuristic arguments follow the proofs (e.g. page 15), an approach that HPS should adopt more frequently. Breiman's presentation of the infinitesimal transitional scheme for Markov processes is more understandable than that of HPS. In fact, here I detected a small mathematical error in HPS (p. 88 following (9)). Unlike the HPS series, Breiman includes a chapter on multivariate normal distributions. It utilizes matrix algebra and complex random variables. When HPS, v. II needs these topics, it provides summaries, respectively, in an Appendix and within the text. In a discussion of Brownian motion, HPS, v. III alludes to the Central Limit Theorem, but fails to remark that the version needed is more general than that presented in the first volume. Breiman's presentation of white noise is better motivated than that of HPS, and he includes a section on ergodicity. Characteristically, Breiman is often the entertainer: "We can't plunk down just any set and say. . ." and "You can't make a silk purse out of a sow's ear."

I conclude with a few general remarks. Both series of texts make infrequent references to historical information: I'd like to see a bit more. Neither series contains references to the vast literature in the areas. Both series contain few inaccuracies of any kind. While the chapters in the texts of both series contain introductions, those of Breiman are more helpful. He also concludes each chapter with a summary of its highlights. HPS includes an enormous number of exercises, nearly all with answers. Some exercises could be more challenging. Breiman's exercises are more interesting, and many also include answers.

Which series should be adopted? I'd choose Breiman for independent reading. I'd choose HPS for use in a course which is based upon carefully motivated lectures designed to add psychological meaning. I wish each volume of the latter series had been about fifty pages longer!

JOHN D. EMERSON, Middlebury College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S*, L*, *Man and the Computer*. Ashley Montagu, Samuel S. Snyder. Auerbach, 1972, vii + 216 pp, $9.95. Attempts to dispel popular apprehensions (and misapprehensions) concerning computers by discussing uses which can potentially contribute to betterment of society, as well as ways of avoiding potential misuses. Also includes survey of previous technological revolutions and their effects, brief history of the evolution of computers, how computers work, and the significance of the phenomenon of interfacing. Philosophical in tone, optimistic in spirit. PJC

GENERAL, S(13), *Mathematical Challenges II, Plus Six*. Ed: Thomas J. Hill. NCTM, 1974, vi + 122 pp, $2.55 (P). Sequel to *Mathematical Challenges*, published in 1965. Collection of 100 problems with solutions, selected primarily from the *Mathematics Student Journal*, plus six articles from this *Journal*, three written by high school students. Problems are in the areas of algebra, geometry, number theory, probability and trigonometry, and are mainly directed at high school students. RSK

GENERAL, S, *The History of Board Games*. Robert McConville. Creat Pub, 1974, 104 pp, $4 (P). Full page board diagrams (with instructions), preceded by a brief informal history, followed by detailed instructions for construction of wooden game boards. Intended for secondary teachers, who are explicitly granted permission to reproduce up to 50 copies "for use in the classroom." LAS

GENERAL, S, P, *Sur La Nature des Mathêmatiques*. C.P. Bruter. Gauthier-Villars, 1973, x + 132 pp, 19F (P). A popular essay on the nature of mathematics and its methodology, illustrated with a non-technical introduction to the nature and usefulness of matroids. PJC

GENERAL, S, L, *Magic Squares*. John Lee Fults. Open Court, 1974, x + 102 pp, $1.95 (P). Designed for self-study by the student knowing some high-school algebra, but requires some mathematical maturity. A simpler and more orderly presentation than Andrews' *Magic Squares and Cubes*; marred, however, by occasional lapses in grammar and syntax. Includes Fults' own method for generating odd-order squares. PJC

GENERAL, S, *Games, Tricks, and Puzzles for a Hand Calculator*. Wallace Judd. Dymax, 1974, 91 pp, $2.95 (P). An automated, empirical exploration of numerical patterns intended to motivate junior high students. Includes simple explanation, with photos, of the construction of hand calculators. LAS

GENERAL, S, *Line Designs*. Dale Seymour, Linda Silvey, Joyce Snider. Creat Pub, 1974, 80 pp, $2.50 (P). Illustrations and instructions for constructing (by paper and pen, and by string) various designs from envelopes of families of lines. Revised from an earlier (1968) edition. LAS

GENERAL, S, L. *The Figure Finaglers.* Robert S. Reichard. McGraw, 1974, xiii + 274 pp, $8.95. Popularized account offering a consumer's guide to statistical reporting, with lots of concrete instances from recent news events. Charts, percentages, interest rates, sampling, averages, indexes. No mathematics beyond percentages. PJC

GENERAL, S. *Raumgeometrie in der Technischen Praxis.* Imre Pál. Akademiai Kiado, 1974, 176 pp, $13. Three dimensional drawings--two colors with decoding glasses (included) of basic figures in geometry, engineering, architecture and chemistry. Sources are given. It would appear that this neat technical trick is little used in English-speaking countries. JAS

GENERAL, P. *Open Questions in Mathematics.* Ed: Dagmar R. Henney. (Dept. of Math., George Washington U.), 1974, 69 pp, $3 (P). A (very) "rough and incomplete" draft of unsolved mathematical problems submitted by members of the Scientific Academies of many different countries. Contributions range from short, cryptic conjectures to extensive papers outlining a problem's complete background. LAS

EDUCATION, S, P*. *Beiträge zum Mathematikunterricht 1973.* Hermann Schroedel, 1974, 312 pp, (P). The papers presented at the Bundestagung für Didaktik der Mathematik in March 1973 in Worms. Forty-four papers cover the gamut from film making to undecidability theorems in foundations courses for teachers. JAS

EDUCATION, S, P. *Mathematische Reflexionen.* Hermann Schroedel, 1973, 200 pp, DM44 (P). A translation of *Mathematical Reflections*, Cambridge U Pr (1970). A collection of papers in memory of A.G. Sillitto whose interest in geometry served as a starting point for reform efforts in the schools of Scotland. The articles included both mathematical and educational material that appears worth reading in any language. JAS

EDUCATION, T(15: 2). *Basic Concepts of Elementary Mathematics, Second Edition.* John M. Peterson. Prindle, 1974, 500 pp, $11.95. First edition reviewed in May 1971. This edition adds statistics, the metric system, and percents. Each chapter now has a section on "common errors and their cures" and each exercise set is accompanied by problems taken from leading elementary texts. Also includes reprints of pages from these texts. RJ

EDUCATION, P. *Neue Methoden im Mathematikunterricht.* Margarita Wittoch. Hermann Schroedel, 1973, 180 pp, DM12,80 (P). Comparative research on achievement, creativity, and motivation among problem-centered, programmed, and Piaget-oriented educational methods. JAS

HISTORY, S*, P, L**. *A History of Mathematical Notations: Volume I, Notations in Elementary Mathematics.* Florian Cajori. Open Court, 1974, xvi + 451 pp, $4.95 (P). Unabridged reproduction of 1928 edition, at an attractive price. Gives first appearance of a symbol and its origin, indicates competition encountered and the spread of the symbol among writers in different countries. Three major divisions: numeral symbols, symbols in arithmetic and algebra, and symbols in geometry. Example: the Mayans were the first to use a symbol for zero, antedating the Hindus by centuries. Large number of figures and illustrations. PJC

HISTORY, S(15-17), P, L**. *Probability Theory: A Historical Sketch.* L.E. Maistrov. Trans. and Ed: Samuel Kotz. Acad Pr, 1974, xiii + 281 pp, $22.50. A modest complement to Todhunter's 1865 classic, especially revealing of the great Russian contributions of Chebyshev, Markov, Kolmogorov. Ranges from "prehistory" (ancient and medieval) to the axiomatizations of the early twentieth century. LAS

HISTORY, P. *Giuseppe Peano.* Hubert C. Kennedy. Birkhauser Verlag, 1974, 31 pp, Sfr. 14,50 (P). A biographical supplement to the journal "Elemente der Mathematik." The fourteenth in a series of short biographies. JAS

HISTORY, S, P. *Hamiltons Entdeckung der Quaternionen.* B.L. van der Waerden. Vandenhoeck and Ruprecht, 1973, 14 pp, DM5 (P). Enlarged version of an historical lecture presented in June 1973 before the Joachim Jungius-Gesellschaft der Wissenschaften. JAS

FOUNDATIONS, P. *Combinators, λ-Terms and Proof Theory.* Sören Stenlund. Reidel, 1972, 184 pp, $13.50. Basic ideas and results in pure combinatory logic. Some material appeared already in the author's *Introduction to Combinatory Logic.* Combinators, Church's λ-calculus, the Church-Rosser theorem, development of arithmetic within the theory of combinators, Gödel's computable functions of finite type, proofs in the theory of species. Author does excellent job of situating the work described in its historical motivation. PJC

FOUNDATIONS, T(17-18: 1), S, P, L. *Modal Logic and Its Applications.* D. Paul Snyder. Van N-Rein, 1971, xiv + 335 pp, $12.50. Extension of Gentzen-style proof technique to modal logics, in a form of reduction called "cancellation" due to Binkley and Clark. Presupposes familiarity with propositional and predicate logic. Discusses a multitude of formal logics, interpreting them via Hintikka's notion of "modal system" (consistency property) and concentrating on the details of the techniques of their manipulation. PJC

FOUNDATIONS, P. *The Logic of Significance and Context, V. 1.* Leonard Goddard, Richard Routley. Halsted Pr, 1973, xi + 641 pp, $32.50. Provides a general formal theory of significance, including a detailed discussion of the semantic basis of the proposed theory in terms of a logic of context. A number of 3-valued logics (T, F, Nonsignificant) are developed at sentential and predicate levels. Volume II will continue the study to the point of including a new foundation for mathematics; it will also contain the index for both volumes. PJC

FOUNDATIONS, P*. *Méthodes Logiques en Géométrie Diophantienne.* Shuichi Takahashi. Pr U Montreal, 1974, 178 pp, $5 (P). A categorical approach to logic, recursivity, model theory and non-standard analysis. Starts with the notion of a topos as an appropriate setting for logic. Includes what looks like non-standard algebraic geometry. PJM

FOUNDATIONS, T(17-18: 1), P*, L*. *Mathematical Logic and Hilbert's ε-Symbol.* A.C. Leisenring. Gordon, 1969, ix + 142 pp, $21. Based on the author's thesis, this book examines the nature of the ε-operator (used, e.g., in Bourbaki's set theory) and demonstrates its use in proving classical theorems and simplifying formulation of logical systems. The ε-operator selects an arbitrary element from a set of objects with a given property (the axiom of choice asserts the existence of a <u>set</u> of such choice elements). The ε-symbol was introduced by Hilbert to provide explicit definitions of $\forall$ and $\exists$; its use is justified by Hilbert's 2nd ε-theorem, which says that ε can be eliminated from the proofs of formulas which do not contain it. The author investigates all aspects, including the relation of ε to axiom of choice, the use of ε in arithmetic, and its connection with Gentzen's sequent calculus. Despite publisher's claim, the book is probably not suitable as supplementary reading for undergraduate logic courses. PJC

COMBINATORICS, T(16-18: 1), P, L*. *Latin Squares and Their Applications.* J. Denes, A.D. Keedwell. Acad Pr, 1974, 547 pp, $24.50. First book

devoted entirely to Latin squares, giving a comprehensive account of mathematical and historical aspects of past and present research. Stresses both combinatorial and algebraic features of the subject, emphasizes applications to statistics and information theory. A list of unsolved problems is provided; and the 50-page bibliography has the novel and useful feature of including page references for each item, citing where it is referred to in the text. A great primer for potential researchers! PJC

NUMBER THEORY, P*, L. *Reviews in Number Theory, 6 Volumes.* Ed: William J. LeVeque. AMS, 1974. *V. 1:* v + 420 pp, $50 (P); *V. 2:* v + 672 pp, $50 (P); *V. 3:* vi + 377 pp, $40 (P); *V. 4:* vi + 582 pp, $50; *V. 5:* v + 470 pp, $40; *V. 6:* ix + 410 pp, $40 (P). Photocopies of 14,426 reviews from V. 1-44 of *Mathematical Reviews*, covering the period 1940-1972, arranged by subject into 20 chapters. Careful subject classification and extensive cross and forward referencing make these volumes a uniquely valuable research tool. LAS

NUMBER THEORY, T(18), P*. *Algebraic Numbers and Diophantine Approximation.* Kenneth B. Stolarsky. Pure and Appl. Math., V. 26. Dekker, 1974, xv + 329 pp, $21.75. A highly readable introduction to the theory of Diophantine approximation. A basic understanding of modern algebra and complex variables is assumed. Concentrates on the famous theorem of Roth and on A. Baker's theorem on linear forms in the logarithms of algebraic numbers. Also includes rudiments of algebraic number theory from a constructive point of view. Lots of exercises of varying degrees of difficulty and an ample bibliography. CEC

NUMBER THEORY, P. *Elliptic Modular Functions: An Introduction.* B. Schoeneberg. Grund. math. Wissenschaften, B. 203. Transl: J.R. Smart, E.A. Schwandt. Springer-Verlag, 1974, viii + 232 pp, $27.90. Based on lectures of Hecke. Useful both as a reference and as a basic introduction to the theory. Topics include the modular group, Eisenstein series, function theory for congruence subgroups, various fields of modular functions, Theta series. SG

NUMBER THEORY, P. *Cubic Forms: Algebra, Geometry, Arithmetic.* Yu. I. Manin. Trans: M. Hazewinkel. Math. Lib., V. 4. North-Holland, 1974, vii + 292 pp, $23.10. Every rational number is a sum of three rational cubes. Natural generalizations of this problem lead to the theory of cubic surfaces, and hence to the study of certain algebraic structures (e.g., quasigroups), algebraic number theory, and algebraic geometry. No exercises, but several expository and historical sections are included. Much of the work is the author's own. Chapters: CH-quasigroups and Moufang loops; classes of points on cubic hyper-surfaces; two-dimensional birational geometry; the 27 lines; minimal cubic surfaces; the Brauer-Grothendieck group. SG

NUMBER THEORY, P. *Lecture Notes in Mathematics-397: The Schur Subgroup of the Brauer Group.* Toshihiko Yamada. Springer-Verlag, 1974, iv + 159 pp, $7.40 (P). An exposition of recent results. The author proves the Brauer-Witt theorem and considers the Schur subgroup of p-adic fields, real and imaginary algebraic number fields. SG

LINEAR ALGEBRA, T(15-16: 1), S, L*. *Linear Algebra and Geometry, A Second Course.* Irving Kaplansky. Chelsea, 1974, xii + 143 pp, $6.75. Corrected reprint of a work first published in 1969. A concise collection of closely related topics on the interface between linear algebra and classical geometry written in a refreshing, informal style that gives each topic a special flavor worth savoring. Assumes the reader to be "comfortably familiar" with coordinate-free linear algebra. LAS

LINEAR ALGEBRA, P. *Indefinite Inner Product Spaces.* János Bognár. Ergebnisse der Math., B. 78. Springer-Verlag, 1974, ix + 223 pp, $19.70. An indefinite inner product space is a vector space with an "inner product" which is not positive definite. This is a thorough exposition of the topological and analytic theory of such spaces. JAS

ALGEBRA, T(18: 1, 2), P. *Abelian Categories with Applications to Rings and Modules.* N. Popescu. Acad Pr, 1973, xii + 467 pp, $32.50. An up-to-date exposition with the new (since Freyd and Mitchell) topics consisting of abundant material on localization, decomposition theories, and new results on duality (colocalization shows up). A limited number of substantial exercises places this as a text only for serious students of mathematics. JAS

ALGEBRA, T*(13-18: 1), S, P*. *Introduction à la Théorie des Sous-Ensembles Flous à l'Usage des Ingénieurs: Tome I: Éléments Théoriques de Base.* A. Kaufmann. Masson, 1973, xxi + 410 pp, (P). First of two volumes, this one abounding in examples and illustrations of basic theoretical concepts of fuzzy subsets ("classes with unsharp boundaries in which the transition from membership to non-membership is gradual rather than abrupt," reflected by characteristic functions which may assume values between 0 and 1). Fuzzy graphs, f. relations, f. logic, f. groupoids, f. categories, f. morphisms. Exercises, too. Preface by L.A. Zadeh in French and English. Second volume will contain applications (including such notions as f. algorithms and f. feedback). PJC

ALGEBRA, P. *Lecture Notes in Mathematics-386: Buildings of Spherical Type and Finite BN-Pairs.* Jacques Tits. Springer-Verlag, 1974, x + 299 pp, $9.90 (P). Notes of a seminar held at Oberwolfach in 1968. The author concentrates on two problems: 1) determination of certain buildings (one associated to simple algebraic or classical groups) of rank 3 or more; 2) determination of isomorphisms between certain buildings. SG

ALGEBRA, T(18), P. *Les formalismes fondamentaux de l'algèbre commutative.* Jean-Pierre Lafon. Hermann, 1974, xii + 260 pp, 68F. Everything you wanted to know about commutative algebra, but were afraid to put in one book. Categories, rings, modules, multilinear algebra, injective, projective and flat modules and elements of homological algebra. Not many examples in the text, but lots of exercises. PJM

ALGEBRA, P. *Groupes de Barsotti-Tate et Cristaux de Dieudonné.* Alexandre Grothendieck. Pr U Montreal, 1974, 155 pp, $5 (P). A Barsotti-Tate group is a "nice" sheaf of commutative groups. A crystal is a "nice" sheaf of modules. These notes discuss the relation between the two notions. PJM

ALGEBRA, T(17-18: 2, 3), P. *Rings and Categories of Modules.* Frank W. Anderson, Kent R. Fuller. Grad. Texts in Math., V. 13. Springer-Verlag, 1974, viii + 339 pp, $13.50 (P). A categorical approach to modules and ring theory. No explicit homological algebra, but Hom and tensor, flat and projective modules are discussed. Ample exercises; good bibliography. PJM

FINITE MATHEMATICS, T(13: 1), L. *Introduction to Finite Mathematics.* Walter Feibes. Hamilton, 1974, xvi + 290 pp, $9.95. Written primarily for social science majors, but all students willing to make an effort to learn some interesting applicable mathematics would profit from this book. Concepts are presented in an honest nonpatronizing way using many engaging examples. Some challenging topics are included but are marked with an asterisk. Chapters: elementary counting: permutations and combinations; probability; expected value and decision making, the straight

line for fun and profit; linear programming; games people play; a birds eye view of high finance. Appendices: interest tables, sets, summation notation. Answers to odd-numbered exercises. RBK

FINITE MATHEMATICS, T(13: 1, 2). *Finite Mathematics: An Introduction to Mathematical Models.* Ruric E. Wheeler, W.D. Peeples, Jr. Brooks/Cole, 1974, xii + 596 pp, $13.95. Logic, sets, matrices, linear programming, descriptive statistics, game theory, Markov chains, intuitive calculus, and the mathematics of finance. FLW

CALCULUS, T**(1-3). *Calculus for the Nonphysical Sciences, with Matrices, Probability and Statistics.* Simeon M. Berman. HR&W, 1974, xvi + 656 pp, $14. From the preface: "The point of the book is to stir the interest of the students by showing how mathematics is motivated by real situations and how it can be used." Features: numerous illustrations attractively drawn; applications to all areas of the natural and social sciences, many with real data; intuitive approach, with a casual style. The author aims at the needs of students of the 70's as he perceives them, and is right on target. Times and students change; so must the textbooks. TAV

REAL ANALYSIS, T(18). *Mésures et Probabilitês.* Charles-Michel Marle. Hermann, 1974, 474 pp, 126F. Chiefly abstract measure theory and integration of functions with values in a Banach space, including integration on a locally compact space with respect to Radon measure (after the Daniell integral has been introduced). The final chapter introduces probability theory. Useful chapter summaries. Challenging exercises.DFA

COMPLEX ANALYSIS, P. *Leçons sur les Familles Normales de Fonctions Analytiques et Leurs Applications, Second Edition.* Paul Montel. Chelsea, 1974, xii + 299 pp, $9.50. A reprint of the 1927 original. One of the *Collection de Monographies sur la Théorie des Fonctions* edited by Émile Borel. JAS

COMPLEX ANALYSIS, P. *Techniques of Extension of Analytic Objects.* Yum-Tong Siu. Lect. Notes in Pure and Appl. Math., V. 8. Dekker, 1974, iv + 256 pp, $14.75 (P). In this book, an analytic object is a holomorphic function or a sheaf, and extension consists in enlarging the domain of definition. PJM

COMPLEX ANALYSIS, S*(16-17), P*, L**. *The Schwarz Function and Its Applications.* Philip J. Davis. Carus Math. Mono., No. 17. MAA, 1974, xi + 228 pp, $4. The Schwarz function for an analytic arc C is the unique analytic function which at each point $z \in C$ takes on the value $\bar{z}$; its conjugate yields reflection in the arc C. This elegant monograph explores variations on this theme, e.g., the nine-point circle, conformal maps, fluid mechanics. LAS

DIFFERENTIAL EQUATIONS, T(17-18), P. *Linear and Nonlinear Waves.* G.B. Whitham. Wiley, 1974, xvi + 636 pp, $22.50. For a course for students of applied mathematics, physics, engineering. Mathematical theory of hyperbolic and dispersive waves, with applications to traffic flow, flood waves, gas dynamics, sonic booms, shock dynamics, water waves, nonlinear optics. Brings together much material previously scattered in the literature. No exercises. DFA

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-377: Almost Periodic Differential Equations.* A.M. Fink. Springer-Verlag, 1974, viii + 336 pp, $10.70 (P). Self-contained notes which comprise an introduction to the subject as well as a source for several recent results. Clear exposition; large bibliography. A useful reference. SG

DIFFERENTIAL EQUATIONS, P. *Second Order Equations with Nonnegative Characteristic Form.* O.A. Oleinik, E.V. Radkevic. Trans: Paul C. Fife. AMS/Plenum, 1973, vii + 259 pp, $20. Presents the basics of the theory of second order pde's with nonnegative characteristic form. Three chapters: the first covers the first boundary value problem; the second discusses local smoothness of weak solutions and hyperrellipticity of second order equations; the third considers the Cauchy problem. Large bibliography. No exercises. SG

DIFFERENTIAL EQUATIONS, P. *Non-Linear Differential Equations of Higher Order.* R. Reissig, G. Sansone, R. Conti. Noordhoff Intern, 1974, xiii + 669 pp, Dfl. 180. General methods of qualitative theory, stability, boundedness and existence of periodic solutions for common third and fourth order equations, non-linear higher order systems with separated variables, Lur'e-type systems. Valuable monograph presenting many results heretofore scattered throughout the literature. DFA

NUMERICAL ANALYSIS, T(16-17: 1), L. *Numerical Quadrature and Solution of Ordinary Differential Equations: A Textbook for a Beginning Course in Numerical Analysis.* A.H. Stroud. Appl. Math. Sci., V. 10. Springer-Verlag, 1974, xi + 338 pp, $9.50 (P). Many tables and Fortran programs implement the well-written sections on quadrature formulas of Newton-Cotes, Gauss and Romberg. The short discussion on merits of the Gauss formulas is very convincing and useful. Standard material: Runge-Kutta, Adams, predictor-corrector methods presented for the numerical solutions of initial-value problems for first order ODE's. No reduction to first order systems, no boundary-value problems. "Additional reading" topics (e.g., cubic splines) included in each chapter. A good one-semester text for students with knowledge of Fortran. I-CH

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-385: Collocation Methods for Parabolic Equations in a Single Space Variable, Based on C^1-Piecewise-Polynomial Spaces.* Jim Douglas Jr., Todd Dupont. Springer-Verlag, 1974, 147 pp, $6.60 (P). Chapters on global error estimates, superconvergence estimates at the knots, local superconvergence by local refinement, a smoothed collocation method and applications to eigensystem approximation. The first consumes half the volume and gives optimal order estimates; the other three consider the two-point boundary problem first, then the parabolic problem. DFA

FUNCTIONAL ANALYSIS, T(18: 1), S. *Topics in Nonlinear Functional Analysis.* L. Nirenberg. Notes: R.A. Artino. Courant Inst, 1974, viii + 259 pp, $6.75 (P). Topological and analytic techniques for the study of nonlinear problems, illustrated with applications to nonlinear differential and integral equations. Assumes basic knowledge of linear operator theory, differentiable manifolds, differential forms, but little topology. Topics: topological degree in a Banach space; bifurcation theory; topological methods; monotone operators and the min-max theorem; generalized implicit function theorems. RBK

FUNCTIONAL ANALYSIS, P. *The Theory of Best Approximation and Functional Analysis.* Ivan Singer. CBMS Reg. Conf. in Appl. Math., No. 13. SIAM, 1974, vii + 95 pp, $7.80 (P). Self-contained but extends with minimum intersection the author's *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, 1970 (TR, August-September 1971). Bibliography of 221 items complements bibliography of 1970 work. Characterizations, existence, uniqueness of elements of best approximation; properties of metric projections; best approximation by elements of nonlinear sets. RBK

FUNCTIONAL ANALYSIS, P. *Locally Compact Semi-Algebras with Applications to Spectral Theory of Positive Operators*. M.A. Kaashoek, T.T. West. Math. Stud., V. 9. North-Holland, 1974, ix + 102 pp, $7.50 (P). Study developed from effort to prove converse to result that a compact linear operator of unit spectral radius whose spectrum contains the point 1 generates a locally compact semi-algebra. Converse was obtained in case of operator with equibounded iterates; further work produced connections between semigroups, semi-algebras, and spectral theory. RBK

FUNCTIONAL ANALYSIS, T(18: 1, 2), P. *The Isometric Theory of Classical Banach Spaces*. H. Elton Lacey. Grund. math. Wissenschaften, B. 208. Springer-Verlag, 1974, x + 270 pp, $32. Presentation of main structure theorems in the isometric theory of classical Banach spaces, i.e., spaces whose dual is linearly isometric to some L_p space. Hahn-Banach theorem and consequent separation theorems are used extensively. Topics: partially ordered Banach spaces; topology and regular Borel measures; Banach spaces of continuous functions; classical sequence spaces; representation theorems; characterizations of abstract M and L_p spaces; L_1-predual spaces. Bibliography of 306 items. RBK

FUNCTIONAL ANALYSIS, P. *Operator Theory of the Pseudo-Inverse*. S.R. Caradus. Pure and Appl. Math., No. 38. Queen's U, 1974, iii + 67 pp, $2.50 (P). Basic properties of the pseudo-inverse of a continuous linear mapping of a Banach space. Much of the material appears for the first time in print. SG

FUNCTIONAL ANALYSIS, S(18), P. *Faltungsgleichungen und Projektionsverfahren zu Ihrer Losung*. I.Z. Gochberg, I.A. Feldman. Math. Reihe, B. 49. Birkhauser, 1974, xi + 275 pp, $19. Translation of a 1971 Russian edition. Wiener-Hopf equations and some other convolution equations are developed into a general scheme based on a non-classical operational calculus. They are solved by projection methods related to the Galerkin method. Assumes a knowledge of functional analysis. The bibliography cites mainly Russian sources. RJ

FUNCTIONAL ANALYSIS, P. *Teoria Operatorilor si Algebre de Operatori*. F.-H. Vasilescu, *et al*. Editura Academiei Romania, 1973, 491 pp, Lei 17,50 (P). Expository articles in Romanian on Hardy spaces, dilation theorems, W*-algebras and ideals in operator algebras. JAS

FUNCTIONAL ANALYSIS, P. *Operatori Neliniari*. Dan Pascali. Editura Academiei Romania, 1974, 282 pp, Lei 11,50 (P). An exposition of the use of monotone mappings in the theory of non-linear operators--in Romanian. JAS

OPTIMIZATION, P. *Dynamic Programming and Markov Potential Theory*. A. Hordijk. Math. Centre Tracts, No. 51. Math Centrum, 1974, vi + 134 pp, Dfl. 15 (P). The author's thesis, in Definition-Theorem-Proof-Corollaries format. The central theme is the existence of optimal strategies in various discrete time dynamic programming problems. Among the results: the optimal stopping time is exponentially bounded under optimal policy for a wide class of sequential decision problems. TAV

OPTIMIZATION, T(14: 1). *Modèles Mathématiques en Science de la Gestion*. Gérald Baillargeon. Pr U Quebec, 1973, xiv + 338 pp, $7 (P). Linear programming, matrix algebra, simplex method, optimization in non-linear models (partial derivatives, Lagrange multipliers), integration, inventory models. Includes text and sample output of several FORTRAN programs. Prior calculus needed for some parts. The chapter on inventory models is particularly good. PJC

ANALYSIS, T(17-18: 1). *Stable Mappings and Their Singularities.* M. Golubitsky, V. Guillemin. Grad. Texts in Math., V. 14. Springer-Verlag, 1973, x + 209 pp, $9.50 (P). Intended for first and second year graduate students. Includes the work of Malgrange and Mather. Chapters on elementary manifold theory, various forms of Sard's theorem, a heuristic proof of the Mather stability theorem on the equivalence of stability and infinitesimal stability, proof of the Malgrange preparation theorem, proof of Mather's theorem, classification schemes for stable singularities including a complete classification of all equidimensional stable maps and their singularities in dimensions ≤ 4. An appendix proves needed facts about Lie groups. RBK

ANALYSIS, P. *Lecture Notes in Mathematics-388: Group Representations: A Survey of Some Current Topics.* Ronald L. Lipsman. Springer-Verlag, 1974, x + 166 pp, $8.20 (P). Surveys representation theory of semisimple Lie groups, nilpotent groups, algebraic groups, and (briefly) solvable groups. Includes Mackey's results on induced representations and representations of group extensions. Examples, exercises, ample bibliography. By no means introductory. DFA

ANALYSIS, T?(15: 2). *Multiple Integrals, Field Theory and Series: An Advanced Course in Higher Mathematics.* B.M. Budak, S.V. Fomin. Trans: V.M. Volosov. MIR, 1973, 640 pp. The treatment is very complete, the style classical: definition-theorem-proof-corollaries. Contains chapters on tensors, improper integrals and asymptotic expansions. No exercises, few examples. First published in 1967, the style seems to be that of the 50's or earlier. TAV

ANALYSIS, S(17), L. *Mathematics for Engineers, Volume 1.* L.J. Nicolescu, M.I. Stoka. Trans: L.J. Nicolescu, A. Georgescu. Abacus Pr, 1974, 460 pp, $28.80. Stiff reference book for engineering graduate students. The first chapter--on vector fields--is elementary, but subsequent ones are demanding. These study functions of a complex variable, vector spaces (including normed ones), tensor calculus, the Lebesgue integral, Fourier series and integrals, Laplace transforms. A few exercises. DFA

ANALYSIS, P. *Lecture Notes in Mathematics-367: Cousinverteilungen und Fortsetzungssätze.* Klaus Langmann, Werner Lütkebohmert. Springer-Verlag, 1974, vi + 151 pp, $6.20 (P). An introduction to the theory of M-coherent sheaves. A coherent subsheaf is M-coherent if every stalk is generated by elements of M. RJ

ANALYSIS, P. *Monotone Matrix Functions and Analytic Continuation.* William F. Donoghue, Jr. Grund. math. Wissenschaften, B. 207. Springer-Verlag, 1974, 182 pp, $19.70. A detailed treatment of Pick functions, i.e., functions analytic on $\mathrm{Im}(z) > 0$, and the related classes of monotone matrix functions. Contains chapters on reproducing kernels, almost positive matrices and the Cauchy interpolation problem. TAV

GEOMETRY, T*(13: 1). *Shapes & Perceptions: An Intuitive Approach to Geometry.* Gail S. Konkle. Prindle, 1974, viii + 248 pp, $9.95. Don't judge a book by its cover (or the color of print). Entertaining and informative, this text would provide a mathematically sound background on informal geometry for any student. Modeled after CUPM guidelines for elementary teachers and influenced by the Nuffield Mathematics Project and recent insights regarding learning, this text stresses experimentation and discovery. Extensive bibliography. JNC

GEOMETRY, S???. *The Hyperbola and the Parabola.* William M. Dawes. Branden Pr, 1974, 112 pp, $9.75. No index, table of contents, or

bibliography. Each "chapter" contains an unexplained geometric figure followed by numerous calculations which appear to be the solution of some unstated problem. JNC

TOPOLOGY, T(16-17: 2, 3), S, L. *Topology and Normed Spaces.* G.J.O. Jameson. Halsted Pr, 1974, xv + 408 pp, $17.75. A two-part book: Part I--Topology, Part II--Normed Linear Spaces. Part I would be an adequate one-term course in general topology, but examples and selection of theorems are directed towards Part II which is concerned primarily with general normed linear spaces, but also considers Banach spaces and normed lattices. PJM

TOPOLOGY, T(18), P*, *Characteristic Classes.* John W. Milnor, James D. Stasheff. Annals of Math. Stud., No. 76. Princeton U Pr, 1974, vii + 330 pp, $10 (P). A reworking of the famous 1957 lecture notes with up-to-date references. A reasonable index and the inclusion of a number of problems make it potentially useful in an advanced course. A welcome addition to many libraries and a replacement for many tattered sets of photocopies. JAS

TOPOLOGY, S(17-18), P, *An Introduction to Topological Groups.* Morikuni Goto. Lect. Notes Ser., No. 40. Aarhus U, 1974, ix + 62 pp, $1.50 (P). Background material for Lie groups. No-nonsense exposition in convenient form of the basic ideas of topological groups, important homotopy concepts for topological groups, and Haar measure. JAS

TOPOLOGY, P, *Topology and Borel Structure.* J.P.R. Christensen. Math. Stud., V. 10. North-Holland, 1974, 133 pp, $7.50 (P). Exposition of some recent applications of the theory of analytic spaces and analytic measurable spaces to potential theory and probability theory. JAS

TOPOLOGY, T(17), *Lecons de Topologie Algêbrique.* Emil Artin, Hel Braun. Pr U Quebec, 1973, 205 pp, $6 (P). Translation from the German of lecture notes from courses by Artin and Braun. PJM

TOPOLOGY, P, *Lecture Notes in Mathematics-375: Topology Conference.* Ed: Raymond F. Dickman, Jr., Peter Fletcher. Springer-Verlag, 1974, 283 pp, $9.90 (P). Mostly point-set topology from the conference at Virginia Polytechnic Institute and State University in March, 1973. JAS

TOPOLOGY, P, *Lecture Notes in Mathematics-373: A Geometrical Study of the Elementary Catastrophes.* A.E.R. Woodcock, T. Poston. Springer-Verlag, 1974, 257 pp, $9.10 (P). A series of related papers providing analytical details and computer-generated projections (including some stereographic ones) of the elementary singularities which form the basis of Rene Thom's catastrophe theory. LAS

PROBABILITY, T(16-17: 1, 2), *Probability.* Marcel F. Neuts. Allyn, 1973, xii + 555 pp, $12.95. Rigorous but not overly abstract treatment, requiring maturity and a good background in analysis. Includes chapters on renewal theory and finite Markov chains. Extensive problem sets. Tables, primarily binomial, take up over 100 pages, and appendices another 50.RSK

PROBABILITY, P, *Stochastic Differential Equations: Theory and Applications.* Ludwig Arnold. Wiley, 1974, xvi + 228 pp, $17.95. For two audiences, the theoretical mathematicians and physicists or engineers who deal with systems on which white noise acts. Emphasis on stochastic integrals, modeling and approximation, applications to stability, control. Extensive bibliography. TAV

PROBABILITY, T(16), P. *Two Stochastic Processes.* John A. Beekman. Almqvist & Wiksell, 1974 (U.S. Distr: Halsted Pr.), 192 pp, Sw.Kr. 85. A concise, lucid treatment of Gaussian Markov and risk stochastic processes. Mathematical prerequisite: probability at the level of Feller's Volume 1. Contains extensive bibliographies, numerous exercises, tables. TAV

STATISTICS, T*(13: 1). *Statistics: A Tool for the Social Sciences.* William Mendenhall, Lyman Ott, Richard F. Larson. Duxbury Pr, 1974, xii + 505 pp, $11. Written at a level between Mendenhall and Ott's new *Understanding Statistics* (TR, March 1974) and Mendenhall's well-known *Introduction to Probability and Statistics, Third Edition* (ER, February 1973). In spirit it is much closer to the former than the latter, which it appears it is intended to supersede. It is split about evenly between descriptive and inferential statistics, with good coverage of both. Very little material on probability. RSK

STATISTICS, T(13-14: 1, 2), S. *Biostatistics: A Foundation for Analysis in the Health Sciences.* Wayne W. Daniel. Wiley, 1974, xvi + 448 pp, $13.95. Presupposes only high school algebra. Many examples. FLW

STATISTICS, T(13: 1). *Basic Statistical Methods, Fourth Edition.* N.M. Downie, R.W. Heath. Har-Row, 1974, viii + 355 pp, $10.95. Presupposes only high school algebra. Descriptive statistics and hypothesis testing. Little on estimation or probability. FLW

STATISTICS, T(14-16: 1, 2). *Introductory Statistics and Probability for Engineering, Science, and Technology.* Elwood G. Kirkpatrick. P-H, 1974, xiv + 446 pp, $16.95. Presupposes calculus. "Not too mathematical." FLW

STATISTICS, T(16-17: 1, 2), S, P, L. *The Analysis of Frequency Data.* Shelby J. Haberman. U of Chicago Pr, 1974, xii + 419 pp, $20. A rigorous general presentation of log-linear models. An appendix summarizes the needed results on linear manifolds and linear operators. Many examples of applications to real data. Maximum likelihood estimation, complete factorial tables, social-mobility tables, incomplete contingency tables, and quantal-response models. FLW

STATISTICS, T(1, 2). *Statistics: An Introductory Analysis, Third Edition.* Taro Yamane. Har-Row, 1973, xx + 1130 pp, $14.95. Revision of the author's 1967 *Second Edition.* First half provides material for a fairly basic freshman course for business and economics students. Last half covers more advanced topics, and some earlier topics at a more advanced level, and so is suitable for upper-division or graduate students. Coverage is very thorough, which results in a voluminous work (all exercises are in a separate *Problems Manual*!), that is really two books in one. RSK

STATISTICS, S(13), L. *Risk, Choice and Prediction: An Introduction to Experimentation.* W.J. Youden. Duxbury Pr, 1974, vii + 81 pp, $1.95 (P). An elementary informal excursion into statistical inference via discussions of experiments with, e.g., dice, marbles, bingo and ESP cards. According to Brian Joiner in the preface, the author is "perhaps the best expositor statistics has ever known." LAS

STATISTICS, T(14-16: 1, 2), S, L. *Applied Statistics: Analysis of Variance and Regression.* Olive Jean Dunn, Virginia A. Clark. Wiley, 1974, xii + 387 pp, $17.95. Assumes some statistics background. Introduces statistical inference, analysis of variance, linear, multiple and polynomial regression, covariance. Discusses practical problems in use of models, relationships between models. Organized around excellent examples. Could use more problems. LH

STATISTICS, T(15-18: 1, 2), S, P, L. *The Spectral Analysis of Time Series.* L.H. Koopmans. Prob. and Math. Stat., No. 22. Acad Pr, 1974, xiv + 366 pp, $26. Spectrum analysis of data from a wide variety of actual applications. Filter design based on estimated spectrum. Continuous and discrete time, single and multivariate models. Linear, digital filtering, linear prediction, real time filtering. Distribution theory of spectral estimates, experimental design and computational methods. LH

STATISTICS, T. *Living Statistics: An Introductory Text for West African Students.* John Bibby. Longman, 1972, iv + 76 pp, (P). Brief, elementary treatment, aimed at sixth-form secondary students. Emphasizes the interpretation of tables and diagrams, mostly based on meaningful West African data. RSK

STATISTICS, P*. *Cluster Analysis.* Brian Everitt. Halsted Pr, vi + 122 pp, $8.95. Short monograph concerned with the general problem of devising a classification scheme for grouping objects, each of which has been measured on the same set of variables, into a number of classes or clusters. First reviews the various clustering techniques now being used, then describes, illustrates and offers suggestions on how to handle the problems associated with these techniques. Good bibliography; no index; no copyright or printing date. RSK

COMPUTER SCIENCE, T(15-17: 1, 2), P, L. *Automata, Languages, and Machines, V. A.* Samuel Eilenberg. Pure and Appl. Math., V. 58A. Acad Pr, 1974, xvi + 451 pp, $24. First book of a four-volume series attempting to give a coherent mathematical presentation of the topics of the title. This volume is concerned with mathematical structures that can be described or recognized by a finite-state device without memory or storage capacity. All arguments and proofs are constructive. Presupposed: "no prior knowledge of the theory of automata or of algebra" but "general mathematical culture on the level of a mathematics major." An attractive text for a course in so-called "applied" abstract algebra alternative (rough going) or subsequent to the usual groups-rings-fields. PJC

COMPUTER SCIENCE, P. *Automata on Infinite Objects and Church's Problem.* Michael O. Rabin. CBMS Reg. Conf. in Math., No. 13. AMS, 1972, 22 pp, $3.30 (P). Expository lectures from regional conference at Morehouse College, Atlanta, Georgia, September 1969. Quick overview of some aspects of mathematical theory of automata, with their application to obtain a new solution to Church's solvability problem for finite automata. The natural setting for treating the problem turns out to be the notion of an automaton on an infinite tree, and the major result is that every automaton-definable non-empty set of (infinite) valued trees contains a regular tree. PJC

COMPUTER SCIENCE, S*, L*. *Computers in Society: The Wheres, Why, and Hows of Computer Use.* Donald D. Spencer. Hayden Book, 1974, 196 pp, $7.50; $4.95 (P). Broad perspective on the uses of computers, delineating present and future applications in medicine, law enforcement, fine arts, engineering, education, business, transportation, and control systems. Good variety of illustrations, plus an indexed "prose glossary" of technical terms. PJC

COMPUTER SCIENCE, T(16-17: 1, 2). *Fundamentals of Pattern Recognition.* Edward A. Patrick. P-H, 1972, xxiv + 504 pp, $18. A Bayesian approach on "how to process data or patterns into classes based on their measurements." Includes preliminaries, estimation of probability densities, decision rules and feature extraction. Practical examples. RWN

COMPUTER SCIENCE, T(13-15: 1, 2), S, P, L. *Fortran for Humans.* Richard L. Didday, Rex L. Page. West Pub, 1974, xv + 430 pp, $8.95 (P). Excellent approach to textbook FORTRAN. Very good sections on problem solving, with solutions. This book continues where most others stop, with fine sections on arrays and sorting and searching. Highly recommended.RB

COMPUTER SCIENCE, T(16-17: 1, 2), L. *Computer-Oriented Approaches to Pattern Recognition.* William S. Meisel. Math. in Sci. and Eng., V. 83. Acad Pr, 1972, xii + 250 pp, $16.50. Except for probability theory, this book includes all necessary mathematical preliminaries. Although not complete, the coverage is broad enough to be useful in several different types of courses. Typical topics: approximation and construction of probability densities, cluster analysis and feature extraction. RWN

COMPUTER SCIENCE, S*, P*, L*. *The Origins of Digital Computers: Selected Papers.* Ed: Brian Randell. Springer-Verlag, 1973, xvi + 464 pp, $21.60. Thirty-two first-hand sources outlining the technical details of the precursors of modern computers through the year 1949. Includes a number never before published, several having been translated into English. Over half the articles are post-World War II. Concludes with an annotated bibliography of 350 items. PJC

COMPUTER SCIENCE, T(13-14: 1), L. *Computers in Society: An Introduction to Information Processing.* Donald H. Sanders. McGraw, 1973, xiii + 372 pp, $9.95. An interesting and readable book for both experienced and non-experienced computer-oriented people. Gives many good applications for, and implications of, information processing using computers. The sections on hardware could be skipped, but the chapters on computers in education, in the humanities, in government are very good. RB

COMPUTER SCIENCE, S, P, L. *Digital Computer Concepts: A Self-Instructional Programmed Manual.* Vester Robinson. Reston, 1974, viii + 295 pp, $7.95. An excellent self-teaching programmed manual which goes into detail on computer hardware and circuits. It does lack as a textbook in not having any problems for the student to work out. The whole book is based upon short sections of instruction, each one followed by questions with one word answers. Recommended highly for personal use. RB

COMPUTER SCIENCE, S, P. *Lecture Notes in Economics and Mathematical Systems-81: Advanced Course on Software Engineering.* Ed: F.L. Bauer. Springer-Verlag, 1973, xii + 545 pp, $11.90 (P). Lecture notes from a two-week course given in Germany in 1972. Quite authoritative. Covers the software considerations of systems, languages, concurrency, modularity, portability, reliability, management, documentation, measurement, evaluation and pricing. RWN

SYSTEMS THEORY, T(17-18: 2), S, P. *System Identification: Paramter and State Estimation.* Pieter Eykhoff. Wiley, 1974, xx + 555 pp, $36.50. Pulls together many aspects of parameter and state estimation. Seeks optimal signal processing to gain information on physical, social or biological systems. Substantial background required. Linear and non-linear models, sampled and continuous signals, simultaneous estimation of parameters and states. LH

APPLICATIONS (NATURAL SCIENCE), T(16-17), L. *Mathematics Applied to Deterministic Problems in the Natural Sciences.* C.C.Lin, L.A. Segel. Macmillan, 1974, xix + 604 pp, $15.95. Construction, analysis, and interpretation of mathematical models (some probabilistic) for problems in physics, engineering, astronomy, chemistry, biology. The mathematics includes topics in ordinary and partial differential equations, Fourier analysis, and continuous field theory. Historical remarks, exercises. Suitable for a variety of introductory "applied" courses. DFA

APPLICATIONS (PHILOSOPHY), P, *Theoretical Concepts.* Raimo Tuomela. Lib. of Exact Philo., V. 10. Springer-Verlag, 1973, xiv + 254 pp, DM 74. Systematic investigation of logic and method of using theoretical concepts in scientific theories. Are theoretical concepts necessary? desirable? when? Various "elimination" results (Craig elimination, Ramsey-elimination) within a setting heavily mathematicized with mathematical logic and model theory. Examples of scientific theories and concepts drawn from social science. PJC

APPLICATIONS (PSYCHOLOGY), P, *Structural Learning I: Theory and Research.* Joseph M. Scandura. Gordon, 1973, xi + 367 pp, $19.50; $11.70 (P). "Structural learning" refers to the acquiring of structured knowledge. Scandura's underlying thesis is that all human behavior (including learning) is rule-governed. The rules turn out to correspond to partial recursive functions, and Scandura goes on to interpret the "behavior potential" (what a "subject" is capable of) in terms of categories and functors (!). These are the high points of his first partial theory (deterministic model) of structural learning, the theory of competence; the second and third take into account (resp.) the "behaving subject" (what rules "may be attributed to the subject"--i.e., what rules does the subject use or know) and the subject's memory and information-processing capacity. Chapter 6 is specifically devoted to a theory of mathematical knowledge: what does it mean to "know" mathematics? Photo-offset from typescript; this reviewer's copy had 2 each of pp. 167-168. PJC

APPLICATIONS (PSYCHOLOGY), S(16-18), P*, L*, *Contemporary Developments in Mathematical Psychology, V. I: Learning, Memory, and Thinking.* Ed: David H. Krantz, *et al.* Freeman, 1974, xiii + 299 pp, $9. Eight 'paradigms' for mathematical psychology by twelve authors intended to illustrate and define the concept of progress ("a growing chain or lattice of development") in mathematical psychology. A bold and courageous attempt by the editors to shape rather than merely to reflect the direction of this rapidly growing field. A remarkably modest price for such a stellar collection. LAS

APPLICATIONS (SOCIAL SCIENCE), P, L, *A Mathematical Theory of Social Change.* Robert L. Hamblin, R. Brooke Jacobsen, Jerry L.L. Miller. Wiley, 1973, xi + 237 pp, $12.50. A series of sociological data sets fitted to various analytical equations together with sociological interpretation of the corresponding differential equations. The authors' purpose (and claim) is to increase the explanation of sociological variance from the usual .30 or so to as high as .98. LAS

APPLICATIONS (SOCIAL SCIENCE), P, L, *Mathematical Models of Social and Cognitive Structures: Contributions to the Mathematical Development of Anthropology.* Ed: Paul A. Ballonoff. U of Ill Pr, 1974, xxi + 122 pp, $5.95 (P). Seven diverse papers from the 1972 Toronto meeting of the Amer. Anthro. Assoc. A sequel to the earlier (1971) similar volume edited by Paul Kay: *Explorations in Mathematical Anthropology.* LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Richard Jensen, Carleton; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Hartwick College: Dr. W. E. Maxey, Purdue University, has been appointed Assistant Professor; Mr. G. E. Stevens, University of Michigan, has been appointed Instructor.

Oakland University: Dr. Jerold Grossman, MIT, has been appointed Assistant Professor; Dr. Richard Molnar, University of North Carolina, Chapel Hill, has been appointed Assistant Professor; Dr. Suzanne Molnar, University of North Carolina, Chapel Hill, has been appointed Lecturer; Assistant Professor Irwin Schochetman has been promoted to Associate Professor

Pace University, New York: Professor Maurice Nadler has been appointed Chairman of the Mathematics Department to succeed Professor Louis Quintas who relinquished the post for teaching and research.

Dr. Fred Frishman, after 14 years of service with the Army Research Office, in Washington, D. C.; London, England; and Durham, North Carolina, has accepted a position with the Internal Revenue Service as Chief, Mathematical Statistics Branch.

Professor Benjamin Lepson, Mathematics Research Center, U. S. Naval Research Laboratory, will be on sabbatical leave from the Laboratory as a Visiting Professor at the University of Maryland, College Park, from July 1974 to June 1975.

Professor W. C. Rheinboldt, University of Maryland, College Park, has been appointed Director of the Interdisciplinary Applied Mathematics Program.

Mr. William R. Burns, Bakersfield, California, died on August 2, 1974. He was a member of the Association for six years.

Professor Puthenpurakkal C. Joseph, Kerala, India, died on December 21, 1973. He was a member of the Association for nine years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of \$100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1974 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Emil Grosswald, and D. E. Richmond, were announced by President R. P. Boas at the business meeting of the Association on January 26, 1975, in Washington, D. C. The recipients of the Ford Awards for articles published in 1973 were the following:

Patrick Billingsley, *Prime Numbers and Brownian Motion*, MONTHLY, 80 (1973), 1099–1115.

Garrett Birkhoff, *Current Trends in Algebra*, MONTHLY, 80 (1973), 760–782.

Martin Davis, *Hilbert's Tenth Problem is Unsolvable*, MONTHLY, 80 (1973), 233–269.

I. J. Schoenberg, *The Elementary Cases of Landau's Problem of Inequalities between Derivatives*, MONTHLY, 80 (1973), 121–158.

L. A. Steen, *Highlights in the History of Spectral Theory*, MONTHLY, 80 (1973), 359–381.

R. J. Wilson, *An Introduction to Matroid Theory*, MONTHLY, 80 (1973), 500–525.

HENRY L. ALDER, *Secretary*

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CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.

FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.

ILLINOIS, Rockford College, Rockford, May 9–10, 1975.

INDIANA

IOWA, Iowa State University, Ames, April 18–19, 1975.

KANSAS

KENTUCKY, Murray State University, April 11–12, 1975.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Madison College, Harrisonburg, Virginia, April 26, 1975.

METROPOLITAN NEW YORK, Brooklyn College, CUNY, April 20, 1975.

MICHIGAN

MISSOURI, Missouri Western State College, St. Joseph, April 18–19, 1975.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.

NEW JERSEY

NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.

PACIFIC NORTHWEST

PHILADELPHIA

ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.

SEAWAY, York University, Toronto, Ontario, April 25–26, 1975.

SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1975.

SOUTHWESTERN, Glendale Community College, Glendale, Arizona, April 11–12, 1975.

TEXAS, Angelo State University, San Angelo, April 11–12, 1975.

WISCONSIN, University of Wisconsin-Superior, April 19, 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Denver, Colorado, April 23–26, 1975.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30–May 2, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

CHOOSE YOUR TEXT FROM THIS WIDE VARIETY

FINITE MATHEMATICS WITH APPLICATIONS, Second Edition A. W. Goodman and J. S. Ratti, both University of South Florida

The second edition of this outstanding text continues as the best available for courses in finite mathematics. It retains the clarity of style and blend of theory and applications in a broad selection of topics exemplified by real life examples and problems.

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- The addition of several easy problems at the end of each section.
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- A direct proof of Gamblers Ruin Theorem has been added in the appendix.
- The chapter on Graph Theory has been expanded to include Koeningsburg Bridge Problem and related material.
- Chapters on probability and statistics have been reorganized for smoother presentation.
- All answers are included in the text.

1975 approx. 512 pages

BASIC IDEAS OF STATISTICS, Bernard W. Lindgren, University of Minnesota

Here is a superb introduction to the principles and uses of statistics for those without an advanced mathematics background; it gives them some appreciation of the scope and limitations of statistics, and it offers them the basic knowledge needed to challenge the statistical bombardments of the media and to sift through reports phrased in statistical terms.

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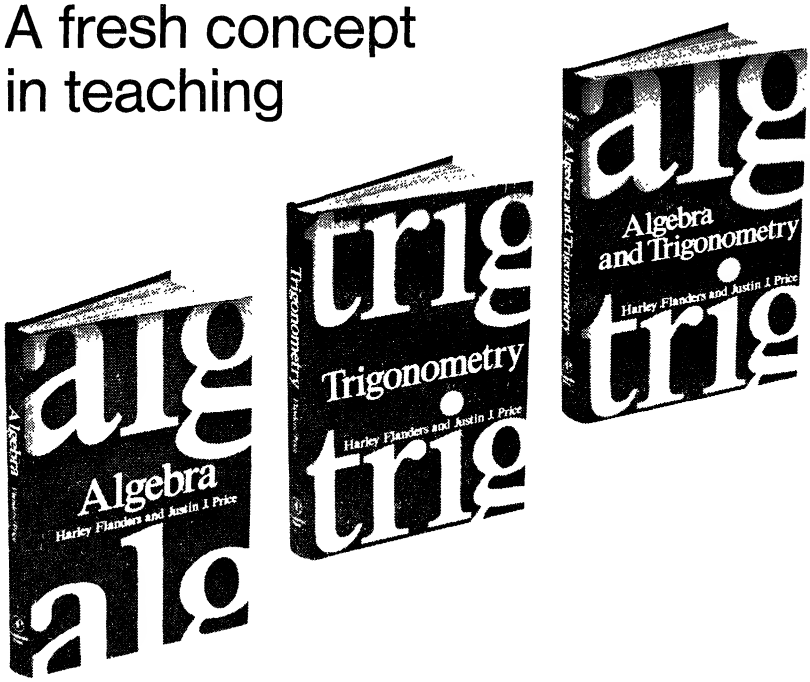

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THE FUTURE OF THE UNIVERSITY IN MATHEMATICS EDUCATION

HARTLEY ROGERS, Jr.

Dedicated to Clarence E. Hardgrove

A distinctive feature of mathematics education in the United States has been the unusually high level of interaction between mathematics education in the schools and mathematical life in the universities. I think it is appropriate, amid this conference's questioning of future directions and possibilities, that we take a moment to celebrate this interaction. In particular, it is fitting to mention a book of lectures first given in this state to an audience that was, I believe, composed largely of secondary school mathematics teachers.

This book is not only an early example of fruitful interaction between university and school mathematics, but also one of the more outstanding indigenous expressions of American mathematics prior to 1930. After 1930, a major infusion of mathematical talent from Europe changed the face of American mathematics. Prior to 1930, America had made two notable contributions to modern mathematics: (1) the work in pure and applied analysis initiated by George David Birkhoff and (2) the concern with axiomatic abstraction illustrated in the work of Oswald Veblen in the first years of this century. This latter, axiomatic, concern appears to have been in advance of similar interest on the continent. It is an approach to mathematical theory in which one considers a list of abstractly formulated axioms and then studies, via structure and representation theorems, the nature and properties of the structures that satisfy these axioms. Perhaps the best early expression of this axiomatic tradition in American mathematics, in my opinion, is the book of lectures that we have already mentioned. Its title is *Lectures on Fundamental Concepts of Algebra and Geometry*. The lectures were given in 1909 by John Wesley Young, then a professor at the University of Kansas, to a summer course at the University of Illinois.

Our title is "The Future of the University in Mathematics Education." Let me begin by commenting in more detail on this title. How distant a future shall we consider? We shall, in fact, take a short-run view based on current trends and pressures, rather than attempt the construction of a long-range model. What shall we take "university" to mean? We shall mean a college-level institution which has graduate as well as undergraduate programs.

Even after we stipulate this much, our title remains somewhat ambiguous. Shall we construe "university" broadly to mean the entire university community, or narrowly to mean the traditional mathematics department within the university? Shall we construe "education" broadly to mean the entire range from kindergarten on up, or narrowly to mean undergraduate and graduate university education?

Obviously our interest is in the broader definitions. As reference point and way into our subject, however, we shall look first at the narrowest case. We shall then, in the last portion of the talk, go on to somewhat broader conclusions about the relation of all mathematics education to all of the university community. We can claim no special authority for these statements and judgments. They are based on visits to a variety of mathematics departments around the country in recent months and on experience in my own department.

What background facts and forces form the setting within which a mathematics department currently must work? This background is having, in my opinion, an increasingly profound effect on what mathematics departments do. It is familiar to most of you, but let us review it briefly. There is, first, the constellation of factors that we may call the economic picture. These factors constitute a familiar and distressing list: (1) The current buyer's market for new Ph.D.'s — an over-production of Ph. D.'s based in part on erroneous predictions as to national needs. (2) The decline, or at least changing pattern, of federal funding for research in the physical sciences and mathematics. (3) The end of the "baby-boom"; the decline, at least in rate of growth, of the national college-undergraduate population; and a foreseeable future decline in absolute numbers. (4) An even greater possible decline in undergraduate mathematics program enrollments. (5) The national economic problems of the moment. (6) The constraints (not often mentioned) placed on a typical mathematics department today by the age and tenure profile of its faculty. During the period of major growth in the 1960's, departments took on many new faculty members. Most were fairly young and are now in their thirties and early forties. All have tenure and none is close to retirement. (7) A further aspect of the economic picture, and, in a sense, a summary of other items mentioned above, is the increasing number of mathematics Ph.D.'s to be found in jobs (both teaching jobs and other jobs) that, until quite recently, would have been filled by individuals with master's or even bachelor's degrees.

An excellent and informative summary of the current economic situation can be found in the articles, "Jobs and Ph.D.'s in the Mathematical Sciences I & II," by Professor R. D. Anderson of Louisiana State University in the *Notices of the American Mathematical Society*, (1973, volume 20, 348–352 and 367–371). We shall come back to the economic picture in a few moments and make a further comment on it.

First, however, there is one other set of background facts that we must note. These have to do with certain changing currents in our culture, currents that involve changing life styles, changing intellectual and cognitive styles, and changing expectations and desires in our students. These last include changing attitudes towards careers and changing views as to what is an appropriate preparation for a career. In particular, they include a growing belief that career choices should be more tentative and reversible than in the past. Such changing attitudes are becoming visible among faculty members as well as students. This is evident at my own institution, where there are signs that at least a small number of faculty want to be able to shift

their interests and to move, if not from department to department (and that sometimes occurs), at least over a rather wide range in their own intellectual areas.

Before turning to a consideration of goals and programs for university mathematics departments, let us now make a further footnote comment on employment prospects and the economic picture. At M.I.T., with our special interest in preparing individuals in areas of science and engineering, we have studied several different models of the employment market over the next decade. The most persuasive model that I have seen differs rather significantly from the model usually adopted in discussions at professional meetings. The latter, commonly accepted, model assumes that without special action, the universities will continue to turn out Ph.D.'s at the present rate. The alternative, and for me more persuasive, model holds that, in the ordinary course of events, a predictable and substantial decline will occur in the number of students choosing to study mathematics at the graduate level. This model is largely due to Professor Herbert Hollomon (formerly president of the University of Oklahoma and now Director of the Center for Policy Alternatives at M.I.T.). According to the historical analysis that supports this model, the statistical pattern of past student career-choices has been directly sensitive to employment prospects. Perhaps it would be better to say that this pattern has been "correlated with" employment prospects. Individual decisions are complex and may often be more closely related, as conscious decisions, to a general sense of life, excitement, and growth, than to cold economic foresight. A special feature of our model is that although the student's career choice is affected, directly or indirectly, by the current job picture, the student himself will not affect this picture until he or she enters the job market some three to five years later. The circumstances are similar to those which economists sometimes call the "corn-hog" cycle (where the market affects decisions about raising hogs and planting corn for feed and where these decisions in turn affect the market after certain intervals of time). In such circumstances, patterns of cyclical oscillation may evidently occur.

Various coefficients in the Hollomon model have been specified so that the model gives a remarkably good description of the job market in physics for the past fifteen years. Applying this specified model to the coming decade, one gets the following result: Provided that graduate programs make no major change in the quality of the persons they admit, the annual production of Physics Ph.D.'s in 1981 will be about one-third of what it is at present. It is tempting to argue that a similar decline can be expected in mathematics. So far, however, such a decline for mathematics has not become evident. For recent figures, see "Mathematical Science Faculty and Enrollments," by Professor J. W. Jewett of Oklahoma State University, in the *Notices of the American Mathematical Society* (1973, volume 20, pages 343–347).

Let us now turn to our major concern: mathematics departments and their programs. Let us try, as a first step, to formulate the proper goal for a mathematics department. Once this is done, other comments and judgments will follow.

I believe that the chief goal of a university mathematics department is to provide

a vigorous intellectual center for the mathematical life of the university. I include in this goal, of course, responsibility for the education of both mathematicians and non-mathematicians, but I foresee greater emphasis on the mathematics department as a center of intellectual life and less on the department as a mere producer of degrees.

Given this goal, what consequences now follow for the various programs and activities of a mathematics department? Let us look first at the graduate program. An active graduate program is essential to the goal that we have just described. Even if the chief educational task of a department is held to be, for example, undergraduate education, proper performance of this task will depend upon the richness of a department's intellectual life. The richness of this life will depend, to an important degree, on the quality of its graduate, and especially its doctoral, programs. At the same time, the economic picture suggests that there will be considerably less emphasis on the traditional strict mathematical Ph.D. A university ought to have this Ph.D. available if it has mathematicians on its faculty qualified to supervise it. I foresee, in every such department, some individuals taking this degree. It will be an unfortunate mistake, however, if the success or quality of a department's graduate program is measured by the number of straight mathematics Ph.D.'s that it produces. What other sorts of graduate program might a department have? In order to develop a vital graduate program, a department must keep in mind the desirable, possible, and likely forms of future employment for its students. Such awareness may lead, for example, to a greater emphasis on the master's degree. It may also lead to greater emphasis on joint degrees with other departments. As a part of mathematics graduate programs, one can also foresee a greater and more genuine emphasis on minor programs in mathematics-related areas that lie outside mathematics. The recently discussed Doctor of Arts degree is another possible form of graduate program, although I tend to think, in view of its controversial history, that it is a less likely development. In his article on employment, Professor Anderson wisely points out that, as university positions become less available, institutions at the secondary and junior-college levels may not welcome our straight mathematics Ph.D.'s into teaching jobs, and may indeed, in view of their own formal or "moral" tenure commitments, have few new positions available. One answer to this, as we have indicated, is to keep the nature and needs of prospective jobs well in mind as we plan our graduate programs and degrees. Such planning should be carried out with the major department goal in mind that we have stated above: not to be a producer of mathematics Ph.D.'s, nor even a producer of mathematical research (though this is certainly more vital), but, rather, to be a center of the university's mathematical life and, thereby, a major focus of the university's intellectual life.

What does our goal imply for the undergraduate program in a university mathematics department? It implies, first, a somewhat less austere interpretation of what mathematics is, a greater willingness to teach mathematics in a variety of more applied forms, and a recognition that intellectual depth does not necessarily require

formal rigor. Secondly, it implies (and I see this coming about in my own department) a greater flexibility in style and atmosphere, with the possibility that larger numbers of students may become involved in joint majors with other departments. This flexibility may include increased use of the research format in undergraduate teaching. Such a format can occur quite early in the undergraduate program. There have been several interesting experiments of this kind carried out recently. One such, at Northeastern University, has been a research seminar for freshman where the chief project of the term was the discovery and classification of the discrete symmetry (crystallographic) groups for the plane. (The "discovery" approach in graduate education has, of course, long been associated with the name of R. L. Moore.) Thirdly, our goal implies an increased concern on the part of the individual faculty member with questions of pedagogy — a concern with the effectiveness of a departmental program. This increased concern is in accord with the general spirit of our time and is, I believe, already becoming visible among younger faculty.

Our goal also implies participation by the department in a third area of university responsibility — an area that we might call "non-regular education." This area, currently in a state of considerable ferment, includes a wide spectrum of such matters as continuing education (professional refurbishment, alumni education, etc.), education within the university's local community, para-professional training (especially in new and developing areas — laser technology, for example), career-development training for university employees, and new forms of education within the community at large. "Open university" programs are an example of this last. (Professor W. T. Martin, who has helped to lead the DeKalb conference, has prepared an informative study of the activities of the British Open University.) In such new ventures of the university, I believe that the crucial responsibility of the mathematics department is not so much one of service and participation (although this has to occur) as it is one of maintaining a close interaction with the substantive content of these various new programs.

Fourthly, our goal has obvious implications for the relationship between a mathematics department and the other departments in a university. We must recognize first, as a standard feature of a healthy university, that non-mathematics departments will make a variety of unreasonable requests of the mathematics department. This will continue to be the case. At the same time, as a mathematics department comes increasingly to view itself as the center of the university's mathematical life, there will be an increasing sensitivity to the needs of other departments. For example (as we have already noted in connection with applied mathematics in the undergraduate mathematics program), one can foresee a much wider range of courses taught within the mathematics department. One important area of interaction with other departments is probability and statistics. Of course, the departmental responsibility for this area varies from university to university, but a close interaction with the mathematics department should exist, and there should be some teaching of it in the mathe-

matics department. Probability and statistics lead to a close interaction with such areas as business administration and the social sciences. A second area of major interaction is computer science, and this leads into the general area of communications science. Changing and growing relationships with other areas will, I believe, lead to significant changes within the mathematics department. To some extent, there will be changes in the mathematician's view of himself. To a more important extent, there will be a change in the style that he develops and the intellectual role that he plays with regard to his university colleagues. A mathematician might expect, for example, to have one or more areas of scientific or quasi-professional interest outside his own purely mathematical interests.

Let us turn next to questions of promotion and tenure in a mathematics department, questions made more difficult by the larger proportion of younger tenured faculty at the present time. What implications does our goal have for these questions? I believe that criteria for promotion and tenure must remain centered on intellectual accomplishment rather than on service. At the same time, we must seek other measures of intellectual competence besides, or in addition to, the customary list of publications and judgments of one's narrowly-construed group of intellectual peers. I also foresee (and I am not sure whether this is good or bad) a decreased concern with proportional representation, within a department, of the various areas of currently active and acceptable mathematics. "Have we got the right number of analysts?" "Do we need more coverage of complex manifolds?" Future decisions within a department may be made in ways more related, perhaps, to the intellectual style and general curiosity of the individual in question than to his or her particular branch of mathematics or to a choice between pure and applied mathematics. An appropriate interactive intellectual style and a curiosity about other areas besides one's own professional area will be important and desired qualities.

Finally, what does our stated goal imply about the relationship of a mathematics department to a department or school of education and to matters of education research and curriculum reform? It certainly implies an increased involvement of professional mathematicians in the exploration of educational questions and in curriculum research and development. (Northern Illinois University, which held the DeKalb conference, is especially fortunate in this regard. Some members of its mathematics department have a primary interest in mathematics education, but pursue this interest within the common administrative framework of the department.) To some extent, this implies a need for in-service involvement on the part of the mathematician. Certainly, for the sake of creative and constructive results, it implies an increased mutuality between the professional educator and the professional mathematician. It is a welcome fact, and part of the spirit of our time, I believe, that one can already discern, in many mathematics departments and on the part of both younger and older faculty, a degree of concern with questions of mathematics education that would not have been thought possible several years ago.

So much for the role of the university in mathematics education as we have narrowly construed it. Let us go on, briefly, to construe our subject more broadly. The essence of the broader subject is implicit in much of what we have already said. For example, we have, in effect, touched on questions of teacher training in our immediately preceding remarks about education, and in our earlier remarks on non-regular education. Several further, and rather obvious, comments can be made. I foresee an increased participation by the university in questions of education and in the general educational process. To this end, I foresee an increased measure of conscious planning on the part of mathematics departments and central university administrations. I foresee changed and improved teacher training. I also foresee improved quality of teachers and teaching at all levels. This improvement will be especially notable if mathematics departments, in order to meet the question raised by Professor Anderson about the suitability of mathematics doctoral graduates for secondary and junior college positions, think through their programs in terms of the ultimate positions that they expect their graduates to hold. Finally, as already indicated, we can expect to see a significantly greater variety and flexibility of intellectual and professional styles on the part of both students and faculty members, and to see increasing numbers of students and faculty participating simultaneously in both university mathematics and in mathematics education outside the university.

Our subject has been the future of the university in mathematics education. We have taken a rather immediate and short-range view of that future. I would like, however, to conclude this discussion of hoped-for change with a cautionary note about long-range prospects. I would urge that universities have a role in our society rather different from that with which they are often credited. Generally, they are viewed as institutions which generate and transmit knowledge. I believe that they have an essential intellectual dimension and a role in our society that go well beyond this. In our busy, and occasionally feverish twentieth century culture, the university is the chief, if not the only, communal focus of speculative and unpressured intellectual life. At other times in history there have been other such foci — in privileged social classes, in religious communities, in special professional enclaves. In our society, there are few, if any, such foci outside our universities. And there is a danger that our society and our universities may get caught up in a variety of immediate concerns (such as some of those implicit in our whole foregoing discussion) in such a way that the university will lose its special (and to my mind, primary) role as a home for speculative intellectual life. If this should occur, our culture will have lost something distinctive, precious, and probably irrecoverable for a long time. Our society and our universities thus face a challenge that is subtle and complex. As a center of intellectual life, the university has a responsibility for major creative interaction with other segments of society. It must meet this responsibility without losing its own essential qualities and character. In the case of mathematics, I believe that the cultural and economic forces of our time can help us to find a greater and healthier interaction

between the university and society, if we can provide guidance that is appropriately wise and sensitive.

An earlier version of this paper was presented as an invited address at the DeKalb Conference on Mathematics Education at Northern Illinois University, November 12–13, 1971.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139.

THE THIRD U.S.A. MATHEMATICAL OLYMPIAD

S. L. GREITZER

The U.S.A. Mathematical Olympiad, now in its third year, has become sufficiently well known, so that inquiries about it, and interest in having students participate in it this year were wide-spread. Participation is by invitation only. In 1974, invitations were sent to 155 students to take part in the Third U.S.A. Mathematical Olympiad. Most students were selected on the basis of their performance on the Annual High School Mathematics Examination. Students from Michigan and Wisconsin, which conduct their own state-wide mathematics contests, were also invited, and a few students were invited on the basis of strong recommendation from school officials. Of these 155 invitations, 149 complete acceptances were received. (An acceptance is considered ''complete'' when the student agrees to participate and the school agrees to administer the test.)

The Third U.S.A. Mathematical Olympiad took place in 119 schools over the nation on May 7, 1974. As before, it consisted of five essay-type problems requiring mathematical power and ingenuity to solve. The allotted time for doing the problems was three hours. The five problems appear below.

THIRD USA MATHEMATICAL OLYMPIAD — MAY 7, 1974

1. Let a, b, and c denote three distinct integers and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

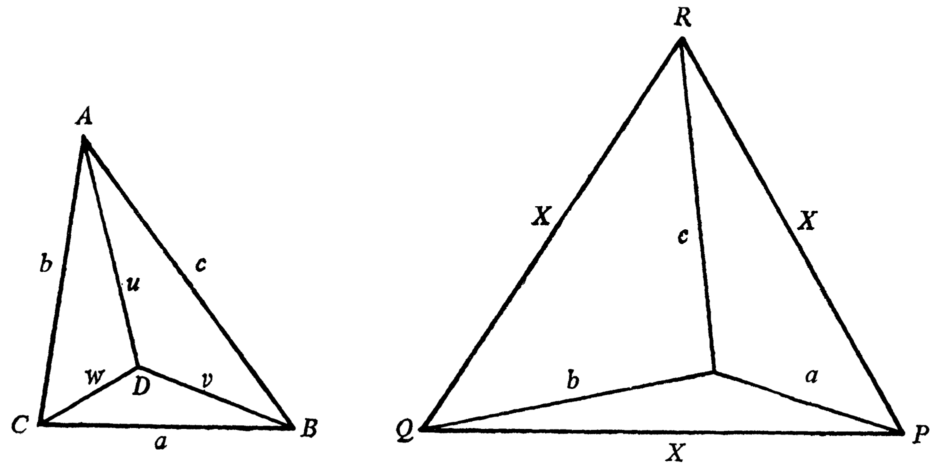
2. Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geqq (abc)^{(a+b+c)/3}$$

3. Two boundary points of a sphere of radius 1 are joined by an interior arc of less than 2. Prove that the arc must lie in some hemisphere of the given sphere.

4. A father, mother and son decide to hold a certain type of board game family tournament. The game is a two-person one with no ties. Since the father is the weakest player, he is given the choice of deciding the two players of the first game. The winner of any game is to play the person who did not play in that game and so on. The first player to win two games wins the tournament. If the son is the strongest player, it is intuitive that the father will maximize his probability of winning the tournament if he chooses to play the first game with his wife. Prove that this strategy is indeed optimal. It is assumed that any player's probability of winning an individual game from another player does not change throughout the tournament.

5. Consider the two triangles $\triangle ABC$ and $\triangle PQR$ shown below. In $\triangle ABC$, $\angle ADB = \angle BDC = \angle CDA = 120°$. Prove that $X = u + v + w$.



All test papers were returned by May 15, and graded on May 20 by a committee of the mathematics faculty at Rutgers. The top thirty papers were regraded at Memphis State University. The results for all participants are shown in Table I. The eight

TABLE I

Results in High School Mathematics Examination and Olympiad

H.S. Exam \ Olympiad	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90
411 – 150	1	1		3	3	4	3		
131 – 140	2	1	6	5	1	1	1	1	1
121 – 130		2	5	7	2	2			1
111 – 120	9	10	10	10	5	3	2		
101 – 110	5	14	9	4	5	1			

top scorers had scores indicated in the rectangle at the upper part of the table. Final results, listing the top twenty-five scorers, were sent to the 119 schools involved on May 25.

As was the case with the first two Olympiads, the correlation between scores on the Annual High School Mathematics Examination and the Third Olympiad is low -0.376; this is even lower than the 0.433 correlation for the second Olympiad. The eight finalists selected to receive awards had scores indicated in the rectangle at the upper part of the table, but even here there was quite a spread in scores. Generally, students who score high on the Olympiad score high on the High School Mathematics Examination, but the converse is not necessarily true.

In the hope that teachers will find it informative, a table of grades attained by participants on each problem is provided in Table II.

TABLE II

Score Per Problem

Score \ No.	1	2	3	4	5
21–25	1	5	4	0	3
16–20	16	22	11	33	15
11–15	2	8	5	33	10
6–10	1	20	12	30	49
1–5	6	58	86	16	35
0	120	33	28	34	34

A list of the eight students who scored highest is given below:

Paul Zeitz	Stuyvesant High School	New York, N.Y.
Stephen Modzelewski	Shady Side Academy	Pittsburgh, Pa.
Gerhard Arenstorf	Peabody Demonstration School	Nashville, Tenn.
Thomas Nisonger	Walt Whitman High School	Bethesda, Md.
Eric Lander	Stuyvesant High School	New York, N.Y.
David Barton	Berkeley High School	Berkeley, Cal.
George Gilbert	Washington Lee H. S.	Arlington, Va.
Paul Herdeg	Hamilton-Wenham H. S.	Hamilton, Mass.

The eight finalists were honored at ceremonial exercises arranged by Dr. Nura Turner. The exercise included a morning program on June 27 at the National Science Foundation, a visit to the Smithsonian Institution in the afternoon, an awards ceremony at the National Academy of Sciences, a reception in the John Quincy Adams Room of the Department of State, and dinner in the Thomas Jefferson Room. These ceremonies were made possible through the generosity of International Business Machines. Among the awards to the students were engraved silver trays and bonds, from IBM, and an HP-35 Calculator through the kindness of the Hewlett-Packard Company.

The Olympiad Committee gratefully acknowledges the help and support of the many individuals and organizations that made the Third Olympiad possible. The problems were prepared by a committee consisting of Murray Klamkin, C. C. Rousseau and P. A. Paige. The Annual High School Mathematics Examination Committee, consisting of R. Artino, A. Gaglione, and N. Shilkret, provided the data needed for selecting and inviting the participants. The papers were graded by Michael Aissen, John Bender, Richard Bumby, Harry Gonshor, Sol Leader, Ben Muckenhoupt, Barbara Osofsky and Hyman Zimmerberg, all on the staff of the Mathematics Department at Rutgers University. The top thirty papers were regraded by Cecil C. Rousseau at Memphis State University.

This year, the U.S.A. was invited to send a team of eight students to Erfurt, in the German Democratic Republic, to participate in the Sixteenth International Mathematical Olympiad. The invitation was accepted and our team was there on July 8 and 9. Thanks are due to the Spencer Foundation for the grant which made it possible for the team to travel to Europe and back.

The Fourth U.S.A. Mathematical Olympiad will take place on Tuesday, May 6, 1975.

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INFINITE FAMILIES OF NONTRIVIAL TRIVALENT GRAPHS WHICH ARE NOT TAIT COLORABLE

RUFUS ISAACS

1. Introduction. Among extant edge coloring problems, the 3-coloring of trivalent graphs is prominent because of its relation to the classic 4-color conjecture on maps. The story is splendidly told by Saaty in [6], but we shall follow a self-contained route for the unitiated by supplying the needed definitions and basic results in Part 2. Well-known theorems there are followed by terse, suggested proofs in parentheses.

To prove the classic 4-color conjecture, it suffices to solve these two problems (See Section 2.1, I and II):

(1) To find all uncolorable trivalent graphs or characterize them in some reasonably constructive way.

(2) To prove all such graphs are not planar.

Problem (2) seems tractable enough as indicated by Section 5.2. Thus (1) can be viewed as a version of the 4-color conjecture unencumbered by the topology of the plane. But aside from this application, (1) is a worthy mathematical problem in itself.

The earliest uncolorable 3-graph, the Petersen graph [1], dates from 1891. It is given and discussed here in Section 2.2. For it, V = number of vertices = 10 which is shown to be as small as possible in Section 5.3.

Between then until the present paper, to my knowledge, only three other nontrivial uncolorable 3-graphs have been found. ("Nontrivial" is an essential, but somewhat elusive, qualifier. Our usage of it is explained and defended in Section 5.1.) Their discoverers, dates, V and references are

Blanuša	1946	18	[2]
Descartes	1948	210	[3]
Szekeres	1973	50	[4]

and the graphs themselves will appear in Section 3.1.

This state of affairs merits two antithetical comments:

1. *It is credible.* Uncolorable 3-graphs are extremely rare in the class of all such. The search for the uncolorable I found a fascinating pastime for spare moments over many months. One who so indulges — and I recommend it as a pleasant diversion for any mathematician — will be vividly impressed with the maddening difficulty of finding a 3-graph he cannot color.

2. *It is incredible.* The main result of this paper is the discovery of an opulent infinite set of uncolorable 3-graphs (Part 3) of which the preceding three examples are members without any special conspicuity. I call it the BDS class after the three authors without whose work this class could not have come into being. In this light it is hard to believe that the three stood alone so long and with interludes of about 50 and 30 years.

I am certainly no authority on graph theory, but Professor W. T. Tutte certainly is. He informs me that the first two are the only uncolorable cases he knows since Petersen's (the third appeared subsequently). Everything I have read or asked elsewhere confirms this situation.

Part 4 supplies an infinite sequence of uncolorable 3-graphs, termed $\{J_k\}$, which I believe new and not in the BDS class. There is also in Part 4 one further uncolorable example, the double star graph, which belongs to a class Q. This class, although well-defined, offers only this one new instance at present.

To assess the plenitude of uncolorable 3-graphs, below is their quantity, as derived from the ideas herein, for each V = number of vertices = necessarily even:

$V < 10$	None	(Sect. 5.3)
$V = 10$	1 (Petersen's)	(Sect. 5.3)
$V \geqq 18$	More than 1	(Sect. 3.2).

I have not attempted accurate counts in the last cases. Whether uncolorable 3-graphs exist (aside from trivial cases as later defined) for $V = 12, 14, 16$ is still open.

2. GENESIS

2.1 Basics. A *cubic*, *trivalent*, or 3-*graph* G is a connected, finite graph with exactly three edges meeting at each vertex. We also require G to satisfy some further conditions which will be stated and discussed in Section 2.4.

A *Tait coloring* or 3-*coloring* of G consists of assigning one of three colors to each edge so that the three edges meeting at a vertex bear distinct colors. In this paper, *graph* or G will mean a 3-graph and coloring (as well as its grammatical variations) a 3-coloring except when otherwise noted; the coloring of a map, for example, will refer to the hues of its countries.

A map, in the classical 4-color conjecture, may, as is well known, be taken so that but three countries meet at each juncture, implying that its edges form a 3-graph. The well-known relation between the two coloring problems we break into two statements.

I. *If a map on a surface of any genus is* 4-*colorable, then its edge graph G is* 3-*colorable.*

(Let A, B, C, D be the colors of the countries. Color an edge of G 1 if the adjoining countries are A, B or C, D; 2, if they are A, C or B, D; 3, if they are A, D or B, C.)

The converse is not true. Thus, Heffter's well-known map on the torus, which requires seven colors, has an edge graph which is 3-colorable.

II. *A planar map is* 4-*colorable if its edge graph is* 3-*colorable.*

(We use Lemma 2.4.2 and the definitions immediately preceding it. If the Tait set has but one cycle, planarity requires it to have an inside and an outside. The countries inside may be colored alternately A and B; those outside, C and D. When there are more Tait cycles, the plane will be divided into more regions. We bicolor the innermost ones first and, if this is done with suitable choices between the two color pairs, we can work outwardly to complete the coloring.)

We shall persevere in denoting the edge colors by 1, 2 and 3. These symbols shall be used in the spirit that a permutation of them is immaterial. Thus, should we write "(1, 1, 2)", what we really mean is "(x, x, y), where x and y are any two distinct colors."

Let V or $V(G)$ denote the number of vertices of G; E or $E(G)$, its edges. Then there is a positive integer λ such that

$$V = 2\lambda, \quad E = 3\lambda \tag{2.1}$$

and if G is colorable, there are λ edges of each color.

(The simple proof is like that of the well-known Euler relation entailing $V - E + F$ (F = number of faces).)

When we speak of the size of a graph (large, small, etc.) we should refer to the magnitude of λ. But we shall follow custom and use the even V instead.

2.2 The Petersen Graph. This primary instance of an uncolorable graph — hereafter denoted by P — is worthy of scrutiny at the outset of our search for more. Figure 2.2 offers three depictions of P. At (a) is the way everyone seems to draw it, but I prefer (b) which I find more wieldy for coloring experiments. At (c) we see P drawn on a torus where it forms the edges of a five-country map requiring five colors. It is not the smallest such map; there is one with V only 8, but its edges are a 3-colorable graph.

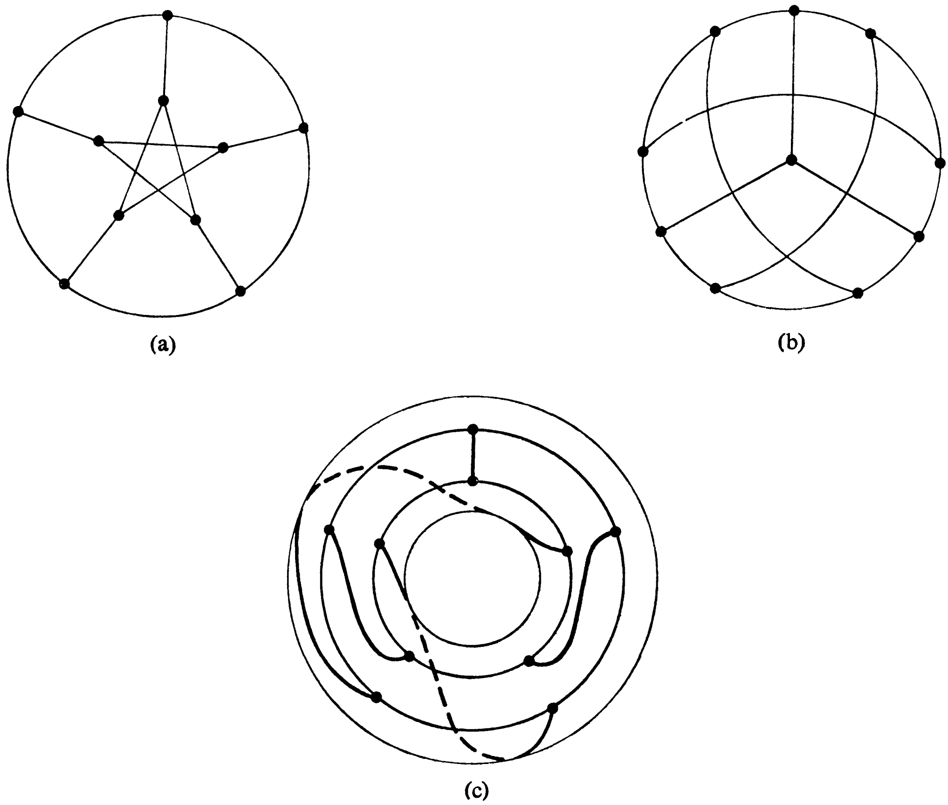


FIG. 2.2

There are several simple ways of showing P to be uncolorable. A formal proof is the case of $k = 3$ of Theorem 4.1.1.

There are further striking advents of P. Whether or not they bear on coloring problems is a tantalizing question.

(Offered by the referee): We obtain P from the graph of edges and vertices of a regular dodecahedron by identifying opposite points.

(Biggs in [8]): The usual $\binom{5}{2} = 10$ pairs from 5 objects can be the vertices of P if an edge between two means "the two pairs are disjoint."

(My observation): The well-known Heawood Graph (see [7], page 61), is a 14-gon with two vertices — i and j under consecutive numeration — also connected when $i - j \equiv 5 \bmod 14$. If any vertex and its three incident edges are removed, P results.

All edges (and also vertices) of P are alike; more precisely, there is an automorphism of P which takes any edge into any other. Such follows from the preceding models; for example, rotations of the dodecahedron.

2.3 Zones. Let G consist of two connected subgraphs, called *zone bounds* or *bounds* and exactly M other edges $Z_1, \cdots, Z_M$ whose end vertices lie one in each bound. If $M \geqq 2$, then for either zone bound A we require that

$$V(A) > M - 2. \tag{2.3}$$

Then the set of Z_i is called a *zone* (or *M-zone*).

Symbols for the latter are Z (or ZM) which may also be used in an adjective or property sense of G to mean "G has a Z (or ZM)."

A zone differs from the familiar cut-set of general graph theory only through the requirement (2.3). "Zone" probably is due to Miss Descartes, although we have extended her definition in [3]. Her term probably has precedence over cut-set.

Now $M - 2$ is the minimal V the bound of an M-zone can have, as Lemma 2.5 will show. For $M = 2$, (2.3) means A is not vacuous; that is, Z_1 and Z_2 cannot be the same edge which enters and then leaves A.

Without (2.3) virtually every G would be ZM for all (sufficiently small) $M > 1$. For example, any three edges incident to a common vertex would comprise a Z3; any edge plus its end vertices would be the zone bound of a Z4.

The *zonality* of a graph G is defined and annotated by

$$\text{Zon}(G) = \min\{M: G \text{ is Z}M\}.$$

2.4 Further graph requirements. Four conditions — indicated collectively by GC — which we shall generally take as part of the definition of a 3-graph, begin with

$$\text{NoZ}M \ (M = 1, 2, 3).$$

They mean: G has no Z1, Z2 or Z3. Current papers require NoZ1, "isthmus" being the usual term for a Z1.

Note that NoZM implies that G has no loops, digons, or triangles (that is, closed circuits with 1, 2 or 3 edges) for $M = 1$, 2 or 3 respectively. The literature does not always require NoZ2 and NoZ3, but the much weaker bans on the latter two configurations are often adopted. We return to this matter in Section 5.1.

The general motive for the GC is to avoid trivially uncolorable cases. How the NoZM do this will emerge from Lemma 2.4.4 with amplification in Section 5.1.

In Part 2 we shall allow some violations of the GC in the early stages of certain constructions, but not in the final resulting graphs. For convenience such violating configurations will be referred to as graphs.

Other grounds for sometimes permitting violations is that they are often easily rectified. For example, the reader can readily see for himself how simple it is to purge digons and triangles in coloration problems.

Squares (closed circuits with 4 edges) can also be purged, as the lemma to follow will show. Thus their presence suggests triviality and hence the fourth GC:

NoSq

which means: G has no squares.

But this GC can be taken as optional. It will play no part in our analyses; the reader who wishes to abide uncolorable graphs with squares is free to do so.*

LEMMA 2.4.1. *If G contains a square, S_1 and S_2 be the graphs obtained by deleting each of its two pairs of opposite sides. Then G is colorable if and only if one of S_1 and S_2 is.*

The reader can prove this more easily by exploring possible coloring cases than I can with text.

Note the rectification done. More may be needed on the S_1 and S_2 generated by the first.

A *Tait cycle* of G is a set of an even number of edges of G and their incident vertices which constitute a simple, closed curve. A *Tait set* is a disjoint set of such cycles which contain all the vertices of G.

Very well known is

LEMMA 2.4.2. *A graph G is colorable if and only if it has a Tait set.*

(If G is colored, starting from a vertex v, form the path lying on edges alternately colored 1 and 2. As G is finite there must be a first recurrence of a path vertex. This can happen only at v. If there are vertices not on the Tait cycle just constructed, start again, etc.

Conversely, if G has a Tait set, color its edges alternately 1 and 2, and the others 3.)

A consequence is: *A Z1 graph is uncolorable.*

This motivates (in part (see Sect. 5.1)) NoZ1.

A Tait set with but one Tait cycle is a Hamilton cycle. Thus when G has a Hamilton cycle it is colorable. The converse — although experience suggests that it holds nearly always — is not true; there are many colorable 3-graphs known which have no Hamilton cycles.

* But the lemma will be used in Part 5 when we are concerned with minimal size graphs.

The following basic lemma appears in both [**2**] and [**3**]. It will be used often in the sequel.

LEMMA 2.4.3. *In a ZM of a colored graph G, let n_j be the number of Z_i of color j. Then*

$$n_1 \equiv n_2 \equiv n_3 \equiv M \qquad (\text{mod } 2).$$

Likewise for cut-sets.

Proof. Those Z_i which are colored 1 and 2 must belong to a Tait set which must cross from one zone bound to the other an even number of times. Thus $n_1 + n_2$ is even. This proves the first congruence and also the second. As to the third:

$$n_1 \equiv 3n_1 \equiv \sum_j n_j = M \qquad \text{mod } 2.$$

Likewise for cut sets as (2.3) is not invoked.

An edge lacking one end vertex will be called a *pendant*. Thus, if we sever an edge of G, we create two pendants; if we remove a vertex, we create three. Each resulting figure is an instance of a *graph with* [M] *pendants* or a G_p [G_pM]. A G_p of course is not a graph. However, if we take one or several G_p and weld each pendant to another, graphhood is usually restored.

These terms are merely a handy locution. If G is Z and we sever the Z_i, G splits into two G_p which do not differ instrumentally from the zone bounds of Z. In fact, when lecture hall and blackboard is the medium, pendants are dispensable.

Every G_p is assumed to satisfy the GC in the sense that it is such a zone bound of some G; likewise it is connected.

Let G be Z2. Sever Z_1 and Z_2 and weld together the two pendants from each zone bound. We then have, after rectifications if needed, two graphs, G_1 and G_2. For a G which is Z3, sever all three Z_i and weld the pendants from each zone bound to a new vertex. Again call the two resulting graphs G_1 and G_2.

LEMMA 2.4.4. *A G which is Z2 or Z3 is colorable if and only if G_1 and G_2 both are.*

Proof. Let G be colored. If it is Z2, from Lemma 2.4.3, Z_1 and Z_2 will be colored alike and the coloration of G carries over into G_1 and G_2. If G is Z3, the lemma tells us that the three Z_i will be colored 1, 2 and 3. Again the coloring of G persists in G_1 and G_2.

Now let G_1 and G_2, arising from a Z2, be colored. If necessary, permute the colors in one G_i so that the two welded edges of G_1 and G_2 match. The colorings now serve for G. The reasoning for the Z3 case is similar.

Hence some motivation for NoZ2 and NoZ3! If we find an uncolorable G, which is Z2 or Z3, we can perform on it the preceding dissection and know that one of G_1 and G_2 is uncolorable. By discarding the other one, we rectify the GC violation.

More forceful is the converse. Suppose we find an uncolorable graph U. By severing an edge and welding the two arising pendants to an arbitrary A, which is a G_p 2, we obtain an uncolorable graph denoted by $U2A$. Similarly $U3A$ arises from removal of a vertex from U and welding the three pendants to A, now a G_p3.

As A is arbitrary either operation yields an infinity of uncolorable graphs. It seems natural to condemn all but U as being trivial; see Section 5.1.

2.5 Minimal G_p. Such are G_pM of least V.

LEMMA 2.5. *If A is a G_pM with $M \geqq 2$, the minimal possible $V(A)$ is $M - 2$.*

Proof. Let $f(M)$ be the sought minimum. For $M = 2$, we accept A's being the vacuous graph and so $f(2) = 0$. For $M > 2$, this — two pendants thought of as lying on one edge with no end vertices — cannot occur, or A would be not connected.

From $M = 3$, clearly $f(M) = 1$. But for $M > 3$ we cannot have three pendants from the same vertex or again A would not be connected.

For $M > 3$, one of these cases must arise: (1) Each pendant is incident to a distinct vertex; (2) A pair of pendants meet a common vertex v. For (1), $V(A) \geqq M$ and $= M$ when A is an M-gon. For case (2), letting A be minimal, we proceed inductively. The remaining edge from v and the $M - 2$ pendants other than the pair meeting v can be considered a fresh set of $M - 1$ pendants from a G_p, which must be minimal if A is and therefore has $f(M - 1)$ vertices. Then

$$V(A) = 1 + f(M - 1) = 1 + ((M - 1) - 2) = M - 2.$$

As such is less than the M ensuing from case (1), the lemma is proved.

3. THE BDS CLASS OF UNCOLORABLE GRAPHS

3.1. The class. The letters in the title stand for Blanuša, Descartes and Szekeres* whose graphs belong to this class and who inspired its construction.

We shall use U and W to denote uncolorable graphs and A an arbitrary pendant graph which may or may not be colorable. Such will be used as components in constructing a final graph G. Pendants will be bestowed on the U and W component graphs and pairs of pendants will be welded together. There will be various *types* of pendant bestowal which will imply rules for pendant colors should G be colorable.

We have already seen two such constructions: $U2A$ and $U3A$. The types used here will be symbolized by (e) and (v). Two other types will suffice for us:

(e, e). Sever *any* two edges of a component U which meet no common vertex. From each we obtain a pair of pendants. Then each pair must bear the same two unlike colors, for any coloring of G.

Proof. The four pendants lie on a $Z4$ of G and we can apply Lemma 2.4.3.

* If, through ignorance, I have omitted other prior discoverers of uncolorable graphs, I beg any writer who adopts my nomenclature to include their initials in the title.

If one pair had matching colors, from the lemma, so must the other. But then the unsevered U would be colorable. Thus we may suppose one pair is colored 1, 2. Again from the lemma, so must the other.

(vev) (or (v_1ev_2)). Here v_1 and v_2 are the end vertices of *any* edge e of a component W of G. Sever the other two edges meeting v_1 and obtain a pendant pair. Obtain another likewise from v_2, discarding e, v_1 and v_2. Then each pair must bear matching colors.

Proof. As before we assume a colored G with a Z4 and may apply Lemma 2.4.3. It tells us that if one pair were colored 1, 2 so would be the other. But then, by letting e be colored 3, the unsevered W would be colored.

We now define our basic operation which we call a dot product. Let U and W be any two graphs. Let four pendants emerge from U of type (e, e) and four from W of type (vev). By $U \cdot W$ we mean a graph obtained by welding the pendants of U to those of W in *any* way as long as pairs — in the sense of definitions of (e, e) and (vev) — weld to pairs.

Figure 3.1(a) diagrams $U \cdot W$.

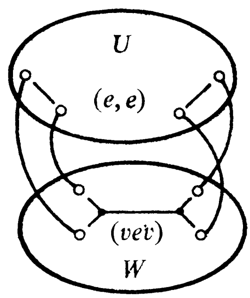


FIG. 3.1 (a)

Observe that $U \cdot W$ is a set of graphs, for in the preceding text there are three usages of "any." They can denote choices, hence different $U \cdot W$. (Not always: if $W = P$, all (vev) are alike if we recall the final paragraph of Section 2.2. However, we shall soon have graphs with A components and the choice will be rich.)

THEOREM 3.1. *If U and W are uncolorable, so is $U \cdot W$.*

Proof. Were $U \cdot W$ colorable, it could be G in the proofs of the (e, e) and (vev) color rules and these rules would hold. But they are incompatible.

We are now in a position to place one of the extant examples.

Blanuša's graph is $P \cdot P$.

(Rather one of the $P \cdot P$. But which one does not seem very consequential.)

There are variations on the dot theme. First, we can inductively compound the operation. Infinite sequences of uncolorable graphs can be built in steps: any uncolorable graphs from earlier steps can be used for U or W in the next.

Szekeres' graph, with $V = 50$, is of this kind. It is

$$P \cdot (P \cdot (P \cdot (P \cdot (P \cdot P))))$$

where, in terms of Figure 2.2(a), he takes the five e in (vev) as the five radial edges of the rightmost P in the formula and the (e, e) of the other P are a side of the star and the opposite arc of the circle.

A second variation arises from letting U violate NoZ2 or NoZ3. If the two e of the (e, e) are chosen on opposite zone bounds of the Z2 or Z3 of U, then the GC of $U \cdot W$ will hold.

We could form, for example, $U2A \cdot W$. Another example is diagrammed in Figure 3.1(b). It is

$$A_1 2U3A_2 \cdot W$$

and is, I hope, self-explanatory.

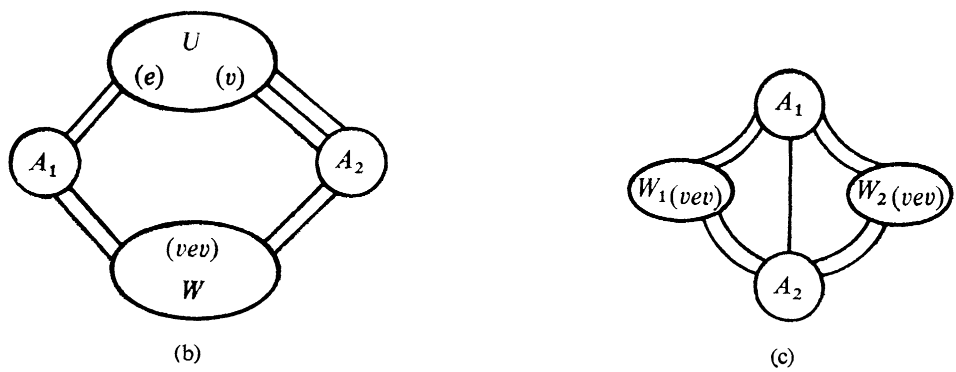


FIG. 3.1

Even a NoZ1 is permissible. Were we to form $A_1 1A_2 \cdot W$, it would be Z3 and so unacceptable. But we can restore the GC by a second dot operation, obtaining

$$(A_1 1A_2 \cdot W_1) \cdot W_2$$

which is depicted in Figure 3.1(c).

Finally, a variation leading to large uncolorable graphs. We first form the graph(s)

$$G_2 = A_3 3U_2 3A_4 \cdot (A_1 3U_1 3A_2 \cdot W).$$

(The operational 3's could be replaced by 2's and so throughout) so that the two edges appearing in the two (vev) of W have a common end vertex v_0. The reader who carries out the construction will see that, after the first dot operation, the remaining edge from v_0 has become an edge WA_2, but it can be used in the second (vev) nontheless. The G_2 he will obtain appears in Figure 3.1(d).

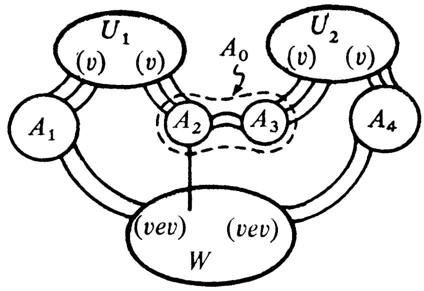


FIG. 3.1 (d)

We now ask: Can A_2 and A_3, connected by two edges, be replaced by a single arbitrary graph A_0, as suggested in the figure? They can. We only need realize that the sole role of the A_i in these constructions is as a vehicle for Lemma 2.4.3. If it holds for A_0, then the two edges A_2A_3 (which is not a Z2 of G_2) can be so colored that it holds for each of A_2 and A_3, while the converse is clear.

We so blend A_2 and A_3 into A_0 and then construct

$$G_3 = A_5 3 U_3 3 A_6 \cdot G_2$$

using for the e of the (vev) in G_2 the third edge incident to v_0, which has by now become the central vertical line in the Figure 3.1(d). We again blend two A_i into one, which now has nine edges emerging from it, three going to each U_j.

We continue this procedure until the edges of W are exhausted. We reach $G_{E(W)}$, from whose diagram W has disappeared. But our final graph is isomorphic to W in the following sense. The (final) A_i correspond to vertices of W and the U_j, to the edges. When a vertex and edge of W are incident, their isomorphs are connected by triples of edges which are of the (v) type in each U_j.

Note that "triples" could as well be "pairs" (of the (e) type at the U_j) which gives a second isomorph of W.

Blanche Descartes' graph is of this (first) isomorphic type with the U_j and W all equal to P and the A_i all "nonagons."

Her derivation (or proof) is quite different from ours. It is not based on repetitions of one operation but proceeds directly (and more simply) to the final graph. She employs reasoning somewhat akin to our approach to Theorem 3.1. We leave to the reader the pleasure of reconstructing her elegant proof; the adaptation to the second kind of isomorph is likewise rewarding.

If our derivation is longer it is because our objective is to unify. The Descartes and dot techniques at first looked very disparate, but unity is obtained by the latter. To recapitulate, it is:

The BDS class consists of results of repeated dot products of given uncolorable graphs. The second operands may violate the NoZM ($M = 1, 2, 3$) *provided that connections can be made which restore ultimate obedience to the GC.*

3.2 Graph sizes in the BDS class. We inquire as to graphs of what V appear in our class. We can see that $V(P \cdot P) = 18$. This is as small as possible, for Theorem 5.3 will show P to be the smallest uncolorable graph.

If the A_i appearing are allowed to be completely arbitrary (see Sections 3.3 and 5.1) we can use this lemma from which we omit a proof:

LEMMA 3.2. *For a fixed $M \geqq 4$, there exists G_pM with V any even number $\geqq M - 2$, except when NoSq holds, $M = 4$ and $V = 4$ or 6.*

Using the preceding examples we find that there are BDS graphs with V = any even number $\geqq 18$.

For certain V they will be especially numerous due to variegated dot products, using, say, the J_k of Part 4 as U or W components, etc. But we have not attempted any precise counting.

3.3 Zonalities in the BDS Class. Clearly

$$\text{Zon}(U \cdot W) = 4 \tag{3.3}$$

for the four edges UW are a Z4 and U and W fulfill the GC, banning lesser zones. The same zonality might appear to hold for any BDS graph G: if G is constructed from repeated dot operations, we can regard (3.3) as valid for the last. But our conclusion is untrue.

We return to G_2, depicted in Figure 3.1(d). Before A_2 and A_3 were blended, Zon $(G) = 4$ as the two edge pairs WA_4 and A_2A_3 are clearly a Z4. But if A_2 and A_3 are blended, the choice of into what A_0 determines the zonality. We see that Zon $(G_2) = 4$ can require that, by severing two edges of A_0, it splits into two parts of which one meets the three edges A_0U_2 and these only. But there are choices of A_0 that require more than two severings to attain this split. If so, Zon $(G_2) = 5$ (the edges WA_1, WA_0, WA_4 are a Z5) provided that the zonalities of U_1, U_2, and W_2 each $\geqq 4$ (such seems true, for example, if all three are P).

But it is also possible to choose A_0 so that one severing will suffice: the zonality is then 3.

Were the zonality of all BDS 4, it would seem to be useful towards characterizing this class. For example, we would know the uncolorable graphs of Part 4 are a distinct set (see Section 4.3).

Despite the preceding arguments, there does seem something intrinsic about zonality 4 for the BDS class. Perhaps in the future of this theory, something like this will be done:

The BDS graphs are divided into equivalence (or something like) subclasses so that all with the same diagram — in the sense of Figure 3.1 — belong to the same subclass. In each there are canonical members determined by some canonical choice of the A_i. A new zonality of G could be defined as the old zonality of a canonical representative of G's class.

This seems a hopeful possibility for canonical A_i: A_i is a polygon such that all edges leading to the same other component of G emanate from consecutive vertices of A_i. Then two severings of A_i would lead to splits suitable for zonal divisions.

4. AN INFINITE SEQUENCE OF UNCOLORABLE GRAPHS, THE Q CLASS AND THE DOUBLE STAR

4.1 The sequence. These graphs will be denoted by J_k for k an odd integer $\geqq 3$. The first three, which are typical, appear in Figure 4.1(a).

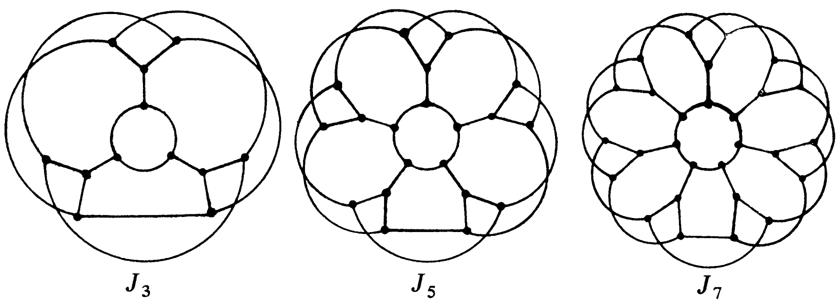


FIG. 4.1 (a)

We see that J_3 is the Petersen graph after rectifying the violation of NoZ3 by replacing the central circle by a single vertex. (Notationally, if T is the triangle, $J_3 = P3T$.)

THEOREM 4.1.1. *The J_k are uncolorable.*

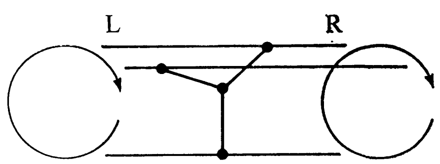


FIG. 4.1 (b)

Proof. Let Y be the G_p shown by Figure 4.1(b) which has three pendants extending to the left and three to the right. If Y is colored, how will the right pendants respond to assigned colors on the left? There are three possibilities for this assignment:

(1) $(1, 1, 1)$ (or all colors on the left alike). This is impossible as one of the three central edges must be colored 1.

(2) $(1, 1, 2)$ (or just two colors alike). The reader who explores the simple possibilities will find there are two. Both yield for colors on the right $(2, 3, 3)$ but in two different orders.

Had we started with $(2,3,3)$ on the left we similarly will conclude with $(1,2,2)$, in some order, on the right. Thus, if we form a chain of replicas of Y by welding the right pendants of one to the left of the next, the two kinds of colorings will alternate. We conclude that a closed circuit so made from an odd number of Y cannot be colored in this fashion.

(3) $(1,2,3)$ (no colors alike). Again we beg the reader to explore. There again result two possibilities. Both are $(1,2,3)$, but in both cyclic order, in the sense of the arrows in Figure 4.1(b), is preserved. So it will be if we weld together two Y when both remain as drawn. But if we were to make one pair of the newly welded edges cross, the cyclic ordering would be reversed. Therefore, if we build a closed circuit of Y with an odd number of such crossovers, it could not be colored in this fashion.

Hence a circuit with an odd number k of Y and an odd number of crossovers cannot be colored at all. But such is J_k — in fact, should be taken as its formal definition.

REMARK. Graphs can be deceptive graphically. Thus, when making the three drawings of Figure 4.1(a), my intent was to depict just one crossover in each on its double outer rim. This was to occur at the bottom! Superficially such seems exactly contrary to appearance. I leave this mild paradox to the reader.

Or we can simply say that an odd number of crossovers is to mean that the two outer rims consist of a single circuit.

4.2 The possible Q class and the double star. The class Q of uncolorable graphs is called possible, because as yet I know of only three members. Of these only one is new; I call it the double star graph and it is depicted in Figure 4.2.

Of a G_p5, from Lemma 2.4.3, the pendants must be colored

$$(1) \qquad 1,1,1,2,3$$

in some order. Now suppose the G_p5 has rotational symmetry, so that, from a suitable color ordering of (1), a cyclic permutation will give another. There are only two possibilities, representable either by $C = (1,1,1,2,3)$ or $S = (1,1,2,1,3)$. In other words, either the three matching pendants are consecutive (C) or they are separated in the only way possible (S).

Let H_S $[H_C]$ be the set of all rotationally symmetric G_p5 which are *not* colorable when the colors of the pendants are $S[C]$. We define: $H = H_S \cup H_C$.

The class Q is the set of G arising in

THEOREM 4.2.1. *From each pair H_1, H_2 (distinct or not) in H we can construct an uncolorable G.*

Proof. If $H_1 \in H_S$, $H_2 \in H_C$, we form G by welding the pendants of H_1 to those of H_2 so as to preserve their cyclic order. The new edges must be colored according to C or S; either way contradicts the definition of H_1 or of H_2.

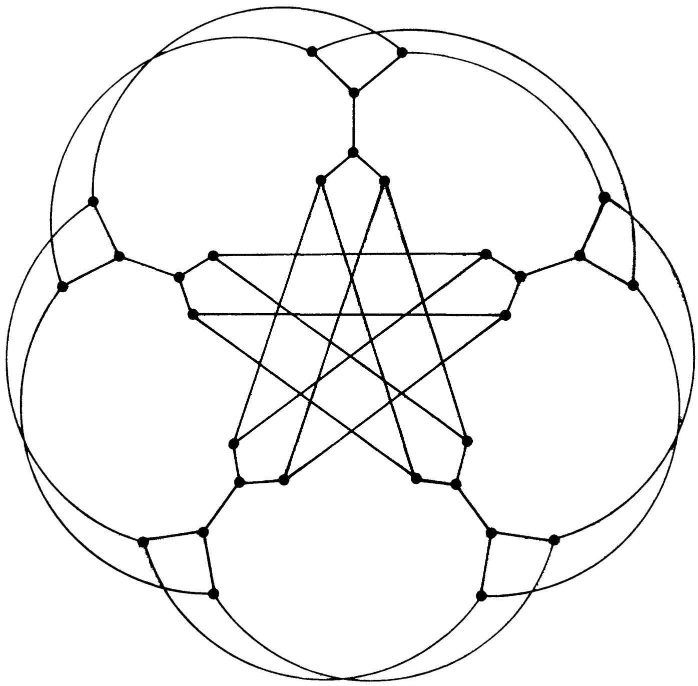


FIG. 4.2

If H_1, H_2 both $\in H_S$ (or H_C), we now weld their pendants so that adjacent ones on H_1 attach to alternate ones on H_2. Then an S (or C) coloring of the pendants of H_1 transfers into a C (or S) coloring on H_2, so that a total coloring is impossible.

$$\text{Notation:} \quad G = \langle H_1, H_2 \rangle.$$

In Figure 4.1(a) sever the five radial edges of J_5, cutting it into two rotationally symmetric G_p5. The inner one, a pentagon, we call V, the outer, we call V^*. Now V can be colored with C holding for the pendants by coloring the sides of pentagon 3, 2, 3, 2, 1. Then V^* cannot be colored with C holding or, by rewelding our severings, we would have colored J_5. Therefore $V^* \in H_C \subset H$.

Observe that P is two copies of V with adjacent pendants of one welded to alternate ones of the other. Were V also colorable with S holding, then P would be colorable. Therefore $V \in H_S \subset H$.

We have established

$$P = \langle V, V \rangle, \quad J_5 = \langle V, V^* \rangle.$$

The third known member of Q is

$$\langle V^*, V^* \rangle$$

which is the double star. Thus we have proved

THEOREM 4.2.2. *The double star graph is uncolorable.*

4.3 Nonmembership in the BDS class. The preceding graphs certainly do not appear to belong to the BDS class, although I do not claim a rigorous proof. The evidence is:

(1) We have seen in Section 3.3 that there is an aura of zonality 4 over BDS graphs, while for Part 4 we appear to have

$$\mathrm{Zon}(Q) = 5$$

$$\mathrm{Zon}(J_k) = 6 \qquad (k > 5).$$

(2) In the BDS class arbitrary A can be inserted into any related pair or triple of edges connecting a U or W. The simplest such nonvoid A is a "rung," that is, a new edge bridging two new vertices inserted in each of two edges (with triangles avoided).

Such a rung cannot be inserted in P without rendering it colorable, as can be learned from exploring all cases. From some sample trials on the J_k, the same appears likely true, but I know of no general proof.

(3) The BDS spring from at least two initial uncolorable graphs. But the J_k and double star do not appear to contain anything of such ilk.

We might note that BDS graphs might be built using a J_k or double star for the various U and W, but U and W themselves are not in BDS.

Also note that we have constructed both BDS graphs as well as the J_k and Q by welding together components, but the underlying ideas seem very different.

5. SOME GENERAL IDEAS

5.1 What means trivial? We have claimed nontriviality in the main title and feel a defense is obligatory. As to the question in the section title, this seems the underlying principle:

If we have found one evasive solution to a problem and others arise from it in an obvious way, we tend to call these others trivial. They can then be banned by adding postulates designed for this purpose.

Thus the answer reduces to that of another question:

What means obvious?

It is, of course, a relative term, depending on our sophistication and that of our mathematical era.

But let us turn to levels of what might be deemed trivia in our current subject.

(1) Graphs with an isthmus or Z1 seem universally banned. There are two reasons. First, if such a graph is planar it cannot be the boundary edges of a map. Second, although there surely are infinitely many $A1A$ graphs, all uncolorable, they are about as obviously so as a 3-graph can be.

(2) We have seen that from one U we can form infinitely many $U2A$ and $U3A$, all uncolorable. Although the literature I have found on Tait coloring is sparse, I think it astonishing that I have not encountered a specific ban on such graphs, aside from the weak special cases of digons and triangles. Nevertheless, I assume the ban is there tacitly. It is hard to imagine anyone seeking uncolorable 3-graphs without becoming aware of the $U2A$ and $U3A$ possibilities. Yet I have read the statement that very few uncolorable G are known. This must be so from either ignorance or a tacit ban; I take the latter as far more likely. Yet "How obvious?" is here a puzzling question.

Note that all this occurs in map coloring too, where the literature is much vaster, although I have read but a bit of it. For example, if U is the edge graph of a map on a torus 7-or-more-colorable and A any planar G_p, then $U2A$ and $U3A$ are edge graphs of other such maps.

(3) If arbitrary A components, as in (2), are to be banned, as we did in the GC, should they not be banned in the BDS class also? Hardly, for then such desirable specimens as Miss Descartes' isomorphs would have to go. We could check a trivial infinitude by only admitting A of some canonical form. But what form? We could, as in Section 3.3, select polygonal A so as to preserve zonality. Minimal G_p is another appealing possibility (but there is more than one minimal G_pM for a given $M \geqq 4$).

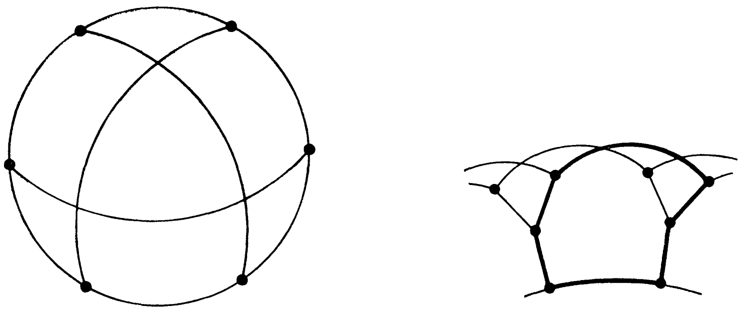
Probably some reckoning by subclasses, as suggested in Section 3.3, is the future course.

The principle given earlier could be pushed further. The BDS class springs from applications of the dot process. We could become so sophisticated as to call this an "obvious way" and hence exile the entire BDS class.

But such exiling, if pushed far enough, would annihilate all mathematics.

5.2 Planarity. We know that "Every uncolorable 3-graph is not planar" implies the classical 4-color conjecture. Let us accordingly look then at the ascertainment of planarity of 3-graphs.

The well-known theorem of Kuratowski (see, for example, [5]) states that a general graph is not planar if it contains one of two particular graphs. Now one of these two is not trivalent, so we need only be concerned with the other. It is shown in Figure 5.2(a) and is called the *utilities graph* — UG for short. By *G possesses UG* is meant that a graph like UG, except that its edges may be replaced by paths, is a subgraph of G. In other words, if we are allowed to alter Figure 5.2(a) by putting new dots on the interior of its arcs, it can become a depiction of a subgraph of G.



(a) (b)

FIG. 5.2

Thus our tool here is

LEMMA 5.2. *G possesses UG if and only if G is not planar.*

THEOREM 5.2.1. *The Petersen graph is not planar.*

Proof. A glance at Figures 2.2(b) and 5.2(a).

THEOREM 5.2.2. *Let G be in the BDS class. Then G is not planar if one of its original U or W components is not.*

Proof. Let U be such a component so that U possesses a UG. The UG will survive the construction of G. Suppose a pair of connecting edges from U to another component C (a U, W or A) if G were of type (e) at U and that the severed edge e of (e) belonged to the UG. We can replace e by a path in G by starting from one end vertex of e, following the connecting edge to C, then, as C is connected, following a path in it to the other connecting edge and back to the other terminal of e. Other types of connecting edges are handled similarly.

THEOREM 5.2.3. *The J_k are not planar.*

Proof. For the outer circle of the UG we can use the hexagon indicated by the heavy lines in Figure 5.2(b). The remaining paths of the UG are not hard to discover.

I have deliberately refrained from thinking about the planarity of the double star graph so that I can bequeath a lottery ticket to the reader, albeit at long odds. Let him investigate the question. If the answer is yes, he will have the glory of having resolved the 4-color conjecture.

5.3. The Petersen graph is the sole smallest uncolorable graph. The *girth* $\gamma(G)$ of G is the smallest n such that G contains an n-gon. Tutte, in [7] Chapter 8, proves that the smallest graph of girth 5 can only be P. As his work is embedded in a framework of broader results, we give a simplified adaptation here.

If $\gamma(G) = 2$, 3 or 4, G would contain a digon, triangle or square, and so, as shown earlier, could not be smallest uncolorable graph. Thus we assume $\gamma(G) \geqq 5$.

Let $\gamma(G) = n$ so that G contains an n-gon H. There is a third edge (not in H) e_i emanating from each vertex v_i of H. Clearly no e_i can join two v_j. Nor can any e_i and e_j terminate at the same vertex (not in H) or e_i, e_j plus the smaller "arc" of H joining v_i, v_j, would lie on an m-gon with $m \leqq 2 + [n/2]$ which would be smaller than H when, as here, $n \geqq 5$. Thus, $V(G) \geqq 2n$. Thus, if $n > 5$, $V(G) > 10$ and G is larger than P.

We now have $\gamma(G) = 5$ with each v_i connected to a new vertex v_i'. If there are to be no further vertices, the five v_i' must, to avoid triangles, etc., belong to a second pentagon. Now the two pentagons must have consecutive vertices of one connected to alternate vertices of the other (again to avoid squares, etc., as the reader can easily see by sketching the various possibilities) and so P, as depicted in Figure 2.2(a) ensues and hence the titular theorem.

5.4 Tutte's conjecture. Throughout our work P seems ubiquitous and Tutte conjectured that this would be so when very few uncolorable graphs were known and to date he seems right.

Our version of his well-known conjecture — which implies the 4-color conjecture — shall be

Every uncolorable graph possesses the Petersen graph.

This form allows us to state

THEOREM 5.4. *A graph in the BDS class satisfies Tutte's conjecture if one of the original V or W does.*

The proof is virtually the same as that of Theorem 5.2.2.

I leave to the reader the hunt for P in the J_k and double star.

References

1. J. Petersen, Die Theorie der regulären Graphen, Acta Math., Stockholm, 15 (1891) 193–220.

2. Danilo Blanuša, Problem Cetiriju Boja, Hrvatsko Prirodoslovno Društvo Glasnik, Mat-Fiz. Astr, Ser II, 1 (1946) 31–42. (In the Croatian tongue with summary in French.)

3. Blanche Descartes, Network-colourings, Math. Gazette, 32 (1948) 67–69.

4. G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Austral. Math. Soc., 8 (1973) 367–387.

5. C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, New York, 1968.

6. T. L. Saaty, Thirteen colorful variations on Guthrie's four-color conjecture, this MONTHLY, 79 (1972) 2–44.

7. W. T. Tutte, Connectivity in Graphs, Univ. of Toronto Press, 1966.

8. Norman Biggs, An edge-colouring problem, this MONTHLY, 79 (1972) 1018–20.

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PRUSSIAN EDUCATION AND MATHEMATICS

MARGUERITE GERSTELL

1. Centers of mathematical activity shift. I *assume* the phenomenon has social causes, even that it reflects the interests of a rising or ruling class. But how can a government, let alone a class, create mathematicians? And why should it, when we know that most of the work for which we remember the great mathematicians remained useless for some time? The second question is the easier of the two: if a class or government succeeds in producing mathematicians who pursue their own interests, some of them will pursue the government's interests as well. The momentum created need not be precisely directed. A simple-minded guess at the answer to the first question is that men need only be educated in mathematics to ensure that a number of them will become competent and creative mathematicians.

Of course every effect has many causes and vice versa. I claim only to have found a case where the progress of mathematical education correlates exceptionally well with the progress of mathematics itself, as measured by the number of notable creative mathematicians working. That is the case of Prussia at the beginning of the nineteenth century. It is more striking than the British or French cases, and too little time has passed for us to consider in this light what has happened in the United States and the Soviet Union in our century.

2. In 1800, the area we now call Germany embraced a collection of small states, the greatest of which was Prussia. She and her allies suffered severe defeats in conflict with Napoleon in 1805 and 1806. When the Prussian king, Frederick William III, accepted the unfavorable Peace of Tilsit in 1807, his ministers, with the support of an increasingly important commercial and industrial class, set out on a program of reform which was intended to result in victory against France. Von Stein and Hardenberg executed bourgeois social and political reforms, while Scharnhorst and Clausewitz reorganized the army. Wilhelm von Humboldt, as minister of education, instituted a national system of primary, secondary, and normal schools, all government-subsidized. He also founded the University of Berlin.

Control of existing schools was centralized, and teachers had to pass rigorous examinations after 1810. Earlier rulers had enunciated the principle of state control, but more effective control was possible as continuing reforms strengthened the state and the bourgeoisie, the government appealing constantly to the patriotic sentiment engendered through conflict with France [**12**].

An unusual feature of the new school system, in comparison with the earlier Prussian schools or the English schools, for example, was the emphasis on mathematics. The curriculum prescribed in the initial reforms required six hours a week of mathematics for all ten classes at the *Gymnasien*, although the shortage of teachers

prevented some schools from meeting the standard immediately. The curricula of 1827 and 1837 reduced the requirement to four hours a week in the lower forms and three in the upper. The newer *Realschulen* for middle-class boys gave even more time to mathematics and especially to science, at the expense of Latin, which the genteel patrons of the *Gymnasien* still valued. But the *Realschulen* had little to do with the development of higher mathematics, as we shall see. The new mathematical program in the *Gymnasien* included the theory of equations, probability, analytic geometry, and mechanics [**12**].

Haines [**5**] calls these reforms part of a conscious effort to "fight fire with fire" and Russell [**12**], who says the reforms "mark the beginning of serious mathematical study in the *Gymnasien*," attributes the change to the "zeal to outdo France." The foundation of the University of Berlin in 1809 and that of Bonn in 1818 had the same nationalistic motivations as the reforms in the school system. Von Humboldt spent the latter part of his term "occupied in negotiations with eminent men of science all over Germany, whose services he hoped to procure" [**12**] for Berlin. Gauss was approached but refused an appointment.

The new school and university plans were only the universal institution of trends that had been growing quietly in a few places for a long time. Pietism, the German analogue of puritanism, propounded the virtue of productive work, and Francke and Thomasius, both pietists, had founded the University of Halle in 1694. The leading professors there were pietists and stressed useful scientific learning. Ludewig, a pietist and Chancellor of Halle, proposed schools of physics, mathematics, and economics, in order to study "how manufacture might be ever more and more improved and excelled," [**8**]. The University of Göttingen was an offshoot of Halle, and Königsberg felt Halle's pietist influence as well. The *Realschulen* were completely a pietist product, according to Merton [**8**].

The study of mathematics otherwise had grown slowly. In 1558, the statutes of the University of Leipzig had required a candidate for the Bachelor's degree to study mathematics during just one of his three terms. The students took four subjects each term. Such attention to mathematics may appear considerable, since one can now get a Harvard degree without studying any mathematics. But only a few of the 16th century preparatory schools in Germany taught even the rudiments of arithmetic or geometry in the upper grades [**9**], so the level of sophistication of the Leipzig Bachelors' course must have been low. Only in the 18th century did elementary schools begin to teach arithmetic and occasionally geometry. More damaging to the study of higher mathematics was the practice, common until well into the 18th century, for German professors to transfer from one chair to another with a larger endowment. One is appalled at the thought of a modern professor of Latin or Greek switching into the mathematics department for a year or two. In Germany, as elsewhere, one did not study "geometry" during the Renaissance, but "Euclid," as much for his historical and literary as for his mathematical significance.

The *Ritter Akademien* of the 17th and 18th centuries, schools founded expressly for the preparation of the younger sons of aristocrats for civil and military service, and some other schools, had taught an élite the fundamentals of arithmetic and geometry, sometimes even trigonometry, with emphasis on practical applications such as surveying. The better *Gymnasien* had required two or three weekly recitations in mathematics by the end of the eighteenth century. But the decision to teach much more mathematics and science to all Gymnasium students, and the implementation of that decision, were sudden. Political reaction as late as the reign (1786–1797) of Frederick William II had led him to dictate that Prussian primary schools must teach no more than "the simplest calculation indispensable for everyday household uses," [9]. This is only a symptom, of indirect relevance, since the author is writing of the *Volksschulen* for children of the working class.

With this background, we should not be surprised to find only scattered mathematical activity in Germany before the beginning of the nineteenth century, and a great deal afterwards. Specifically, I should expect to find a number of mathematicians in the generations beginning with those born in 1795, at the earliest.

Since I constructed this argument with prior knowledge of the biographical facts, it is unfair to spring the trap without an apology. But a look at the following table is suggestive. I have chosen mathematicians in as fair a way as I could devise, which is by simply listing everyone mentioned in Carl Boyer's *History of Mathematics* [4]. To check against Boyer's possible bias, I scanned half of Debus's biographical dictionary [1]. Each German mathematician I found there, other than those Boyer had named, I have indicated with an asterisk. I do not name them because their names are less familiar than most of those that appear in Boyer.

Birthdates	Birthdates
1556–65	1566–75 Kepler
1576–85*	1586–95*
1596–05	1606–15
1616–25	1626–35
1636–45	1646–55 Leibniz, von Tschirnhaus
1656–65	1666–75
1676–85*	1686–95 Goldbach
1696–05*	1706–15
1716–25*	1726–35
1736–45 Wessel	1746–55
1756–65	1766–75
1776–85 Gauss,*	1786–95 Möbius,*
1796–05 Von Staudt, Feuerbach, Plücker, Jacobi, Dirichlet, ******	
1806–15 Listing, Grassmann, Kummer, Weierstrass, **	
1816–25 Heine, Seidel, Kronecker,**	
1826–35 Riemann, Dedekind, **	
1836–45 Hankel, Cantor, ***	

1846–55 Klein, Frege, Lindemann, ******
1856–65 Hilbert, ****
1866–75 Hausdorff, Zermelo, ***

3. It remains to be seen whether the majority of the mathematicians born after 1795 were indeed educated in the institutions I have discussed. First we have the geographical question. The Prussian school reforms were followed closely in Hanover, Hesse, Brunswick, and Baden, but not in Saxony or the south German states of Bavaria and Württemberg. To confirm my hypothesis, then, I hoped to find that most of the twenty-two mathematicians named on the chart and born after 1795 had come from Prussia or states that adopted Prussian reforms. Fifteen of them, it turns out, were born in Prussia, four in other "reform" states, and only two in states without the school reforms. Cantor was born in Russia but immigrated to Prussia at age twelve.

Unfortunately, I have not been able to discover how all these men were educated before they came to the universities. In the early part of the nineteenth century, compulsory school attendance was rigorously enforced nowhere but in the United States, though statutes had existed in some German states as early as 1716. So there is no guarantee that our mathematicians, even the Prussians, were alumni of von Humboldt's reformed system.

From Bell [2] and Klein [6], we learn a little about the early lives of nine of the twenty-two in question, namely, Cantor, Plücker, Jacobi, Kummer, Weierstrass, Kronecker, Riemann, Dedekind, and Hilbert. Each of these attended a *Gymnasium* in or near the city where he lived, seven in Prussia, two in other "reform" states.

Bell makes much of the inadequacy of the instruction, but his description of their experiences suggests to me that the *Gymnasien* had changed since the eighteenth century. At the gymnasium in Brunswick, Dedekind favored his physics and chemistry, which I suspect, were new to the curriculum. Riemann attended the Lüneburg *Gymnasium*, and I doubt that an eighteenth-century director of the same would have had the tastes of Riemann's headmaster, who, when the standard course failed to challenge Riemann, supplied from his private library copies of Legendre's *Theory of Numbers* and some works of Euler.

While the evidence of these last paragraphs is scanty, I am pleased with it, because every bit of it is consistent with my thesis; nor am I guilty of withholding any contrary evidence.

Kronecker studied under Kummer at the *Gymnasium* in Liegnitz; of the nine mathematicians on the chart born between 1795 and 1815, at least eight, that is, all but Jacobi, taught in secondary schools at some time, at least six in *Gymnasien*. I take this as a sign of the improving quality of secondary education. Weierstrass, for example, did creative scientific work while teaching in a secondary school. The point to be emphasized is not only that such men were willing and capable teachers, but that they were indeed hired. The advent of this new mathematical occupation

also made mathematics, as a field of study, an economically feasible choice for a wider class of people. In this connection we should take account not only of the *Gymnasien* and *Realschulen*, but also of the large number of technical high schools which grew up during the middle and later parts of the century. Curiously, I know of only two mathematicians on the chart born *after* 1815 who taught outside the universities, Dedekind and Klein; both taught in technical high schools outside Prussia. Perhaps by mid-century the crop of potential mathematics teachers was large enough that the Prussian government no longer had to make the occupation very attractive, and the more competent mathematicians taught only at universities.

4. Ben-David [3] attributes the favorable conditions for research and education in medicine at German universities in the nineteenth century to decentralization of the university system. He suggests that competition to attract scientists, who were relatively free to move around, naturally led the universities to expand laboratory facilities and to create new scientific jobs. The argument sounds reasonable; I don't believe it applies to mathematics, however. Berlin, Halle, and Göttingen established a fairly effective oligopoly on mathematicians.

Broader than the hegemony of Berlin, Halle, and Göttingen was the general hegemony of the Prussian universities, along with Göttingen, which became Prussian only after 1866. The earliest statistics I could find [7] show that by 1865, the Prussian universities of Berlin, Bonn, Breslau, Halle, and Königsberg all received over three-fourths of their funds from state subsidies. (Göttingen's main source of support was the Hanoverian monastic fund.) Therefore I assume the state control of which various authors speak was a reality. If so, the decision to spend on particular fields of study would have arisen from political, ideological, economic, or philosophical considerations more than on competition among the universities in question.

One can imagine competition between the Prussian universities on the one hand and those of Bavaria, Baden and Saxony on the other, but the evidence is that no such effective competition existed. The combined student population of the Prussian universities and the University of Göttingen in 1850 was about the same as that of all other German universities combined. Yet among our famous mathematicians born after 1795, who collectively spent at least five hundred years studying or teaching in universities, eighty percent of that time found them in Prussia or at Göttingen. More significant, professors rarely moved around, even though students nearly always did. Such moving as went on among teachers was typically from one Prussian university to another, or to or from Göttingen. Thus, since their funds were all ultimately controlled by the same state, the competition if it existed would not have taken the form of an expansion of facilities for mathematical study at one university at the expense of other universities or departments, unless such a change were a design of that state. One may allow for competition between Prussia and Göttingen up until 1865. Unfortunately I have no idea who ran the Hanoverian monastic fund or why.

5. Now I must put everything in the context of my disclaimers. I am anxious nobody should think I intend these arguments to "explain" the German mathematicians of the nineteenth century. It would be an especially bad mistake to say that education makes the difference between the genius of Gauss or Newton and the lesser talent of any of a hundred other European mathematicians of their respective times. Nor do I assert that the things I have written about determined the direction of mathematical progress. The inspiration for this paper was the tendency of some authors to dismiss social history for mathematics by dismissing social origins for particular mathematical innovations. If the origins of mathematical problems are not directly related to science, industry, or accounting, some authors conclude that their *raison d'être* is internal to mathematics. This may be true of the problems themselves, but not of the men who solve them.

Education is only one of many avenues through which political rulers or powerful social groups can exert their influence on intellectual history, and I find no case so pure as the German, where I believe the educational factor can be isolated with no more distortion than the isolation of a single cause inevitably produces in historical writing.

References

1. A. G. Debus, ed., World Who's Who in Science, A. N. Marquis, Chicago, 1968.
2. E. T. Bell, Men of Mathematics, Simon & Schuster, New York, 1937.
3. J. Ben-David, Scientific productivity and academic organization in nineteenth-century medicine, *Science and Society*, Rand McNally, Chicago, 1965.
4. C. B. Boyer, A History of Mathematics, Wiley, New York, 1968.
5. G. Haines, IV, German Influence Upon English Education and Science, 1800–1866, Conn. Coll. Monograph, New London, 1957.
6. Felix Klein, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, Julius Springer, Berlin, 1926.
7. W. Lexis, A General View of the History and Organization of Public Education in the German Empire, transl. Tamson, A. Asher, Berlin, 1904.
8. R. K. Merton, Science, Technology, and Society in Seventeenth Century England, Fertig, New York, 1970.
9. F. Paulsen, German Education, Past and Present, transl. T. Lorenz, Unwin, New York, 1908.
10. A. D. C. Peterson, A Hundred Years of Education, Humanities Press, New York, 1952.
11. J. C. Poggendorff, Biographisch-Literarische Handwörterbuch zur Geschichte der exacten Wissenschaften, J. A. Barth, Leipzig, 1863 and supplements.
12. J. E. Russell, German Higher Schools, Longmans Green, New York, 1899.

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THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) nondegenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

1. Relatively parallel fields. We say that a normal vector field M along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to the given curve. Indeed, if γ is a curve, considered as a displacement vector function of a parameter t, then if M is relatively parallel, the curve with displacement vector $\gamma + M$ has velocity $(\gamma + M)' = (v + f)T$, where T is the unit tangent vector field of γ, v is the speed of γ, and $M' = fT$. Thus the segment between the two curves is perpendicular to both. Whether or not this segment is locally a segment of minimum length between the two curves depends on the curvature and the length of M. It is easily verified that this segment locally minimizes length if M is short enough. Conversely, a curve which runs at constant distance from γ must be given by $\gamma + M$, where M is relatively parallel.

A single normal vector M_0 at a point $\gamma(t_0)$ generates a unique relatively parallel field M such that $M(t_0) = M_0$. The uniqueness is trivial: the difference of two relatively parallel fields is obviously relatively parallel, so if two such coincide at one point, their difference has constant length 0. To show existence one takes auxiliary adapted frames; the Frenet frame would do if it exists, but we want existence even for degenerate curves, that is, those which have curvature vanishing at some points. Such frames can be constructed locally by applying the Gram-Schmidt process to T and two parallel fields. If T, N_1, N_2 is an adapted frame, then we have

$$T' = p_{01}N_1 + p_{02}N_2, \quad N_1' = -p_{01}T + p_{12}N_2, \quad N_2' = -p_{02}T - p_{12}N_1.$$

Now we find the condition for a normal field of constant length L to be relatively

parallel. There is a smooth angle function θ such that $M = L(\cos\theta N_1 + \sin\theta N_2)$. Differentiating, we have

$$M' = L[(\theta' + p_{12})(-\sin\theta N_1 + \cos\theta N_2) - (p_{01}\cos\theta + p_{02}\sin\theta)T].$$

From this we see that M is relatively parallel if and only if $\theta' + p_{12} = 0$. Since there is a solution for θ satisfying any initial condition, this shows that local relatively parallel normal fields exist. To get global existence we can patch together local ones, which exist on a covering by intervals. Smoothness at the points where they link together is a consequence of the uniqueness part.

We define a tangential field to be relatively parallel if it is a constant multiple of the unit tangent field T. An arbitrary field is relatively parallel if its tangential and normal components are relatively parallel. We spell out the complete hypotheses for the existence and uniqueness of these fields as follows.

THEOREM 1. *Let γ be a C^k curve in euclidean 3-space which is regular, that is, the velocity never vanishes ($k \geqq 2$). Then for any vector V_0 at $\gamma(t_0)$ there is a unique C^{k-1} relatively parallel field V along γ such that $V(t_0) = V_0$ and the scalar product of two relatively parallel fields is constant.*

Proof. The first part is merely a matter of checking that the smoothness hypothesis has been correctly determined and that the normal and tangential parts add together without difficulty. To prove that the scalar product $\langle V, W\rangle$ of two relatively parallel fields V, W is constant, we observe that it is trivial for tangential ones and may be verified for the tangential and normal parts separately. Thus we assume V and W are normal, with derivatives fT and gT. Then the derivative of $\langle V, W\rangle$ is $\langle fT, W\rangle + \langle V, gT\rangle = 0$, as desired.

2. Special adapted frames. It should be clear that the relatively parallel fields on a C^2 regular curve form a 3-dimensional vector space over R with distinguished subspaces consisting of an oriented 1-dimensional tangential part and a 2-dimensional normal part, and there is a scalar product inherited from the pointwise scalar product on the ambient euclidean space. We call an orthonormal basis of this vector space which fits the two subspaces a **relatively parallel adapted frame,** or **RPAF.** If we assume that the ambient euclidean space has a preferred orientation, then so does the normal space of the curve, and we may refer to a **properly oriented** RPAF. The totality of RPAF'S are in the form of two circles, one in each orientation class, since they can be parametrized by the 2-dimensional orthogonal group, according to the following obvious result.

THEOREM 2. *If (T, M_1, M_2) is a RPAF, then the totality of RPAF's consists of frames of the form $(T, aM_1 + bM_2,\ cM_1 + dM_2)$, where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ runs through orthogonal matrices having constant entries.*

Now if (T, M_1, M_2) is a RPAF, denoting derivatives with respect to arc length by a dot, we have

$$\begin{aligned} T^{\cdot} &= k_1 M_1 + k_2 M_2 \\ M_1^{\cdot} &= -k_1 T \\ M_2^{\cdot} &= -k_2 T. \end{aligned}$$

This shows that we have accomplished our original goal of showing that there are other adapted frames which have only two nonzero entries in their *Cartan* matrices. (See [**1**] or [**2**, pp. 85–97] for a more general discussion of Cartan matrices.) In fact, given one such RPAF, Theorem 2 tells us that the possible Cartan matrices for RPAF's are

$$\begin{bmatrix} 0 & ak_1 + bk_2 & ck_1 + dk_2 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix},$$

where * denotes an entry which can be determined by using skew-symmetry. The Frenet frame has Cartan matrix

$$\begin{bmatrix} 0 & \kappa & 0 \\ * & 0 & \tau \\ 0 & * & 0 \end{bmatrix},$$

and is unique once the orientation of the ambient space and a convention on the sign of the torsion τ have been chosen. The only other possibility for a Cartan matrix with one entry vanishing would be

$$\begin{bmatrix} 0 & 0 & f \\ 0 & 0 & g \\ * & * & 0 \end{bmatrix},$$

but this is easily seen to be merely a remake of the Frenet frame, having $f = \pm\kappa$, $g = \pm\tau$.

It is simple to relate the entries of the various Cartan matrices. Indeed, $\kappa = |T^{\cdot}| = |k_1 M_1 + k_2 M_2| = (k_1^2 + k_2^2)^{\frac{1}{2}}$. Writing the principal normal as

$$N = \cos\theta M_1 + \sin\theta M_2 = (k_1/\kappa)M_1 + (k_2/\kappa)M_2,$$

and differentiating we obtain

$$N^{\cdot} = -\kappa T + \tau B = -\kappa T + \theta^{\cdot}(-\sin\theta M_1 + \cos\theta M_2).$$

If (T, M_1, M_2) is properly oriented, we conclude that $B = -\sin\theta M_1 + \cos\theta M_2$ and

hence $\tau = \theta^{\cdot}$. Thus κ and an indefinite integral $\int \tau ds$ are polar coordinates for the curve (k_1, k_2).

3. The normal development of a curve. We want to view (k_1, k_2) as a sort of invariant of the curve γ. This is slightly more difficult to conceive than in the case of (κ, τ), since the RPAF is not unique. However, we have spelled out what degree of freedom there is in Theorem 2: (k_1, k_2) is determined up to an orthogonal transformation in the nonoriented case and up to a rotation about the origin in the oriented case. Thus we must think of (k_1, k_2) as a parametrized (by an arclength parameter for γ) continuous curve in a centro-euclidean plane, that is, a euclidean plane having a distinguished point. When conceived of in this way we call (k_1, k_2) the *normal development* of γ. The situation is not really so different from the case of the Frenet invariants (κ, τ), because in the nonoriented case (κ, τ) and the Frenet frame are determined only up to an action by the two-element group, with the non-identity changing the sign of τ and B. That is, (κ, τ) cannot be distinguished from $(\kappa, -\tau)$. The standard facts about the relation between (κ, τ) and the curve γ as an object of euclidean geometry correspond to similar facts about (k_1, k_2) and γ. The proofs are identical with the Frenet case, and in fact are partly given in unified form in [**2**, p. 121].

THEOREM 3. *Two C^2 regular curves in euclidean space are congruent if and only if they have the same normal development. For any parametrised continuous curve in a centro-euclidean plane there is a C^2 regular curve in euclidean space having the given curve as its normal development.*

The modifications for the oriented case are clear: make both the euclidean space and the centro-euclidean plane be oriented and the congruences be proper.

4. Comparison and applications. The uniqueness and the ease of geometrical interpretation of the Frenet invariants will clearly make them retain their favored position. Their existence, however, requires a "more generally positioned" curve than does the existence of a normal development. Specifically, for a Frenet apparatus the curve γ is required to be C^3 and γ', γ'' must be linearly independent (nondegeneracy). For the normal development we need only C^2 and γ' nonzero.

Just as with κ and τ, the normal development frequently has simple properties for special types of curves. Clearly a straight line has its normal development equal to the constant curve 0. When κ is constant and $\tau = 0$ we get a circle of radius $1/\kappa$; for the normal development we get the constant curve $(\kappa, 0)$, since the distance from the origin is κ and the polar angle θ satisfies $\theta^{\cdot} = \tau = 0$. The other curves obtained as orbits of a one-parameter subgroup of euclidean motions are helices, and they have normal development which is obtained by a one-parameter subgroup of the centro-euclidean plane group, namely, a central circle with constant speed.

A plane nondegenerate C^3 curve is characterized by $\tau = 0$. This means that the normal development has $\theta =$ constant, that is, lies on a line through the origin.

This fact generalizes: a C^2 regular curve lies in a plane if and only if its normal development lies on a line through the origin. If we take a curve whose normal development moves through the origin on a line, it switches from bending one way in the plane to the other. If we take a normal development which is C^1 and moves to the origin on one line and out on another line, then the corresponding space curve has $\tau = 0$ whenever it is defined, but is not planar.

The condition for a C^3 regular curve in space to lie on a sphere in terms of κ and τ is so complicated that the various elementary texts are filled with versions which are either incomplete or erroneous. Usually this involves something about satisfying the differential equation $(\kappa^\cdot/\tau\kappa^2)^\cdot = \tau/\kappa$. The difficulty arises from the fact that τ and $\kappa^\cdot$ may vanish at some points of the curve but not everywhere. Correct versions which cover this difficulty are given in [**3**, **4**]. If we take into account the relation between (κ, τ) and (k_1, k_2), the version in [**4**] is essentially the following simple geometric characterization:

A C^2 regular curve is spherical if and only if its normal development lies on a line not through the origin. The distance of this line from the origin and the radius of the sphere are reciprocals.

Proof. If γ lies on a sphere with center P and radius r, then $\langle \gamma - P, \gamma - P\rangle = r^2$. Differentiating with respect to arc length gives $\langle T, \gamma - P\rangle = 0$, so $\gamma - P = fM_1 + gM_2$ for some functions f, g. But

$$f^\cdot = \langle \gamma - P, M_1\rangle^\cdot = \langle T, M_1\rangle + \langle \gamma - P, -k_1 T\rangle = 0;$$

similarly, g is also constant. Then differentiating $\langle T, \gamma - P\rangle$, we get

$$\langle k_1 M_1 + k_2 M_2, \gamma - P\rangle + \langle T, T\rangle = fk_1 + gk_2 + 1 = 0.$$

That is, (k_1, k_2) is on the line $fx + gy + 1 = 0$. Moreover, $r^2 = \langle \gamma - P, \gamma - P\rangle = f^2 + g^2 = 1/d^2$, where d is the distance of this line from the origin.

Conversely, suppose that $fk_1 + gk_2 + 1 = 0$, where f and g are constant. Let $P = \gamma - fM_1 - gM_2$; then $P^\cdot = T + (fk_1 + gk_2)T = 0$, so P is constant. A similar derivation shows that $r^2 = \langle \gamma - P, \gamma - P\rangle$ is constant, so γ lies on a sphere of radius r and center P.

More generally, the radius of the osculating sphere of a C^3 regular space curve is given by the reciprocal of the distance from the origin to the tangent line of the normal development.

Now take a C^1 normal development which runs along one line in the plane, stops, turns a corner and runs along another line, with neither line through the origin. Thus the corresponding space curve passes from one sphere to another, possibly running along the circle of intersection at the join. The torsion and curvature of this curve satisfy the differential equation $(\kappa^\cdot/\tau\kappa^2)^\cdot = \tau/\kappa$ whenever it makes sense, but the curve is not spherical.

5. Generalizations. For curves in n-dimensional oriented euclidean space, with multiple assumptions of higher-order nondegeneracy and C^n smoothness, we can define a unique Frenet frame; all but $n-1$ of the independent entries of its Cartan matrix vanish and $n-2$ of these are positive. These entries are a complete set of absolute differential euclidean invariants of the curve, having orders 2 through n [**1**, p. 159].

The normal development of a curve in n-dimensional oriented euclidean space is defined by a straightforward generalization of the 3-dimensional case. It exists for any C^2 regular curve and is a continuous parametrized curve in oriented centro-euclidean space of dimension $n-1$; that is, it is defined up to a proper orthogonal transformation. It arises from a RPAF whose Cartan matrix has all but $n-1$ independent entries zero (the first row). Curves in n-space which lie in a lower-dimensional flat or sphere are characterized simply as those which have their normal development lying in a lower-dimensional flat of the same codimension.

The corresponding idea for affine geometry seems to be hinted at in the first paragraph on page 172 of [**1**].

References

1. Heinrich W. Guggenheimer, Differential Geometry, McGraw-Hill, New York, 1963.
2. Barrett O'Neill, Elementary Differential Geometry, Academic Press, New York, 1966.
3. Yung-chow Wong, A global formulation of the condition for a curve to lie on a sphere, Monatsh. Math., 67 (1963) 363–365.
4. ———, On an explicit characterization of spherical curves, Proc. Amer. Math. Soc., 34 (1972) 239–242.

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CORRECTION TO: "INNER PRODUCT SPACES"

STANLEY GUDDER

Professor Jürg Rätz, University of Bern, recently pointed out an error in my paper "Inner Product Spaces," (this MONTHLY, 81 (1974) 29–36). There is a gap in the proof of Theorem 3.2. In this proof, the existence of the vector z has not been justified. In particular, it has not been shown that the series $\sum d_i x_i$ (or the series $\sum c_i y_i$) converges. As far as I know, whether this theorem is true or not is open.

The given proof is valid under some stronger hypotheses. For example, the following weaker results can be proved using similar methods.

THEOREM. *Let V be an inner product space. If for every closed subspace M every maximal orthonormal set in M is basic in M, then V is complete.*

THEOREM. *Let V be an inner product space. If every maximal orthonormal set in V is basic and every closed subspace of V has a basis, then V is complete.*

Using the Gram-Schmidt orthonormalization process, every closed subspace of a separable inner product space has a basis. We then get the following corollary.

COROLLARY. *Let V be a separable inner product space. If every maximal orthonormal set in V is basic, then V is complete.*

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

Reply to Query 9. Two references were received dealing with alternatives to the "standard" class-lecture format for teaching mathematics:

Bagnato, Educational Studies in Mathematics, 5, (1973), 185–192.

Dubin and Taveggia, *The Teaching Learning Paradox*, Center for the Advanced Study of Educational Administration, University of Oregon, Eugene, Oregon.

19. Rami S. Rao. In multivariate normal statistical theory, suppose a problem is invariant under an operator T. As usual an operator is called a discrete integral operator (I.O.) if TX has discrete range with probability at least 1/2. I am under the impression that there is some literature on this subject and would like to receive references. In particular,is it necessary that an I.O. not be discrete if it grows without bound?

MATHEMATICAL NOTES

EDITED BY R. A. BRUALDI

Material for this Department should be sent to R. A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

A CONTINUOUS MODULUS OF CONTINUITY

STEPHEN B. SEIDMAN AND J. A. CHILDRESS

Let X and Y be metric spaces. A function $f\colon X \to Y$ is called *continuous* if for each $\varepsilon > 0$ and $x \in X$, there is a $\delta > 0$ (depending on x and on ε), such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. (We ambiguously use "$d$" for all metrics.) For each fixed $\varepsilon > 0$, we may think of δ as a function from X to the positive real numbers. One might ask the following questions:

(1) Under what conditions may the function δ be chosen to be a *continuous* function?

(2) Under what conditions may the function δ be chosen to be a *constant* function (i.e., δ is a positive number independent of x)? In this case, f is called *uniformly continuous.*

The following partial answer to (2) is well known:

THEOREM 1. *If $f\colon X \to Y$ is a continuous function and if X is compact, then f is uniformly continuous.*

We shall show, in contrast, that the answer to (1) is "always." More generally, we show that this holds when ε is not just a constant but any continuous function into the positive reals.

DEFINITION. Let $f\colon X \to Y$ be a continuous function between metric spaces, and let $\varepsilon\colon Y \to (0, \infty)$ be continuous. A continuous function $\delta\colon X \to (0, \infty)$ is called a *continuous modulus of continuity* for f and ε if for each $x, y \in X$ such that $d(x, y) < \delta(x)$, we have $d(f(x), f(y)) < \varepsilon(f(x))$.

The following theorem answers question 1.

THEOREM 2. *For each continuous function $f\colon X \to Y$ and continuous $\varepsilon\colon Y \to (0, \infty)$ there exists a continuous modulus of continuity $\delta\colon X \to (0, \infty)$.*

Before giving the proof, we introduce some definitions. Let $\{U_\alpha\}$ be a cover of the metric space X. The cover $\{U_\alpha\}$ is said to be *locally finite* if for each $x \in X$, there exists an open set V containing x such that V meets only finitely many U_α. If f is a map into R (= real line), the *support of f* is the set closure $\overline{\{f^{-1}(R-\{0\})\}}$. A family $\{\pi_\alpha\}$ of continuous functions from X to I (= $[0,1]$) is called a *partition of unity on X* if

(1) {support (π_α)} forms a locally finite cover of X, and

(2) $\Sigma_\alpha \pi_\alpha(x) = 1$ for each $x \in X$.

If $\{U_\beta\}$ is an open cover of X, we say that a partition of unity $\{\pi_\beta\}$ is *subordinated to the cover* $\{U_\beta\}$ if each support $(\pi_\beta) \subset U_\beta$.

If $x \in X$ and $\delta > 0$, then we let

$$N_\delta(x) = \{y \in X \mid d(x,y) < \delta\}.$$

Proof of Theorem 2. Let $f: X \to Y$ and $\varepsilon: Y \to (0, \infty)$ be continuous functions. Define the function $\mu_{f,\varepsilon}: X \to (0,\infty)$ by

$$\mu_{f,\varepsilon}(x) = \sup\{\delta \mid d(x,y) < \delta \text{ implies } d(f(x), f(y)) < \varepsilon(f(x))\}$$

if this supremum is less than or equal to 1, and $\mu_{f,\varepsilon}(x) = 1$ otherwise. Suppose (temporarily) that ε is a *constant*. In this case, it is easily verified that

$$\mu_{f,\varepsilon}(x) \geqq \tfrac{1}{2}\mu_{f,\varepsilon/2}(x_0)$$

for all $x \in N_r(x_0)$, where $r = \frac{1}{2}\mu_{f,\varepsilon/2}(x_0)$.

Now for an arbitrary continuous ε, let $N(x_0) = f^{-1}\{y \mid \varepsilon(y) > \frac{1}{2}\varepsilon(f(x_0))\}$, an open neighborhood of x_0. Let $r = \frac{1}{2}\mu_{f,\varepsilon/4}(x_0)$, and let $V(x_0) = N_r(x_0) \cap N(x_0)$. Let $\varepsilon_* = \frac{1}{2}\varepsilon(f(x_0))$, and note that for $z \in V(x_0)$, $\mu_{f,\varepsilon}(z) \geqq \mu_{f,\varepsilon^*}(z) \geqq \varepsilon_0$, where $\varepsilon_0 = \frac{1}{2}\mu_{f,\varepsilon^*/2}(x_0)$. Then the collection $\{V(x)\}_{x \in X}$ is an open cover of X. Since all metric spaces are paracompact ([**2**], p. 186, [**3**], [**4**]) we can refine this cover to a locally finite open cover $\{U_\beta\}$, and we have the corresponding positive numbers ε_β as (ε_0) above. Let $\{\pi_\beta\}$ be a partition of unity subordinate to $\{U_\beta\}$, and define $\delta(x) = \Sigma_\beta \pi_\beta(x)\varepsilon_\beta$. It is clear that δ is continuous.

Consider $x, x' \in X$ with $d(x,x') < \delta(x)$. Then

$$\delta(x) \leqq \max_{x \in U_\beta} \{\varepsilon_\beta\},$$

so that for some U_β, we have $x \in U_\beta$ and $d(x,x') < \varepsilon_\beta$. But $\varepsilon_\beta \leqq \mu_{f,\varepsilon}(x)$ since $x \in U_\beta$, so that we have $d(x,x') < \mu_{f,\varepsilon}(x)$. Thus $d(f(x), f(x')) < \varepsilon(f(x))$, which verifies that δ is a continuous modulus of continuity for f with respect to ε.

It should be noted that if ε is constant, Theorem 2 can be proved without the use of partitions of unity. For an application of Theorem 2 for non-constant ε, see [**1**, p. 401].

References

1. J. A. Childress and S. B. Seidman, Topological properties of the space of homeomorphisms of n-dimensional Euclidean space, Michigan Math. J., 20 (1973) 397–402.

2. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

3. E. Michael, A note on paracompact spaces, Proc. Amer. Math. Soc., 4 (1953) 831–838.

4. A. H. Stone, Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948) 977–982.

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A NOTE ON NON-ASSOCIATIVE ALGEBRAS DERIVED FROM GRAPHS

D. J. RODABAUGH

In a recent article in this MONTHLY [**1**], Jenner gave a construction that associated an algebra with any finite graph in such a way that the algebra was simple if and only if the graph was connected. In this article we shall show the strong connection between associativity, power-associativity and a condition on adjacent points.

For convenience, we will say that p and q are adjacent if either $p = q$ or if p and q are adjacent in the usual sense. As we use the term, p adjacent to q is a relation that is reflexive and symmetric.

In the definitions, all that is really needed is a relation ρ that is reflexive and symmetric. We say that Γ is ρ-connected if for any two of its vertices p and q there is a sequence of vertices beginning with p and terminating with q such that any two successive vertices in the sequence are related. Our definition of $A_\rho(\Gamma)$ follows that of Jenner [**1**]. In the multiplication $e_{pq} \cdot e_{rs} = 0$ if $q \neq r$ and $e_{pq} \cdot e_{qs} = e_{ps}$ if either $p\rho q$ or $q\rho s$; otherwise the product is zero. Jenner's results easily follow for $A_\rho(\Gamma)$.

In any algebra A, we define $x^{k+1} = x^k \cdot x$ and say that A is power-associative if $x^m x^n = x^{m+n}$ for all pairs m, n of natural numbers. Define $(x, y, z) = (xy)z - x(yz)$. We say that A is third power-associative if $x^2x = xx^2$ and for characteristic not 2 or 3 this is equivalent to the multilinear identity (see [**2**], p. 129)

(1) $g(x, y, z) = 0$ where

(2) $g(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) + (z, y, x) + (y, x, z) + (x, z, y)$.

If $p\rho q$ and $q\rho r$ but p not related to r then

(3) $g(e_{pq}, e_{qr}, e_{rp}) = e_{rr} - e_{pp} \neq 0$.

Thus $A_\rho(\Gamma)$ need not even be third power-associative.

LEMMA. *If Γ is a finite graph with components Γ_i then $A(\Gamma)$ is associative if and only if each $A(\Gamma_i)$ is associative.*

Proof. Assume each $A(\Gamma_i)$ is associative. As in [**1**],

$$A(\Gamma) = A(\Gamma_1) + \cdots + A(\Gamma_t) + N,$$

where N is spanned by $\{e_{pq} \mid p, q \text{ are in different components}\}$. The algebra $S = A(\Gamma_1) + \cdots + A(\Gamma_t)$ is semisimple as in [**1**]. Also, $A(\Gamma_i)A(\Gamma_j) \subseteq \delta_{ij}A(\Gamma_i)$ (Kronecker delta). Thus, S is associative. We next note that N is a zero algebra and hence associative. Also, $A(\Gamma)$ is a direct sum of S and N so $A(\Gamma)$ is associative.

THEOREM. *If Γ is a finite graph and if the characteristic is not 2 or 3, then the following are equivalent:*

(a) *$A(\Gamma)$ is associative.*

(b) *$A(\Gamma)$ is third power-associative.*

(c) *ρ is transitive.*

Proof. Clearly, ρ is transitive on Γ if and only if ρ is transitive on each component Γ_i. Following the proof of the lemma, we can show that (b) holds on $A(\Gamma)$ if and only if it holds on each $A(\Gamma_i)$. Thus it suffices to assume that Γ is connected. It is immediate that (a) implies (b). If $p\rho q$ and $q\rho r$ but p not related to r, then $g(e_{pq}, e_{qr}, e_{rp}) = e_{rr} - e_{pp} \neq 0$, so that (b) implies (c). Finally, if ρ is transitive, then any two vertices in the connected graph Γ are related so $A_\rho(\Gamma)$ is isomorphic to the standard matrix algebra and (c) implies (a).

References

1. W. E. Jenner, On non-associative algebras derived from graphs, this MONTHLY, 80 (1973) 288–289.

2. R. D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, New York, 1966.

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GENERALIZED FUNCTIONS ON THE UNIT CIRCLE

T. K. BOEHME AND G. WYGANT

1. Introduction. An elementary introduction to generalized functions can be obtained by restricting consideration to functions on the unit circle, or to periodic functions. The Mikusiński operational calculus is developed on the half-line in the books by Mikusiński [2] and Erdélyi [1]. The Mikusiński operational calculus works out nicely on the unit circle and illustrates some of the concepts at work in distribution theory as well as in the Mikusiński operator calculus. The resulting theory is a good example for an analysis course studying Fourier series or generalized functions or as an example of an application of the concepts of ring theory for an algebra course. The development here needs no background from [1] or [2].

The ring of Mikusiński operators on the unit circle turns out to be isomorphic with the ring of all formal trigonometric series $\Sigma_{-\infty}^{\infty} \alpha_n e^{in\theta}$. Every formal trigonometric series is the Fourier series of an operator and each operator X is the sum of its Fourier series,

$$X = \sum_{-\infty}^{\infty} c_n(X) e^{in\theta},$$

where convergence is in the ring of operators. The usual operator convergence for Mikusiński operators turns out to be equivalent to convergence of the Fourier coefficients. Thus $X_n \to X$ as $n \to \infty$ in the ring of operators if and only if $c_m(X_n) \to c_m(X)$ as $n \to \infty$ for each $m = 0, \pm 1, \pm 2, \cdots$. It follows that convergence in the field of operators is topological and in fact metrizable.

The only fact from Fourier series which we shall need in developing the Mikusiński operators is that if all the Fourier coefficients of a continuous function are zero then the function is identically zero.

The analogous development in several variables is essentially the same and with the obvious modifications the theorems discussed here are true in several variables. Some possible applications are outlined in Section 5.

2. The ring of operators. Let C be the space of continuous functions on the unit circle, or equivalently, the space of continuous functions on the real line with period 2π.

For f and g in C we write the convolution as

$$r(\theta) = f * g(\theta) = \int_{-\pi}^{\pi} f(\theta-\phi)g(\phi)d\phi.$$

The Fourier coefficients of r are then given by the equation

$$c_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\theta)e^{-in\theta}d\theta = 2\pi c_n(f)c_n(g) \tag{1}$$

for all $n = 0, \pm 1, \pm 2, \cdots$. With convolution as the product operation and with the usual addition of continuous functions C is a commutative ring. The ring has no identity element, and it has divisors of zero, e.g., $e^{in\theta} * e^{im\theta} = 0$ if $n \neq m$.

Let us denote by D the subset of C which consists of the continuous functions all of whose Fourier coefficients are non-zero,

$$D = \{g \in C\colon \text{for every } n,\ c_n(g) \neq 0\}. \tag{2}$$

Then D is closed under convolution and consists of those functions g such that $f \in C$ and $f * g = 0$ implies $f = 0$.

We can define an equivalence relation on $C \times D$ by the statement

$$(f,g) \sim (h,k), \quad f,h \in C, \quad g,k \in D,$$

if and only if $f * k = h * g$. A fraction f/g with $f \in C$ and $g \in D$ is defined to be the equivalence class $[(f,g)]$ in $C \times D$ containing (f,g). The set of fractions is denoted by $\mathfrak{M}(\Gamma)$ or just $\mathfrak{M}$, and elements of $\mathfrak{M}$ are called operators. Thus an operator $X \in \mathfrak{M}$ is a fraction $X = f/g$; and two fractions f/g and f_1/g_1 are the same operator if and only if $f * g_1 = f_1 * g$.

The usual rules for addition and multiplication of fractions are used to extend addition and convolution (ring multiplication) to $\mathfrak{M}$;

$$\frac{f}{g} + \frac{h}{k} = \frac{f * k + h * g}{g * k}$$

and

$$\frac{f}{g} * \frac{h}{k} = \frac{f * h}{g * k} \quad \text{for } f,h \in C,\ g,k \in D.$$

With these operations $\mathfrak{M}$ is a commutative ring with unity which contains a subring

isomorphic to C. For fixed $g \in D$ the mapping

$$\Phi_g : f \to X_f = \frac{f * g}{g}$$

is a ring isomorphism of C onto the subring $\{X_f : f \in C\}$. For any two $g_1, g_2 \in D$, $\Phi_{g_1} = \Phi_{g_2}$. We complete the identification of $\Phi_g(C)$ and C by writing $X = f$ if $X = X_f$. For students familiar with Lebesgue integration or with the theory of distribution, the mapping Φ_g is also seen to embed the spaces $L[-\pi, \pi]$ and $\mathfrak{D}'(\Gamma)$ into $\mathfrak{M}$. In the latter case g should be taken in $D \cap C^{\infty}$.

The identity element of $\mathfrak{M}$ is usually denoted by δ (the Dirac delta function), and $\delta = g/g$ for any $g \in D$. For any $X \in \mathfrak{M}$, $\delta * X = X$ and in particular for any $f \in C$, $\delta * f = f$.

Let $\psi \in D \cap C^{\infty}$ (e.g., $\psi(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$ where $a_n = (|n| + 1)^{-|n|}$ for $n = 0, \pm 1, \pm 2, \cdots$). Then the operator $\delta^{(k)} = \psi^{(k)}/\psi$, $k = 0, 1, 2, \cdots$ has the property that for any k-times continuously differentiable function f

$$\delta^{(k)} * f = \frac{\psi^{(k)} * f}{\psi} = \frac{\psi * f^{(k)}}{\psi} = f^{(k)}$$

since differentiation under the integral sign yields $\psi^{(k)} * f = \psi * f^{(k)}$. In general we make the following definition

Definition. For $X \in \mathfrak{M}$ the kth derivative of X, $X^{(k)}$, is the operator

$$X^{(k)} = \delta^{(k)} * X.$$

3. The Fourier series of an operator. A formal trigonometric series $\sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$ is said to be the Fourier series of a continuous function f if the complex numbers α_n are generated by the relationship $2\pi\alpha_n = \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$ for $n = 0, \pm 1, \pm 2, \cdots$. We will define the nth Fourier coefficient $c_n(X)$ for an $X \in \mathfrak{M}$ as follows:

Definition. If $X \in \mathfrak{M}$ and $X = f/g$ for $f \in C$ and $g \in D$, then $c_n(X) = c_n(f)/2\pi c_n(g)$ for $n = 0, \pm 1, \cdots$. A trigonometric series $\sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$ is the Fourier series of X if $\alpha_n = c_n(X)$ for each n. In this case we write

$$X \sim \sum_{n=-\infty}^{\infty} c_n(X) e^{in\theta}. \tag{3}$$

If $f/g = f_1/g_1$ then $f * g_1 = f_1 * g$ and it follows from equation (1) and the fact that g and g_1 are in D that the Fourier coefficients are well determined by the above definition. If the formal trigonometric series are given an algebraic structure by the definitions

$$\sum \alpha_n e^{in\theta} + \sum \beta_n e^{in\theta} = \sum (\alpha_n + \beta_n) e^{in\theta}$$

and

$$(\sum \alpha_n e^{in\theta})(\sum \beta_n e^{in\theta}) = \sum 2\pi\alpha_n\beta_n e^{in\theta}$$

then the mapping $X \to \sum_{-\infty}^{\infty} c_n(X)e^{in\theta}$ maps $\mathfrak{M}$ into the ring of formal trigonometric series.

THEOREM A. *The ring of operators is isomorphic to the ring of formal trigonometric series.*

Proof. The above mapping is clearly a homomorphism of $\mathfrak{M}$ since $c_n(X * Y) = 2\pi c_n(X)c_n(Y)$ for all $X, Y \in \mathfrak{M}$ and all n. Moreover, if $X = f/g$ and $c_n(X) = 0$ for all n then $c_n(f) = 0$ for all n, and a continuous function all of whose Fourier coefficients are zero must be zero. Thus $f = 0$ and therefore $X = 0$. Thus the mapping is an isomorphism of $\mathfrak{M}$ into the ring of formal trigonometric series. To see that the mapping is onto (surjective) consider a formal trigonometric series $\sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$. Take

$$g = \sum_{-\infty}^{\infty} [2\pi(n^2+1)(1+|\alpha_n|)]^{-1}e^{in\theta} \text{ and } f = \sum_{-\infty}^{\infty} \alpha_n[(1+|\alpha_n|)(n^2+1)]^{-1}e^{in\theta}.$$

Then $f \in C$ and $g \in D$ and $c_n(f/g) = \alpha_n$ for each n. Thus the proof of Theorem A is complete.

It can be seen from this characterization of $\mathfrak{M}$ that an operator $X \in \mathfrak{M}$ has an inverse in $\mathfrak{M}$ if and only if no Fourier coefficient of X is zero.

Another method of calculating the Fourier series of an operator is obtained from the fact that for $f \in C$

$$c_n(f)e^{in\theta} = f * e^{in\theta}/2\pi. \tag{4}$$

It follows from Theorem A that $c_n(X)e^{in\theta} = X * e^{in\theta}/2\pi$ for any X in $\mathfrak{M}$. Thus

COROLLARY 1. *For* $X \in \mathfrak{M}$,

$$X \sim \sum_{-\infty}^{\infty} X * e^{in\theta}/2\pi. \tag{5}$$

Since for any L^1 function or any distribution equation (4) holds when f is replaced by the L^1 function or the distribution and $c_n(f)$ is calculated in the usual manner for L^1 functions or distributions we have from (4) and (5):

COROLLARY 2. *If $X \in \mathfrak{M}$ is a continuous function (L^1 function, or a distribution) then the Fourier series as an operator is the same as its Fourier series as a continuous function (L^1 function or distribution).*

4. Convergence in $\mathfrak{M}$. The following definition is the analogue for the circle of that given in [1] and [2] for convergence of Mikusiński operators.

DEFINITION. A sequence of operators X_n, $n=1,2,\cdots$, is said to converge in $\mathfrak{M}$ if there is a $g \in D$ such that

$$X_n = f_n/g, \quad f_n \in C, \quad n = 1,2,\cdots,$$

and the functions f_n converge uniformly on $[-\pi, \pi]$. If the f_n converge uniformly to $f \in C$ then we say $X_n \to X = f/g$ as $n \to \infty$. The function $g \in D$ is called a convergence factor for the sequence (X_n).

If a sequence of continuous functions (f_n) converges uniformly on $[-\pi, \pi]$ to a function f then $c_m(f_n) \to c_m(f)$ as $n \to \infty$ for each m. Thus if $X_n \to X$ as $n \to \infty$ and g is a convergence factor for (X_n) then $c_m(X_n) = c_m(f_n)/2\pi c_m(g) \to c_m(f)/2\pi c_m(g)$ as $n \to \infty$ for each m. This proves one half of the following theorem.

THEOREM B. *$X_n \to X$ in $\mathfrak{M}$ if and only if for each m, $c_m(X_n) \to c_m(X)$ as $n \to \infty$. In particular, every trigonometric series is convergent in $\mathfrak{M}$ and for any $X \in \mathfrak{M}$*

$$X = \sum_{-\infty}^{\infty} c_m(X)e^{im\theta}.$$

Completion of proof. Because of linearity, it is sufficient to suppose $X = 0$. Thus we shall suppose $c_m(X_n) \to 0$ as $n \to \infty$ for each m and show that $X_n \to 0$ as $n \to \infty$.

Since a uniformly convergent trigonometric series has itself for its Fourier series, it follows from Corollary 2 above that for any $X \in \mathfrak{M}$ if $\sum_{-\infty}^{\infty} c_m(X)e^{im\theta}$ is uniformly convergent on $[-\pi, \pi]$ then $X \in C$ and $X = \sum c_m(X)e^{im\theta}$.

Now let $A_m = (1 + m^2)\operatorname{Sup}_n(1 + |c_m(X_n)|)$ for $m = 0, \pm 1, \pm 2, \cdots$. If $g = \sum e^{im\theta}/(2\pi A_m)$ we have $g \in D$ and for each n, the Fourier series of $X_n * g$ is uniformly convergent. Thus

$$f_n = X_n * g = \sum_{-\infty}^{\infty} \frac{c_m(X_n)}{A_m} e^{im\theta}.$$

For $\varepsilon > 0$ and M such that

$$\sum_{|m|>M} \frac{1}{1 + m^2} < \varepsilon/2$$

we have

$$|f_n(\theta)| \leqq \sum_{|m| \leqq M} |c_m(X_n)| + \sum_{|m|>M} (1 + m^2)^{-1} \leqq \sum_{|m| \leqq M} |c_m(X_n)| + \varepsilon/2.$$

Now if N is large enough so that $n > N$ implies $|c_m(X_n)| < \varepsilon(4M + 2)^{-1}$ whenever $|m| < M$ then $n > N$ implies $|f_n(\theta)| \leqq \varepsilon$ for all θ. Thus $X_n \to 0$ as $n \to \infty$.

The last statement in the theorem follows from the fact that the partial sums $S_N \doteq \sum_{-N}^{N} \alpha_n e^{in\theta}$ of any trigonometric series converge to $\sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$ in the ring of formal trigonometric series.

5. Examples.

1. An application can be made to the Dirichlet problem on the unit disc. Any harmonic function on the disc $u(re^{i\theta}) = u_r(\theta)$ defines a collection of operators

$X_r = u_r(\theta)$ and $u_r(\theta) \to X \in \mathfrak{M}$ as $r \to 1$. The harmonic function can be recovered from its boundary value by Poisson's formula $u(re^{i\theta}) = X * P_r(\theta)$, where $P_r(\theta)$ is the Poisson kernel. A necessary and sufficient condition that an operator be the boundary value of a harmonic function is that

$$\limsup_{|n| \to \infty} |c_n(X)|^{1/|n|} \leqq 1.$$

2. For any operator X, $c_n(X^{(k)}) = (in)^k c_n(X)$ and in view of the Riemann-Lebesgue lemma a necessary and sufficient condition that an operator be a derivative of some order of a continuous function is that $c_n(X) = O(|n|^k)$ as $|n| \to \infty$ for some k. These operators are exactly the distributions. Also, since $\phi \in C^\infty$ if and only if $c_n(\phi) = O(|n|^{-k})$ as $n \to \infty$ for every k, the distributions are exactly those operators such that $X * \phi \in C$ for every $\phi \in C^\infty$.

3. Simple examples of applications to partial differential equations can be made by applying the theory in the case of several variables. For example, if $g(\theta, \phi)$ is a C^∞ function of θ and ϕ which is periodic with period 2π in each of θ and ϕ, an operator $X = X(\theta, \phi)$ which satisfies the non-homogeneous Laplace equation $\Delta X = g$ must be a C^∞ function and thus must satisfy the equation in the usual sense. In particular any twice continuously differentiable function which satisfies this equation must in fact be infinitely differentiable.

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References

1. A. Erdélyi, Operational Calculus and Generalized Function, Holt, Rinehart and Winston, New York, 1962.

2. J. Mikusiński, Operational Calculus, Pergamon Press, New York, 1959.

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WHAT IS THE PROBABILITY AN AUTOMORPHISM FIXES A GROUP ELEMENT?

GARY SHERMAN

Introduction. Let the finite group G operate on the finite non-empty set X (i.e., G is represented as a group of permutations of X). We ask: What is the probability an element chosen at random from G fixes an element chosen at random from X? If this probability is denoted by $P_G(X)$, then

$$P_G(X) = \frac{|\{(g, x) \mid gx = x \text{ for } g \in G \text{ and } x \in X\}|}{|G|\,|X|}.$$

Setting $G_x = \{g \in G \mid gx = x\}$ we find that

$$\left|\{(g,x) \mid gx = x \text{ for } g \in G \text{ and } x \in X\}\right| = \sum_{x \in X} |G_x| = \sum_{i=1}^{k} [G:G_{x_i}]\,|G_{x_i}| = k \cdot |G|,$$

where $\{x_1, \cdots, x_k\}$ is a set of representatives of the distinct orbits of X under G. Thus, $P_G(X) = k/|X|$; the ratio of the number of orbits in X under G to the order of X.

Of special interest, and the motivation for the preceding remarks, is when G operates on itself by conjugation. In this case one obtains the probability of two group elements commuting (see [1]). It is easy to show this probability to be at most 5/8 for nonabelian groups. Recently, Gustafson [2] has shown the bound 5/8 to be valid when G is an infinite nonabelian compact group (of course the counting measure is replaced by the normal left Haar measure on G).

The purpose of this note is to consider $P_A(G)$ where G is a finite abelian group and A is its group of automorphisms. We show:

(i) $P_A(G) = 1$ if and only if $G = Z_2$,

(ii) $P_A(G) \leqq 3/4$ if $G \neq Z_2$,

(iii) there is only a finite number of groups with a given probability,

(iv) $P_A(G) \to 0$ as $|G| \to \infty$.

Finite abelian groups. It is trivial that $P_A(Z_2) = 1$. Conversely, if $P_A(G) = 1$, then G is an elementary abelian 2-group since the automorphism $x \to -x$ must be the identity mapping. Viewing G as a Z_2 space it follows that any two nontrivial elements of G are in the same orbit. Thus, $2/2^j = 1$ where $|G| = 2^j$. This implies $j = 1$; i.e., $G = Z_2$.

From the introductory remarks we have $|G| = |F| + |O_1| + \cdots + |O_r|$, where F is the subgroup of trivial orbits and $O_1, \cdots, O_r$ are the nontrivial orbits. Hence, $(|G| - |F|)/2 \geqq r$ since $|O_i| \geqq 2$ for $i = 1, 2, \cdots, r$. From $k = r + |F|$ we obtain $k \leqq (|G| + |F|)/2$. If $G \neq Z_2$, then $F \neq G$ and therefore $[G:F] \geqq 2$; i.e., $|F| \leqq |G|/2$. Thus $k \leqq (3/4) \cdot |G|$, so $P_A(G) \leqq 3/4$ proving (ii). Observe that $P_A(Z_4) = 3/4$.

If G is decomposable, say

$$(*) \qquad G \cong \bigoplus_{i=1}^{s} G_i,$$

then it is routine to verify

$$(**) \qquad P_A(G) \leqq \prod_{i=1}^{s} P_{A_i}(G_i),$$

where, for $i = 1, 2, \cdots, s$, A_i is the group of automorphisms of G_i (equality prevails in (**) if G is decomposed into a direct sum of its Sylow subgroups). In view of this inequality, to obtain a bound for $P_A(G)$ it suffices to obtain a bound for the Sylow subgroups. To this end, assume $|G| = p^n$ for some prime p. If G is elementary abelian, then $P_A(G) = 2/p^n$. This follows since G is a Z_p-space. For cyclic G there is at least one element of order p^m for $m = 0, 1, \cdots, n$. As elements in the same

orbit must have equal orders, G has at least $n+1$ orbits. Since elements of the same order can be written as powers of elements with orders prime to p, the elements of a particular order form an orbit. Thus $P_A(G) = (n+1)/p^n$ when G is cyclic. The following lemma is helpful in establishing a general bound for $P_A(G)$ when G is a p-group.

LEMMA. *Let n be a positive integer greater than 1. The maximum value of $\prod_{i=1}^k n_i$, for $\sum_{i=1}^k n_i = n$, is*

$$3^{(n-i)/3} \cdot 2^{i/2}, \text{ where } \begin{cases} i = 4 \text{ if } n \equiv 1 \pmod 3 \\ i = 2 \text{ if } n \equiv 2 \pmod 3 \\ i = 0 \text{ if } n \equiv 0 \pmod 3). \end{cases}$$

Proof. The maximum occurs when no $n_i = 1$. Further, $(m-2)\cdot 2 > m$ if, and only if, $m > 4$. Thus each $n_i > 4$ can be replaced by $(n_i - 2) + 2$ in the partition and the associated product will be increased. If some $n_i = 4$, replacing it by $2+2$ leaves the product unchanged. If $2+2+2$ occurs in the partition, replacing it by $3+3$ increases the product. Hence the maximum occurs when each n_i is a two or a three. The conclusion of the lemma follows immediately.

We observe that the maximum product associated with the partitions of n is smaller than the corresponding product obtained from m if $n < m$.

PROPOSITION 1. *If $|G| = p^n$, then $P_A(G) \leqq 2\cdot(3/p^2)^{n/2}$.*

Proof. Suppose the invariants of G are $m_1, \cdots, m_k, 1, \cdots, 1$, where $\sum_{i=1}^k m_i = m$ and $i < j$ implies $m_i \geqq m_j$. Let G_e denote the summands of G of order p and G' denote the summands of G of order at least p^2. Thus $G \cong G' \oplus G_e$. From (**) we get

$$P_A(G) \leqq P_A(G')\cdot P_A(G_e) \leqq \left(\prod_{i=1}^k \frac{m_i+1}{p^{mi}}\right)\cdot\left(\frac{2}{p^{n-m}}\right)$$

$$({}^*_*) \qquad \leqq (2/p^n)\cdot \prod_{i=1}^k (m_i+1).$$

Since $\sum_{i=1}^k (m_i+1) = m+k$, maximizing k maximizes the sum. The largest value for k occurs when the m_i's are smallest (all twos, except for one three if m is odd). Taking k' to be the integer of $\{m/2, (m-1)/2\}$ and applying the lemma to $({}^*_*)$ wé have

$$P_A(G) \leqq \frac{2}{p^n}\cdot 3^{(m+k')/3} \leqq \frac{2}{p^n}\cdot 3^{(m+\frac{1}{2}m)/3}$$

$$\leqq \frac{2}{p^n}\cdot 3^{m/2} \leqq 2\cdot\left(\frac{3}{p^2}\right)^{n/2} .$$

PROPOSITION 2. *If* $0 < \rho < 1$, *then there is only a finite number of finite abelian groups* G *with* $P_A(G) \geqq \rho$.

Proof. We may choose a positive integer N and a prime q both so large that $2 \cdot (3/4)^{N/2} < \rho$ and $2 \cdot (3/q^2)^{1/2} < \rho$. If $P_A(G) \geqq \rho$ and p^j divides $|G|$, where p is a prime, then $j < N$ and $p < q$. This condition imposes an upper bound on the order of G. The result is clear.

PROPOSITION 3. *If* $\{G_n\}$ *is a sequence of finite abelian groups for which* $|G_n| \to \infty$ *as* $n \to \infty$, *then* $P_A(G_n) \to 0$ *as* $n \to \infty$.

Proof. Since $\{P_A(G_n)\}$ is bounded above by 1, the limit superior of $\{P_A(G_n)\}$ is finite. Indeed, $\limsup P_A(G_n) = 0$, otherwise we contradict Proposition 2. Thus $0 \leqq \liminf P_A(G_n) \leqq \limsup P_A(G_n) = 0$.

A problem. As a final remark we pose the following problem. Suppose G is a finite (not necessarily abelian) group and S is its set of subgroups. Let G operate on S by conjugation and consider $P_G(S)$. It is clear that $P_G(S) = 1$ if, and only if, each subgroup of G is normal. This is equivalent to G being abelian or Hamiltonian. The problem then is to determine if there exists some real number ρ, where $0 < \rho < 1$, for which $P_G(S) \leqq \rho$ when G is neither abelian nor Hamiltonian. The author conjectures $\rho = 2/3$. If this conjecture is true, the bound is sharp since $P_{S_3}(S) = 2/3$.

References

1. P. Erdös and P. Turan, On some problems of a statistical group-theory, IV, Acta Math. Acad. Sci. Hung., 19 (1968) 413–435.

2. W. H. Gustafson, What is the probability that two group elements commute? this MONTHLY, 80 (1973) 1031–1034.

3. S. Lang, Algebra, Addison-Wesley, Reading, Mass. 1965.

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SOME SUMS INVOLVING FRACTIONAL PARTS

L. CARLITZ

1. Put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}), \end{cases} \tag{1.1}$$

where x is real and $[x]$ denotes the greatest integer $\leqq x$. Berndt [**1**] has proved the following result:

THEOREM 1. *Let a, c be real, $a > 0$, $0 \leqq c < 1$ and n a positive integer. Then*

$$\tag{1.2} \tfrac{1}{2}\{g(0+0) + g(n-0)\} + \sum_{r=1}^{n-1} ((ar+c)) + \sum_{r=1}^{[d]} \left(\left(\frac{r-c}{a}\right)\right)$$
$$= \tfrac{1}{2}n(d+c-1) + \tfrac{1}{2}[d]([d]-2d+1)/a,$$

where $g(x) = ((ax+c))$ *and* $d = an + c$.

It is noted that

$$g(0+0) = \begin{cases} ((c)) & (c \neq 0) \\ -\frac{1}{2} & (c = 0) \end{cases}$$

and

$$g(n-0) = \begin{cases} ((d)) & (d \neq \text{integer}) \\ \ \frac{1}{2} & (d = \text{integer}). \end{cases}$$

Berndt gives two analytic proofs of this theorem and as corollaries obtains results due to Gauss, Eisenstein, Sylvester, Hacks and Cesaro (for references, see [**1**]).

It may be of interest to indicate a simple elementary proof of (1.2). It is convenient to use the Bernoulli function

$$\tag{1.3} \bar{B}_1(x) = x - [x] - \tfrac{1}{2}.$$

Consider

$$\tag{1.4} S = \sum_{r=1}^{n} \bar{B}_1(ar+c) + \sum_{s=1}^{m} \bar{B}_1\left(\frac{s-c}{a}\right),$$

where $m = [an+c]$. Using (1.3) we get

$$\tag{1.5} \begin{aligned} S &= \sum_{r=1}^{n} (ar + c - \tfrac{1}{2} - [ar+c]) + \sum_{s=1}^{m} \left(\frac{s-c}{a} - \tfrac{1}{2} - \left[\frac{s-c}{a}\right]\right) \\ &= \sum_{r=1}^{n} (ar+c-\tfrac{1}{2}) + \sum_{s=1}^{m} \left(\frac{s-c}{a} - \tfrac{1}{2}\right) - \sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} 1 - \sum_{s=1}^{m} \sum_{r=1}^{[(s-c)/a]} 1. \end{aligned}$$

Since

$$\sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} 1 = \sum_{s=1}^{[an+c]} \sum_{(s-c)/a \leqq r \leqq n} 1,$$

it follows that

$$\sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} 1 + \sum_{s=1}^{m} \sum_{r=1}^{[(s-c)/a]} 1 = \sum_{s=1}^{m} \left\{ \sum_{1 \leqq r \leqq (s-c)/a} 1 + \sum_{(s-c)/a \leqq r \leqq n} 1 \right\}$$

$$= \sum_{s=1}^{m} \sum_{r=1}^{n} 1 + \sum_{\substack{s-c=ar\\ 1\leqq r\leqq n\\ 1\leqq s\leqq m}} 1 = mn + \rho,$$

where ρ is the numbers of ordered pairs of integers r, s such that

$$1 \leqq r \leqq n,\ 1 \leqq s \leqq m,\ ar + c = s. \tag{1.6}$$

In the next place

$$\sum_{r=1}^{n} (ar + c - \tfrac{1}{2}) = \tfrac{1}{2}an(n+1) + (c - \tfrac{1}{2})n,$$

$$\sum_{s=1}^{m} \left(\frac{s-c}{a} - \frac{1}{2}\right) = \frac{1}{2a} m(m+1) - \left(\frac{c}{a} + \frac{1}{2}\right) m.$$

Substituting in (1.5) we get $S = \tfrac{1}{2}an(n+1) + (c-\tfrac{1}{2})n + (1/2a)m(m+1) - (c/a + \tfrac{1}{2})m - mn - \rho$. This completes the proof of

THEOREM 2. *Let* $a > 0$, $0 \leqq c < 1$, n *a positive integer and* $m = [an+c]$. *Then*

$$\sum_{r=1}^{n} \bar{B}_1(ar + c) + \sum_{s=1}^{m} \bar{B}_1\left(\frac{s-c}{a}\right) \tag{1.7}$$

$$= \tfrac{1}{2}an(n+1) + (c - \tfrac{1}{2})n + \frac{1}{2a} m(m+1) - \left(\frac{c}{a} + \frac{1}{2}\right) m - mn - \rho$$

where ρ *is the number of ordered pairs of integers* r, s *satisfying* (1.6).

It follows easily from (1.1), (1.3) and (1.7) that

$$\sum_{r=1}^{n} ((ar + c)) + \sum_{s=1}^{m} \left(\left(\frac{s-c}{a}\right)\right) \tag{1.8}$$

$$= \tfrac{1}{2}an(n+1) + (c - \tfrac{1}{2})n + \frac{1}{2a} m(m+1) - \left(\frac{c}{a} + \frac{1}{2}\right) m - mn$$

and it is not difficult to verify that (1.8) is equivalent to (1.2).

2. We recall ([**2**, Ch. 2]) that the Bernoulli polynomials may be defined by

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1} \qquad (|z| < 2\pi). \tag{2.1}$$

The Bernoulli function $\bar{B}_n(x)$ is defined by

$$\bar{B}_n(x) = B_n(x - [x]). \tag{2.2}$$

It follows from (2.1) that

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

so that

$$B_n(x+k) - B_n(x) = n \sum_{j=0}^{k-1} (x+j)^{n-1}. \tag{2.3}$$

Generalizing (1.4) we consider

$$S_p = \sum_{r=1}^{n} \bar{B}_p(ar+c) + (-a)^{p-1} \sum_{s=1}^{m} \bar{B}_p\left(\frac{s-c}{a}\right) \quad (p > 1). \tag{2.4}$$

By (2.2) and (2.3) we have

$$\begin{aligned}
\sum_{r=1}^{n} \bar{B}_p(ar+c) &= \sum_{r=1}^{n} B_p(ar+c-[ar+c]) \\
&= \sum_{r=1}^{n} \left\{ B_p(ar+c) - p \sum_{s=1}^{[ar+c]} (ar+c-s)^{p-1} \right\} \\
&= \sum_{r=1}^{n} B_p(ar+c) - p \sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} (ar+c-s)^{p-1},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{s=1}^{m} \bar{B}_p\left(\frac{s-c}{a}\right) &= \sum_{s=1}^{m} B_p\left(\frac{s-c}{a} - \left[\frac{s-c}{a}\right]\right) \\
&= \sum_{s=1}^{m} \left\{ B_p\left(\frac{s-c}{a}\right) - p \sum_{r=1}^{[(s-c)/a]} \left(\frac{s-c}{a} - r\right)^{p-1} \right\} \\
&= \sum_{s=1}^{m} B_p\left(\frac{s-c}{a}\right) - p(-a)^{-p+1} \sum_{s=1}^{m} \sum_{r=1}^{[(s-c)/a]} (ar+c-s)^{p-1}.
\end{aligned}$$

Thus, by (2.4),

$$\begin{aligned}
S_p = {} & \sum_{r=1}^{n} B_p(ar+c) + (-a)^{p-1} \sum_{s=1}^{m} B_p\left(\frac{s-c}{a}\right) \\
& - p \sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} (ar+c-s)^{p-1} - p \sum_{s=1}^{m} \sum_{r=1}^{[(s-c)/a]} (ar+c-s)^{p-1}.
\end{aligned}$$

Now

$$\begin{aligned}
& \sum_{r=1}^{n} \sum_{s=1}^{[ar+c]} (ar+c-s)^{p-1} + \sum_{s=1}^{m} \sum_{r=1}^{[(s-c)/a]} (ar+c-s)^{p-1} \\
&= \sum_{s=1}^{m} \left\{ \sum_{1 \leqq r \leqq (s-c)/a} (ar+c-s)^{p-1} + \sum_{(s-c)a \leqq r \leqq n} (ar+c-s)^{p-1} \right\} \\
&= \sum_{s=1}^{m} \sum_{r=1}^{n} (ar+c-s)^{p-1};
\end{aligned}$$

since $p > 1$, the overlap

$$\sum_{ar+c=s} (ar + c - s)^{p-1} = 0.$$

We have accordingly proved

$$S_p = \sum_{r=1}^{n} B_p(ar + c) + (-a)^{p-1} \sum_{s=1}^{m} B_p((s-c)/a) \tag{2.5}$$

$$- p \sum_{r=1}^{n} \sum_{s=1}^{m} (ar + c - s)^{p-1} \qquad (p > 1).$$

Since $p\ \sum_{s=1}^{m}(ar + c - s)^{p-1} = B_p(ar + c) - B_p(ar + c - m)$, (2.5) reduces to

$$S_p = \sum_{r=1}^{n} B_p(ar + c - m) + (-a)^{p-1} \sum_{s=1}^{m} B_p\left(\frac{s-c}{a}\right) \qquad (p > 1). \tag{2.6}$$

To carry out the summations on the right of (2.6) we make use of Bernoulli polynomials of order 2 which may be defined by [2, Ch. 6]

$$\frac{\omega_1\omega_2 z^2 e^{xz}}{(e^{\omega_1 z} - 1)(e^{\omega_2 z} - 1)} = \sum_{p=0}^{\infty} B_p^{(2)}(x \mid \omega_1, \omega_2)\frac{z^p}{p!} \tag{2.7}$$

$$(|\omega_1 z| < 2\pi, |\omega_2 z| < 2\pi).$$

It follows from (2.7) that

$$\sum_{p=0}^{\infty} \{B_p^{(2)}(x + \omega_2 \mid \omega_1, \omega_2) - B_p^{(2)}(x \mid \omega_1, \omega_2)\}\frac{z^p}{p!} = \frac{\omega_1\omega_2 z^2 e^{xz}}{e^{\omega_1 z} - 1},$$

so that, by (2.1), $B_p^{(2)}(\omega_1 x + \omega_2 \mid \omega_1, \omega_2) - B_p^{(2)}(\omega_1 x \mid \omega_1, \omega_2) = p\omega_1^{p-1}\omega_2 B_{p-1}(x)$. This gives

$$(p + 1)\omega_1^p\omega_2 \sum_{r=0}^{n-1} B_p\left(x + \frac{r\omega_2}{\omega_1}\right) = B_{p+1}^{(2)}(\omega_1 x + n\omega_2 \mid \omega_1, \omega_2) \tag{2.8}$$

$$- B_{p+1}^{(2)}(\omega_1 x \mid \omega_1, \omega_2).$$

Hence

$$\sum_{r=1}^{n} B_p(ar + c - m) = \frac{1}{(p + 1)a}\{B_{p+1}^{(2)}((n+1)a + c - m \mid 1, a) - B_{p+1}^{(2)}(a + c - m \mid 1, a)\},$$

$$\sum_{s=1}^{m} B_p\left(\frac{s-c}{a}\right) = \frac{1}{(p + 1)a^p}\{B_{p+1}^{(2)}(m + 1 - c \mid a, 1) - B_{p+1}^{(2)}(1 - c \mid a, 1)\}.$$

Since $B_n^{(2)}(x \mid \omega_1, \omega_2) = B_n^{(2)}(x \mid \omega_2, \omega_1)$, (2.6) becomes

$$S_p = \frac{1}{(p+1)a}\{B_{p+1}^{(2)}((n+1)a+c-m\,|\,1,a) - B_{p+1}^{(2)}(a+c-m\,|\,1,a)\} + \frac{(-1)^{p-1}}{(p+1)a}\{B_{p+1}^{(2)}(m+1-c\,|\,1,a) - B_{p+1}^{(2)}(1-c\,|\,1,a)\}. \tag{2.9}$$

This proves

THEOREM 3. *Let* $a > 0$, $0 \leqq c < 1$, $n \geqq 1$, $p > 1$ *and* $m = [an + c]$. *Then* $S_p = \sum_{r=1}^{n} \bar{B}_p(ar + c) + (-a)^{p-1} \sum_{s=1}^{m} \bar{B}_p((s-c)/a)$ *satisfies* (2.6) *and* (2.9).

The special case an equal to an integer is of some interest. Clearly $m = an$ and we may assume (without loss of generality) that $(m,n) = 1$.

Then [**2**, p. 21]

$$\sum_{r=1}^{n} \bar{B}_p\left(\frac{m}{n}r + c\right) = \sum_{r=1}^{n} \bar{B}_p\left(\frac{r}{n} + c\right) = n^{-p+1}\bar{B}_p(cn),$$

$$\sum_{s=1}^{m} \bar{B}_p\left(\frac{n}{m}(s-c)\right) = \sum_{s=1}^{m} \bar{B}_p\left(\frac{s}{m} - \frac{nc}{m}\right) = m^{-p+1}\bar{B}_p(-cn).$$

Also $\bar{B}_p(-x) = (-1)^p\bar{B}_p(x)$. This yields

THEOREM 4. *Let* m, $n \geqq 1$, $(m,n) = 1$, $p \geqq 1$ *and odd. Then*

$$n^{p-1}\sum_{r=1}^{n} \bar{B}_p\left(\frac{mr}{n} + c\right) + m^{p-1}\sum_{s=1}^{m} B_p\left(\frac{n}{m}(s-c)\right) = 0. \tag{2.10}$$

For $p = 1$ see Corollary 6 of [**1**].

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References

1. B. C. Berndt, A generalization of a theorem of Gauss on sums involving $[x]$, this MONTHLY, 82 (1975) 44–51.

2. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924.

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The proof is left to the interested reader.

7. Questions. The reader may have formed many questions in his own mind on the basis of the material presented. We offer the following general questions as typical of the ones we find interesting.

1. (The converse of (*I*)). For what finite groups G is it true that if $o\colon X(G) \to D(o(G))$ is an injection, then o is a lattice homomorphism?

2. Which finite groups are solvable and satisfy (SN)?

3. What are necessary and sufficient conditions for a solvable group to satisfy (N)? In this connexion, we note that if for a group G normality of subgroups is transitive, then $N(G) = SN(G)$. The solvable finite groups for which normality of subgroups is transitive have been characterized by G. Zacher [**5**].

4. What can be obtained by restricting the range of the mapping o, instead of restricting its domain? For example $D(n)$ possesses the sublattice $H(n)$ of all Hall divisors of n (a **Hall divisor** d of n is a divisor of n for which $(d, n/d) = 1$). We note that $H(n)$ is the maximal Boolean sublattice of $D(n)$. P. Hall [**2**, p. 199] showed that $H(o(G))$ is a subset of the range of o, for a finite group G, if and only if G is solvable.

5. Do there exist analogs of Lagrange's theorem for algebraic systems other than groups?

References

1. R. D. Carmichael, Introduction to the theory of groups of finite order, Dover, New York, 1956.

2. P. Hall, A characteristic property of soluble groups, J. London Math. Soc., 12 (1937) 198–200.

3. D. J. McCarthy, A survey of partial converses to Lagrange's theorem on finite groups, Trans. New York Acad. Sci. (2) 33 (1971) 586–594.

4. O. Ore, Structures and group theory II, Duke Math. J., 4 (1938) 247–269.

5. G. Zacher, Caratterizzazione dei t-gruppi finiti risolubili, Ricerche Mat., 1 (1952) 287–294.

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TESSELLATION OF THE PLANE WITH CONVEX POLYGONS HAVING A CONSTANT NUMBER OF NEIGHBORS

L. FEJES TÓTH

A set of convex polygons is said to form a **tessellation** if the polygons fill the Euclidean plane without overlapping and without interstices. Two polygons having a boundary point in common are called neighbors. If each polygon has the same number n of neighbors we speak of an n-neighbor tessellation. We define $K(n)$ as the least number of convex polygons whose congruent replicas can be put together to form an n-neighbor tessellation. It is known [**5**] that for any integer $n > 5$ the function $K(n)$ exists and we have $K(n) \leqq (n + 1)/2$.

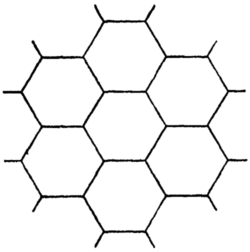

FIG. 1

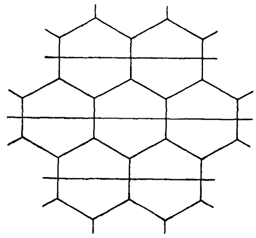

FIG. 2

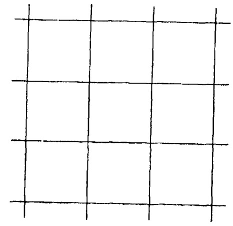

FIG. 3

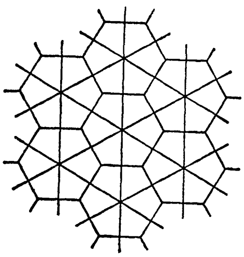

FIG. 4

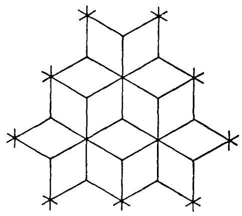

FIG. 5

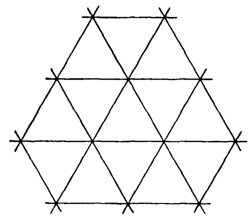

FIG. 6

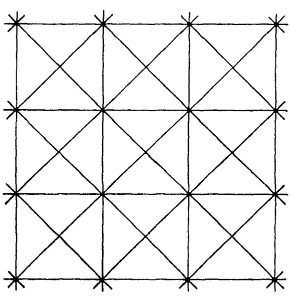

FIG. 7

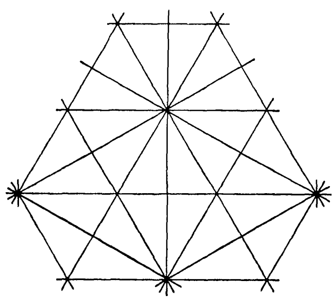

FIG. 8

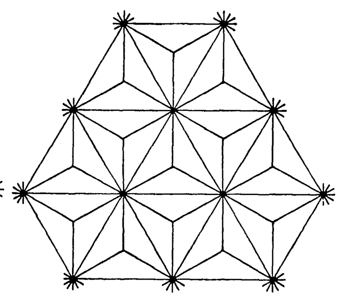

FIG. 9

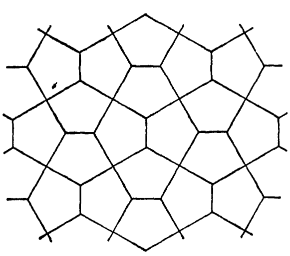

FIG. 10

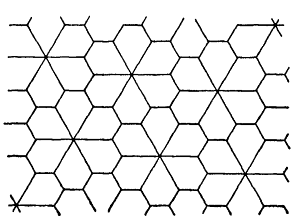

FIG.11

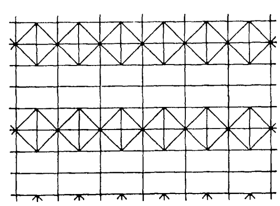

FIG. 12

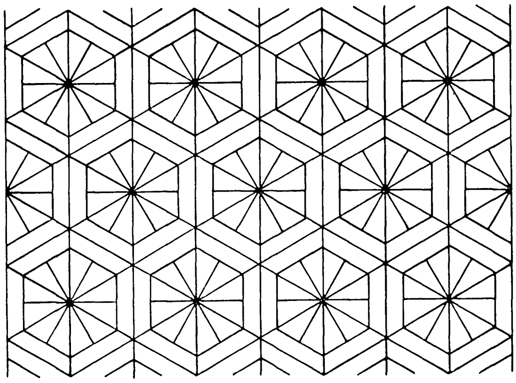

FIG. 13

The tessellations exhibited in Figs. 1–9 show that for

$$(*) \qquad n = 6, 7, 8, 9, 10\,12, 14, 16, 21$$

we have $K(n) = 1$. Does there exist an n-neighbor tessellation with congruent convex polygons for a value of n other than those enumerated in (*)? The author would be surprised if the answer turned out to be "Yes".

Figs. 12 and 13 show 11- and 13-neighbor tessellations with two kinds of convex polygon constructed by G. Fejes Tóth [5] and Wegner [4] respectively. Thus we have $K(11) \leqq 2$ and $K(13) \leqq 2$. This being all we know about $K(n)$, we phrase the following problem:

Try to give any further information about the function $K(n)$.

It would be especially interesting to know whether $K(n) \to \infty$ with n. It seems to be likely that the answer is "Yes," though the following example, due to H. Hanani, makes the conjecture doubtful if we drop the condition of the convexity of the faces.

In the 8-neighbor tessellation with squares, inscribe in each square a regular $4n$-gon ($n \geqq 4$) so that each $4n$-gon has a side in common with the four adjacent $4n$-gons. Decompose each $4n$-gon in $4n$ isosceles triangles. These triangles along with the concave $(4n-4)$-gons lying between the regular $4n$-gons constitute a tessellation consisting of two kinds of polygons. Though the number of neighbors is not constant, each face has approximately the same number of neighbors, namely either $4n$ or $4n+2$ or $4n+4$.

Further problems and results about packing of convex plates with certain conditions imposed on the number of neighbors of each plate are contained in [**1**, $\cdots$, **11**].

I thank Raphael Robinson for the following remarks. In all the first 9 examples, with one kind of tile, the tiles meeting at a point all have equal angles. Consider such a tiling, using a tile with s sides and angles $2\pi/a_k$ ($k = 1, 2, \cdots, s$). Then we must

have

$$\sum_{k=1}^{s} (a_k - 2) = n, \quad \sum_{k=1}^{s} \frac{a_k - 2}{a_k} = 2.$$

Figures 1–9 correspond to the following values of the parameters: $(3,3,3,3,3,3)$, $(3,3,3,4,4)$, $(4,4,4,4)$, $(3,4,6,4)$, $(3,6,3,6)$, $(6,6,6)$, $(4,8,8)$, $(4,6,12)$, $(3,12,12)$. There is also an alternative solution for $n = 7$ with the parameters $(3,3,4,3,4)$ in that order. (Fig. 10). On the other hand, $(3,4,4,6)$, and $(3,3,6,6)$ are impossible. It is not hard to find all solutions of the second equation displayed above. Besides the 9 mentioned, there are 8 others, namely $(3,3,3,3,6)$, $(3,3,4,12)$, $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$, $(4,5,20)$, $(5,5,10)$. However, all of the triangles are impossible, since, as is easily seen, we can have a_1 odd only if $a_2 = a_3$, and the same for other positions. The case $(3,3,4,12)$ is also found to be impossible in all permutations, but $(3,3,3,3,6)$ is possible, and leads to an alternative solution for 8 neighbors (Fig. 11). In this case, unlike all the others, there are right-hand and left-hand versions. Thus the problem with the stated restriction can be completely solved.

[I believe I recall seeing $(3,3,4,3,4)$ among the many unusual tessellations at the Taj Mahal—R. K. G.]

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References

1. L. Fejes Tóth, Scheibenpackungen konstanter Nachbarnzahl, Acta Math. Acad. Sci. Hungar., 20 (1969) 375–381.

2. R. M. Robinson, Finite sets of points on a sphere with each nearest to five others, Math. Ann., 179 (1969) 296–318.

3. L. Fejes Tóth, Remarks on a theorem of R. M. Robinson, Studia Sci. Math. Hungar., 4 (1969) 441–445.

4. G. Wegner, Bewegungsstabile Packungen konstanter Nachbarnzahl, Studia Sci. Math. Hungar., 6 (1971) 431–438.

5. G. Fejes Tóth, Über Parkettierungen konstanter Nachbarnzahl, Studia Sci. Math. Hungar., 6 (1971) 133–135.

6. K. Böröczky, Über die Newtonsche Zahl regulärer Vielecke, Periodica Math. Hungar., 1 (1971) 113–119.

7. J. Linhart, Über einige Vermutungen von L. Fejes Tóth, Acta Math. Acad. Sci. Hungar., 24 (1973).

8. ———, Endliche n-nachbarn Packungen in der Ebene und auf der Kugel, Periodica Math. Hungar.

9. P. Gács, Packing convex sets in the plane with a great number of neighbours, Acta Math. Acad. Sci. Hungar., 23 (1972) 383–388.

10. L. Fejes Tóth, Five-neighbour packing of convex discs, Periodica Math. Hungar., 4 (1973), 221–229.

11. E. Makai, Jr., Five-neighbour packing of convex discs, Periodica Math. Hungar., 5 (1974).

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BOUNDS FOR DIFFERENCE QUOTIENTS AND DERIVATES

ROGER CHALKLEY

1. Introduction. For a continuous real-valued function defined on an interval of the real line, the difference quotients and the derivates have the same lower bounds and the same upper bounds. Information is given in [1], [3], [4], [5], and [6].

We present a simple proof for the following reformulation of this important result.

THEOREM. *Suppose f is a real-valued function defined and continuous on an interval I of the real line. Let T_1 be the range of the extended-real-valued function D^+f, let T_2 be the range of D_+f, let T_3 be the range of D^-f, let T_4 be the range of D_-f, and set*

$$S = \{(f(b)-f(a))/(b-a)\colon a, b \in I;\ a < b\}.$$

Then, $\inf T_n = \inf S$ *and* $\sup T_n = \sup S$, *for* $n = 1,2,3,4$.

2. Explanation. We use the notation of [2], where

$$(D^+f)(x) = \limsup_{h\to 0^+}\frac{f(x+h)-f(x)}{h},\quad (D_+f)(x) = \liminf_{h\to 0^+}\frac{f(x+h)-f(x)}{h},$$

$$(D^-f)(x) = \limsup_{h\to 0^-}\frac{f(x+h)-f(x)}{h},\quad (D_-f)(x) = \liminf_{h\to 0^-}\frac{f(x+h)-f(x)}{h}.$$

If the interval I has zero length, then the sets S, T_1, T_2, T_3, T_4 are vacuous and the conclusion is: $\infty = \infty$ and $-\infty = -\infty$. Thus, we shall suppose I has positive (finite or infinite) length.

We establish the conclusion for $n = 1$. Only slight changes are needed for $n = 2,3,4$. Our argument is based upon the following easily-verified lemma.

LEMMA. *If F and G are real-valued functions defined on an interval $(0,r)$ and if there exists a real number L with $\lim_{h\to 0^+}G(h) = L$, then*

$$\limsup_{h\to 0^+}(F+G)(h) = \limsup_{h\to 0^+}F(h) + \lim_{h\to 0^+}G(h).$$

3. Proof of $\inf T_1 \leqq \inf S$ and $\sup S \leqq \sup T_1$. Suppose s belongs to S. We write $s = (f(b)-f(a))/(b-a)$, where $a, b \in I$ and $a < b$. We define a function g on the closed interval $[a,b]$ by

$$g(x) = f(x) - sx + sa - f(a).$$

Since g is continuous on $[a, b]$ and $g(b) = g(a)$, there exists a number c in $[a, b)$ at which g assumes its maximum value and there exists a number d in $[a, b)$ at which g assumes its minimum value. We find $c, d \neq \sup I$,

$$0 \geqq \limsup_{h \to 0^+} \frac{g(c+h) - g(c)}{h} = (D^+f)(c) - s,$$

$$0 \leqq \limsup_{h \to 0^+} \frac{g(d+h) - g(d)}{h} = (D^+f)(d) - s, \text{ and}$$

$$\inf T_1 \leqq (D^+f)(c) \leqq s \leqq (D^+f)(d) \leqq \sup T_1 .$$

Thus, we obtain $\inf T_1 \leqq \inf S$ and $\sup S \leqq \sup T_1$.

4. Proof of inf $S \leqq$ inf T_1 and sup $T_1 \leqq$ sup S.

(i) Suppose M is a real number and $\inf T_1 < M$. Then, numbers x_1 in I and $h_1 > 0$ exist and satisfy

$$(D^+f)(x_1) < M, \; x_1 + h_1 \in I, \text{ and } \frac{f(x_1 + h_1) - f(x_1)}{h_1} < M.$$

This yields $\inf S < M$.

(ii) Suppose m is a real number and $\sup T_1 > m$. Then, numbers x_2 in I and $h_2 > 0$ exist and satisfy

$$(D^+f)(x_2) > m, \; x_2 + h_2 \in I, \text{ and } \frac{f(x_2 + h_2) - f(x_2)}{h_2} > m.$$

This gives $\sup S > m$.

The arguments of (i) and (ii) establish the desired conclusion.

5. Several applications. *Suppose f satisfies the hypothesis of the theorem, and let J denote the interval*

$$J = I - \{\sup I\}.$$

We notice: $\inf S \geqq 0$ if and only if $\inf T_1 \geqq 0$; thus, *f is non-decreasing on I if and only if D^+f is non-negative on J*. Also, $\sup S \leqq 0$ if and only if $\sup T_1 \leqq 0$; thus, *f is non-increasing on I if and only if D^+f is non-positive on J*. In particular, *f is a constant function on I if and only if D^+f is the zero function on J*.

We observe: $\inf S = \sup S$ if and only if $\inf T_1 = \sup T_1$; thus, *the graph of f is a subset of a straight line if and only if D^+f is constant on J*.

References

1. U. Dini, Fondamenti per la Teorica delle Funzioni di Variabili Reali, Nistri, Pisa, 1878, 178–187.

2. L. Scheeffer, Allgemeine Untersuchungen über Rektifikation der Kurven, Acta Math. 5 (1884) 52–53.

3. Ch. -J. de la Vallée Poussin, Cours d'Analyse Infinitésimale, 1 (3rd. ed.), Gauthier-Villars, Paris, 1914, 98–99.

4. E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. 1 (3rd. ed.), Cambridge, 1927, (reprinted by Dover, New York, 1957), 380–383.

5. C. Carathéodory, Vorlesungen über Reelle Funktionen (2nd. ed.), Teubner, Leipzig, 1927, (reprinted by Chelsea, New York, 1948), 530–534.

6. R. P. Boas, Jr., A Primer of Real Functions, Carus Mathematical Monograph 13, 1960, 122–124.

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AN ELEMENTARY APPROACH TO THE PROBLEM OF EXTENDING CONFORMAL MAPS TO THE BOUNDARY

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Let Ω be a bounded, simply connected region in the complex plane $\mathbf{C}$. A well-known theorem of C. Carathéodory ([**1**, p. 86], [**2**, p. 96], [**3**, p. 367]) states that if $\partial\Omega$ is a simple closed curve, then every conformal one-to-one map of Ω onto the unit disc Δ can be extended to a homeomorphism of their respective closures. This result has many applications, one of the most common being a solution of the Dirichlet problem for Jordan regions, given that the problem has been solved for discs.

Most introductory textbooks on complex analysis include a proof of the Riemann mapping theorem but stop short of investigating, in any depth, the boundary behavior of the mapping functions. One reason for the latter is that anything approaching a self-contained account of the boundary behavior of a mapping function necessitates delving into some of the deeper aspects of the topology of the plane. Few authors of these introductory textbooks (as well as the teachers who use their books) feel that the return from such an investigation is worth the investment in space and time that is required. In this note we discuss an alternative which is more accessible than that of studying boundary behavior in the context of Jordan regions. It is that of considering bounded, simply connected regions whose boundary consists entirely of *simple* boundary points as defined in [**4**, p. 279].

A point $\beta \in \partial\Omega$ is called *simple* if to each sequence $\{\alpha_n\}$ in Ω such that $\alpha_n \to \beta$, there corresponds a curve (= continuous function) γ on $[0,1]$ and a strictly increasing sequence $\{t_n\}$ in $[0,1)$ such that $t_n \to 1$, $\gamma(t_n) = \alpha_n$, and $\gamma(t) \in \Omega$ for $0 \leqq t < 1$. For a certain Ω, it may well be that a straightforward verification will show that each boundary point of Ω is simple. In that case the following theorem can be applied.

THEOREM. *Let Ω be a bounded, simply connected region such that each boundary point of Ω is simple. Then each conformal equivalence of Ω onto Δ can be extended to a homeomorphism of $\overline{\Omega}$ onto $\overline{\Delta}$.*

The proof (as it appears in [**4**, pp. 279–282]) is accomplished by stages. The first of these—which in my opinion is the only one which might pose a problem of presentation in an introductory course—is as follows.

Step 1. Let Ω be a bounded, simply connected region and suppose that $\beta \in \partial\Omega$ is simple. If f is a conformal equivalence of Ω onto Δ, then f has a continuous extension to $\Omega \cup \{\beta\}$.

The method of proof which appears in [**4**, pp. 280, 281] requires a certain amount of knowledge about the boundary behavior of bounded analytic functions on the disc Δ. In particular, Fatou's theorem on the existence almost everywhere of radial limits for such functions is used, as are other results which require some background in the Lebesgue theory. Our principal aim, then, is to give a proof of Step 1 which is elementary in that, first, no knowledge of measure theory is assumed and, second, only a small amount of topology enters into the discussion. Our argument uses ideas that appear in [**1**] and [**4**], and will require the following lemma.

LEMMA. *Suppose Ω is an open set in the plane, $z_0 \in \Omega$, and the circle with center z_0 and radius r has an arc lying in the complement of $\bar{\Omega}$ which subtends an angle greater than π at z_0. Let g be a continuous function on $\bar{\Omega}$ which is analytic on Ω. If $|g(\zeta)| \leqq M$* for *all $\zeta \in \bar{\Omega}$ while $|g(\zeta)| \leqq \varepsilon$ for all $\zeta \in \{z: |z - z_0| < r\} \cap \partial\Omega$, then $|g(z_0)| \leqq \varepsilon^{\frac{1}{2}} M^{\frac{1}{2}}$.*

Proof. Assume, without loss of generality, that $z_0 = 0$. Put $U = \Omega \cap (-\Omega) \cap \{z: |z| < r\}$ and define h on $\bar{U}$ by $h(\zeta) = g(\zeta)g(-\zeta)$. It is readily verified that $\bar{U} \subset \bar{\Omega} \cap (-\bar{\Omega}) \cap \{z: |z| < r\}$ and hence if $\zeta \in \partial U$, then ζ or $-\zeta$ is in $\{z: |z| < r\} \cap \partial\Omega$. Consequently, $|h(\zeta)| \leqq \varepsilon M$ for $\zeta \in \partial U$; thus, by the maximum principle, $|h(\zeta)| \leqq \varepsilon M$ for all $\zeta \in \bar{U}$. The desired conclusion now follows by taking $\zeta = 0$.

Proof of Step 1. Assume the statement of Step 1 is false. This implies that there is a simple boundary point β, a sequence $\{\alpha_n\}$ in Ω which converges to β, and distinct complex numbers w_1 and w_2 of modulus 1, such that $f(\alpha_{2j-1}) \to w_1$ while $f(\alpha_{2j}) \to w_2$. Let p be the midpoint of the positively oriented arc of $\partial\Delta$ from w_1 to w_2 and choose points a and b in $\partial\Delta$ equidistant from p and close enough to p for figure 1 to obtain. Let γ and $\{t_n\}$ be as in the definition of simple boundary point. It results in no loss of generality to assume that $f(\alpha_{2j-1}) \in W_1$ and $f(\alpha_{2j}) \in W_2$ for all j, and that $|f(\gamma(t))| > 1/2$ for all t. Next, a routine argument shows that for each j, there exist x_j and y_j with $t_{2j-1} < x_j < y_j < t_{2j}$ and one of the following holding:

(1) $$f(\gamma(x_j)) \in (0, a),\ f(\gamma(y_j)) \in (0, b),\ \text{and } f(\gamma(t))$$

is in the open sector $ab0a$ for $x_j < t < y_j$, or

(2) $$f(\gamma(x_j)) \in (0, d),\ f(\gamma(y_j)) \in (0, c)\ \text{and } f(\gamma(t))$$

is in the open sector $d0cd$ for $x_j < t < y_j$.

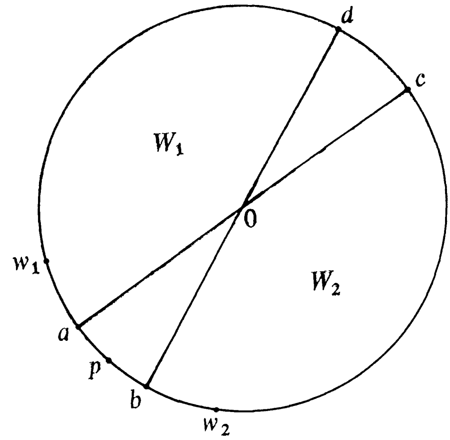

FIG. 1

So (1) holds for infinitely many j or (2) holds for infinitely many j. Assume that the former is the case and let J be the infinite set $\{j:(1) \text{ is true}\}$. For $j \in J$ define γ_j on $[0,1]$ by the rule

$$\gamma_j(t) = \begin{cases} (t/x_j)f(\gamma(x_j)), & 0 \leqq t \leqq x_j \\ f(\gamma(t)), & x_j \leqq t \leqq y_j \\ \dfrac{1-t}{1-y_j}f(\gamma(y_j)), & y_j \leqq t \leqq 1. \end{cases}$$

Use γ_j^* to denote the range of γ_j and Ω_j to denote that component of $\mathbf{C}\backslash\gamma_j^*$ such that $\frac{1}{2}p \in \Omega_j$. Obviously, $\partial\Omega_j \subset \gamma_j^*$ and one can show that $\Omega_j \subset \Delta$. (Here is one argument: the winding number of γ_j with respect to $\frac{1}{2}p$ is 1 while the winding number with respect to any point in $\mathbf{C}\backslash\Delta$ is 0.) Let r be a positive number less than $\frac{1}{2}|a-b|$ and choose a point z_0 on the open radius $(0,p)$ and so near to p that the circle with center z_0 and radius r meets $\mathbf{C}\backslash\bar{\Delta}$ in an arc of length greater than πr. Call the inside of this circle $D(z_0,r)$. Then for sufficiently large $j \in J$, $|f(\gamma(t))| > |z_0|$ for all t in $[t_{2j-1}, t_{2j}]$; so for these j we have $z_0 \in \Omega_j$. Further, if ζ is in $\partial\Omega_j \cap D(z_0,r)$, then ζ is in $\{f(\gamma(t)): t_{2j-1} \leqq t \leqq t_{2j}\}$ and hence $f^{-1}(\zeta)$ belongs to $\gamma([t_{2j-1}, t_{2j}])$. Define $\varepsilon_j = \sup\{|f^{-1}(\zeta) - \beta| : \zeta \in \partial\Omega_j \cap D(z_0,r)\}$ and $M = \sup\{|f^{-1}(z) - \beta| : z \in \Delta\}$. The lemma implies that

$$|f^{-1}(z_0) - \beta| \leqq \varepsilon_j^{\frac{1}{2}} M^{\frac{1}{2}}.$$

Since ε_j can be made as small as we please by taking $j \in J$ sufficiently large, we have $f^{-1}(z_0) = \beta$. This is absurd and so the proof of Step 1 is now complete.

The object of this note, which is an elementary proof of the theorem dealing

with boundary behavior, can now be accomplished by proceeding exactly as in [4, p. 281, 282]. We refer the reader there for the remaining details.

References

1. C. Carathéodory, Conformal Representation, 2nd ed., Cambridge Tracts in Mathematics and Mathematics Physics no. 28, Cambridge University Press, London, 1952.
2. ———, Theory of Functions of a Complex Variable, vol. II, Chelsea, New York, 1960.
3. E. Hille, Analytic Function Theory, vol. II, Ginn and Co., Boston, 1962.
4. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

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COMPLETIONS OF GROUPS

T. K. TEAGUE AND C. L. COX

A group is called **complete** (or **radicable** or **divisible**) if each of its elements is an nth power for each integer n. Properties of complete Abelian groups are well known [1, p. 168]. Methods usually employed to embed an arbitrary group in a complete group require either wreath products [3, p. 99] or free products with amalgamated subgroups [2, p. 254]. We present here several embedding methods which can be introduced in a beginning abstract algebra course.

Denote the natural numbers by N and the first m natural numbers by X_m.

Let S be a nonempty set. Set $S^1 = S$ and recursively define S^{n+1} to be $S^n \times S$. Denote the group of permutations on S by $P(S)$.

We shall prove that $P(S)$ can be embedded in a complete group. Since any group G is isomorphic to its right regular representation in $P(G)$, the desired embedding for G follows.

THEOREM 1. *There is a monomorphism from $P(S)$ into $P(S \times N)$ where every element in the image of $P(S)$ is an nth power in $P(S \times N)$ for every integer n.*

Proof. Define $\theta : P(S) \to P(S \times N)$ by $(s,i)(\lambda\theta) = (s\lambda, i)$ for $s \in S$, $i \in N$, and $\lambda \in P(S)$. Then θ is a monomorphism. Now let $\lambda \in P(S)$ and let n be an integer greater than zero. Define $\sigma \in P(S \times N)$ in the following way. For $s \in S$, $i \in N$ write $i = kn + r$, where $1 \leqq r \leqq n$. Put $(s,i)\sigma = (s, i+1)$ if $r \leqq n-1$; while if $r = n$, put $(s,i)\sigma = (s\lambda, kn+1)$. Then $\sigma^n = \lambda\theta$.

By modifying the preceding proof, we may construct a similar embedding of $P(S)$ into $P(S^\infty)$, where S^∞ denotes all sequences $(s_1, \cdots, s_i, \cdots)$ from S. Define $\theta : P(S) \to P(S^\infty)$ by

$$(s_1, \cdots, s_i, \cdots)(\lambda\theta) = (s_1\lambda, \cdots, s_i\lambda, \cdots)$$

for $\lambda \in P(S)$. Then θ is a monomorphism. For $\lambda \in P(S)$, $n \in N$, define $\sigma \in P(S^\infty)$ by

$$(s_1, \cdots, s_i, \cdots)\sigma = (s_n\lambda, s_1, s_2, \cdots, s_{n-1}, \cdots, t_i, \cdots).$$

To define t_i, write $i = kn + r$, where $1 \leqq r \leqq n$. Set $t_i = s_{i-1}$ if $r > 1$ and $t_i = [s_{(k+1)n}]\lambda$ if $r = 1$. Then $\sigma^n = \lambda\theta$.

No full permutation group $P(S)$ is complete if S has at least 2 elements. To prove this, assume δ^2 is a 2-cycle for $\delta \in P(S)$. Then δ has order 4 and thus possesses an orbit with 4 elements. This implies that δ^2 has at least 2 orbits with 2 elements each. This contradiction shows that no 2-cycle in $P(S)$ is a square in $P(S)$.

Our embedding of $P(S)$ in a complete group is similar to the proof of [3, Theorem III.5.h]. Let $G_1 = P(S)$ and, for each positive integer k, inductively define G_{k+1} to be a group containing G_k in which every element of G_k is an nth power for every integer n. Such a group exists by Theorem 1. In fact, if $G_k \cong P(S \times N^{k-1})$, then we may take $G_{k+1} \cong P(S \times N^k)$. Alternatively, G_{k+1} may be chosen isomorphic to $P(G_k \times N)$ regardless of the construction of G_k. In any case, the union $\bigcup_{k=1}^{\infty} G_k$ of the resulting ascending chain is a complete group containing $P(S)$.

We say a group G can be **n-embedded** in the group H if there is a monomorphism $\theta: G \to H$ where $g\theta$ is an nth power for each g in G.

The proof of the following theorem is similar to the proof of Theorem 1.

THEOREM 2. *Let S be a nonempty set and n a positive integer. Then $P(S)$ can be n-embedded in $P(S \times X_n)$.*

An embedding into $P(S^n)$ follows similarly.

If we set $S = X_m$ in Theorem 2, we see that the symmetric group $S_m = P(X_m)$ can be n-embedded in S_{mn}. Thus every group of order m can be n-embedded in S_{mn}.

To apply Theorem 2 to finite symmetric groups, it is helpful to exhibit an embedding from S_m into $S_{mn} = P(X_{mn})$. Any element of X_{mn} can be written uniquely in the form $km + r$, where $0 \leqq k \leqq n-1$ and $1 \leqq r \leqq m$. Define $\theta: S_m \to S_{mn}$ by $(km + r)(\lambda\theta) = km + r\lambda$ for each λ in S_m. Then θ is a monomorphism. For $\lambda \in S_m$, define $\sigma \in S_{mn}$ in the following way. Put $(km + r)\sigma = (k+1)m + r$ if $k < n-1$; while if $k = n-1$, put $(km + r)\sigma = r\lambda$. Then $\sigma^n = \lambda\theta$.

For example, $(1,2,3)(4,5)$ in S_5 is mapped to

$$\begin{aligned}&(1,2,3)(4,5)(6,7,8)(9,10)(11,12,13)(14,15)\\ &\quad = [(1,6,11,2,7,12,3,8,13)(4,9,14,5,10,15)]^3\end{aligned}$$

in S_{15} and to $(1,2,3)(4,5)(6,7,8)(9,10) = [(1,6,2,7,3,8)(4,9,5,10)]^2$, in S_{10}.

References

1. A. G. Kurosh, The Theory of Groups, Vol. I, Chelsea, New York, 1955.
2. ———, The Theory of Groups, Vol. II, Chelsea, New York, 1956.
3. E. Schenkman, Group Theory, Van Nostrand, Princeton, N. J. 1956.

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NUMERICAL DIFFERENTIATION FOR CALCULUS STUDENTS

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It is becoming fashionable to make the standard calculus course more "relevant" or "applied" by the addition of numerical methods, perhaps at the expense of more traditional topics, or in a supplementary or "laboratory" course attached to the regular course. Of course, the student must use these methods to solve problems with the aid of a computer, or else they will be meaningless to him. The first trickles of an anticipated flood of books presenting supplementary numerical material for calculus courses have already appeared. In such books the subject of integration is handled in predictable fashion: upper and lower sums, the trapezoidal rule, Simpson's rule, and perhaps some discussion of error estimates. Indeed, these subjects are also in most of the standard calculus books, but the problems posed there for the student are necessarily contrived or trivial when it cannot be assumed that there is a computer handy with which to do the arithmetic.

In the same manner, the method proposed in supplementary texts for numerical differentiation is usually the method first presented in the standard text: compute a limit of difference quotients. The real objective of such material in computer-oriented texts is to *illustrate* the concepts of limit and derivative, but nevertheless this is usually the only differentiation method presented. Difference quotients have a disturbing tendency to turn out to be 0 when Δx gets small enough that the computer cannot tell the difference between $f(x)$ and $f(x+\Delta x)$. The instructor must then explain that the terminal zeros in the computed sequence are to be ignored, and the "real" limit is the last non-zero entry (unless $f'(x) = 0$, of course). This procedure may lead the perceptive student to conclude that a computer is not a very useful tool for differentiating a function.

The purpose of this note is to point out that simple, effective numerical differentiation formulas are readily accessible at the first-year calculus level, and that the derivation of these formulas *and* their error estimates can reveal a useful application of a very elementary result, namely Rolle's Theorem.

Let us suppose that our problem is to compute $f'(0)$. This choice is a matter of convenience for the derivation, and the result is easily translated elsewhere on the x-axis later. We shall approximate $f'(0)$ by first fitting a polynomial $p(x)$ to conveniently chosen points on the graph of f, and then computing $p'(0)$, a trivial task. In deriving Simpson's rule, the instructor will have already carried out the fit of a quadratic polynomial to the points at $-h, 0, h$. Without repeating the details, we simply note that one finds easily:

$$p'(0) = \frac{1}{2h}[f(h) - f(-h)]. \tag{1}$$

(This is also the slope of the chord joining the points at $-h$ and h.) Similarly, one

may fit a quartic polynomial $p(x)$ to five points at $x = 0, \pm h, \pm 2h$. This leads to five linear equations in the coefficients of p. Only the coefficient of x (i.e., $p'(0)$) is needed, and one easily solves for this coefficient:

$$p'(0) = \frac{1}{12h}[f(-2h) - 8f(-h) + 8f(h) - f(2h)]. \tag{2}$$

Formula (2) is presumably "better" than formula (1), but how much better? How does one choose h in each case to get an acceptable approximation for $f'(0)$? The answers to such questions depend on having a reasonable error estimate, which we now provide in a form for mathematicians. For students, one would want to consider only the cases $k = 1$ and 2 of the following theorem. Even in the $k = 1$ case, the theorem says that quite good answers for $f'(0)$ can be obtained from formula (1) for values of h that are only moderately small (on the order of, say, 10^{-4}) provided f is reasonably "nice". Such an h is large enough to avoid the numerical anomalies inherent in computation of $f'(0)$ via a "limit" of difference quotients.

THEOREM. *Let f be a function with a continuous $(2k + 1)$-th derivative on the interval $[-kh, kh]$, where k is a positive integer. Let p be a polynomial of degree $2k$ that agrees with f at the $2k + 1$ points $0, \pm h, \cdots, \pm kh$. Then*

$$|f'(0) - p'(0)| < Kh^{2k} \Big/ \binom{2k}{k}(2k + 1), \tag{3}$$

where K is any bound for $|f^{(2k+1)}(x)|$ on $[-kh, kh]$.

Proof. Let $E(x) = f(x) - p(x)$, and note that we seek a bound for $|E'(0)|$. E has as roots at least the $2k + 1$ numbers $0, \pm h, \cdots, \pm kh$, and so does the polynomial $P(x) = \prod_{j=-k}^{k}(x - jh)$. Hence these numbers are also roots of any function of the form

$$F(x) = E(x) - cP(x), \tag{4}$$

where c is a constant. It follows from Rolle's Theorem, for any such F, that $F'(x)$ has $2k$ roots other than 0. The choice of $c = E'(0)/P'(0)$ makes 0 a root of F' as well, so that F' has $2k + 1$ distinct roots in $(-kh, kh)$. Note that $P'(0)$ is the coefficient of x in $P(x)$, i.e., $\prod_{j \neq 0}(-jh)$, so that

$$c = (-1)^k E'(0)/(k!)^2 h^{2k}. \tag{5}$$

By another (multiple) application of Rolle's Theorem to F', $F^{(2k+1)}$ must have at least one root in $(-kh, kh)$, say t. Now one may compute directly from (4) and the definitions of E and P that

$$F^{(2k+1)}(x) = f^{(2k+1)}(x) - c(2k + 1)!. \tag{6}$$

Setting $x = t$, and using (5), we obtain from (6) the equation

$$0 = f^{(2k+1)}(t) \pm E'(0)(2k+1)!/(k!)^2 h^{2k}. \tag{7}$$

Finally, solving (7) for $E'(0)$ and using the assumed bound K, we conclude that

$$\left|E'(0)\right| \leqq K(k!)^2 h^{2k}/(2k+1)!,$$

which is equivalent to (3).

The questions raised earlier may now be answered. The error in (1) is roughly "proportional" to h^2 while that in (2) is "proportional" to h^4. The proportionality constants cannot reasonably be evaluated, of course, since they depend on bounds for the third and fifth derivatives, while we are still computing a single value of the first derivative. Nevertheless, the theorem provides strong evidence that values of h large enough to avoid numerical difficulties due to cancellation will produce highly accurate values of the derivative. The usual pragmatic approach to determining how small h should be is to successively halve h, recomputing (1) or (2) until halving h no longer makes a difference in the desired number of decimal places.

```
 10•            LET F(X)=SQRT(COS(X*X-1)+X*X)/(3*X*X+17)**(1/3);
 20•            LET DF(X,H)=(F(X-2*H)-8*F(X-H)+8*F(X+H)-F(X+2*H))/(12*H);
 30•            GET LIST(A,B,S);
 40•            PUT EDIT('X','F(X)','DF(X)','H')(X(3),A,(3) (X(8),A));
 50•            DO X=A TO B BY S;
 60•            H=•25;
 70•            Y=DF(X,H);
 80•            DO  WHILE(H>•ØØØØØØØ1);
 90•            H=H/2;
100•            Z=DF(X,H);
110•            IF ABS(Y-Z)>=•ØØØØØ1 THEN Y=Z; ELSE GO TO OUT;
120•            END ;
130•            PUT EDIT(X,F(X))(F(6,2),F(12,5));
140•            GO TO END;
150•    OUT:    PUT EDIT(X,F(X),Y,2*H)(F(6,2),(2) F(12,5),E(13,3));
160•    END:    END ;
170•            STOP ;
?EXECUTE
A
?Ø
B
?5
S
?•5
   X       F(X)        DF(X)        H
  Ø•ØØ    Ø•28587     Ø•ØØØØØ     2•5ØØE-Ø1
  Ø•5Ø    Ø•37983     Ø•3Ø393     6•25ØE-Ø2
  1•ØØ    Ø•521ØØ     Ø•2Ø84Ø     3•125E-Ø2
  1•5Ø    Ø•55721    -Ø•Ø5376     3•125E-Ø2
  2•ØØ    Ø•5647Ø     Ø•24438     3•125E-Ø2
  2•5Ø    Ø•78937     Ø•4321Ø     1•562E-Ø2
  3•ØØ    Ø•84288    -Ø•1119Ø     1•562E-Ø2
  3•5Ø    Ø•9369Ø     Ø•39413     1•562E-Ø2
  4•ØØ    Ø•97Ø94    -Ø•Ø3Ø38     7•812E-Ø3
  4•5Ø    1•Ø78Ø4     Ø•Ø15Ø3     1•562E-Ø2
  5•ØØ    1•11693     Ø•29717     7•812E-Ø3
```

The following example illustrates this via a short *PL/I* program (see p. 286) that uses formula (2) and tabulates f and f' for

$$f(x) = \frac{[\cos(x^2 - 1) + x^2]^{\frac{1}{2}}}{(3x^2 + 17)^{1/3}}.$$

The example is artificially complicated, of course, as textbook examples are often artificially simple. By calculating $f'(x)$, the diligent student will discover $f'(0) = 0$, but not much else about this function. The tabulation shows the existence of at least seven relative extrema on $[0, 5]$ and also provides evidence of an absolute minimum at 0 (note symmetry). A closer tabulation would reveal another pair of extrema in $[4.5, 5]$, plus approximate locations of extrema and inflection points.

Remarks and acknowledgments. It has been pointed out by Allen Ziebur and by the referee that formulas (1) and (2) and error estimates of the form $O(h^{2k})$ can be derived easily from Taylor's formula. We prefer the approach above for freshmen because the error estimate is more explicit and the proof requires only Rolle's Theorem.

Thanks are due to David Hayes for suggestions that led to several improvements in an earlier version of this note, including a somewhat cleaner proof of the Theorem.

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A STRONG SECOND DERIVATIVE TEST

J. H. C. CREIGHTON

DEFINITION. A *dense zero* of a function $g(x)$ is a point x_0 such that any interval containing x_0 contains a zero of $g(x)$ other than x_0. That is, x_0 is a limit point of the zeros of $g(x)$.

Except for the trivial case that $g(x)$ is zero on an entire interval, this phenomenon is a pathology for one variable calculus and accounts for much that is unsatisfactory in the usual presentation. For example, in the absence of dense zeros of the first derivative an easy proof of the chain rule is possible (see David Gans [1] p. 150; note that $y' \neq 0$ implies $\Delta y \neq 0$) and in the absence of dense zeros of the second derivative, a simple and natural definition of inflection point is possible (see below). Furthermore, the possibility of a dense zero of the second derivative precludes a satisfactory classification of critical points. By ruling out this pathology, we obtain in Theorem A of this note a complete classification in terms of the second derivative.

Let $f(x)$ be a real valued function of a real variable. We assume throughout that $f''(x)$ is continuous. Consequently, if x_0 is not a dense zero of $f''(x)$, the sign of $f''(x)$ is constant on sufficiently small intervals of the form $(x_0, x_0 + \varepsilon)$ or $(x_0 - \varepsilon, x_0)$. Then we define x_0 to be an *inflection point* of $f(x)$ if this sign changes; that is, if

the concavity of $f(x)$ changes at x_0. This definition is equivalent, *in the absence of dense zeros of the second derivative,* to the various inequivalent standard definitions. Thus the disagreement among the standard definitions revolves about the pathology of dense zeros. Of course, one might not adopt our viewpoint that dense zeros are pathological in which case he must choose among the standard definitions of inflection point (see the articles of G. M. Ewing [**1**] p. 155 and A. W. Walker [**1**] p. 161). By a *strict extremum* of $f(x)$ we mean the defining inequality to be strict.

THEOREM A. *If $f'(x_0) = 0$ and x_0 is not a dense zero of the (continuous) second derivative, then x_0 is a strict relative extremum or an inflection point. Furthermore, the second derivative distinguishes which:*

(a) *x_0 is a strict relative minimum $\Leftrightarrow f''(x) \geqq 0$ on a neighborhood of x_0.*

(b) *x_0 is a strict relative maximum $\Leftrightarrow f''(x) \leqq 0$ on a neighborhood of x_0.*

(c) *x_0 is an inflection point $\Leftrightarrow f''(x)$ changes sign at x_0.*

THEOREM B. *If x_0 is a dense zero of $f(x)$ then x_0 is a dense zero of $f'(x)$.*

Thus in Theorem A, x_0 is not a dense zero of $f'(x)$ nor of $f(x)$ nor is $f(x)$ constant at x_0 (i.e., constant on a neighborhood of x_0).

THEOREM C. *If x_0 is a relative extremum of $f(x)$ and not a dense zero of the first derivative, then x_0 is a strict relative extremum.*

Theorems B and C are elementary applications of Rolle's Theorem and the Mean Value Theorem, respectively.

Proof of Theorem A. The hypotheses guarantee that one of the conditions on $f''(x)$ given in (a), (b) or (c) holds, thus we need only prove these equivalences. We prove only part (a), assuming $f''(x_0) = 0$. The other cases are immediate from standard theorems or follow by analogy to this case.

Suppose $f''(x) \geqq 0$ on (a, b). Then since x_0 is an isolated zero of $f''(x)$, we may assume equality holds only at x_0. If $x_1 \in (x_0, b)$, by the Mean Value Theorem there is a $\xi \in (x_0, x_1)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Since $f''(x) \geqq 0$, $f'(x)$ is increasing on (a, b) thus $f'(\xi) \geqq 0$. But $f'(\xi) \neq 0$, for otherwise by Rolle's Theorem $f''(x)$ has a root between x_0 and ξ. Hence $f(x_1) > f(x_0)$. Similarly, if $x_2 \in (a, x_0)$, then $f(x_2) > f(x_0)$. Hence x_0 is a strict relative minimum.

Conversely, suppose that in every neighborhood of x_0 $f''(x) < 0$ for some x. Then by continuity, $f''(x) \leqq 0$ in a neighborhood of x_0 or $f''(x)$ changes sign at x_0 and we are reduced to part (b) or (c). This completes the proof of part (a).

We now show that the existence of a dense zero of the *second* derivative is indeed the pathology to be avoided. That is, we provide a counter example to Theorem A, where it is assumed only that x_0 is not a dense zero of $f'(x)$. Define $g(x)$ on $[0, 1]$ to

be the saw-tooth function that is zero at $\{1/n\}$ and whose value at the midpoint of $[1/(2n+1), 1/2n]$ is $1/n$ and at the midpoint of $[1/2n, 1/(2n-1)]$ is $-1/2n$. Extend to $[-1,1]$ by $g(x) = x$ for $x \leqq 0$. Then $g(x)$ is continuous and has the following two properties:

(1) $\int_0^t g(x)dx > 0, \qquad t \neq 0,$

(2) *on any interval of the form* $(0,\varepsilon)$, $g(x)$ *takes on both positive and negative values.*

Now let $f(u) = \int_0^u \int_0^t g(x)dx\, dt$. Then by (1) zero is an *isolated* zero of $f'(u)$ but is not a relative extremum of $f(u)$ since in fact $f(u)$ is strictly increasing. Further, by (2), zero is a dense zero of $f''(u)$ and therefore is not an inflection point of $f(u)$.

Reference

1. T. M. Apostol et al., Selected Papers on Calculus, MAA, 1968.

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A RECURSION FORMULA FOR THE SEQUENCE OF ODD PRIMES

MAKIS PAPADIMITRIOU

This note shows how Bertrand's postulate can be used to obtain a simple recursion formula for the sequence of odd primes. It expresses p_{n+1} in terms of p_n.

If x is an integer $\geqq 3$, let

$$f_x = \operatorname{sgn}\left(\frac{2(x-1)!}{x} - \left[\frac{2(x-1)!}{x}\right]\right).$$

It is easily verified that this is the characteristic function of the odd primes; that is

$$f_x = \begin{cases} 1 & \text{if } x \text{ is prime, } x \geqq 3, \\ 0 & \text{if } x \text{ is composite.} \end{cases}$$

By Bertrand's postulate there is a prime between p_n and $2p_n$. If $p_n = p \geqq 3$, then p_{n+1} must be the first prime among the integers $p+2$, $p+4, \cdots, 2p-1$, so we must have

$$p_{n+1} = (p+2)f_{p+2} + (p+4)f_{p+4}(1-f_{p+2}) + (p+6)f_{p+6}(1-f_{p+2})(1-f_{p+4}) + \cdots + (2p-1)f_{2p-1}(1-f_{p+2})(1-f_{p+4})\cdots(1-f_{2p-3}).$$

This recursion formula expresses p_{n+1} in terms of p_n.

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EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

ONE APPROACH TO COMPUTER SCIENCE EDUCATION AS A FORM OF GENERAL EDUCATION

ROBERT A. ORCHARD

...and suggests that if computer science education is developed as a form of general (liberal arts?) education for a technological society, then we need not be as concerned with questions of supply and demand for computer scientists.

P. Wegner, A View of Computer Science Education, this MONTHLY, 79 (1972) 168–179.

Wegner's philosophy of computer science has been known for some time among computer scientists and mathematicians interested in the field of computer science. During the period 1967–1971, the author and several other faculty members, all of whom had industrial computer science experience, implemented B. S. and M. S. curricula in computer science at Fairleigh Dickinson University. The undergraduate program reflects to a degree Wegner's suggestion for a "general (liberal arts?) education for a technological society." It also reflects the biases of mathematicians working in computer science.

The purpose of this paper is to outline the philosophy, development, and curricula for a mathematically oriented computer science program. It is hoped that the program integrates to a reasonable degree the three computer cultures [**1**] associated with the study of digital computer systems: computer technology, computer mathematics and computer science. Wegner has commented on these three cultures in [**2**]. It is felt that the undergraduate program presented here is a good balance of mathematics, computer science and general education. A modern systemic flavor [**3**], which follows naturally from the mathematics and computer science core, is imparted in the program wherever possible.

1. Program Philosophy. The basic philosophy underlying the B. S. and M. S. degree programs in computer science is the presentation of a well-balanced program in computer architecture, systems software, computer applications and theoretics. The undergraduate program requires a solid grounding in the natural sciences (biology, chemistry, mathematics, and physics) and in the liberal arts and humanities. A minor field of study is accommodated through the use of liberal arts and free electives and is strongly advised for the computer science major.

We view computer science as the study of computer systems from the perspectives of computer architecture, systems software and theoretics. It includes the study of the organization and administration of information through the design, analysis, representation, and application of algorithms on computers. The applications of computers are considered to be an integral part of the program and, in particular, the applications of dedicated minicomputer systems are stressed.

Independent student projects are considered by the faculty to be a vital component in computer science education and interdisciplinary exposure is stressed. For this reason, the curriculum allows much freedom to the student in choosing elective courses, while at the same time it does not sacrifice the "discipline of a discipline." The student majoring in computer science must develop the facility for moving into and out of differing problem environments.

We have attempted to give the student a broad education in the sciences, mathematics, computer science, and a sufficient amount of elective freedom to branch out into a particular area of application. This area is left totally unspecified and it remains for the student to specify it. This requires a certain breadth of vision on the part of the student and his faculty adviser.

Minicomputer systems are of particular interest at Fairleigh Dickinson. The potential for application of these small-to-medium-sized computers is virtually unlimited. They offer the student more direct contact with the computer than is normally possible with multi-million dollar systems, and their quick response, great adaptability and low cost make them particulary attractive for colleges and universities with smaller budgets. We believe the minicomputer approach taken is characterized by cost effectiveness and is a well-balanced, scholarly activity (educationally sound and capable of sustaining masters theses and faculty research). It also has the nice attribute that the system is dedicated solely to the support of the computer science program, and is therefore available twenty-four hours a day.

Both the B. S. and M. S. degree programs have been operative since the Fall of 1971. Prior to this time, computer science was offered as an option within the B. S. and M. S. programs in mathematics. The computer science program also seeks to make available elective sequences in computer science to interested students in other disciplines.

2. The B. S. Curriculum. Much of the content of [4] is included within the framework of the program, but not necessarily organized in the same manner. There are also components of the program which lean more heavily toward Wegner's philosophy. For example, our course CS 252, *Foundations of Computer Science*, intersects Wegner's Course 7, *Mathematical and Computational Model Theory* [2]. An elective course in mathematical logic (for computer science majors) attempts a similar orientation as Wegner's Course 6, *Mathematical Properties of Information Structures.*

TABLE 1

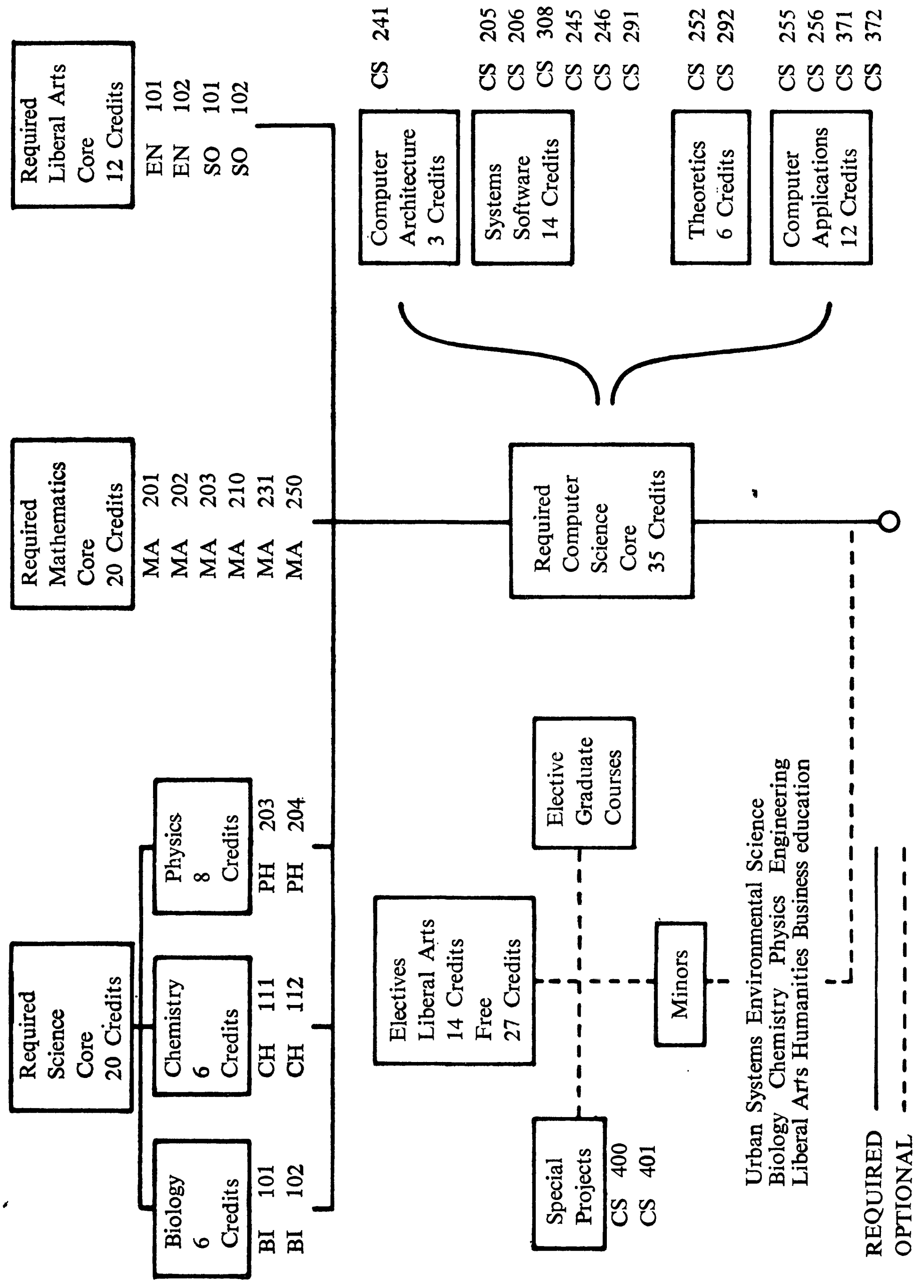

Required
Liberal Arts
Core
12 Credits
EN 101
EN 102
SO 101
SO 102
Required
Mathematics
Core
20 Credits
MA 201
MA 202
MA 203
MA 210
MA 231
MA 250
Required
Science
Core
20 Credits
Biology
6
Credits
BI 101
BI 102
Chemistry
6
Credits
CH 111
CH 112
Physics
8
Credits
PH 203
PH 204
Computer
Architecture
3 Credits
CS 241
Systems
Software
14 Credits
CS 205
CS 206
CS 308
CS 245
CS 246
CS 291
Theoretics
6 Credits
CS 252
CS 292
Computer
Applications
12 Credits
CS 255
CS 256
CS 371
CS 372
Required
Computer
Science
Core
35 Credits
Elective
Graduate
Courses
Electives
Liberal Arts
14 Credits
Free
27 Credits
Minors
Special
Projects
CS 400
CS 401
Urban Systems Environmental Science
Biology Chemistry Physics Engineering
Liberal Arts Humanities Business education
REQUIRED
OPTIONAL

Table 1 indicates the functional organization of the B. S. computer science degree and Table 2 is a listing of the curriculum. Table 3 is a suggested semester-by-semester outline.

TABLE 2. B. S. Computer Science Curriculum

COMPUTER SCIENCE MAJOR

Required Computer Science Courses

COURSE No.	CREDITS
CS 205 Computer Laboratory	1
CS 206 Computer Laboratory	1
CS 241 Fundamentals of Digital Computer Systems	3
CS 245 Computer Systems	3
CS 246 Computer Systems	3
CS 252 Foundations of Computer Science	3
CS 255 Applied Mathematics for Computer Science	3
CS 256 Applied Mathematics for Computer Science	3
CS 291 Current Topics in Computer Science	3
CS 292 Current Topics in Computer Science	3
CS 308 Digital and Analog Computer Programming	3
CS 371 Mathematical Modeling and Computer Simulation (Continuous)	3
CS 372 Mathematical Modeling and Computer Simulation (Discrete)	3
	35

Required Mathematics Courses

COURSE NO.	CREDITS
MA 201 Analytic Geometry and Calculus	4
MA 202 Analytic Geometry and Calculus	4
MA 203 Analytic Geometry and Calculus	3
MA 210 Differential Equations	3
MA 231 Advanced Calculus	3
MA 250 Numerical Analysis	3
	20

N.B. Selected topics in probability, statistics, matrix theory, combinatorics and operations research are covered in CS 255 and CS 256. Selected algebraic topics are incorporated in CS 252.

Required Liberal Arts Courses

COURSE NO.	CREDITS
EN 101 English Composition	3
EN 102 English Composition	3
SO 101 Sociology	3
SO 102 Sociology	3
	12

Required Science Courses

COURSE NO.	CREDITS
BI 101 General Biology	3
BI 102 General Biology	3
CH 111 Physical Principles of Chemistry	3
CH 112 Physical Principles of Chemistry	3
PH 203 General Physics	4
PH 204 General Physics	4
	20

Electives	CREDITS
Liberal Arts Electives	14
Free Electives	27
	41

TABLE 3. Suggested semester-by-semester outline

1st Semester		*2nd Semester*	
MA201 Analytic Geometry & Calc.	(4)	MA202 Analytic Geometry & Calc.	(4)
CS205 Computer Laboratory	(1)	CS206 Computer Laboratory	(1)
BI101 General Biology	(3)	BI102 General Biology	(3)
CH111 General Chemistry	(3)	CH112 General Chemistry	(3)
EN101 English Composition	(3)	EN102 English Composition	(3)
Electives	(2)	Electives	(2)
	16		16
3rd Semester		*4th Semester*	
MA203 Analytic Geometry & Calc.	(3)	Elective	(3)
MA210 Differential Equations	(3)	CS241 Fund. Dig. Comp. Sys.	(3)
CS308 Digital & Analog Prog.	(3)	MA250 Numerical Analysis	(3)
PH203 General Physics	(4)	PH204 General Physics	(4)
SO101 Sociology	(3)	SO102 Sociology	(3)
	16		16
5th Semester		*6th Semester*	
MA231 Advanced Calculus	(3)	CS252 Foundations of Comp. Sci	(3)
CS255 Appl. Math. for Comp. Sci.	(3)	CS256 Appl. Math. for Comp. Sci.	(3)
CS245 Computer Systems	(3)	CS246 Computer Systems	(3)
Electives	(7)	Electives	(7)
	16		16
7th Semester		*8th Semester*	
CS371 Math Modeling & Comp. Sim.	(3)	CS372 Math Modeling & Comp. Sim.	(3)
CS291 Current Top. in Comp. Sci.	(3)	CS292 Current Top. in Comp. Sci.	(3)
Electives	(10)	Electives	(10)
	16		16

The 128 credits required for the degree are distributed as follows:

Mathematics	15.6%	Liberal Arts	
Computer Science	27.4%	and Humanities	20.4%
Science	15.6%	Free Electives	21 %

It is noted that 32% of the degree credits are elected by the student allowing preparation for particular career objectives other than purely computer science.

3. The M. S. Computer Science Curriculum. Fairleigh Dickinson University does not have a doctoral program in computer science. The masters program must serve the needs of students entering industry, doctoral programs in computer science and the teaching profession. The philosophy of a computer science education as a general education on the graduate level obviously cannot work. In this regard the author believes that a "general (liberal arts?) education for a technological society" on the graduate level would have to be carried out within the framework of general systems theory [**9**]. The flavor of Wegner's proposed course contents generalized to the general systems level may be the harbinger of things to come.

The M. S. in computer science has been the subject of controversy [**5, 6, 7**] and the subject of at least one workshop at a major conference on computer science education [**8**]. The program outlined in Tables 4 and 5 attempts a fine balance in the required core between computer architecture, software and theoretics. After completion of the core, specialization in one of three areas is chosen. The following objection raised in [**6,7**] was recognized early and was a major criterion in the curriculum design.

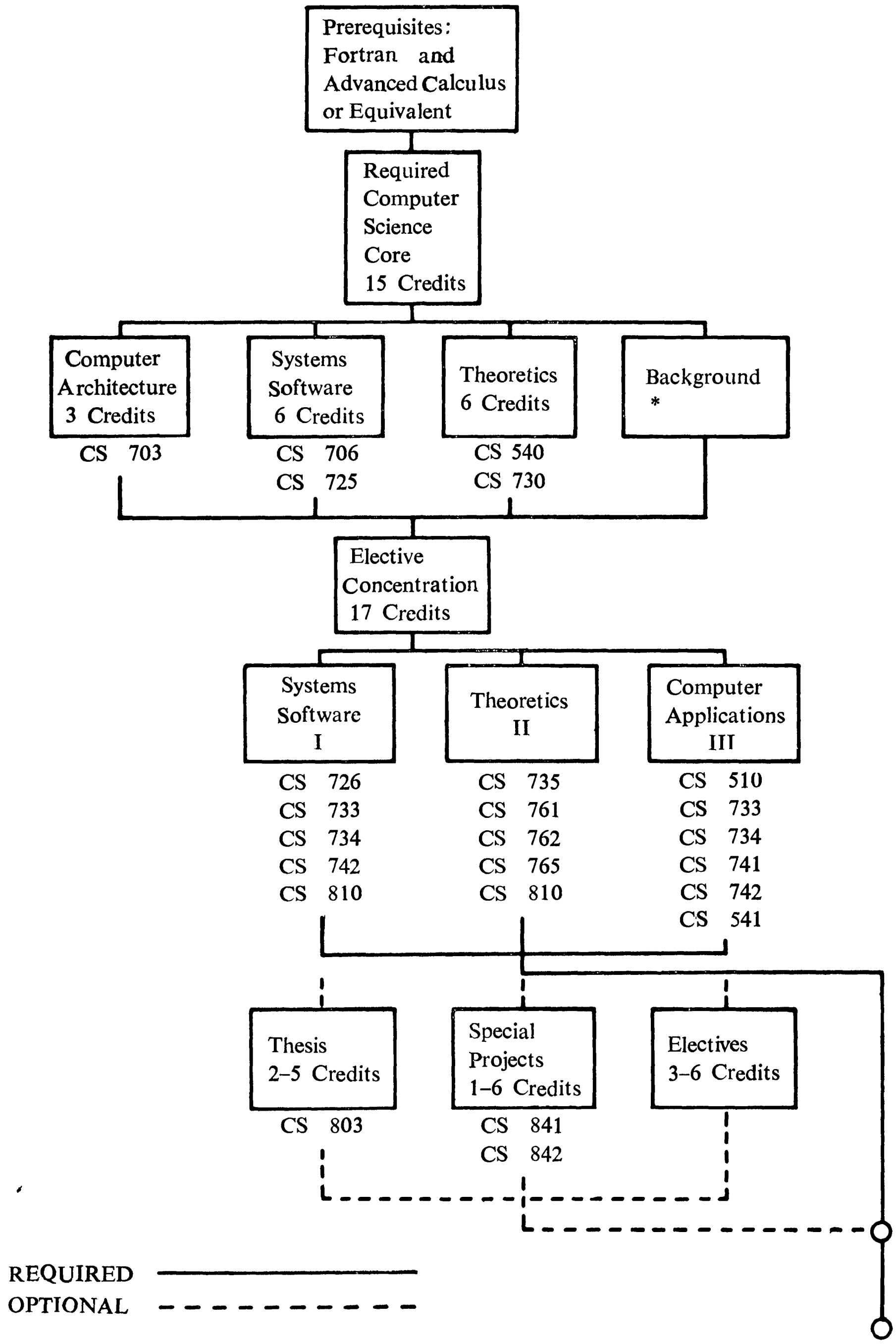
> . . .graduates (of computer science departments) emerge thoroughly unprepared to tackle the intricacies associated with design work in the real-life world–both in software and hardware. We must take the right steps in training computer science students to benefit the industry, now. . . by combining theoretical courses with a practical approach to computer engineering.
>
> The industry requires not only people who can fill jobs that have sophisticated requirements but also people who have the knowhow to implement their background in the most practical way. [**6**].

Hopefully, special projects in computer science requiring a more direct exposure to the computer system will alleviate some of these problems. This was another motivation behind the choice of a minicomputer system to support the computer science program. A student can become familiar with all aspects of the system in a reasonable period of time.

Table 4 indicates the functional organization of the masters program and Table 5 a listing of the curriculum. Further details on particular courses can be obtained from the university graduate bulletin.

It should be noted that in some instances courses have been listed under more than one category. For example, CS841 and CS842 (Special Projects courses) are listed under both *Systems Software* and *Computer Applications*, indicating that

TABLE 4. Functional Organization M. S. Computer Science



* additional core requirements may be needed for students not meeting program prerequisites.

TABLE 5. M. S. Computer Science Curriculum

PROGRAM PREREQUISITES: Advanced Calculus and CS 705 Introduction to Computer Programming or their equivalent.

Required Core Courses

CS 703 Fundamentals of Computer Architecture
CS 706 Introduction to Computer Programming
CS 725 Systems Programming
CS 730 Mathematical Foundations of Computer Science
CS 540 General Systems Theory

Choose six elective courses for a concentration

I Systems Software
CS 726 Systems Programming
CS 733 Applied Mathematics for Computer Science
CS 734 Applied Mathematics for Computer Science
CS 742 Mathematical Modeling and Computer Simulation (Discrete)
CS 810 Special Topics in Computer Science (repeatable)
CS 841 Special Projects in Computer Science
CS 842 Advanced Special Projects

II Theoretics
CS 735 Mathematical Logic
CS 761 Theory of Automata
CS 762 Theory of Automata
CS 765 Theory of Recursive Functions
CS 810 Special Topics in Computer Science

III Computer Applications
CS 510 Numerical Techniques
CS 733 Applied Mathematics for Computer Science
CS 734 Applied Mathematics for Computer Science
CS 741 Mathematical Modeling and Computer Simulation (Continuous)
CS 742 Mathematical Modeling and Computer Simulation (Discrete)
CS 841 Special Projects in Computer Science
CS 842 Advanced Special Projects
CS 541 Seminar in General Systems Theory
Electives (3–6 credits in fields of computer application)

projects may be done in either area. Similarly, CS733 and CS734 (Applied Mathematics) are listed in both places since they are *vital* courses for both categories.

On the other hand, while CS741 and CS742 (Mathematical Modeling) are both listed under *Computer Applications*, only the second semester is listed under *Systems Software.* The reason is that CS742 is quite valuable in the analysis and design of computer systems (both software and hardware) and in the evaluation of their

performance. The first semester of the course is currently of marginal interest in this area although future mathematical developments may change this. CS810 (Special Topics) is listed under *Systems Software and Theoretics*, indicating that the content of the "seminar" can be in either category and the appropriate one would be elected.

4. Computer Science Readings for Mathematicians. Tables 1 and 4 were presented with the thought in mind that the old cliché, *a picture is worth a thousand words*, is still valid. For the benefit of those mathematicians who may have been attracted to some region of the figures, selected readings which may be of especial appeal to the mathematical mind are suggested in the bibliography. The texts indicated by no means exhaust the books available in the area of computer science but are only given as a starting point. The Computing Surveys of the Association for Computing Machinery are also highly recommended as introductory material as well as the Communications of the ACM.

We do not claim that these readings will convert mathematicians into computer scientists. However, they can motivate interested mathematicians to become active in selected portions of the computer science area. Hopefully, this may lead to more computer science programs stemming from mathematics departments where computer scientists can feel at home.

References

1. P. Wegner, Three computer cultures: computer technology, computer mathematics, and computer science, Advances in Computers, 10 (1970).

2. ———, A view of computer science education, this MONTHLY, 79 (1972) 168–179.

3. R. A. Orchard, On an approach to general systems theory, Trends in General Systems Theory, G. Klir, Ed., Wiley, New York, 1972.

4. Curriculum 68, Communications of the ACM, 11,3 (March 1968) 151–197.

5. A. Finerman, ed., University Education in Computing Science, Academic Press, New York, 1968.

6. A. Kandel, Computer science — a vicious circle, Communications of the ACM, 15,6 (June 1972) 470–471.

7. S. E. Blount and L. Fein, The practical aspect of computer science — discussion, Communications of the ACM, 16,1 (Jan. 1973) 45–46.

8. R. L. Ashenhurst, The master's program in computing science, University Education in Computing Science, A. Finerman, ed., Academic Press, New York, 1968.

9. R. A. Orchard, A flexible master's program in general systems theory, unpublished communication, Department of Mathematics and Computer Science, Fairleigh Dickinson University, Teaneck, N.J.

10. A. T. Berztiss, Data structures, theory and practice, Academic Press, New York, 1971.

11. P. C. Brillinger and D. J. Cohen, Introduction to data structures and non-numeric computation, Prentice-Hall, Englewood Cliffs. N.J. 1972.

12. L. R. Johnson, System structure in data, programs and computers, Prentice-Hall, Englewood Cliffs, N.J. 1970.

13. P. Wegner, Programming languages, information structures, and machine organization, McGraw-Hill, New York, 1968.

14. H. Hellerman, Digital computer system principles, McGraw-Hill, New York, 1967.

15. R. L. Nolan, Introduction to computing through the Basic language, Holt, Rinehart and Winston, New York, 1969.

16. B. A. Trakhtenbrot, Algorithms and automatic computing machines, D. C. Heath & Co., Boston, 1963.

17. E. Engeler, Formal Languages: automata and structures, Markham Publishing Co., Chicago, 1968.

18. J. E. Hopcroft and J. D. Ullman, Formal language and their relation to automata, Addison Wesley, Reading, Mass. 1969.

19. J. R. Emshoff and R. L. Sisson, Design and use of computer simulation models, Macmillan, New York, 1970.

20. J. G. Cox and J. H. Mize, Essentials of simulation, Prentice-Hall, Englewood, N. J., 1968.

21. J. Reitman, Computer simulation applications, Wiley, New York, 1971.

22. H. Rogers, Theory of recursive functions and effective computability, McGraw-Hill, New York, 1967.

23. A. F. Cardenas, L. Presser, and M. A. Marin, Computer Science, Wiley Interscience, New York, 1972.

24. G. Birkhoff and T. C. Bartee, Modern Applied Algebra, McGraw-Hill, New York, 1970.

25. W. C. Gear, Computer organization and programming, McGraw-Hill, New York, 1969.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before June 30, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2522. *Proposed by Joel Spencer, Massachusetts Institute of Technology*

A theorem due to van der Waerden asserts that if the set N of positive integers is partitioned into a finite number of subsets, then at least one of the subsets contains arbitrarily long arithmetic progressions.

An infinite subset $S = \{s_1, s_2, \cdots\}$ of N $(s_1 < s_2 < \cdots)$ *has bounded gaps* if $(s_{n+1} - s_n)$ is bounded. Show that if S has bounded gaps, then it contains arbitrarily long arithmetic progressions.

(Remark: This result is a special case of a theorem of E. Szemeredi which states that $S \subseteq N$ contains arbitrarily long arithmetic progressions if it has positive upper density.)

E 2523. *Proposed by K. P. Kerney, Naval Ship Research and Development Center, Bethesda, Maryland*

Evaluate the following integral exactly:

$$\int_0^1 \log(1+x)\log(1-x)dx.$$

(This integral arose in connection with a problem in wing theory.)

E 2524*. *Proposed by T. H. Foregger, University of Wisconsin*

Show that 41 $1 \times 2 \times 4$ bricks can be packed in a $7 \times 7 \times 7$ box. Is there a packing of 42 such bricks into this box?

E 2525. *Proposed by D. Ž. Djoković, University of Waterloo*

Let A be a complex $n \times n$ matrix, let $\bar{A}$ be its complex conjugate, and let I be the $n \times n$ identity matrix. Prove that $\det(I + A\bar{A})$ is real and nonnegative.

E 2526*. *Proposed by Paul Smith, University of Victoria*

Call a set $\{a_1, \cdots, a_n\}$ of positive integers *sum-distinct* if the 2^n possible sums $\sum \varepsilon_i a_i$ (with $\varepsilon_i = 0$ or 1) are all distinct.

Obviously for any n, the set $\{1, 2, 4, \cdots, 2^{n-1}\}$ is an n-element sum-distinct set. Do n-element sum-distinct sets exist with $a_i < 2^{n-1}$ for every i? (For example, $\{3, 5, 6, 7\}$ is a 4-element sum-distinct set with this property.)

Cf. P. Erdös, Problem 220, Canad. Math. Bull., 16 (1973) p. 463.

E 2527. *Proposed by F. D. Hammer, Stockton State College (N. J.)*

(a) A finite number of pennies are placed flat in the plane. Prove that these (non-overlapping) pennies can be painted with at most four colors so that touching pennies bear different colors.

(b) Prove the same result for an infinite collection of pennies in the plane.

(c)* What is the minimum number of pennies which require four colors?

SOLUTIONS OF ELEMENTARY PROBLEMS

Area Enclosed by a Jordan Curve

E 2456 [1974, 169]. *Proposed by H. H. Johnson, University of Washington*

Let C be a simple closed rectifiable curve in the plane, parametrized with respect to arc length s by the vector function $\mathbf{P}(s)$ $(0 \leqq s \leqq L)$. Assume that $\mathbf{P}$ has three continuous derivatives, and that the curvature $k(s)$ never vanishes. If $R(s) = 1/k(s)$ denotes the radius of curvature and if the area enclosed by C is A, show that

$$A \leqq \tfrac{1}{2} \int_0^L R(s)ds + \frac{L}{8} \int_0^L |R'(s)|\, ds.$$

For which curves does equality hold?

Solution by L. E. Mattics, University of South Alabama, and the proposer (independently). It is known that since the arc length of C is L, necessarily C can be covered by a circular disk of radius $r \leqq L/4$ and that a disk of radius $r = L/4$ is necessary only if C is a "needle," i.e., a line segment of length $L/2$ traversed twice. Since our curve C is smooth, we see that C can be covered by a disk of radius $r < L/4$. (This intuitively plausible theorem has received considerable attention; see the Editor's comment below. — Ed.) We can assume that the center of this disk is the origin so that we have

$$|\mathbf{P}(s)| \leqq r < L/4 \quad \text{for } 0 \leqq s \leqq L. \tag{1}$$

Letting $\mathbf{P}(s) = (x(s), y(s))$ we note that since the parametrization is with respect to arc length, we have that the normal to C is given by $\mathbf{N}(s) = (-y'(s), x'(s))$. (This will be the *inward* normal if we assume that C is traversed in a counterclockwise direction, as we do now.) Now by Green's Theorem

$$A = \tfrac{1}{2} \int_0^L (xy' - yx')ds = -\tfrac{1}{2} \int_0^L \mathbf{N} \cdot \mathbf{P}\, ds. \tag{2}$$

By the Frenet equations, $\mathbf{P}''(s) = k(s)\mathbf{N}(s)$ so that $\mathbf{N}(s) = R(s)\mathbf{P}''(s)$. Substituting this in (2), we have that

$$A = -\tfrac{1}{2} \int_0^L (\mathbf{P}'' \cdot \mathbf{P}) R(s)\, ds = \tfrac{1}{2} \int_0^L \left[\mathbf{P}' \cdot \mathbf{P}' - \frac{d}{ds}(\mathbf{P}' \cdot \mathbf{P})\right] R(s)\, ds.$$

We now break this into two integrals, and noting that $\mathbf{P}'$ is a unit vector and using an integration by parts on the second integral, we obtain

$$A = \tfrac{1}{2}\int_0^L R(s)ds + \tfrac{1}{2}\int_0^L (\mathbf{P}'\cdot\mathbf{P})R'(s)ds.$$

Since $|\mathbf{P}'\cdot\mathbf{P}| \leqq |\mathbf{P}'|\,|\mathbf{P}| \leqq r < L/4$ by (1), we see that

$$A \leqq \tfrac{1}{2}\int_0^L R(s)ds + \frac{L}{8}\int_0^L |R'(s)|\,ds$$

with equality if and only if $R'(s) \equiv 0$, i.e., if and only if C is a circle.

Editor's Comment. The covering theorem mentioned in the above proof generalizes to higher dimensions and appears to be due originally to Segre [6] and (independently) to Rutishauser and Samelson [5]. Newer proofs can be found in [2], [4], and [7], and interesting related results in [1] and [3]. Your editors were rather disappointed that so few solutions were received to this problem.

References

1. G. D. Chakerian and M. S. Klamkin, Inequalities for sums of distances, this MONTHLY, 80 (1973) 1009–1017.

2. ———, Minimal covers for closed curves, Math. Mag., 46 (1973) 55–61.

3. ———, Minimum triangles inscribed in a convex curve, Math. Mag., 46 (1973) 256–260.

4. J. C. C. Nitsche, The smallest sphere containing a rectifiable curve, this MONTHLY, 78 (1971) 881–882.

5. H. Rutishauser and H. Samelson, Sur le rayon d'une sphère dont la surface contient une courbe fermée, C. R. Acad. Sci. Paris, 227 (1948) 755–757.

6. B. Segre, Sui circoli geodetici di una superficie a curvatura totale costante, che contengono nell'interno una linea assegnata, Boll. Un. Mat. Ital., 13 (1934) 279–283.

7. J. E. Wetzel, Covering balls for curves of constant length, L'Enseignement Math., 17 (1971) 275–277.

An n-ary Sheffer Function

E 2458 [1974, 169]. *Proposed by E. O. Buchman, California State University at Fullerton*

An *n-place truth function* is a function of n variables whose arguments and values come from the set $\{T, F\}$. Show that the number of n-place truth functions which are capable of generating all n-place truth functions is $2^t(2^t - 1)$, where $t = 2^{n-1} - 1$. (Remark: For $n = 2$, the two functions are "not or" and "not and," otherwise known as *Sheffer's stroke.*)

Editor's comment. It appears that the problem is well known. Several readers gave references (all in 1962!) for its solution. Joel Berman found an elementary solution in R. L. Goodstein, *Truth tables*, Math. Gaz. 46 (1962), 18–23. Paul Cull notes that a solution appears in Manuel Blum, Properties of a neuron with many inputs, in *Principles of Self Organization*, H. von Foerster and R. Zopf (eds.), Pergamon Press (1962). V. Frederick Rickey writes that the problem is elegantly solved in

T. W. Scharle, *Note to my paper: "A diagram of the functors of the two-valued propositional calculus,"* Notre Dame J. Formal Logic 3 (1962), 287–288.

Solutions were submitted by Sylvan Greene, L. F. Meyers, D. J. Samuelson, Kenneth Schilling, W. R. Westphal, and the proposer.

Appearance of Integers in Pythagorean Triples

E2460 [1974, 170]. *Proposed by D. Meyers and C. F. Pinzka, University of Cincinnati*

A *Pythagorean triple* is a solution of $x^2 + y^2 = z^2$ in natural numbers with $x < y$. Show that, given any $n \geqq 0$, there exists a natural number k such that k appears in precisely n distinct triples.

I. *Solution by W. R. Westphal, Montclair State College.* Given $n \geqq 0$, the number 2^{n+1} appears in precisely n distinct triples. For $n = 0$, $2^1 = 2$ appears in no triple at all. Continuing by induction, assume the claim is true for the nonnegative integer $n - 1$. Then 2^n appears in exactly $n - 1$ distinct triples, producing $n-1$ distinct nonprimitive triples each containing 2^{n+1}. But 2^{n+1} appears exactly once in a primitive triple for the following reason: All primitive triples are given by the well-known formulas

$$x = u^2 - v^2, \ y = 2uv, \quad z = u^2 + v^2,$$

where u and v have no common factor and are not both odd, and where $u > v$. Since x and z are both odd, we can have only $y = 2uv = 2^{n+1}$ and since u and v have opposite parity, necessarily $u = 2^n$ and $v = 1$. Hence 2^{n+1} appears in exactly n distinct triples as was claimed.

II. *Solution by Helen M. Marston, Douglass College.* If $T(m)$ is the number of Pythagorean triples containing m, and if $P(m)$ is the number of these triples which are primitive, then

$$T(m) = \sum_{d|m} P(d).$$

Since $P(1) = P(2) = 0$ and $P(2^t) = 1$ for $t \geqq 2$ [as in Solution I], it follows that $T(2^{n+1}) = n$, so that 2^{n+1} appears in exactly n Pythagorean triples.

III. *Solution by R. B. Eggleton, Weizmann Institute of Science, Rehovot, Israel.* Let p be any prime of the form $4t + 3$. We shall show that p^n appears in precisely n Pythagorean triples and is moreover always the smallest member of any triple in which it appears.

No power of p can be the largest member of a Pythagorean triple by the well-known result of Fermat. Suppose that p^n is a member of a triple. Then there exist natural numbers $c > d$ such that $(c-d)(c + d) = p^{2n}$, hence $c-d = p^m$ and $c + d = p^{2n-m}$ for some $m = 0, 1, \cdots, n-1$. Clearly every such m corresponds to

a solution so that there exist precisely n triples containing p^n, namely

$$x = p^n, \; y = \tfrac{1}{2}(p^{2n-m} - p^m), \quad z = \tfrac{1}{2}(p^{2n-m} + p^m)$$

for $m = 0, 1, \cdots, n-1$. Note that we have avoided appealing to the parametric representation of Pythagorean triples.

Also solved by T. S. Bolis, David Cok, Kay Dundas, T. E. Elsner, Karl Heuer (West Germany), Carl Hurd, N. J. Kuenzi & Bob Prielipp, T. C. La Brenz, Carolyn Mac Donald, L. E. Mattics, L. F. Meyers, Andrew Rich, Joanne Rudnytsky, R. W. Sielaff, Paul Smith, G. W. Valk, Charles Wexler, K. L. Yocom, and the proposer.

Editor's comment. Several of these solvers did not adequately dispose of the impossibility of $z = 2^{n+1}$ or $z = p^n$. Five incorrect solutions were received; in each case not enough triples were counted — some multiples of primitive triples were overlooked. Curiously, two of the five attempted solutions specifically allowed for such multiples, but then ignored them.

Kay Dundas comments that if p is a prime of the form $4t + 1$, then p^m appears in precisely $2m$ distinct triples, and Joanne Rudnytsky shows that if N is odd and has n distinct prime factors, then it is a "leg" in precisely 2^{n-1} primitive Pythagorean triples. A corollary of this result is that every number N of the following form appears as a leg in precisely n (not necessarily primitive) triples: N is odd and $N = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, where $2^{m-1} \leqq n$, and where $\Pi\,(2e_i + 1) = 2n + 1$. For example, $N = p^n$ (p an odd prime) is of this form.

Elsner observes that a more difficult problem is to find for each $n \geqq 0$ the least number $L(n)$ which appears in precisely n triples. He gives the first few values of $L(n)$, obtained by trial: 1, 3, 5, 16, 12, 15.

A Congruence Modulo $n!$

E 2461 [1974, 170]. *Proposed by Andreas Zachariou, University of Athens, Greece*

Let n and x be natural numbers such that x is divisible by only primes which are larger than n. Show that

$$(x-1)(x^2 - 1) \cdots (x^{n-1} - 1) \equiv 0 \pmod{n!}.$$

I. *Solution by D. M. Bloom, Brooklyn College.* Let (*) denote the given congruence and let $p \leqq n$ be prime. Since $p \nmid x$, Fermat's Little Theorem gives $x^{k(p-1)} \equiv 1 \pmod{p}$ for all k; hence at least $[(n-1)/(p-1)]$ of the $n-1$ factors on the left-hand side of (*) are divisible by p. (The square brackets indicate the greatest integer function.) But the highest exponent to which p divides $n!$ is

$$e = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right].$$

Since only finitely many summands in this formula are nonzero, e is *strictly less* than

$$\sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1},$$

and hence $e \leqq [(n-1)/(p-1)]$. Thus the left-hand side of (*) is divisible by p^e; since this holds for all primes $p \leqq n$, the result follows.

II. *Comment by Tor Bu, University of Bergen, Norway.* Suppose that $x = p^t$ for some prime $p > n$. If G denotes the group of nonsingular $n \times n$ matrices over the Galois field of order x, then it is known that

$$o(G) = x^{n(n-1)/2}(x-1)(x^2-1)\cdots(x^n-1).$$

Let S be the subgroup of G consisting of those matrices with precisely one nonzero element in each row and each column. (I.e., S is the subgroup generated by the diagonal matrices and the permutation matrices.) The order of S is known to be

$$o(S) = (x-1)^n n!,$$

so by Lagrange's Theorem, it follows that

$$(x-1)(x^2-1)\cdots(x^n-1) \equiv 0 \pmod{(x-1)^n n!}.$$

Also solved by W. B. Adams, Anders Bager (Denmark), Richard Bauer, The Bennett College Team, David Cok, Brian Conrey, Gene Gale, R. A. Gibbs, M. G. Greening (Australia), Emil Grosswald, G. A. Heuer (Germany), Carl Hurd, L. Kuipers, Graham Lord, O. P. Lossers (Netherlands), Helen Marston, L. E. Mattics, M. R. Murty & V. K. Murty, Andrew Rich, J. D. Rogawski, V. V. S. Sastry, Veber Shlom (Israel), J. B. Staton, Temple University Problem Solving Group, Charles Wexler, and the proposer.

Editor's comment. F. W. Saunders notes that if x and $n > 2$ meet the conditions of the problem, then x must be of the form $kn! \pm 1$ (consider a complete residue system (mod n) of the form $kn! \pm j$). Therefore we have that if p is a prime which divides $n!$, then $x^2 - 1 \equiv 0 \pmod{p}$; this of course does not imply that $x^2 - 1$ is necessarily divisible by $n!$.

Bloom observed the result of Bu for the particular case x a prime.

A Consequence of Wolstenholme's Theorem

E 2463* [1974, 281]. *Proposed by R. N. Gupta, Punjab University, India.*

Let p be a prime greater than 3. Since the set $S = \{1, 2, \cdots, p-1\}$ forms a group under multiplication mod p, for every $k \in S$ there exists a (unique) $x_k \in S$ such that $kx_k \equiv 1 \pmod{p}$. Thus integers $n_1, n_2, \cdots, n_{p-1}$ are specified such that

$$kx_k = 1 + n_k p$$

for $k = 1, 2, \cdots, p-1$. Show that

$$\sum_{k=1}^{p-1} kn_k \equiv \tfrac{1}{2}(p-1) \pmod{p}.$$

I. *Solution by Joe Flowers, Northeast Missouri State University.* We have that $n_{p-k} = n_k + p - k - x_k$ since $(p-k)(p-x_k) = kx_k + p^2 - pk - px_k = 1 + (n_k + p - k - x_k)p$. Now

$$\sum_{k=1}^{p-1} kn_k = \sum_{k=1}^{p-1} (p-k)n_{p-k} = \sum_{k=1}^{p-1} (p-k)(n_k + p - k - x_k)$$
$$\equiv \sum_{k=1}^{p-1} (-kn_k + k^2 + 1) \pmod p.$$

Therefore

$$2\sum_{k=1}^{p-1} kn_k \equiv (p-1) + \sum_{k=1}^{p-1} k^2 \pmod p,$$

and the conclusion follows since

$$\sum_{k=1}^{p-1} k^2 = \frac{p(p-1)(2p-1)}{6} \equiv 0 \pmod p$$

when p is prime and greater than 3.

The above method of proof also gives the following generalization: If x_k and n_k are such that $kx_k = j + n_k p$ for some fixed j, $1 \leqq j \leqq p-1$, and all $k = 1, 2, \cdots, p-1$, then

$$\sum_{k=1}^{p-1} kn_k \equiv \tfrac{1}{2}j(p-1) \equiv \tfrac{1}{2}(p-j) \pmod p.$$

II. *Solution by K. A. Ribet, Princeton University.* For $k = 1, 2, \cdots, p-1$, let k^{-1} denote the (multiplicative) inverse of $k \pmod{p^2}$. By Wolstenholme's Theorem (Theorem 116, p. 89, of G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd Ed., Oxford, 1954), we know that

$$1^{-1} + 2^{-1} + \cdots + (p-1)^{-1} \equiv 0 \pmod{p^2}.$$

But from the definition of n_k, we see that $k^{-1} \equiv x_k - n_k x_k p \pmod{p^2}$. Therefore

$$p\sum_{k=1}^{p-1} n_k x_k \equiv \sum_{k=1}^{p-1} x_k = \sum_{k=1}^{p-1} k = \tfrac{1}{2}p(p-1) \pmod{p^2},$$

from which the desired result follows.

Also solved by Renato Acampora (Switzerland), W. B. Adams, Bonnie Averbach, S. Baskaran (India), E. P. Bauhoff (West Germany), The Bennett College Team, K. A. Beres, David Bienenfeld (Israel), H. W. Borchers (West Germany), Marian Brashears, Robert Breusch (New Zealand), Saundra Bright, Kevin Brown, L. Carlitz, T. B. Carroll, E. S. Eby, Rev. John Fischer, Peter Garst, Irving Gerst, M. G. Greening (Australia), Sidney Heller, B. W. Hogan, Myron Hlynka, Carl Hurd, R. D. James, Vojtech László (Czechoslovakia), Albert Leisinger, Marijo LeVan, Richard Levaro, O. P. Lossers (Netherlands), Marc Low, William Margulies, Helen M. Marston, L. E. Mattics, J. G. Mauldon, N. S. Mendelsohn & Barry Wolk, Tauno Metsänkylä (Finland), Wanda J. Mourant, M. R. Murty & V. K. Murty, J. E. Pasciak, Carl Pomerance, Bart Rice, J. D. Rogawski, Jonathan Ryshpan, Harvey Schmidt, Jr., G. J. Simmons, Stephen Spindler, F. B. Strauss, Charles Ullery, A. J. Vince, Wolfgang Vogt (West Germany), Roland Wais (West Germany), and the proposer.

Editor's comment. Levaro, Simmons, and Vogt all obtain the same generalization as did Flowers. Using the same technique, both James and Murty & Murty observe that if $(m, 6) = 1$ (m not necessarily prime) then $\Sigma\, kn_k \equiv \frac{1}{2}\varphi(m) \pmod{m}$, the sum being taken over those k, $1 \leqq k \leqq m-1$, which are relatively prime to m; φ denotes Euler's totient function. This latter generalization can be derived using the technique of Ribet using Leudesdorf's generalization of Wolstenholme's Theorem (Th. 128, p. 101, Hardy & Wright, *op. cit.*).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before June 30, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6018*. *Proposed by Antonio Marquina, Valencia, Spain*

Does there exist a real-valued function $f(x, y)$ defined at every point of R^2, satisfying the following properties:

(i) For every point (x, y), $f(x, y)$ is continuous.

(ii) For every point (x, y), there exist the two partial derivatives $D_x f$ and $D_y f$.

(iii) The function $f(x, y)$ is not differentiable in (x, y) for every point of R^2?

6019. *Proposed by R. E. Shafer, Berkeley, California*

K. Chandrasekharen, *Introduction to Analytic Number Theory.* Chap. VII, Section 3 proves that

$$\frac{2^{2n}}{2\sqrt{n}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{2n}}$$

for all integers $n \geqq 2$. Prove for all positive numbers n that

$$\frac{2^{2n}}{\sqrt{\pi}(n^2 + n/2 + 1/8)^{\frac{1}{4}}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{(n + 1/4)\pi}}.$$

6020*. *Proposed by C. W. Anderson and Dean Hickerson, University of California, Berkeley*

A pair of distinct numbers (k, m) is called a *friendly* pair (k is a friend of m) if $\Sigma(k) = \Sigma(m)$, where $\Sigma(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n. (For example, (4320, 4680) is a friendly pair.) Show that almost all numbers have friends, i.e., the natural (asymptotic) density of numbers with friends is unity. Equivalently, the density of solitary numbers (numbers without friends) is zero.

6021*. *Proposed by C. R. Diminnie and Albert White, St. Bonaventure University*

In l^p, $p > 2$, does $\|x - y\|\,\|x + y\| = \big|\|x\|^2 - \|y\|^2\big|$, with $x, y \neq 0$, imply that $y = \alpha x$ for some real α?

6022. *Proposed by Neal Felsinger, Yale University*

Given a collection X of subsets of S, no one containing another, let $C(X)$ consist of all minimal subsets of S which intersect every member of X. (For properties of $C(X)$ see Problem 5883 [1974, 293].) Show that if S is infinite, $C(X)$ does not necessarily exist.

6023*. *Proposed by S. J. Sidney, University of Connecticut*

If for each k in the uncountable index set K, I_k denotes a copy of $[0,1]$ and U_k denotes the copy of $(0,1]$ contained therein, prove or disprove that $\prod_k U_k$ is a Borel set in the compact space $\prod_k I_k$.

SOLUTIONS OF ADVANCED PROBLEMS

Norms in a Barreled Space

5937 [1973, 1067]. *Proposed by Albert Wilansky, Lehigh University*

Give an example of a vector space with two comparable (unequal) norms such that it is barreled with the larger norm and a Banach space with the smaller. [A barreled (locally convex topological vector) space is one such that every closed absolutely convex absorbing set is a neighborhood of 0.]

Solution by A. C. Cochran and T. K. Muckherjee, University of Arkansas. Let $(E, \| \ \|)$ be an infinite dimensional Banach space and let f be a discontinuous linear functional on E (such a functional always exists since the topological dual is a *proper* subspace of the algebraic dual). Let E_0 denote the kernel of f and let x be an element in E such that $f(x) = 1$ and $\|x\| = 1$ (such a choice is possible by taking a suitable multiple of f). Consider the space $E_0 \oplus [x]$ (where $[x]$ denotes the one-dimensional subspace generated by x) with the topology determined by the norm $p(y + \lambda x) = \|y\| + |\lambda|$, $y \in E_0$. Then $(E_0 \oplus [x], p)$ is nothing but the direct sum topology induced by $(E_0, \| \ \|)$ and is clearly strictly finer than the $\| \ \|$-topology (f is discontinuous). By a result of Dieudonné [*Sur les propriétés de permanence de certains espaces vectoriels topologiques*, Ann. Soc. Polon. Math. 25 (1952), 50–55] E_0 is barreled; so the direct sum $E_0 \oplus [x]$ is also barreled. Since E_0 is of co-dimension one in E, we may identify as sets E and $E_0 \oplus [x]$ and we have the desired example.

Also solved by Antonio Marquina (Spain), J. H. Webb (South Africa), John Horváth, and the proposer.

Avoiding the Axiom of Choice

5941 [1973, 1146]. *Proposed by Jan Mycielski, University of Colorado*

Prove (without using the Axiom of Choice) that $\mathbf{R}/\mathbf{Q}$ is of the same cardinality as $\mathbf{B}/\mathbf{F}$, where $\mathbf{R}$ is the additive group of real numbers, $\mathbf{Q}$ is the additive group of

rational numbers, **B** is the Boolean algebra of all subsets of the set of integers, and **F** is the ideal of finite sets of integers.

Solution by Barbara L. Osofsky, Rutgers University. For $n \in \mathbf{N}$, let $[n] \equiv \{0,1,2,\cdots,n-1\}$. Let $\langle a_i \rangle$ be a typical element in the cartesian product $\prod_{i=k}^{\infty} A_i$. Let $\mathbf{A}'$ be the set $\mathbf{Z} \times \prod_{n=2}^{\infty}[n]$, $\mathbf{F}'$ the equivalence relation defined on $\mathbf{A}'$ by $(\langle a_i \rangle, \langle b_i \rangle) \in \mathbf{F}'$ if and only if $a_i = b_i$ for all but a finite number of i. Set $\mathbf{A} = \mathbf{A}' - \mathbf{F}'$ equivalent class $\langle 0 \rangle$. Then the map ϕ: $\mathbf{A} \to \mathbf{R}$ defined by $\phi \langle a_i \rangle = \sum_{i=1}^{\infty} a_i/i!$ is a one-one correspondence such that $(\langle a_i \rangle, \langle b_i \rangle) \in \mathbf{F}'$ if and only if $\phi \langle a_i \rangle - \phi \langle b_i \rangle \in \mathbf{Q}$.

Thus we have an effective one-one correspondence between $\mathbf{R}/\mathbf{Q}$ and $\mathbf{A}/\mathbf{F}'$. Moreover, $\mathbf{A}/\mathbf{F}'$ contains a copy of $\mathbf{B}/\mathbf{F}$ which is obtained by interpreting elements of **B** not in the **F**-class $\langle 0 \rangle$ as having entries in $\{0,1\} \subsetneqq [n]$, and sending **F**-class $\langle 0 \rangle$ to class $\langle n-1 \rangle$.

The map χ': $\mathbf{A} \to \mathbf{B}$ defined by $\chi' \langle a_i \rangle = \langle b_i \rangle$, where $\langle b_{2^i+1}, \cdots, b_{2^{i+1}} \rangle$ is the binary representation of a_{i+2} for all $i \geqq 0$, $b_1 = b_0 = 0$, defines a one-one map χ: $\mathbf{A}/\mathbf{F}' \to \mathbf{B}/\mathbf{F}$. That $\mathbf{B}/\mathbf{F}$ is effectively equivalent to $\mathbf{R}/\mathbf{Q}$ follows from the existence of an effective proof of the Cantor-Bernstein-Schroeder Theorem.

Also solved by J. G. Mauldon, J. G. Wendel, and the proposer.

Transitive Automorphisms

5943 [1973, 1147]. *Proposed by L. J. Wallen, University of Hawaii*

Let V be a vector space over some field and let $L(V)$ denote the algebra of all endomorphisms of V. A set $\phi \subset L(V)$ is transitive if $\phi x = V$ for each $x \in V$, $x \neq 0$. Let Ω be a transitive subalgebra of $L(V)$. Determine the automorphisms α of Ω having the property that whenever $\phi \subset \Omega$ is transitive, so is $\alpha(\phi)$.

Solution by the proposer. For $x \in V$, $x \neq 0$, define $J_x = \{A \in \Omega: Ax = 0\}$. Then J_x is a left ideal in Ω and in fact is maximal (if $Bx = y \neq 0$ and C is given, pick D such that $DBx = Cx$ so $DB - C \in J_x$). If J is a left ideal such that $J \not\subset J_x$ for all $x \neq 0$, then J is transitive. Hence if $\beta = \alpha^{-1}$, $\beta(J_{x_0})$ is an intransitive maximal left ideal, so $\beta(J_{x_0}) = J_{x_1}$.

Define a transformation T by $TAx_0 = \beta(A)x_1$. T is well-defined since $Ax_0 = Bx_0 \Rightarrow A - B \in J_{x_0} \Rightarrow \beta(A) - \beta(B) \in J_{x_1} \Rightarrow \beta(A)x_1 = \beta(B)x_1$. Clearly $T \in L(V)$ and T is invertible. Finally $TA(Bx_0) = \beta(AB)x_1 = \beta(A)\beta(B)x_1 = \beta(A)TBx_0$ so $\beta(A) = TAT^{-1}$.

Conversely, any automorphism of Ω induced by an inner automorphism of $L(V)$ preserves transitivity.

Weak Sequential Closure of a Class of Operators

5944 [1973, 1147]. *Proposed by L. J. Wallen, University of Hawaii*

Let H be a separable, complex, infinite-dimensional Hilbert space. A venerable

theorem of Halmos states that every contraction is the weak limit of a sequence of unitaries. What is the weak sequential closure of the class of operators similar to unitaries?

Solution by the proposer. Every operator! Let A_n be the upper left-hand $n \times n$ corner of the matrix for A (with respect to some orthonormal basis) and let I_n be the $n \times n$ identity matrix. Define B_n via the matrix

$$\left[\begin{array}{c|c} \begin{array}{c|c} A_n & I_n \\ \hline I_n - A_n^2 & -A_n \end{array} & 0 \\ \hline 0 & \begin{matrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{matrix} \end{array}\right]$$

B_n is evidently an involution and so is similar to a reflection (a unitary involution). Since $\| B_n \|$ is bounded $B_n \to A$ weakly.

Intersection of Finitely Generated Groups

5946 [1974, 89]. *Proposed by Andrzej Ehrenfeucht, University of Colorado*

Find a sequence of groups $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ such that each G_n is a finitely generated group but $\bigcap_{n=1}^{\infty} G_n$ is not a finitely generated group.

Solution by R. G. Burns, York University, Ontario. In the free group freely generated by a, b define, for each positive integer n, the subgroup

$$G_n = \langle a^{2^n}, a^i b a^{-i} \,|\, -2^n < i < 2^n \rangle .$$

Then $G_1 \supset G_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} G_n$ is free on the set $\{a^i b a^{-i} \,|\, i \in Z\}$.

(We remark that examples are not rare, since the only finitely generated solvable groups *not* containing such a sequence of finitely generated subgroups are the polycyclic ones. For, if a finitely generated solvable group G is not polycyclic then it has an infinitely generated normal subgroup N with polycyclic, and therefore residually finite, quotient. Thus there exists a sequence $G_1 \supset G_2 \supset \cdots$ of subgroups of G of finite index and therefore finitely generated, whose intersection is N.)

Also solved by D. R. Anderson, James Buddenhagen, P. M. Cohn (England), D. Ž. Djoković, C. R. Hampton, A. A. Jagers (Netherlands), J. I. Miller, E. T. Ordman, W. R. Scott, C. Y. Tang, T. K. Teague, Z. Z. Uoiea, C. N. Winton, Kenneth Yanosko, and the proposer.

Note. Jagers notes that a solution is contained in Problem # 4, p. 112 of Magnus, Karrass and Solitar, *Combinatorial Group Theory*. Interscience (1966).

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Finite Mathematics and Calculus with Applications to Business and the Social Sciences. By R. V. Hogg, R. H. Randles, A. J. Schaeffer and J. C. Hickman. Cummings Publishing, Menlo Park, California, 1974. 308 pp. $11.00.

I had the opportunity of teaching finite mathematics courses for several semesters in the past, and I found this book quite interesting because of a different flavor and unique style it possesses compared with some other books that I used. This text comes in two different formats; one without calculus and the other with two additional chapters on calculus. Neither volume contains a bibliography.

The aim of the authors is well achieved. As the title indicates, the contents and problems are all heavily geared toward the applications to business and social sciences. However, a lot of problems deal with realistic situations in the areas of pollution control, industrial production and population, etc. Each section begins with one or more examples to give students some motivation. Neither formal definitions, theorems nor proofs are presented; instead relevant material is developed through examples.

The first three chapters, which lead to the solutions of linear programming problems in three variables by use of the simplex method, are well organized. The examples and problems for Order Exploding in Chapter 2 are very interesting, but I feel that they are rather too long, and students may tend to lose their sense of direction. In Chapter 6, Sets and Probability, Equally Likely Outcomes, Additional Properties ($P(A^c) = 1 - P(A), P(A \cup B) = P(A) + P(B) - P(A \cap B)$) are all treated in the separate sections in detail and these sections are somewhat repetitious. I found them quite tedious to read.

Discrete random variables are introduced hastily in Chapter 7, which begins with an example employing a single random variable, followed by a second example for two random variables, and then immediately followed by a third one for three random variables.

A major innovation, however, can be found in the last chapter on integral calculus. The differential calculus is treated informally throughout, and material covered is

just a bare bones minimum. Yet it is enough for the purpose in the last chapter. The main goal of the authors in the integral calculus is applications to the probability of continuous random variables. Therefore, the introduction of the integral calculus itself ends with the Fundamental Theorem of Integral Calculus. The remainder is a derivation of cumulative probability functions from probability density functions, and a study of mean and variance.

In closing, I feel that the authors are successful in presenting finite mathematics in a new and stimulating direction for applications.

CHURL S. KIM, Indiana University Southeast

FILM

Statistics at a Glance. Written, produced and directed by Robert Johnson and Lynn Hill. Consultants: David Kanouse and Thomas Wickens, Department of Psychology, U.C.L.A.; Kenneth MacCorquodale, Department of Psychology, University of Minnesota. 16 mm. sound and color, 26 minutes. Available for rent ($25) or purchase ($295) from John Wiley and Sons, Inc.

Getting across a few basic ideas of descriptive statistics in a clear yet nontechnical way is not easy. But this film does it, and throws in a good deal of entertainment to boot.

Furthermore, using a variety of devices (e.g., animation, old silent film clips, and scenic panoramas) to provide data for analysis, and displaying nary a formula, the film convinces its audience that knowing some of this mathematical stuff just might be valuable.

The beginner is told about frequency distributions (histograms, polygons, "normal curves" and "skewed curves"), measures of central tendency (mean, median, mode) and of variability (range, standard deviation, "Z-scores", percentiles), and correlation. Throughout, there is an effective use of color in the carefully drawn sketches, and the sound track, which includes some lively music, makes a very definite contribution. And the message is always clear: *Statistical notions are useful.*

The film is of very high technical quality. It is definitely recommended for elementary classes and even for campus-wide groups. Although the audience will suffer through one weak attempt at humor (a simulated experiment about reaction times of chickens), its patience will be rewarded. For not only will viewers be entertained while learning and appreciating some statistical ideas, but they will also get to see and hear on film that kid who's making so much noise eating popcorn.

DAVID F. APPLEYARD, Carleton College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T**(13-15: 1, 2), S, L. *Matters Mathematical.* I.N. Herstein, I. Kaplansky. Har-Row, 1974, viii + 246 pp, $10.95. A fascinating, well-written, and (probably) teachable introduction to abstract mathematics. Suitable for non-majors (with only high school mathematics) and possibly for some majors (especially teachers). Topics from number theory, group theory, finite geometry, game theory, and transfinite arithmetic which really do give the flavor of modern mathematics as the authors intend. FLW

GENERAL, S*, L*. *The Golden Mean: Mathematics and the Fine Arts.* Charles F. Linn. Doubleday, 1974, xvii + 131 pp, $4.95. A mostly attractive book on an enthralling subject. Little of the book deals specifically with the golden ratio, but musical scales, Birkhoff's mathematical theory of aesthetics, perspective drawing, symmetry and patterns in ornament, and "proper deception" in architecture all come in for mention. There are also superficial treatments of computer art and of Escher's paintings. No topic is investigated thoroughly; for example, Poe and Birkhoff are mentioned in connection with poetry and mathematics, but Scott Buchanan's book *Poetry and Mathematics* isn't. In short: an *hor d'oeuvre*, if not the hoped-for feast. PJC

GENERAL, S, P*. *Proceedings of the Conference on the Application of Undergraduate Mathematics in the Engineering, Life, Managerial, and Social Sciences.* Ed: P.J. Knopp, G.H. Meyer. Georgia Inst of Tech, 1973, 298 pp, free (P). Papers and transcripts of panel discussions; a rich source of uncommon applications. LAS

GENERAL, L. *Standard Mathematical Tables, Twenty-second Edition.* Ed: Samuel M. Selby. CRC Pr, 1974, xii + 706 pp, $9.95. Major change from earlier edition: greater emphasis on metric system. Includes some material (e.g., combinatorial analysis) not found in the larger and more expensive *Handbook of Tables for Mathematics.* LAS

GENERAL, L. *Handbook of Tables for Mathematics, Fourth Edition.* Ed: Samuel M. Selby. CRC Pr, 1970, xvi + 1120 pp, $26.50. Very similar to the better known *Standard Mathematical Tables*, but in larger (8" x 11") format. Examples of differences: astrodynamic formulas, financial tables from 8% to 20%, 30 pp. on finite difference methods. LAS

GENERAL, S*(13), L. *Mathematics on the Geoboard.* John Niman, Robert Postman. Cuisenaire Co, 1974, 123 pp, $4.95 (P). Fascinating, nontrivial enrichment projects: Pick's theorem, random walk, Pascal's triangle, networks, Euler's theorem, homeomorphisms, rotation groups--all from rubber band patterns on a 5 x 5 lattice of pegs. LAS

GENERAL, P*, L. *The Proceedings of the Bertrand Russell Memorial Logic Conference.* Ed: John Bell, *et al.* (Available from A. Slomson, School

of Mathematics, The University, Leeds, LS2 9JT, England). 1973, 404 pp, $12 (P). Proceedings of "alternative" logic conference held in Uldum, Denmark, in August 1971, simultaneously with the NATO Advanced Study Institute at Cambridge, England. The alternative conference was held by 50 logicians dissociating themselves from the aims of NATO. In addition to history of the birth of the conference, the volume contains critical papers on scientism, mathematical practice, NATO, and philosophy of mathematics, as well as mathematical papers in set theory and model theory. PJC

BASIC, T(13: 1). *Intermediate Algebra for College Students, Fourth Edition.* Thurman S. Peterson, Charles R. Hobby. Har-Row, 1974, xii + 372 pp, $9.95. For students with no more than one year of high school algebra. Some of its definitions and explanations lack clarity. FWL

BASIC, T(13: 1). *Intermediate Algebra for College Students.* Louis Leithold. Macmillan, 1974, ix + 518 pp, $9.95. A book suitable for students studying algebra for the first time as well as for those who need a review. Includes a good assortment of word problems. LLK

BASIC, T(13: 1), S. *Quick Arithmetic.* Robert A. Carman, Marilyn J. Carman. Wiley, 1974, xii + 252 pp, $4.95 (P). An attractive treatment of basic arithmetic. For self-study or classroom use. FLW

PRECALCULUS, T(13). *Essentials of Technical Mathematics.* Richard S. Paul, M. Leonard Shaevel. P-H, 1974, xii + 689 pp, $11.95. Written for students in engineering technology programs. Covers basic precalculus material: real number system; functions; trigonometry; logarithmic and exponential functions; complex numbers; analytic geometry; sequences and series. Contains 3500 exercises including many word problems. An appendix discussing the slide rule is included. SG

PRECALCULUS, T(13: 2). *Precalculus Mathematics with Applications.* John M. Peterson, Floyd E. Haupt. Prindle, 1974, viii + 536 pp, $12.50. College algebra, trigonometry, complex numbers, sequences and series, introduction to coordinate geometry and probability. Includes matrices and linear algebra. Intuitive. Very few proofs and theorems. Many examples, problems. LH

PRECALCULUS, T(13: 1). *College Algebra, Third Edition Revised.* Gordon Fuller. Van N-Rein, 1974, x + 333 pp, $9.95. Clear presentation of topics with many examples, exercises and word problems. LLK

PRECALCULUS, T(13: 1). *College Algebra and Trigonometry.* Robert D. Bechtel, Arthur A. Finco, Robert B. Kane. Van N-Rein, 1974, x + 422 pp, $8.95. Precalculus topics designed for classroom use but with some self-test methods included. This would allow for flexibility, i.e., students working independently to cover topics needed. A supplementary study guide is available. LLK

PRECALCULUS, T(13: 1). *Functional Approach to Precalculus, Second Edition.* Mustafa A. Munem, James P. Yizze. Worth, 1974, xi + 569 pp, $10.95; *Study Guide to Accompany Functional Approach to Precalculus, Second Edition,* 364 pp, $3.95 (P). This second edition of a very well received text has few revisions other than color highlights and a rearranging of problem sets (even numbered problems for concepts, odd numbers for drill). LLK

EDUCATION, T(15-16: 1), S, P*, L*. *Mathematics, Society and Curricula.* H.B. Griffiths, A.G. Howson. Cambridge U Pr, 1974, xv + 423 pp, $22.50. A fascinating comprehensive survey of the relation between societal

influence and mathematical curriculum development at the high school and undergraduate level in Britain and the U.S. Plentiful illustrations, thought-provoking exercises, extensive bibliography, name and subject indices, sample (British) examination papers. LAS

EDUCATION, S(15-16), P. *A Metric Handbook for Teachers.* Ed: Jon L. Higgins. NCTM, 1974, iv + 137 pp, $2.40 (P). A joint venture of NCTM and ERIC, this handbook contains 17 articles; 10 are reprints from *The Arithmetic Teacher.* Includes excellent suggestions for introducing and teaching the metric system K-12. Presents curriculum guidelines, psychological implications, and a good discussion of measure functions. Good reading for teachers. Should be on all methods course reading lists. PSJ

EDUCATION, T(13-14: 1, 2). *A Second Mathematics Course for Elementary Teachers.* Eugene F. Krause. Har-Row, 1974, xiv + 493 pp, $12.95. Main content is geometry and measurement. Also includes algebra, probability, functions and applications, with some attention to sets and logic, informal inductive and deductive reasoning, combinatorics and graph theory. Integrates topics in useful manner. Many and varied exercises. Informal geometry, broad coverage, including transformations. PSJ

EDUCATION, S(15-16), P, L. *The Common Sense of Teaching Mathematics.* Caleb Gattegno. Educ Solutions, 1974, ix + 129 pp, $4.95 (P). One of a series on making the study of education properly scientific. Considers elementary algebra and theory of numbers as the author supports his theory that only awareness is educable. Uses fingers and Algebricks to generate numerals and algebra. Concludes with consideration of the teaching of mathematics. For elementary teachers. PSJ

EDUCATION, T(13-14: 1). *Mathematics for Elementary Teachers.* Lyle J. Dixon. Merrill, 1974, xv + 296 pp, $9.95. Includes numeration systems, set theory, whole numbers and algorithms, other bases, fractions, decimals, geometry (no transformations), metric system (5 pages) and simple statistics (6 pages). PSJ

EDUCATION, S(16-17), P, L. *Curriculum Development in Elementary Mathematics: 9 Programs.* Kathleen Devaney, Lorraine Thorn. Far West Lab, 1974, 246 pp, $7.95 (P). Analysis of nine major elementary math programs developed via government financing since 1950's. Eight American programs (Madison, Arithmetic Project, IPI, IMS, PIA, MINNEMAST, USMES, DMP), one United Kingdom (Nuffield). Reports include goals, content, teaching strategy, implementation, history of development and evaluation results. Appears excellent. PSJ

EDUCATION, S, P, L. *The Child's Discovery of Space: From Hopscotch to Mazes: An Introduction to Intuitive Topology.* Jean and Simonne Sauvy. Penguin, 1974, 92 pp, $2.25 (P). A discussion of elementary topological properties (e.g., order, interior, exterior, intersection, connectivity, circuits) with suggestions for activities for school children that will encourage intuitive understanding of these concepts. An attempt to balance the traditional school emphasis on measurement with a focus on place and position. LAS

HISTORY, S, P*, L**. *Dictionary of the History of Ideas: Studies of Selected Pivotal Ideas.* Ed: Philip P. Wiener. Scribner's, 1968-71. 4 volumes, xxxi + 677 pp, 696 pp, 677 pp, 537 pp, $155. Includes many articles of interest to mathematicians, on such topics as abstraction, axiomatization, game theory, infinity, rigor, number, probability, symmetry. Authors include K.J. Arrow, S. Bochner, Banesh Hoffmann, Maurice Kendall, O. Morgenstern, C.J. Scriba, G.J. Whitrow, R.L. Wilder. KOM

HISTORY, P, L. *Oeuvres Choisies, Tome I.* Waclaw Sierpiński. PWN, 1974, 300 pp, $18. The first of three volumes, this one contains selected works of Sierpiński in number theory and analysis, a biography by Kuratowski, and a complete bibliography of Sierpiński's publications. All articles appear in French. The articles are selected for their importance in the development of mathematics. CEC

HISTORY, S*(15-17), P*, L*. *The Life and Times of the Central Limit Theorem.* William J. Adams. Kaedmon, 1974, 119 pp, $6.95. An historical profile, ranging from seeds sown in the seventeenth century to the abstract formulation established at the beginning of the twentieth century by Markov and Lyapunov. Includes portraits from D.E. Smith's collection. LAS

HISTORY, P*, L**. *A Source Book in Medieval Science.* Ed: Edward Grant. Harvard U Pr, 1974, xviii + 864 pp, $32.50. An incomparable resource in a distinguished series: 190 selections (10% in mathematics) by 85 authors mostly from the Latin West, half translated here for the first time. Includes mathematics selections from Boethius, Bacon, Al-Khwārizmī, Nicole Oresme and Leorando of Pisa (Fibonacci). LAS

HISTORY, L*. *Ramanujan, The Man and the Mathematician.* S.R. Ranganathan. Asia Pub, 1967, 138 pp, $4.75. A personal, non-technical biography, largely comprised of diverse vignettes from Ramanujan's friends and wife. Includes bibliographic information complete only to 1940. LAS

FOUNDATIONS, T(17-18: 1), P, L. *Modal Logic: The Lewis-Modal Systems.* J. Jay Zeman. Oxford U Pr, 1973, x + 302 pp, $21. Comprehensive study of formal modal propositional logic. Uses Łukasiewicz (Polish) notation and presupposes prior acquaintance with classical propositional logic. Both Hilbert- and Gentzen-style formulations are presented for the logics discussed, and model theory is developed via Beth-Kripke semantic tableaux. PJC

FOUNDATIONS, T*(16-18: 1), P, L*. *Computability and Logic.* George Boolos, Richard Jeffrey. Cambridge U Pr, 1974, x + 262 pp, $10.95. Enlightening text for a second course in mathematical logic, too heady for a first exposure. Presents the principal metalogical results concerning computability, first order logic, and arithmetic, plus some other topics. More and better exposition than usual, but no historical perspective. Incompleteness of arithmetic is proved in an unusual setting but with a particularly sharp result. The approach through computability especially recommends the book to computer scientists. PJC

FOUNDATIONS, P. *A Metaphysics of Elementary Mathematics.* Jeffrey Sicha. U of Mass Pr, 1974, x + 444 pp, $10 (P). An exercise in systematic philosophy in which a metaphysics of arithmetic (a "mild" nominalism) is derived from Wilfrid Sellars' general theory of abstract entities. LAS

FOUNDATIONS, T, S*, L. *Naive Set Theory.* Paul R. Halmos. Springer-Verlag, 1974, vii + 104 pp, $6.80. Reprint of the well-known Van Nostrand monograph. This and other old Halmos texts form the core of Springer's new series of Undergraduate Texts. LAS

FOUNDATIONS, P, L. *Elements of Combinatory Logic.* Frederic B. Fitch. Yale U Pr, 1974, viii + 162 pp, $6.95. Presents a specific system Q of combinatory logic based on introduction and elimination rules (the "method of subordinate proofs"; see Fitch's *Symbolic Logic* (1952)). System Q's main attraction is that with its unrestricted quantifiers plus combinators, it can serve as a foundation for arithmetic. It

contains all of elementary arithmetic but can be shown to be free from contradiction; Gödel's incompleteness-inconsistency dilemma is skirted by means of an infinitary elimination rule. There are some exercises and a bibliography but no general index; an earlier version appeared as a technical report in 1960. Prior study in mathematical logic is necessary for a reader to realize the book's contribution, as the author provides no historical lead-in. PJC

COMBINATORICS, P. *Combinatorics, Parts I-III.* Ed: M. Hall, Jr., J.H. Van Lint. Math. Centre Tracts, No. 55, 56, 57. Math Centrum, 1974. *Part I: Theory of Designs, Finite Geometry and Coding Theory,* i + 196 pp, Dfl. 20 (P); *Part 2: Graph Theory: Foundations, Partitions and Combinatorial Geometry,* i + 118 pp, Dfl. 13 (P); *Part 3: Combinatorial Group Theory,* iv + 161 pp, Dfl. 17 (P). Instructional lectures given at the Advanced Study Institute on Combinatorics in the Netherlands, July 8-20, 1974 (prompt publication!). Overviews of specialized subjects, with indications of present frontiers of research. PJC

LINEAR ALGEBRA, T(14: 1). *A Primer of Linear Algebra.* Gerald L. Bradley. P-H, 1975, xvi + 382 pp, $11.95. Sound basic development of topics with geometric arguments preceding abstraction. Excellent choice of applications, optional sections and supplementary exercises. LLK

LINEAR ALGEBRA, T(14: 1). *Lessons in Linear Algebra.* James R. Wesson. Merrill, 1974, x + 277 pp, $10.95. Mathematical presentation of topics with little use of geometric motivation or applications. LLK

LINEAR ALGEBRA, T(15-16: 1), S, L. *Finite-Dimensional Vector Spaces.* Paul R. Halmos. Springer-Verlag, 1974, viii + 200 pp, $7.95. Reprint of the second Van Nostrand Edition; part of Springer's new series of undergraduate texts. LAS

ALGEBRA, P. *Lecture Notes in Mathematics-396: The Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications.* Karl Heinrich Hofmann, Michael Mislove, Albert Stralka. Springer-Verlag, 1974, xvi + 122 pp, $7.40 (P). A semi-lattice is a commutative, idempotent monoid. The order is given by $s \leq t$ if $st = s$. This monograph is an excellent example of using category theory to exploit a duality between discrete (i.e., untopologized) semilattices and compact 0-dimensional semilattices. More precisely, the two categories are dual. Applications are to lattice theory and to structure of compact 0-dimensional semilattices. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-380: Rings and Semigroups.* Mario Petrich. Springer-Verlag, 1974, viii + 182 pp, $7.40 (P). Theory of rings and semigroups with applications primarily to rings and semigroups of linear functions of a topological vector space. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-389: Théorie de la Descente et Algèbres d'Azumaya.* Max-Albert Knus, Manuel Ojanguren. Springer-Verlag, 1974, iv + 163 pp, $8.20 (P). The theory of descent is concerned with problems of the following type: when is an element of $N \otimes S$ of the form $x \otimes 1$, for N an S module, S a commutative ring, and 1 the unit of S? More general problems of similar nature are also handled by theories of descent. These notes are applications of the theory of descent to various algebras, among others, Azumaya algebras. An exposition and development of work of Grothendieck. PJM

ALGEBRA, P. *Homologie des algèbres commutatives.* Michel André. Grund. math. Wissenschaften, B. 206. Springer-Verlag, 1974, xv + 338 pp, $40.

A study of certain homology and cohomology modules (which are derived from modules of differentials and derivations) associated with an algebra over a ring. Symmetric, exterior, and Hopf algebras are among those considered. Of special interest to algebraic geometers. SG

REAL ANALYSIS, T(15-17: 1), S*, L. *Introduction to Measure Theory.* G. DeBarra. Van N-Rein, 1974, ix + 287 pp, $6.50 (P). A concrete undergraduate treatment featuring dozens of worked examples as well as extensive hints and answers (over 20% of the book) to the many excellent, specific exercises. Includes L^p spaces, signed measures and product spaces. An attractive book at a very attractive price. LAS

REAL ANALYSIS, T(17: 1), S, L. *Measure Theory.* Paul R. Halmos. Grad. Texts in Math., V. 18. Springer-Verlag, 1974, xi + 304 pp, $12. Reprint of well-known original Van Nostrand edition. LAS

DIFFERENTIAL EQUATIONS, P. *Systems of Linear Partial Differential Equations and Deformation of Pseudogroup Structures.* A. Kumpera, D.C. Spencer. Pr U Montreal, 1974, 100 pp, $5 (P). Essentially a summary of the authors' 1972 *Annals of Mathematics Studies* monograph *Lie Equations: Volume I, General Theory.* Basically, the authors redevelop deformation theory which was developed earlier by B. Malgrange, among others, in connection with his proof of the existence of local coordinates for almost- structures defined by elliptic Lie equations satisfying an appropriate integrability or compatibility condition. I-CH

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-365: Constructive Methods for Elliptic Equations.* Robert P. Gilbert. Springer-Verlag, 1974, vii + 397 pp, $10.70 (P). An exposition of recent applications of function theoretic methods to the theory of partial differential equations. Topics include: second order equations in the plane and applications to boundary value problems; higher order equations in the plane and in R^n, $n \geq 3$; singularity theory, initial value problems for semilinear equations; weak solutions of equations in the plane. SG

DIFFERENTIAL EQUATIONS, P. *Theory of Branching of Solutions of Non-Linear Equations.* M.M. Vainberg, V.A. Trenogin. Trans: Israel Prog. for Sci. Trans. Noordhoff Intern, 1974, xxvi + 485 pp, Dfl. 112. Exposition of the branching theory of non-linear integral and integro-differential equations with analytic operators. Discusses classical problems of implicit functions, classical integral, differential, and integro-differential equations, equations in Banach spaces, perturbation theory. Large bibliography. No exercises. SG

DIFFERENTIAL EQUATIONS, P. *Integral Geometry and Inverse Problems for Hyperbolic Equations.* V.G. Romanov. Tracts in Nat. Philo., V. 26. Springer-Verlag, 1974, vi + 151 pp, $23.80. The determination of differential equations from known functionals of their solutions is called an 'inverse problem.' The chapter on the integral-geometric inverse problems mainly is the author's own contribution to the field. The rest of the book is devoted to the analysis of inverse problems for second-order hyperbolic linear PDE's and application to the study of the earth's structure. I-CH

DIFFERENTIAL EQUATIONS, T(17-18: 1, 2), S, P. *Gewöhnliche Differentialgleichungen.* H.W. Knobloch, F. Kappel. Teubner Stuttgart, 1974, 332 pp, DM 48 (P). A flexible volume designed to serve as an advanced text (many problems and some solutions given) or as a handy reference for the professional. The theory is presented with applications in mind, but this is no solutions manual. JAS

NUMERICAL ANALYSIS, P. *Software for Numerical Mathematics*. Ed: D.J. Evans. Acad Pr, 1974, xi + 451 pp, $28. 25 survey papers from an April, 1973 conference in Loughborough, Leicestershire. LAS

FUNCTIONAL ANALYSIS, P. *Semigrupuri de Contractii Neliniare in Spatii Banach*. Viorel Barbu. Editura Academiei Romania, 1974, 342 pp, Lei 16,50 (P). Systematic exposition of the theory of nonlinear contraction semigroups--in Romanian. JAS

OPTIMIZATION, S(18), P. *Introduction to Minimax*. V.F. Dem'yanov, V.N. Malozemov. Trans: D. Louvish. Halsted Pr, 1974, vii + 307 pp, $20. The authors use the inductive method of exposition, first particular cases of minimax problems then the more general ones. A translation from Russian, this book consists of three almost independent parts: 1) polynomial approximation, 2) discrete minimax problems and nonlinear programming, and 3) the general minimax theory with applications. Historical information and bibliographic references can be found at the end of the book. I-CH

OPTIMIZATION, T(14-16: 1), S. *Studies in Optimization*. D.M. Burley. Halsted Pr, 1974, xii + 228 pp, $12.95. Mainly a nonlinear optimization textbook, covers material flexible for sophomores to seniors; free and constrained hill climbing techniques, variational methods, dynamic programming, optimal control theory. Includes some recently well-used minimization methods and applications to physical problems. Gives the author's experienced view of topics for further study. I-CH

OPTIMIZATION, T(14-16: 1), S, L. *Calculus of Variations with Applications to Physics and Engineering*. Robert Weinstock. Dover, 1974, x + 326 pp, $4 (P). Corrected republication of the 1952 McGraw-Hill original edition. "A very useful book"--J.L. Synge. LAS

ANALYSIS, P. *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*. Laurent Schwartz. Stud. in Math., V. 6. Tata Inst (Oxford U Pr), 1973, xii + 393 pp, $15. A modified theory of Radon measures is developed and then applied to probability theory. Minimal indices in two separate locations make this hard to use as a quick reference. JAS

ANALYSIS, T*(13-15), S*, P, L. *Le Livre du Problème, Volume 4: La Convexité*. I.R.E.M. de Strasbourg. CEDIC, 1974, 108 pp, (P). A problem-oriented approach to convexity that emphasizes creative mathematics and touches such diverse fields as topology, geometry, analysis, and linear programming. An excellent book. PJM

ANALYSIS, T(17-18), P, L. *Topological Transformation Groups*. Deane Montgomery, Leo Zippin. Krieger, 1974, xi + 289 pp, $15.25. Unaltered reprint of the 1964 corrected reprinting of the well-known 1955 Interscience tract. LAS

ANALYSIS, P. *Generic Hamiltonian Dynamical Systems are Neither Integrable nor Ergodic*. L. Markus, K.R. Meyer. Memoirs No. 144. AMS, 1974, iv + 52 pp, $2.80 (P).

ANALYSIS, P. *Lie Algebras: Applications and Computational Methods*. Ed: Bernard Kolman. SIAM, 1973, 155 pp, $12.50. 12 papers from a June 1972 conference sponsored by Drexel University, reprinted from the September 1973 issue of *SIAM J. Appl. Math*. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-393: Stability of Unfoldings*. Gordon Wassermann. Springer-Verlag, 1974, ix + 164 pp, $8.20 (P). An

unfolding of a smooth function f is, roughly a germ of a smooth family of germs which comprise a neighborhood of f. Wassermann shows that there is only one reasonable notion of stability for unfoldings by showing that seven different natural definitions are equivalent. As a corollary he states precisely and proves René Thom's statement that there are exactly seven elementary catastrophes. LAS

ANALYSIS, P. *Recent Advances in the Representation Theory of Rings and C*-Algebras by Continuous Sections.* Ed: Karl Heinrich Hofmann, John R. Liukkonen. Memoirs No. 148. AMS, 1974, x + 182 pp, $4 (P). Papers from a 1973 seminar at Tulane on representation by continuous sections in sheaves and bundles. LAS

GEOMETRY, S(17-18), P. *Tensor Calculus.* Stanislaw Golab. Trans: Eugene Lepa. PWN, 1974, xviii + 371 pp, $32.70. A moderately classical view of the uses of tensors in differential geometry presuming a background in linear algebra, analysis, and the fundamentals of differential geometry. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-364: Proceedings on Infinite Dimensional Holomorphy.* Ed: T.L. Hayden, T.J. Suffridge. Springer-Verlag, 1974, vii + 212 pp, $7.70 (P). Invited addresses from the May 1973 International Conference at the University of Kentucky together with referred papers and a result obtained by Waelbroeck at the conference.JAS

TOPOLOGY, P. *Topics in Topology.* Ed: Á. Császár. North-Holland, 1974, 643 pp, $50. 67 papers from a June, 1972 Colloquium on Topology at Keszthely, Hungary. LAS

PROBABILITY, P, L. *Stochastic Point Processes and Their Applications.* S.K. Srinivasan. Griffin's Stat. Mono., No. 34. Hafner Pr, 1974, xi + 174 pp, $13.95 (P). A heuristic but thorough presentation of the theory with applications to statistical physics, management science, and biology. Prerequisite: probability at the level of Feller, V. I. JAS

STATISTICS, T(13-14: 1, 2), S. *Statistics for Education, With Data Processing.* David White. Har-Row, 1973, xv + 382 pp, $10.95. Presupposes only high school algebra. Reviews the needed mathematics and introduces the needed statistical programs (in FORTRAN). Many good problems using given educational data. No Bayesian methods. FLW

STATISTICS, P. *Exploring Data Analysis: The Computer Revolution in Statistics.* Ed: W.J. Dixon, W.L. Nicholson. U of Calif Pr, 1974, xvi + 409 pp, $10. Eight papers, plus discussions, presented at a conference on statistical computing at UCLA in September, 1971, by a select group of data analysts. Each involves a real problem, with a large data set, requiring the use of a computer. Topics range from clinical medicine to rainfall prediction, and the methodological techniques used are equally varied. RSK

STATISTICS, P. *Automatische Klassifikation.* Hans Hermann Bock. Vandenhoeck and Ruprecht, 1974, 480 pp, DM 82. An extensive introduction to cluster-analysis. JAS

STATISTICS, P. *Selected Tables in Mathematical Statistics, Volume II.* Coed: H.L. Harter, D.B. Owen. AMS, 1974, viii + 388 pp, $14.10. In a series sponsored by the Institute of Mathematical Statistics (Vol. I, TR, April 1971 and November 1974). Contains tables of the probability integral for both the doubly noncentral t and doubly noncentral F distributions, expected sample size for curtailed fixed sample size tests of a Bernoulli parameter, and zonal polynomials of order 1 through 12. RSK

STATISTICS, P. *Efficient Estimation with A Priori Information.* Thomas J. Rothenberg. Cowles Found. Mono., 23. Yale U Pr, 1973, viii + 180 pp, $10. "An attempt at unifying certain aspects of econometric theory by embedding them in a more general statistical framework." Concerned with questions of the value of a priori information and efficient methods of incorporating it into point estimation procedures. In particular covers the problem of estimating the parameters of simultaneous equation systems. Uses both classical and Bayesian approaches. Presumes background in mathematical statistics and matrix algebra. RSK

STATISTICS, P. *Compstat 1974: Proceedings in Computational Statistics.* Ed: Gerhart Bruckmann, Franz Ferschl, Leopold Schmetterer. Physica Verlag, 1974, 539 pp, DM 48 (P). Papers (all in English) from a symposium at the University of Vienna focused on the computer revolution in statistical techniques and models. LAS

COMPUTER SCIENCE, S*(15-16), P, L. *Introduction to Computer Science: BASIC Language Programming.* Terry M. Walker. Allyn, 1972, x + 241 pp, $4.95 (P). An excellent programming text with many complete programmed examples and applications. Meant to be used as a supplement to the author's *Introduction to Computer Science* (TR, August-September, 1972), it could be used to supplement the main text in any introductory computer science course. RB

COMPUTER SCIENCE, T(13: 2), S, P, L. *Computer Mathematics Handbook.* Jerrold R. Clifford, Martin Clifford. Allyn, 1974, xiv + 338 pp, $14.95. A very thorough and handy book for anyone working with computers. As a textbook the first eight chapters give a very simple introduction to number systems, Boolean algebra, flow charting and algebra. The ninth chapter with charts and tables is an excellent reference. Recommended for use as a textbook and as a reference book. There are answers to all problems in the back. RB

COMPUTER SCIENCE, P. *Handbook of Computer Management.* Eds: R.B. Yearsley, G.M.R. Graham. Halsted Pr, 1973, xxvii + 328 pp, $15.50. Collection of practical essays for data processing managers, divided into four parts: survey of computing systems, buying computer services, managing the data processing function, and applications of computers in business. RSK

COMPUTER SCIENCE, S. *Programarea in Limbajul PL/I.* Horia Georgescu. Editura Academiai Romania, 1973, 225 pp, Lei 13 (P). PL/I in Romanian; a language manual. JAS

COMPUTER SCIENCE, S. *Computers in Action: How Computers Work.* Donald D. Spencer. Hayden, 1974, 150 pp, $4.95 (P); $7.50. Written as a guide for the layman. Interesting chapters on the evolution of computers and on computers in general. Later chapters present more technical information on input/output, computer storage, and computer programming. Unfortunately several of the sample programs contain errors. RSK

COMPUTER SCIENCE, T?(13), S. *Introduction to Computers.* Keith London. Third revised edition. Transatlantic Arts, 1973, 272 pp, $4.50 (P). Completely revised, including new illustrations. How a computer works, how it can be used to solve simple problems, how it is used in commerce and science. Distinguishable from other similar books by the large number of illustrations and the omission of time-sharing. Includes glossary. PJC

SYSTEMS THEORY, S*(17-18), P*. *Hyperstability of Control Systems.* V.M. Popov. Grund. math. Wissenschaften, B. 204. Springer-Verlag,

1973, 400 pp, $26.20. Hyperstable elements are those that can be combined without loss of stability. Self-contained, comprehensive presentation of results up to 1966. Characterization of hyperstability discussed for single-input, multi-input and discrete systems. Applications to recent problems. LH

SYSTEMS THEORY, T(14-16: 2), S, L. *Mathematical Foundations for Management Science and Systems Analysis.* J. William Schmidt. Acad Pr, 1974, xiii + 581 pp, $15.95. Excellent. Will prepare student with calculus background for advanced topics in operations research or systems analysis. Large problem sets covering theory and mechanics. Many references. Includes distribution theory, matrix algebra, classical optimization, calculus of finite differences, complex variables and transform methods. LH

SYSTEMS THEORY, S(13-16). *A Practical Approach to Computer Simulation in Business.* L.R. Carter, E. Huzan. Halsted Pr, 1973, 298 pp, $15.95. Introduction to simulation based on realistic management problems. Queueing, forecasting, inventory and production control, profitability analysis. Non-mathematical. Reviews BASIC, FORTRAN, CSL, lists all programs described. Virtually no exercises. LH

APPLICATIONS (PHYSICS), P*. *Applications of Global Analysis in Mathematical Physics.* Jerry Marsden. Publish or Perish, 1974, 273 pp, $10 (P). Notes based on lectures at Carleton University, Ottawa in the summer of 1973. Applications of general analysis (in the sense of Morse theory) to very mathematical physics. Nicely written with lots of references but, unfortunately, no index. JAS

APPLICATIONS (PHYSICS), P, L. *The Physics of Time Asymmetry.* P.C.W. Davies. U of Calif Pr, 1974, xviii + 214 pp, $15.75. A synthesis of diverse but related strands of research on the enigma of how to account for the difference between past and future. Assumes only special relativity and electrodynamics. Although focused on physics, it does discuss related philosophical and psychological issues. LAS

APPLICATIONS (PHYSICS), T(18), P. *Statistical Mechanics: A Set of Lectures.* R.P. Feynman. Benjamin, 1972, xii + 354 pp, $8.95 (P); $17.50.

APPLICATIONS (PHYSICS), S, P, L*. *From Stonehenge to Modern Cosmology.* Fred Hoyle. Freeman, 1972, 96 pp, $4.95. Four public lectures (together with some supplementary material) delivered in 1971 at S.U.N.Y. at Buffalo. General observations on science, politics, religion and society preceed a careful analysis of Stonehenge and two chapters which unfold details and dilemmas of contemporary cosmology. LAS

APPLICATIONS (PHYSICS), S?(17-18), P. *Elements of Group Theory for Physicists.* A.W. Joshi. Halsted Pr, 1973, xi + 316 pp, $5.95. Basic theory of classical groups through normal subgroups and homomorphisms in twenty-nine pages. Then Hilbert spaces and operators along with representation theory of finite and continuous groups leads quickly to the latter half of the book which treats quantum mechanics, crytallography, and solid state physics. Lack of exercises (there are none), cursory treatment of mathematical ideas and emphasis on research make this unsuitable for a text. JAS

APPLICATIONS (PHYSICS), S(16), P, L. *Lecture Notes in Mathematics-361: Foundations of Special Relativity: Kinematic Axioms for Minkowski Space-Time.* John W. Schutz. Springer-Verlag, 1973, xx + 314 pp, $10.10 (P). Axiomatic development of Minkowski space-time using undefined elements

called "particles", a single undefined relation, the "signal relation", and eleven axioms. Rectilinear and three-dimensional kinematics. The main concerns are careful comparisons and contrasts to theorems of absolute geometry. DFA

APPLICATIONS (PHYSICS), P. *Louis de Broglie: Sa Conception du Monde Physique.* Ed: Maurice Druon. Gauthier-Villars, 1973, xxviii + 387 pp, 160F. Biography of the man and expositions of the influence of his work on the occasion of his eightieth birthday. A nicely done volume though more a memorial than a scholarly work. Also rather expensive. JAS

APPLICATIONS (PHYSICS), P. *The Boltzmann Equation: Theory and Applications.* Ed: E.G.D. Cohen, W. Thirring. Springer-Verlag, 1973, xi + 642 pp, $60.70. Historical, biographical and technical papers from an international symposium in Vienna on the occasion of the centennial of the Boltzmann equation. Extravagantly bound and printed from typewritten copy; similar material in the publisher's Lecture Note series costs only one-fifth as much. LAS

APPLICATIONS (PHYSICS), S*(13-14), P*. *Higher Mathematics for Beginners and Its Application to Physics.* Ya. B. Zeldovich. Trans: George Yankovsky. MIR, 1973, 494 pp. In two parts. The first is a short, very terse treatment of calculus of one variable through Taylor series; the second is a collection of applications of the elementary calculus to a variety of physics problems including mechanics, thermal motion of molecules, nuclear fisson and electric circuits. The examples in the second half are thoroughly developed, and should provide calculus instructors a rich source of complete examples. TAV

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-28: Lectures in Statistical Physics.* Ed: W.C. Schieve, J.S. Turner. Springer-Verlag, 1974, vi + 342 pp, $9.90 (P). Seven major lecture series--ranging from astrophysics to biophysics--from an Advanced School for Statistical Mechanics and Thermodynamics held in Austin, Texas in 1971. LAS

APPLICATIONS (PHYSICS), S(15-17), P*, L*. *The Many-Worlds Interpretation of Quantum Mechanics.* Ed: Bryce S. DeWitt, Neill Graham. Princeton U Pr, 1973, viii + 252 pp, $12.50; $5.50 (P). The conventional non-deterministic quantum theory produces severe paradoxes whenever it must deal with more than one observer or measuring apparatus. Hugh Everett III showed in a 1957 Princeton thesis that these paradoxes may be avoided if one assumes that quantum observations yield all possible outcome states simultaneously--instead of, as in the conventional interpretation, one state randomly chosen with predictable probability. In Everett's bold proposal, reality is rigorously deterministic, but composed of many worlds; his thesis is, according to his mentor John Wheeler, "a step [which] can be matched but few times in history: when Newton...; when Maxwell...; when Einstein...." Everett's brief thesis, his own lengthy exposition of it, together with several papers with reactions and variations are collected in this volume. LAS

APPLICATIONS (PHYSICS), P. *Conformal Algebra in Space-Time and Operator Product Expansion.* S. Ferrara, R. Gatto, A.F. Grillo. Tracts in Mod. Phys., No. 67. Springer-Verlag, 1973, 64 pp, $14.70. First assumes conformal invariance on the light cone and studies effects of this on equal-time commutators of local fields and on operator product expansion. Then examines consequences of the application of strict conformal symmetry on operator products. DFA

APPLICATIONS (PHYSICS), P. *Graph Theory and Feynman Integrals.* Noboru Nakanishi. Math. and its Appl., V. 11. Gordon, 1971, xi + 223 pp, $17.50. A detailed survey of the graph-theoretical aspects of Feynman integrals. Contains an introduction to graph theory followed by a quick introduction to the Feynman integral and then a lot of theoretical physics. JAS

APPLICATIONS (PHYSICS), T(16-17), S, P, L. *Symmetry and its Applications in Science.* A.D. Boardman, D.E. O'Connor, P.A. Young. Halsted Pr, 1973, xiii + 305 pp, $14.95 (P). The classical applications of group theory written for undergraduate science students having a basic knowledge of quantum mechanics. The mathematics is "unencumbered by rigorous proofs"; the "frustration quotient" is likely to remain quite high. LCL

APPLICATIONS (PSYCHOLOGY), P*, L. *Contemporary Developments in Mathematical Psychology, Volume II: Measurement, Psychophysics, and Neural Information Processing.* Ed: David H. Krantz, *et al.* Freeman, 1974, xv + 468 pp, $14. Ten examples by fifteen authors providing an ostensive definition of cumulative progress in mathematical psychology. Together with Vol. I (TR, February 1975) it offers a grand vision of the research frontier in mathematical psychology. LAS

APPLICATIONS (SOCIAL SCIENCE), S(15-17), P, L*. *Game Theory as A Theory of Conflict Resolution.* Ed: Anatol Rapoport. Reidel, 1974, 283 pp, $31 (P). 11 essays on current research in 2- and n-person games by mathematicians and behavioral scientists, introduced by an excellent general survey by the editor. Intended as supplementary reading for students in behavioral science. LAS

APPLICATIONS (SOCIAL SCIENCE), T*(15-17: 2), S**, L**. *Mathematical Sociology: An Introduction to Fundamentals.* Thomas J. Fararo. Wiley, 1973, xxvi + 802 pp, $18.95. A unique survey, loaded with concrete examples, of the techniques, methods and achievements of mathematical sociology. In four parts: formal methods, measurement, social processes and game theory. Addressed to the reader with no calculus background: only occasional "notational usage" of derivatives occur. Includes an extensive bibliography and a brief guide to the literature, but no exercises. LAS

APPLICATIONS (SOCIAL SCIENCE), S, L*. *Genetics and Social Structure: Mathematical Structuralism in Population Genetics and Social Theory.* Ed: Paul Ballonoff. Dowden, Hutchinson & Ross, 1974, xv + 504 pp, $22. Reprints of landmark papers in the formal, algebraic approach to kinship, genetics and related anthropological concerns. Of primary interest to demographers and social scientists, it includes many classic examples of mathematical models in the social sciences which should be of considerable interest to teachers of college mathematics. LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; Kenneth O. May, University of Toronto, J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Colgate University: Associate Professor W. E. Mastrocola has been named Chairman of the Department of Mathematics; Assistant Professor T. K. Frutiger has been promoted to Associate Professor; Drs. Linda Sue Esch, Boston University, C. H. Nevison, Stanford University, and D. H. Saracino, Yale University, have been appointed Assistant Professors.

SUNY College at Fredonia: Assistant Professors Frank Chimenti and James McKenna have been promoted to Associate Professors.

Wartburg College: Assistant Professor W. E. Beck, University of Wisconsin, Whitewater, has been appointed Assistant Professor of Mathematics and Director of the Computer Center; Professor J. O. Chellevold has retired after 45 years of teaching, with the title of Professor Emeritus.

Assistant Professor D. M. Davis, Northwestern University, has been appointed Assistant Professor at Lehigh University.

Assistant Professor Morton Goldberg, Broome Community College, has been promoted to Associate Professor.

Professor Morris Kline, Courant Institute of Mathematical Sciences, New York University, has been appointed Visiting Distinguished Professor at Brooklyn College of the City University of New York.

Assistant Professor J. L. Martin, Missouri Southern State College (Joplin), has been promoted to Associate Professor.

Dr. D. L. Parker, University of Indiana, has been appointed Assistant Professor at Salisbury State College.

Professor Carl Barnett Allendoerfer, University of Washington, died on September 29, 1974, at the age of 63. He was a member of the Association for thirty-five years. He served as Editor-in-Chief of the MONTHLY from 1952 to 1956, and as President of the Association from 1959 to 1960; he received the Lester R. Ford Award for expository writing in 1965, and the Award for Distinguished Service to Mathematics in 1972.

Gerhard C. Arenstorf, a member of the first U. S. team participating in the International Mathematical Olympiad in 1974, with a first place tie among the members of that team, died on August 28, 1974, at the age of 17.

Dr. Marion M. Bontrager, Goshen College, died on February 5, 1974, at the age of 50. He was a member of the Association for eight years.

Dr. Mary Louise Buchanan, Adelphi University, died on August 23, 1974, at the age of 43. She was a member of the Association for twenty-one years.

Dr. Saul Kravetz, Torrance, California, died on August 31, 1974. He was a member of the Association for thirty years.

REPORT: MATHEMATICS DEPARTMENT SPEAKERS BUREAU, SAINT PETER'S COLLEGE

Five members of the mathematics department at Saint Peter's College — Professors A. C. Calianese, B. M. Kiernan, Eileen L. Poiani, L. E. Thomas, and F. A. Varrichio — participated in a Speakers Bureau for New York and New Jersey high schools during the 1974 Spring term. Presentations were made by faculty/student teams. The service was provided without charge. The purpose of the program was to present new applications of mathematics in computer science and in the social and life sciences, as well as to look at some classical mathematical problems from a fresh point of view.

Response to the program was gratifying. Forty-nine talks were requested, and fifty-four presentations were actually given. Audiences ranged in size from 10 to 350, with 40 as an approximate average. Several requests for repeat performances, as well as several new requests for the Fall, have been received. A list of the topics which were offered is available from Associate Professor Eileen L. Poiani, Department of Mathematics, Saint Peter's College, Jersey City, New Jersey 07306.

Response to an article by Mary Gray, this MONTHLY, Vol. 81, No. 5, May 1974.

THE UNIVERSITY OF TEXAS: ELEVENTH SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES

The University of Texas Health Science Center at Houston, Division of Continuing Education and the Region V of IEEE, announces the ELEVENTH SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES, April 3, 4, 5, 1975, at the Marriott Motor Hotel, Houston, Texas, under the direction of General Chairman, Dr. S. O. Zimmerman.

The program will emphasize mathematical, statistical, bioengineering and computing applications in biology and medicine.

An exhibit for instruments, computer components, etc., relevant to biomathematics and bioengineering will be arranged. Exhibitors are welcome to direct their questions to the Exhibition Chairman, Dr. A. S. Badger, Mandrel Products, 6909 Southwest Freeway, P. O. Box 36827, Houston, Texas 77036.

For further information write: Dr. S. O. Zimmerman, c/o Office of the Director, The University of Texas Health Science Center at Houston, Division of Continuing Education, P.O. Box 20367, Houston, Texas 77025.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE KENTUCKY SECTION

The fifty-seventh annual meeting of the Kentucky Section was held at Eastern Kentucky University, Richmond, on Friday and Saturday, April 12–13, 1974, Chairman J. E. Mack presiding. The meeting was postponed from the originally scheduled dates, April 5–6, due

to the April 3 tornado; 59 persons registered at the meeting, including 45 members of the Association.

Friday evening a panel discussed "College Mathematics Curriculum — The First Two Years." Members of the panel were Professor Stephen Langston, University of Kentucky; Professor Peter Moore, Northern Kentucky State College; Professor Jacqueline Moss, Paducah Community College; and Professor John Mack, University of Kentucky, Moderator.

Dr. A. B. Willcox, Executive Director of the Association, gave the invited address entitled "Elementary bridges to and from mathematics."

Saturday morning there were two parallel sessions for contributed papers.

Professor Ed Curtis, Maysville Community College, presided over the Mathematics Education section during which the following papers were presented:

1. *Elementary Statistics and the Computer*, by Andre Brousseau, Centre College, and Robert Piziak, Centre College (Presenter).
2. *Teaching Cultural Mathematics for Liberal Arts Students*, by W. O. Nowell, Jr., Georgetown College.
3. *Mathematics for the Liberal Arts*, by Sara Penry, Paducah Community College.
4. *Improving the Teaching of Mathematics for Elementary Teachers*, by W. C. Jones, Western Kentucky University.
5. *A Mathematical Laboratory for a Two-Year College*, by George Livingston, Ashland Community College.

Professor P. E. Bland, Eastern Kentucky University, presided over the session for Contributed Mathematics Papers during which the following papers were presented:

1. *The Remainder Term in Taylor's Formula*, by Harold Robertson, Murray State University.
2. *Counting Automorphisms of Finite p-groups — Some Recent Developments*, by R. M. Davitt, University of Louisville.
3. *Some Operational Formulas for DE's*, by J. B. Barksdale, Jr., Western Kentucky University.
4. *An Introduction to Shape Theory*, by J. H. LeVan, Eastern Kentucky University.
5. *A Topological Characterization of the Irrationals*, by Carl Eberhart, University of Kentucky.
6. *Application of Manifold Theory in Continuum Mechanics*, by David Ng, Eastern Kentucky University.

At the business meeting the following officers were elected: Vice Chairman, Professor Sara A. Penry, Paducah Community College; Secretary-Treasurer, Dr. T. M. Jenkins, University of Louisville. The Chairman, Dr. J. E. Mack, University of Kentucky and the Chairman-Elect, Dr. H. G. Robertson, Murray State University, continue in office.

It was agreed to hold the next annual meeting at Murray State University, Murray, Kentucky.

Dr. J. E. Simpson, University of Kentucky, Chairman of the "Pre-calculus" committee, presented a report outlining objectives of the high school mathematics curriculum for students intending to take a college calculus course. After discussion and amendment the report was adopted by the section.

Dr. Amy C. King, Eastern Kentucky University, Kentucky chairman of the MAA High School Contest, introduced the top two contestants from Kentucky on the 1974 High School Contest: Roger C. Callahan, Lexington, Kentucky, and William L. Boston, Murray, Kentucky.

T. M. Jenkins, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.

FLORIDA

ILLINOIS, Rockford College, Rockford, May 9–10, 1975.

INDIANA, Purdue University, Fort Wayne, April 26, 1975.

IOWA, Iowa State University, Ames, April 18–19, 1975.

KANSAS

KENTUCKY, Murray State University, Murray, April 11–12, 1975.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA Madison College, Harrisonburg, Virginia, April 26, 1975.

METROPOLITAN NEW YORK, Brooklyn College, CUNY, April 20, 1975.

MICHIGAN, General Motors Institute, Flint, May 2–3, 1975.

MISSOURI, Missouri Western State College, St. Joseph, April 18–19, 1975.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.

NEW JERSEY

NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Bowling Green State University, Bowling Green, May 2–3, 1975.

OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.

PACIFIC NORTHWEST

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 22, 1975.

ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.

SEAWAY, York University, Toronto, Ontario, April 25–26, 1975.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, Glendale Community College, Glendale, Arizona, April 11–12, 1975.

TEXAS, Angelo State University, San Angelo, April 11–12, 1975.

WISCONSIN, University of Wisconsin-Superior, April 19, 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Denver, Colorado, April 23–26, 1975.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30-May 2, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

THE AMERICAN
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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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CONTENTS

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NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

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K_0 AND K_1 — AN INTRODUCTION TO ALGEBRAIC K-THEORY

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Introduction. The last decade has witnessed the birth of a new subject: **Algebraic K-Theory.** This article purports to discuss: what is it about, what are some of its major examples and theorems, how can these examples be generalized, and how can the theorems be applied?

To dispel some mystery, let us first explain the term **K-theory.** Generally speaking, 'K-theory' refers to the study of certain groups which happen to be called K_i. *Topological K-theory* refers to the study of these K_i-groups defined for topological spaces, and *Algebraic K-theory* refers to the study of 'similar' K_i-groups defined for associative rings. There is, however, one interesting historical difference between the 'topological' and the 'algebraic'. In topology the groups K_i came into existence *all at once* [1]. They found immediate applications, and their important role was quickly established. In algebra, however, the groups K_0 and K_1 came into existence first [2], only to be followed by an experimental period during which people tried to search for the right definition of the 'higher' K_i's, in vain. It is only in the last few years that these higher algebraic K_i's were finally discovered; their existence and applicability have now justified the name of the subject—algebraic K-theory.

Because of the nature of this article it is not our purpose to survey the entire spectrum of the subject matter. In particular, it is hardly appropriate for us to attempt a general discussion of *all* the algebraic K_i's. We shall, therefore, only treat the more 'classical' aspects of the subject, concentrating on the two basic groups K_0 and K_1. This approach is, of course, testrictive in scope and old-fashioned in content, but it still serves the purpose of conveying a general flavor of the subject, as well as describing certain basic techniques used during its initial stage of development. In this approach, however, the theory of the higher K_i's will be largely ignored.

The organization of this article is as follows. Part I deals with K_0 and Part II deals with K_1. Part III relates K_0 to K_1, with connecting homomorphisms and exact sequences. Part IV discusses the Stability Theorems, showing a glimpse of the

great harmony between algebra and topology. Further remarks concerning the perspective of the subject are grouped together in a final section called 'Historical Notes'.

The prerequisites or the article are sometimes uneven. By and large, the text requires only first year graduate algebra, but some of the examples given require slightly more. As a rule, however, the more difficult examples (as well as some of the more technical footnotes) can be skipped without affecting the understanding of the later material. Thus, the reader who is stuck at one particular point should by all means go ahead — cheerfully — to read the rest.

PART I. THE FUNCTOR K_0

1. The Grothendieck group. We shall begin our exposition with the introduction of the so-called 'Grothendieck construction'. For concreteness, let us first present this basic construction in the setting of rings and modules. Let R be an arbitrary ring, and $\mathscr{M}(R)$ be the family of all left R-modules. Roughly speaking, *the* **Grothendieck construction** *is a process which, to any subfamily $\mathscr{C}$ in $\mathscr{M}(R)$, assigns a certain abelian group, denoted by $K_0\mathscr{C}$.* In detail, the construction works as follows.

For a left R-module M in the given family $\mathscr{C}$, let (M) denote the isomorphism class of M. Let F be the free abelian group on the basis $\{(M): M\in\mathscr{C}\}$, and let H be the subgroup of F generated by expressions $(M_2)-(M_1)-(M_3)$, where

$$(*) \qquad 0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

ranges over all exact sequences in $\mathscr{C}$ (i.e., $M_i\in\mathscr{C}$; f, g are R-homomorphisms; f is injective, g is surjective, and $\ker(g) = \operatorname{im}(f)$). We then define

$$K_0\mathscr{C} = F/H \text{ (the } \textbf{Grothendieck group } \textit{of } \mathscr{C}\text{)}.$$

For $M\in\mathscr{C}$, the image of (M) in $K_0\mathscr{C}$ will be denoted by $[M]$ (or, if necessary, $[M]_{\mathscr{C}}$). Thus, whenever we have an exact sequence (*) in $\mathscr{C}$, there results an equation $[M_2] = [M_1] + [M_3]$ in $K_0\mathscr{C}$. To simplify the language, it is usually permissible to say that *$K_0\mathscr{C}$ is generated by the symbols $[M]$ $(M\in\mathscr{C})$, with relations given by $[M_2] = [M_1] + [M_3]$ where $M_i\in\mathscr{C}$ are as in (*).*

Like many other objects defined in mathematics, the group $K_0\mathscr{C}$ satisfies a certain **'universal property'**, which is easily described as follows. Let χ be a mapping from $\mathscr{C}$ to some abelian group $(G, +)$ such that

(1) For $M\in\mathscr{C}$, the image $\chi(M)$ depends only on the isomorphism class of M;

(2) χ is 'additive over short exact sequences', i.e., whenever (*) is an exact sequence in $\mathscr{C}$, we have $\chi(M_2) = \chi(M_1) + \chi(M_3)$.

Then, there exists a unique group homomorphism $\tilde{\chi}: K_0\mathscr{C} \to G$ such that $\chi(M) = \tilde{\chi}([M])$ for all $M\in\mathscr{C}$. The proofs for the existence and uniqueness of $\tilde{\chi}$ are both very easy.

In the definition of $K_0\mathscr{C}$, we have not imposed any substantial restriction on the subfamily $\mathscr{C} \subset \mathscr{M}(R)$. In practice, however, it is desirable to require that $\mathscr{C}$ satisfy certain mild conditions, such as

(A) *$\mathscr{C}$ is closed under finite direct sums;*

(B) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{M}(R)$, then, $Y, Z \in \mathscr{C} \Rightarrow X \in \mathscr{C}$.*

If $\mathscr{C}$ satisfies both (A) and (B), we shall say that $\mathscr{C}$ is an *admissible subfamily* of $\mathscr{M}(R)$. Note that, granting (A), the sum of $[M]$, $[N]$ in $K_0\mathscr{C}$ may be taken as $[M \oplus N]$, since there exists an exact sequence $0 \to M \to M \oplus N \to N \to 0$. This implies, in particular, that *any element in $K_0\mathscr{C}$ may be expressed in the form $[M_1] - [M_2]$, with $M_i \in \mathscr{C}$.* As for the condition (B), one trivial consequence is that $\mathscr{C}$ contains the zero module $\mathbf{0}$ of $\mathscr{M}(R)$, and that $[\mathbf{0}] = 0$ in $K_0\mathscr{C}$. Another (more interesting) consequence is that, whenever we have a long exact sequence in $\mathscr{C}$:

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \longrightarrow 0,$$

($M_i \in \mathscr{C}$; f_1 is injective, f_{n-1} is surjective, and $\ker(f_{i+1}) = \operatorname{im}(f_i)$), then *the alternating sum $\Sigma(-1)^i[M_i]$ vanishes in $K_0\mathscr{C}$.* This follows by an obvious induction on n.

To illustrate the structure of $K_0\mathscr{C}$, let us examine a few elementary examples.

(1.1) The simplest case is given by letting R be a field, and $\mathscr{C}$ be the (admissible) family of all *finite dimensional* vector spaces over R. The mapping χ from $\mathscr{C}$ to $\mathbb{Z}$ given by 'dimension' is clearly additive over short exact sequences in $\mathscr{C}$. Hence it induces $\tilde{\chi}: K_0\mathscr{C} \to \mathbb{Z}$ such that $([M]) = \dim M$. This homomorphism $\tilde{\chi}$ is *surjective* since $\tilde{\chi}([R]) = 1$. It is also *injective*, since $\tilde{\chi}([M]-[N]) = 0 \Rightarrow \dim M = \dim N \Rightarrow M \cong N \Rightarrow [M]-[N] = 0$. Thus, $K_0\mathscr{C} \cong \mathbb{Z}$ by the dimension map.

(1.2) A similar example is given by letting R be the ring of integers and $\mathscr{C}$ be the (admissible) family of all *finite* abelian groups. The mapping χ from $\mathscr{C}$ to $\mathbb{Q}^+$ ($=$ the multiplicative group of positive rationals) given by 'cardinality' is clearly multiplicative over short exact sequences in $\mathscr{C}$. Hence it induces $\tilde{\chi}$: $K_0\mathscr{C} \to \mathbb{Q}^+$ such that $\tilde{\chi}([M]) = |M|$. This homomorphism $\tilde{\chi}$ is *surjective*, since $\tilde{\chi}([\mathbb{Z}/p\mathbb{Z}]) = p$ and $\mathbb{Q}^+$ is generated by the positive prime numbers. Furthermore, $\tilde{\chi}$ is *injective*, since $\tilde{\chi}([M]-[N]) = 1 \Rightarrow |M| = |N| \Rightarrow M$ and N have the same composition factors C_i (by 'Jordan-Hölder') $\Rightarrow [M] = \Sigma\,[C_i] = [N] \in K_0\mathscr{C}$. Thus, $K_0\mathscr{C} \cong \mathbb{Q}^+$ by the cardinality map, and it follows that $K_0\mathscr{C}$ is a *free* abelian group on the basis $\{[\mathbb{Z}/p\mathbb{Z}]: p = \text{prime}\}$.

(1.3) Again with $R = \mathbb{Z}$, let $\mathscr{C}'$ be the (admissible) family of all *finitely generated* abelian groups, which contains the family $\mathscr{C}$ in (1.2). The structure of $K_0\mathscr{C}'$, however, turns out to be completely different from $K_0\mathscr{C}$. If $0 \neq n \in \mathbb{Z}$, we have the following exact sequence in $\mathscr{C}'$:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0, \qquad i(x) = nx.$$

Thus, $[\mathbb{Z}/n\mathbb{Z}]_{\mathscr{C}'} = [\mathbb{Z}] - [\mathbb{Z}] = 0 \in K_0\mathscr{C}'$ (in contrast to $[\mathbb{Z}/n\mathbb{Z}]_{\mathscr{C}} \neq 0$ in $K_0\mathscr{C}$ for $n > 1$). It follows (from the Fundamental Theorem of Abelian Groups) that $K_0\mathscr{C}'$ is a cyclic group generated by the single element $[\mathbb{Z}]$. Now, by the universal property,

$M \in \mathscr{C}' \mapsto \operatorname{rank} M \in \mathbb{Z}$ defines a homomorphism *rank*: $K_0\mathscr{C}' \to \mathbb{Z}$. Since rank $\mathbb{Z} = 1$, we conclude that $K_0\mathscr{C}' \cong \mathbb{Z}$ by 'rank'.

In the second example above we saw that $[M] = [N]$ need not imply $M \cong N$. In general, given a subfamily $\mathscr{C} \subset \mathscr{M}(R)$, we can establish the following criterion for $[M] = [N]$ in $K_0\mathscr{C}$:

(1.4) PROPOSITION. *Suppose $\mathscr{C}$ is closed under finite direct sums, and $M, N \in \mathscr{C}$. Then $[M] = [N]$ in $K_0\mathscr{C}$ if and only if there exist $U, V, W \in \mathscr{C}$, together with exact sequences*

$$0 \to U \to M \oplus W \to V \to 0, \quad 0 \to U \to N \oplus W \to V \to 0.$$

Proof. For the 'if' part we have $[M] + [W] = [U] + [V] = [N] + [W]$. Since $K_0\mathscr{C}$ is an (abelian) group, cancellation yields $[M] = [N]$. Conversely, suppose $[M] = [N]$. Then, $(M) - (N)$ lies in the relation subgroup $H \subset F$ introduced in the definition of $K_0\mathscr{C}$. Say

$$(M) - (N) = \Sigma_i\{(T_i) - (T_i') - (T_i'')\} - \Sigma_j\{(S_j) - (S_j') - (S_j'')\} \in F,$$

where $0 \to T_i' \to T_i \to T_i'' \to 0$, $0 \to S_j' \to S_j \to S_j'' \to 0$ are exact sequences in $\mathscr{C}$. By transposition,

$$(M) + \Sigma_j(S_j) + \Sigma_i\{(T_i') + (T_i'')\} = (N) + \Sigma_j\{(S_j') + (S_j'')\} + \Sigma_i(T_i) \in F.$$

By the freeness of F, the isomorphism classes of M, S_j, T_i', T_i'' must be a permutation of the isomorphism classes of N, S_j', S_j'', T_i. In particular, we have an isomorphism

$$(*) \quad M \oplus \underbrace{\Sigma_j^{\oplus} S_j}_{S} \oplus \underbrace{\Sigma_i^{\oplus} T_i'}_{T'} \oplus \underbrace{\Sigma_i^{\oplus} T_i''}_{T''} \cong N \oplus \underbrace{\Sigma_j^{\oplus} S_j'}_{S'} \oplus \underbrace{\Sigma_j^{\oplus} S_j''}_{S''} \oplus \underbrace{\Sigma_i^{\oplus} T_i}_{T}.$$

Define $S, T', T'', \cdots$ as indicated above, and note that there are exact sequences $0 \to S' \xrightarrow{\alpha} S \to S'' \to 0$ and $0 \to T' \xrightarrow{\beta} T \to T'' \to 0$. Let W be any module in $\mathscr{C}$ isomorphic to the two sides of $(*)$, and let $U = S' \oplus T'$, $V = M \oplus N \oplus S'' \oplus T''$. Since $M \oplus W \cong M \oplus N \oplus S' \oplus S'' \oplus T$, we may imbed U into $M \oplus W$ via $1_{S'} \oplus \beta$, with a cokernel $\cong M \oplus N \oplus S'' \oplus (\operatorname{coker} \beta) \cong V$. Similarly, since $N \oplus W \cong N \oplus M \oplus S \oplus T' \oplus T''$, we may imbed U into $N \oplus W$ via $\alpha \oplus 1_{T'}$, with a cokernel $\cong N \oplus M \oplus (\operatorname{coker} \alpha) \oplus T'' \cong V$. Q.E.D.

(1.5) COROLLARY. *Suppose $\mathscr{C}$ above is such that all short exact sequences in $\mathscr{C}$ split. (Recall that a short exact sequence $0 \to X \to Y \xrightarrow{g} Z \to 0$ splits iff there exists an R-homomorphism $h: Z \to Y$ such that $g \circ h = 1_Z$, so in particular $Y \cong X \oplus Z$.) Then $[M] = [N]$ in $K_0\mathscr{C} \Leftrightarrow \mathscr{C}$ contains an R-module W such that $M \oplus W \cong N \oplus W$.*

Proof (*of Necessity*). Keeping the notations of (1.4), we have

$$M \oplus W \cong U \oplus V \cong N \oplus W. \qquad \text{Q.E.D.}$$

So far, we have discussed the Grothendieck group $K_0\mathscr{C}$ in the concrete setting of $\mathscr{C} \subset \mathscr{M}(R)$, where R is a ring. It is, however, clear from the discussion that we never really need the ring R itself, except to speak about its modules, and the homomor-

phisms between them. This observation brings in the modern notion of a 'category'. Roughly speaking, a **category** $\mathscr{A}$ consists of a collection of 'objects', together with sets of 'morphisms' (maps) between these objects. (Our $\mathscr{M}(R)$ is a quintessential example of such a 'category'.) If a given category $\mathscr{A}$ is 'rich' enough, we may be able to talk about the 'kernel' and the 'image' of the morphisms in $\mathscr{A}$. In such a nice category $\mathscr{A}$, we shall then have a viable notion of 'exact sequences'. If $\mathscr{C}$ is a **subcategory** of $\mathscr{A}$ (i.e., a subfamily of objects in $\mathscr{A}$), we may then repeat the Grothendieck construction word for word in this abstract setting (with 'modules' replaced by objects in $\mathscr{C}$), to define a *Grothendieck group* $K_0\mathscr{C}$. Predictably, this group $K_0\mathscr{C}$ enjoys the same universal property as discussed before, and results such as (1.4), (1.5) also carry over *verbatim* to the abstract case.

The rationale of the above digression is that the general formulation of $K_0\mathscr{C}$ for *categories* turns out to be vital for many important applications, and is not prompted by a blind impulse to generalize. The context of some of these applications will be briefly sketched in the section of Historical Notes. For our purposes, however, it will be sufficient to work with categories of modules most of the time; *thus, we will not assume the reader to know the ingredients of category theory.* Nevertheless, we will sometimes use words such as 'object', 'category', 'admissible subcategory' (same definition as admissible subfamily of modules), etc, to conform with current mathematical terminology. For example, when we discuss $K_1\mathscr{C}$ in Part II (Section 5), we will try to give its definition in terms of general categories, and not just module categories. Needless to say, the reader may interpret everything said just in terms of module categories, whenever he so wishes.

2. K_0 of a ring. Given a ring R, we shall now apply the K_0 construction to a *specific* subcategory $\mathscr{C}$ of $\mathscr{M}(R)$. In fact, we take $\mathscr{C}$ to be $\mathscr{P}(R)$: *the family of all projective, finitely generated* (left) *R-modules.* The Grothendieck group $K_0\mathscr{P}(R)$ will be denoted simply by $K_0(R)$.

Recall that a (left) R-module P is called *projective* if P is a direct summand of some free R-module. Another characterization of P being projective is that any short exact sequence $0 \to X \to Y \to P \to 0$ in $\mathscr{M}(R)$ must split. The latter characterization has the following important consequences:

(A) If a projective P is f.g. (shorthand for 'finitely generated', i.e., $P = \sum_{i=1}^{n} R \cdot p_i$ for suitable $p_1, \cdots, p_n$ in P), then P is in fact a direct summand of some f.g. free R-module (e.g., R^n !).

(B) $\mathscr{P}(R)$ is an *admissible* subcategory of $\mathscr{M}(R)$.

(C) "K_0 is a **functor** on the category of rings" (or, as some people say, $K_0(R)$ is 'functorial' in R). The meaning of all this jargon is actually more concrete than it sounds. Namely, it just refers to the property that, if we have a ring homomorphism $f\colon R \to S$, then, we also have, in a natural way, an induced group homomorphism $f_*\colon K_0(R) \to K_0(S)$. To construct f_*, consider the mapping χ from $\mathscr{P}(R)$ to $K_0(S)$ sending $P \in \mathscr{P}(R)$ to $[S \otimes_R P] \in K_0(S)$. If $0 \to P_1 \to P_2 \to P_3 \to 0$ is an exact sequence

in $\mathscr{P}(R)$, then it *splits*, and so $0 \to S \otimes_R P_1 \to S \otimes_R P_2 \to S \otimes_R P_3 \to 0$ is also (split) exact. It follows that $\chi(P_2) = \chi(P_1) + \chi(P_3)$, so, by the universal property, χ defines a unique homomorphism f_*, such that $f_*([P]) = [S \otimes_R P]$. Note that we have

$$(f \circ g)_* = f_* \circ g_*, \text{ and } (1_R)_* = 1_{K^0(R)}.$$

The study of projective modules is not only a crucial topic in the modern chapter of ring theory, it has also found important applications to representation theory of groups and algebras, algebraic geometry, and topology. Unfortunately, given an arbitrary ring R, there is no known algorithm whereby to calculate its (say, f.g.) projective modules. On the other hand, it turns out that the abelian group $K_0(R)$ is somewhat more amenable to computations. Indeed, recent advances in algebraic K-theory have yielded various efficient means and techniques which help calculate $K_0(R)$.

To set matters straight once and for all, we should note carefully that a full computation of $K_0(R)$ does not necessarily lead to the determination of all the isomorphism classes in $\mathscr{P}(R)$. In fact, for $P_i \in \mathscr{P}(R)$, $[P_1] = [P_2] \in K_0(R)$ implies only that there exists $W \in \mathscr{P}(R)$ (which may be assumed *free* and f.g.) such that $P_1 \oplus W \cong P_2 \oplus W$ (see (1.5)). In this event, we say that P_1 and P_2 are **stably isomorphic.** Thus, a full computation of $K_0(R)$ leads to a determination of all *stable isomorphism classes* in $\mathscr{P}(R)$. We say that $P \in \mathscr{P}(R)$ is *stably free* if P is stably isomorphic to some free module R^n. In general, this need not imply $P \cong R^n$. An intriguing case in point is the polynomial ring $R = k[X_1, \cdots, X_n]$ (k = a field). For this ring, it can be shown that $K_0(R) \cong \mathbb{Z}$, with the generator $[R]$.* This says that any $P \in \mathscr{P}(R)$ is stably free. Yet, it has remained a major unsolved problem (**'Serre's Conjecture'**) whether such P need actually be free.**

On the other hand, it is not difficult to construct a (commutative) ring A whose $\mathscr{P}(A)$ contains a module P which is *stably* free but *not* free. In fact, let A be the coordinate ring of the real sphere S^2, i.e., $A = \mathbb{R}[x, y, z]$ with the relation $x^2 + y^2 + z^2 = 1$. Let $\varepsilon\colon A^3 \to A$ be the A-linear functional given by $\varepsilon(\alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$. Since $\varepsilon(x, y, z) = 1$, we have a splitting $A^3 \cong \ker \varepsilon \oplus A$, so $P = \ker \varepsilon$ is stably free. Assume, for the moment, that P is actually free, so (necessarily) $P \cong A^2$. Then, A^3 will have a new basis consisting of (x, y, z) and two other triples (f, g, h), (f', g', h'). The matrix

$$\begin{pmatrix} x & f & f' \\ y & g & g' \\ z & h & h' \end{pmatrix}$$

is therefore invertible over A, and has determinant equal to a unit $e \in A$. We think of f, g, h, e, $e^{-1}, \cdots$ as functions on S^2 (they are polynomial expressions in the

*This follows, for example, from Hilbert's Syzygy Theorem which says that any f. g. R-module has a finite resolution by f. g. free R-modules.

**For $n = 1$, the answer is YES since R is a *PID*. For $n = 2$, the answer is YES by a theorem of Seshadri. For $n = 3$ and k algebraically closed, the answer is also YES by recent work of Murthy and Towber. Further results have been announced by Roitman, Swan, Souslin and Vasserstein.

'coordinate functions' x, y, z). Consider the continuous vector field on S^2 given by $v \in S^2 \mapsto (f(v), g(v), h(v)) \in \mathbb{R}^3$. Since $e = f' \cdot (yh - zg) - g' \cdot (xh - zf) + h' \cdot (xg - yf)$ is clearly nowhere zero on S^2, the vector $(f(v), g(v), h(v))$ is nowhere collinear with the vector v. Taking the orthogonal projections onto the tangent planes of S^2, we obtain a continuous vector field of nowhere vanishing tangents. This is well known to be impossible by elementary topology (see, e.g., [**20**, Cor. 5]). It follows that P cannot be free over A.

Coming back to a general ring R, it might be wondered whether the study of $K_0(R)$ is of any practical value since $[P] \in K_0(R)$ fails to pinpoint the isomorphism type of P. Remarkably enough, the answer is still an emphatic 'yes'. This is due to the fortunate fact that, in application, various useful invariants take their well-defined values in $K_0(R)$ (or in the quotient $K_0(R)/\mathbb{Z} \cdot [R]$), and *not* in the set of isomorphism classes of projective R-modules.*

To present a few basic computations of $K_0(R)$, let us study the class of rings R for which $\mathscr{P}(R)$ has the so-called **Krull-Schmidt property.** This means that any nonzero $Q \in \mathscr{P}(R)$ admits a decomposition $Q = Q_1 \oplus \cdots \oplus Q_r$ where the $Q_i (\neq 0)$ are indecomposable, and the isomorphism types $\{(Q_1), \cdots, (Q_r)\}$ are uniquely determined up to a permutation. Before giving examples, we first establish the following elementary but remarkable result.

(2.1) PROPOSITION. *If $\mathscr{P}(R)$ satisfies the Krull-Schmidt Property, then $K_0(R)$ is a free abelian group generated by the finite set of distinct* f.g. *indecomposable projectives.*

Proof. Let $\{P_i\}$ be a full set of f.g. indecomposable projectives. Form a free abelian group G whose basis $\{e_i\}$ are in one-one correspondence with $\{P_i\}$. Define a homomorphism $\phi: K_0(R) \to G$ by $\phi([Q]) = \Sigma\phi_i(Q) \cdot e_i$, where $\phi_i(Q)$ is the number of indecomposable constituents of Q $(\in \mathscr{P}(R))$ which are $\cong P_i$. Then, ϕ is clearly an isomorphism, with the inverse map induced by $e_i \mapsto [P_i]$. It only remains to show that the $\{P_i\}$ are finite in number. Let $R = I_1 \oplus \cdots \oplus I_r$ be a 'Krull-Schmidt decomposition' of the (left) module R itself. Any P_i is an indecomposable constituent of some free module $R^n \cong n \cdot I_1 \oplus \cdots \oplus n \cdot I_r$. Hence $P_i \cong I_j$ for some suitable j. (For this reason, the I_j's are sometimes called the 'principal indecomposables'. Note, however, that one could have $I_j \cong I_k$ even though $j \neq k$.)

We shall now give examples via the following corollaries.

(2.2) COROLLARY. *If R is local, or else a PID, then $K_0(R) \cong \mathbb{Z}$ with generator*

*Without giving details, let us mention a specific instance of this. In [**18**], Wall investigated a topological question raised by Milnor: if a complex X is a homotopy retract of a finite complex, is X then homotopy equivalent to a finite complex? Letting π be the fundamental group of X, and R be the group ring $\mathbb{Z}\pi$, Wall constructed an invariant of (the homotopy type of) X lying in $K_0(R)/\mathbb{Z} \cdot [R]$. It is shown that this invariant is trivial iff X is homotopy equivalent to a finite complex; further, the invariant can take any value in the quotient group $K_0(R)/\mathbb{Z} \cdot [R]$. In the special case when X is simply connected $(\pi = 1)$, this quotient group is trivial (see (2.2) below), so X is automatically homotopy equivalent to a finite complex.

$[R]$. (In particular, this covers division rings (hence fields), as well as $R = \mathbb{Z}$, $R = k[t]$, $R = k[[t]]$ (k a field).)

Proof. First, note that (f.g.) projective modules are free. (For a PID, it is well known that submodules of free modules are free. For a local ring (not necessarily commutative), see [**14**, p. 93].) With this information, the Krull-Schmidt Property for $\mathscr{P}(R)$ boils down to the **invariant basis property:** $R^n \cong R^m \Rightarrow n = m$. This is true for any commutative ring and for any local ring. The conclusion of the Corollary follows since R is clearly the unique principal indecomposable.

(2.3) COROLLARY. *If R is left artinian, then $K_0(R)$ is a free abelian group, generated by the finite set of distinct principal indecomposables.* (In particular, this covers all finite dimensional algebras over fields.)

Proof. Any f.g. (left) R-module possesses a composition series (see, e.g. [**12**, p. 69]). Thus, the Krull-Schmidt Property holds for all f.g. modules [**12**, p. 24], and not just for the projective ones!

EXAMPLE. If D is a division ring and R is the matrix ring $\mathbf{M}_n(D)$, then $K_0(R) \cong \mathbb{Z}$, with generator given by $[D^n]$ ($R = \mathbf{M}_n(D)$ acts on the column vectors of D^n by left multiplication). Note that $[R] = n \cdot [D^n]$ in $K_0(R)$.

3. K_0 of commutative rings. Throughout this section, let R denote a *commutative* ring. One significant consequence of this is that the tensor product ($\otimes = \otimes_R$) of two R-modules is again an R-module ($r \cdot (x \otimes y) = rx \otimes y = x \otimes ry$). Since $R^m \otimes R^n \cong R^{mn}$, it is clear that $\mathscr{P}(R)$ is closed under $\otimes$. From this, we conclude easily that $[P] \cdot [Q] = [P \otimes Q]$ induces a multiplication on $K_0(R)$, which makes $K_0(R)$ into a commutative *ring*, with identity $[R]$.

Another advantage of having R commutative is, of course, that we may consider localizations of R and their modules. Note that if $P \in \mathscr{P}(R)$ and if $\mathfrak{p}$ is a prime ideal in R, then the localization $P_{\mathfrak{p}}$ must be a f.g. *free* module over the local ring $R_{\mathfrak{p}}$. We shall say that P is of *constant rank* if there exists an integer $n \geqq 0$ such that $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ for all primes $\mathfrak{p}$. In this case, we write $\operatorname{rk} P = n$.

Of special interest are those $P \in \mathscr{P}(R)$ which have constant rank 1. The set of isomorphism classes of such modules has a commutative monoid structure under $(P) \cdot (Q) = (P \otimes Q)$. One can show that this monoid is, in fact, a group. To see this, observe first that $\operatorname{rk} P = 1 \Rightarrow \operatorname{rk} P^* = 1$, where $P^* = \operatorname{Hom}_R(P, R)$ is the dual of P. The evaluation map $P^* \otimes P \to R$ is an isomorphism upon localization at any prime $\mathfrak{p}$; hence $P^* \otimes P \cong R$. The inverse of (P) is thus furnished by (P^*). The group $\{(P): P \in \mathscr{P}(R),\ \operatorname{rk} P = 1\}$, constructed in this way, is denoted by $\operatorname{Pic}(R)$ (the **Picard group** of R).* Its relationship with $K_0(R)$ is given as follows:

(3.1) PROPOSITION. *The map $(P) \mapsto [P]$ determines a group monomorphism from* $\operatorname{Pic}(R)$ *into the group of units of* $K_0(R)$.

*The basic idea used in this construction pervades several different branches of mathematics. In algebraic geometry, one considers the group of 'invertible sheaves' over an algebraic variety. In a somewhat analogous way, the topologist studies the group of line bundles over a space (Cf. Section 10).

Proof. Injectivity is the only thing to show. For P_1, P_2 of constant rank 1, suppose $[P_1] = [P_2] \in K_0(R)$. Say $P_1 \oplus R^n \cong P_2 \oplus R^n$. Consider the exterior algebras for the modules P_i. We have $\Lambda^\beta(P_i) = 0$ for all $\beta \geqq 2$, since this holds 'locally'. Now consider the exterior power Λ^{n+1} of $R^n \oplus P_i$:

$$\Lambda^{n+1}(R^n \oplus P_i) \cong \sum_{\alpha+\beta=n+1}^{\oplus} \Lambda^\alpha(R^n) \otimes \Lambda^\beta(P_i) \cong \Lambda^n(R^n) \otimes \Lambda^1(P_i) \cong R \otimes P_i \cong P_i.$$

Thus, $P_1 \oplus R^n \cong P_2 \oplus R^n$ clearly implies $P_1 \cong P_2$ by taking the above exterior power. Q.E.D.

Here is a remarkable application which demonstrates very well the distinct advantage of working with $K_0(R)$:

(3.2) THEOREM. *Suppose $P \in \mathscr{P}(R)$ has $\operatorname{rk} P = 1$. If P has a 'finite free resolution', i.e. if there exists a long exact sequence*

$$0 \to F_m \to F_{m-1} \to \cdots \to F_0 \to P \to 0,$$

where F_i are f.g. free modules, then $P \cong R$.

Proof. In $K_0(R)$, we have an equation $[P] = \sum_{i=0}^m (-1)^i [F_i]$ (see Section 1). By transposition, we get $[P \oplus R^r] = [R^s]$ for suitable integers r, $s \geqq 0$. Changing r, s if necessary, we may suppose that $P \oplus R^r \cong R^s$. By rank consideration, $s = r+1$. Thus, $[P] = [R]$, and the previous Proposition implies $P \cong R$. Q.E.D.

As a useful illustration, let us compute the ring structure of $K_0(R)$ in the case when R is a Dedekind domain. This computation is a rather nice translation of the well-known Steinitz theory (see [**10**, Section 22]) for f.g. modules over such domains.

(3.3) THEOREM. *Let R be Dedekind, and L be its quotient field. Let J be the kernel of j: $K_0(R) \to K_0(L) \cong \mathbb{Z}$. Then $K_0(R) \cong \mathbb{Z} \oplus J$, and $J \cong \operatorname{Pic}(R)$. Further $J^2 = 0$ (which completely determines the ring structure of $K_0(R)$).*

Proof. Note that any rank 1 projective is isomorphic to a non-zero ideal, and conversely. Consider the map i: $\operatorname{Pic}(R) \to J$ given by $i(P) = [R]-[P]$. We claim that i is a *homomorphism*, i.e., $i(P) + i(Q) = i(P \otimes Q)$. Think of P, Q as (nonzero) ideals in R. Then we have an isomorphism $P \oplus Q \cong R \oplus P \cdot Q$ [**10**, p. 150]. Granting this, we obtain $i(P) + i(Q) = 2[R] - [P \oplus Q] = [R]-[P \cdot Q] = i(P \otimes Q)$ (since $P \otimes Q \cong P \cdot Q$). By Proposition (3.1), we know already that i is injective. To show surjectivity, suppose $j[P_1] = j[P_2]$, where $P_i \in \mathscr{P}(R)$. By the Steinitz theory, $P_i \cong R^{n_i} \oplus \mathfrak{A}_i$ where $\mathfrak{A}_i$ are nonzero ideals. Since $j[P_i] = (n_i + 1) \cdot [L]$, we have $n_1 = n_2$. Therefore $[P_1]-[P_2] = [\mathfrak{A}_1]-[\mathfrak{A}_2] = i(\mathfrak{A}_2) - i(\mathfrak{A}_1) \in \operatorname{Im}(i)$. Finally, $J^2 = 0$ follows from $i(\mathfrak{A}) \cdot i(\mathfrak{B}) = [R]-[\mathfrak{A}]-[\mathfrak{B}] + [\mathfrak{A} \cdot \mathfrak{B}] = 0$. Q.E.D.

In the above situation, $\operatorname{Pic}(R)$ is exactly the group of fractional ideals in L modulo the subgroup of principal fractional ideals. As such, it is often called the *ideal class group* of the (Dedekind) domain R. The formation of this group originates from the classical context in which one studies ideals in a full ring R of algebraic

integers in a given number field L. A central result in number theory states that, in this classical case, Pic(R) is always *finite*—its cardinality h is called the 'class number' of L. Inasmuch as R is a UFD $\Leftrightarrow R$ is a PID $\Leftrightarrow \mathrm{Pic}(R) = 0$, the value of the integer h is an effective numerical measure of how much R fails to satisfy unique factorization, and how far the R-ideals are from being principal.* For an explicit example, the class number for $L = \mathbb{Q}(\sqrt{-5})$ is 2, whence $K_0(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.**

Hardly surprisingly, the 'Finiteness of Class Number' theorem admits appropriate generalizations to the *noncommutative* case. The precise statement of such a generalization is given by the so-called *Jordan-Zassenhaus Theorem* (see [**10**, §79], [**3**, p. 544]). Instead of stating this result in detail, however, suffice it for us to record what it does give in K-theory:

(3.4) Corollary (to 'Jordan-Zassenhaus'). *Let A be any ring (commutative or otherwise) whose underlying additive group is finitely generated. Then, the group $K_0(A)$ is also finitely generated* (see [**3**, p. 547]).

Resuming the discussion of Pic(R) in the commutative case, we append the observation that, unfortunately, the 'Finiteness of Class Number' theorem does not hold for a *general* Dedekind domain. In fact, an ingenious construction of L. Claborn [**9**] has shown that, given *any* abelian group G whatsoever, there exists a Dedekind domain R with $\mathrm{Pic}(R) \cong G$! This says that nothing more can be added to (3.3) in so far as the additive structure of the K_0 of Dedekind domains.

4. G_0 of a ring. For a ring R, let $\mathscr{M}_{fg}(R)$ denote the subcategory of $\mathscr{M}(R)$ consisting of all f.g. left R-modules. Note that if we assume R to be left noetherian, $\mathscr{M}_{fg}(R)$ will be an *admissible* subcategory of $\mathscr{M}(R)$, in the sense discussed in Section 1.

(4.1) Definition. *For R left noetherian, define $G_0(R) = K_0\mathscr{M}_{fg}(R)$. The inclusion of categories $\mathscr{P}(R) \subset \mathscr{M}_{fg}(R)$ induces a map $c: K_0(R) \to G_0(R)$ (by $c([P]_{\mathscr{P}(R)}) = [P]_{\mathscr{M}_{fg}(R)}$), called the Cartan homomorphism over R.*

We observe, however, that G_0 does not possess the functorial property of K_0. In other words, if $f: R \to R'$ is a homomorphism of left noetherian rings, the rule $M \in \mathscr{M}_{fg}(R) \mapsto R' \otimes_R M \in \mathscr{M}_{fg}(R')$ in general *does not* induce a well-defined map from $G_0(R)$ to $G_0(R')$. The trouble is that a short exact sequence in $\mathscr{M}_{fg}(R)$ may not stay short exact *after* application of the 'tensor product functor' $R' \otimes_R -$.

*This second statement is to be taken with a grain of salt, as any R-ideal $\mathfrak{A} \neq 0$ can always be generated by *two* elements! (This follows, for instance, from the observation that $\mathfrak{A} \oplus \mathfrak{A}^* \cong R \oplus (\mathfrak{A} \otimes \mathfrak{A}^*) \cong R^2$.) One way to justify the statement is to note that, for $\mathfrak{A}$ above, the power $\mathfrak{A}^h$ is always a *principal* ideal.

**Generators for the two summands are, e.g. $[R]$ and $[R] - [\mathfrak{p}]$, where $R = \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{p} = (2, 1 - \sqrt{-5})$. An easy calculation shows $\mathfrak{p}^2 = (2)$.

In dealing with G_0, keep in mind that we are working with *more modules* but meanwhile with *more relations*. This often tends to make G_0 somewhat more unpredictable than K_0. For example, a novice will likely be surprised by an equation such as $[R/(\mathfrak{p}, x)] = 0 \in G_0(R)$, where $x \notin \mathfrak{p}$, and $\mathfrak{p}$ is a prime ideal in a commutative (noetherian) ring R. Nevertheless, the equation follows from the R-isomorphism $R/\mathfrak{p} \cong (\mathfrak{p}, x)/\mathfrak{p}$ which sends 1 to x. Let us now examine several computations of G_0.

(4.2) EXAMPLE. R = *Dedekind domain.* It turns out that $c: K_0(R) \to G_0(R)$ is an isomorphism, so $G_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R)$. The *surjectivity* of c is clear. For, if a module M can be generated by r elements, we have an exact sequence $0 \to P \to R^r \to M \to 0$, where $P \in \mathscr{P}(R)$ since P is torsion-free (see [**14**, Theorem 4.22]). In $G_0(R)$, we have therefore $[M] = [R^r] - [P] \in \mathrm{Im}(c)$. We shall omit the proof of the *injectivity* of c, which depends on 'Schanuel's Lemma' (essentially to the effect that $[M] \in G_0(R) \mapsto [R^r] - [P] \in K_0(R)$ gives a *well-defined* inverse for c). In case R is a PID, clearly the surjectivity of c already implies the injectivity, so we conclude: $G_0(\mathrm{PID}) \cong \mathbb{Z}$, which generalizes (1.3).

(4.3) EXAMPLE. R = *left artinian ring.* As observed before, the modules in $\mathscr{M}_{fg}(R)$ are precisely those which have (finite) composition series. Thus, every $[M]$ equals the sum of its own composition factors in $G_0(R)$. Using the Jordan-Hölder Theorem, and proceeding as in the proof of (2.1), one sees easily that $G_0(R)$ is a free abelian group with basis $[S_i]$, where $\{S_i\}$ are a full set of distinct simple left R-modules. The S_i's are finite in number, and are, in fact, in one-one correspondence with the I_i's which are the principal indecomposables.* If I_i has c_{ij} composition factors isomorphic to S_j, we get the equations $[I_i] = \Sigma_j c_{ij}[S_j] \in G_0(R)$. The map $c: K_0(R) \to G_0(R)$ is then essentially given by the square matrix (c_{ij}), which is known classically as the **Cartan matrix** of R. (This explains why c is called the Cartan homomorphism in general.) Note that $\det(c_{ij}) \neq 0 \Leftrightarrow c$ is injective $\Leftrightarrow \operatorname{coker} c$ is finite. This happens, for instance, for group algebras $R = kG$ (k a field, G a finite group)—by a celebrated theorem of Brauer. While we shall not prove Brauer's theorem here, let us explain what happens in the simplest possible case—when G is a p-group, and $\mathrm{char}(k) = p$. In this case, $R = kG$ is a local (artinian) ring, so the only principal indecomposable is $I_1 = R$, and the only simple module is $S_1 = R/\mathrm{rad}\, R \cong k$ (with the trivial G-action). The Cartan matrix is 1×1, with single entry $= \dim_k R = |G|$.

(4.4) EXAMPLE. R = *an algebra over a field* k, $\dim_k R < \infty$. The Grothendieck group $G_0(R)$ is intimately related to the characters of k-representations of R. For $M \in \mathscr{M}_{fg}(R)$ which affords such a representation, the *character* of M is a map $\chi_M: R \to k$ defined by $\chi_M(r) = \mathrm{trace}(L_r(M))$, where $L_r(M)$ is the k-endomorphism

*One may take $S_i = I_i/(\mathrm{rad}\ R)\ I_i$ (see [**10**], p. 374).

on M given by left multiplication by r. Clearly, χ_M belongs to the k-dual $R^* = \mathrm{Hom}_k(R, k)$. Given an exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{M}_{fg}(R)$, we may express $L_r(M)$ schematically as

$$\begin{pmatrix} L_r(M') & * \\ 0 & L_r(M'') \end{pmatrix}$$

(for a suitable choice of basis in M), hence $\chi_M = \chi_{M'} + \chi_{M''}$ on taking trace. By the universal property of $G_0(R)$, the rule $[M] \mapsto \chi_M$ defines a homomorphism $\chi\colon G_0(R) \to R^*$. It turns out that *$\chi$ is, in fact, injective if k has zero characteristic.* This means that, in characteristic zero, *the character of M determines the class* $[M] \in G_0(R)$ (though not necessarily the *isomorphism type* of M). To prove this, let $\{S_i\}$ be the simple R-modules. Say $[M] = \sum m_i[S_i]$. The semi-simple ring $R/\mathrm{rad}\, R$ has idempotents $\{e_i\}$ such that e_j acts on S_i as δ_{ij} (Kronecker delta). Let a_j be a lifting of e_j to R itself. Then,

$$\chi_M(a_j) = \sum_i m_i\chi_{S_i}(a_j) = \sum_i m_i\delta_{ij}\dim_k S_i = m_j \dim_k S_j.$$

If $\mathrm{char}(k) = 0$, the integers $m_j = \chi_M(a_j)/\dim_k S_j$ are thus uniquely determined by χ_M.*

(This last example is especially important in the theory of group representations. If R is a complex group algebra $\mathbb{C}G$, G a finite group, then the character map χ induces an isomorphism of $\mathbb{C} \otimes_Z G_0(\mathbb{C}G)$ with the $\mathbb{C}$-space of all 'class functions' on G, i.e., functions $\alpha\colon G \to \mathbb{C}$ such that $\alpha(g^{-1}hg) = \alpha(h)$ for all $g, h \in G$. In particular, the rank of $G_0(\mathbb{C}G)$ equals the number of conjugacy classes in G.)

Lastly, we mention without proof the following Finiteness Theorem for G_0 which is the analog of (3.4):

(4.5) THEOREM. *Let R be any ring whose underlying additive group is finitely generated. Then, $G_0(R)$ is also finitely generated.*** (See [3, p. 547].)

PART II. THE FUNCTOR K_1

5. Two definitions of K_1. Having studied K_0 and G_0 to some extent, we shall now begin a new line of investigation by introducing the 'next' functor, K_1. In studying K_1, there are two goals which we must try to accomplish. On the one hand, we must demonstrate the intrinsic interest of this new mathematical object K_1; on the other hand, we should try to exhibit the intimate relationship between K_1 and K_0 which is supposed to justify the nomenclature. For exposition purposes, it will be convenient for us to separate the above two goals. In Part II, therefore, we

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*In case char $(k) = p \neq 0$, χ is clearly *not* injective, for $\chi\,(p \cdot G_0\,(R)) \subset p \cdot R^* = 0$. For example, M always has the same character as $p+1$ copies of M!

**On the contrary, if R is only f.g. *as a ring* over $\mathbb{Z}$, it is apparently not known whether $G_0(R)$ need be f.g.

shall first study K_1 in its own right. The job of relating K_1 to K_0 will be postponed to Part III.

The simplest possible definition of K_1 is given in terms of matrices, and, in this form, it was first conceived by the late topologist J. H. C. Whitehead. In [**19**], one of the pioneering papers in geometric topology, Whitehead noticed that, if π is the fundamental group of some space X, and if $R = \mathbb{Z}\pi$ (the integral group ring of π) the elementary row and column transformations for matrices over the ring R have certain natural topological interpretations. Prompted by this, Whitehead introduced a certain abelian group, $\mathrm{Wh}(\pi)$ (the 'Whitehead group' of π), to study homotopies between spaces. For every homotopy equivalence $f: X \to Y$, Whitehead defined an invariant $\tau(f) \in \mathrm{Wh}(\pi)$, which is such that $\tau(f) = 0$ if and only if f is a 'simple homotopy equivalence'. In topology, this invariant $\tau(f)$ has come to be known as the 'Whitehead torsion' of f.

If one examines carefully Whitehead's definition of $\mathrm{Wh}(\pi)$ via elementary row and column operations on matrices, one will easily see that most of the steps taken depend only on the *ring* $R = \mathbb{Z}\pi$, and not on the *group* π. Thus, repeating these steps for an *arbitrary* ring R, one obtains a certain abelian group determined by R, free from the topological context. This group is precisely the $K_1(R)$, which we shall now describe in detail.

Let $GL_n(R)$ denote the group of invertible $n \times n$ matrices over an arbitrary ring, R. For $i \neq j$, and $\lambda \in R$, let e_{ij}^{λ} be the 'elementary matrix' with 1's down the diagonal, λ at the ij-entry, and zeros elsewhere. We have $e_{ij}^{\lambda} \in GL_n(R)$, since $(e_{ij}^{\lambda})^{-1} = e_{ij}^{-\lambda}$. By easy inspection, we see that left and right multiplications by e_{ij}^{λ} correspond to elementary row and column operations on matrices. Let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by all e_{ij}^{λ}, $\lambda \in R$, $1 \leqq i \neq j \leqq n$. Note that, if τ is any rectangular matrix, then any $n \times n$ matrix in block form $\sigma = \begin{pmatrix} I & \tau \\ 0 & I \end{pmatrix}$ is automatically in $E_n(R)$.* It was further discovered by Whitehead that, if n is even, $E_n(R)$ contains another interesting class of matrices. This important discovery is known nowadays as

(5.1) WHITEHEAD'S LEMMA. *If* $\alpha \in GL_m(R)$, *then*

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2m}(R).$$

If, also, $\beta \in GL_m(R)$, *then*

$$\begin{pmatrix} [\alpha, \beta] & 0 \\ 0 & I_m \end{pmatrix} \in E_{2m}(R).$$

($[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$ *is the commutator of* α *and* β.)

Proof. The first assertion is proved by finding a sequence of elementary 'block'

*An obvious sequence of elementary row transformations brings σ to the identity matrix.

transformations which brings $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$ to the identity. For example, one such sequence of row transformations is given by

$$\begin{pmatrix} I & 0\\ -I & I\end{pmatrix}\begin{pmatrix} I & I\\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0\\ -I & I\end{pmatrix}\begin{pmatrix} I & -\alpha\\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0\\ \alpha^{-1} & I\end{pmatrix}\begin{pmatrix} I & -\alpha\\ 0 & I\end{pmatrix}\begin{pmatrix} \alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}.$$

The second assertion follows from the first, with the opportune observation that

$$\begin{pmatrix} [\alpha,\beta] & 0\\ 0 & I\end{pmatrix}=\begin{pmatrix} \alpha^{-1}\beta^{-1} & 0\\ 0 & \beta\alpha\end{pmatrix}\begin{pmatrix} \alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix} \beta & 0\\ 0 & \beta^{-1}\end{pmatrix}. \qquad \text{Q.E.D.}$$

Let us now think of $GL_n(R)$ as a subgroup of $GL_{n+1}(R)$ by identifying any $\sigma\in GL_n(R)$ with

$$\begin{pmatrix} \sigma & 0\\ 0 & 1\end{pmatrix}\in GL_{n+1}(R).$$

In this way, it is meaningful to talk about the ascending union $GL(R) = \bigcup_{n=1}^{\infty} GL_n(R)$, called the *infinite general linear group* over R. Since clearly $E_n(R)\subset E_{n+1}(R)$ under the above identification, we may also form $E(R) = \bigcup E_n(R)$. The Whitehead Lemma may now be rehashed as follows.

(5.2) COROLLARY. $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$.

Proof. A short commutator calculation shows that e_{ij}^{λ} equals $[e_{ik}^{\lambda}, e_{kj}^{1}]$ if i, j, k are distinct. Thus, we have $E_n(R) = [E_n(R), E_n(R)]$ whenever $n \geqq 3$. Whitehead's Lemma says $[GL_m(R), GL_m(R)] \subset E_{2m}(R)$ which then completes the proof.

One consequence of (5.2) is, of course, that $E(R)$ is a *normal* subgroup of $GL(R)$. *We now define $K_1(R)$ to be the quotient group $GL(R)/E(R)$*, which, in view of (5.2), is just the 'abelianization' of $GL(R)$. Whitehead himself, however, did not use an explicit notation for the $K_1(R)$ just defined. In his topological investigations, where $R = \mathbb{Z}\pi$, the group $\mathrm{Wh}(\pi)$ he introduced is *not* our $K_1(\mathbb{Z}\pi)$, but is, rather, a quotient of $K_1(\mathbb{Z}\pi)$, modulo the subgroup formed by classes of the 1×1 matrices $\{(\pm g)\colon g\in\pi\}$.

Note that, as is the case with K_0, our new K_1 is also a 'functor' on the category of rings. A ring homomorphism $f\colon R\to S$ induces a group homomorphism $GL(R)\to GL(S)$, which obviously sends $E(R)$ into $E(S)$. Hence, f induces a group homomorphism $f_*\colon K_1(R)\to K_1(S)$. Again, we have the 'functorial properties' $(f\circ g)_* = f_*\circ g_*$, and $(1_R)_* = 1_{K_1(R)}$.

The purely algebraic formulation of $K_1(R)$ for a general ring R was first given by Bass [2]. More significantly, Bass discovered that *it is possible to give a 'Grothendieck style' definition for $K_1(R)$ which puts it on an equal footing with $K_0(R)$*. In fact, for any reasonable category $\mathscr{C}$, Bass defined an abelian group, called $K_1\mathscr{C}$. This definition works in such a way that, for any ring R, the group $K_1\mathscr{P}(R)$ coincides with the $K_1(R)$ defined above (see (5.4)).

Let us now explain how Bass defined $K_1\mathscr{C}$. Let $\mathscr{C}$ be an admissible subcategory in some 'nice' category $\mathscr{A}$ in which there is a notion of exact sequences. We define a new category, $\tilde{\mathscr{C}}$, as follows. 'Objects' of $\tilde{\mathscr{C}}$ are of the form (M,α) where $M \in \mathscr{C}$, and α is an automorphism of M. 'Morphisms' f from (M,α) to (N,β) are just those morphisms $f\colon M \to N$ in $\mathscr{C}$ such that $f \circ \alpha = \beta \circ f$. A sequence

$$(*) \qquad 0 \longrightarrow (M_1,\alpha_1) \xrightarrow{f} (M_2,\alpha_2) \xrightarrow{g} (M_3,\alpha_3) \longrightarrow 0$$

in $\tilde{\mathscr{C}}$ is called exact if $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is exact in $\mathscr{A}$. By definition, $K_1\mathscr{C}$ is the abelian group with generators $[(M,\alpha)]$ in correspondence with isomorphism types of $\tilde{\mathscr{C}}$, and with relation: (i) $[(M_2,\alpha_2)] = [(M_1,\alpha_1)] + [(M_3,\alpha_3)]$ for every exact sequence (*), (ii) $[(M,\alpha\beta)] = [(M,\alpha)] + [(M,\beta)]$.

In other words, $K_1\mathscr{C}$ is a quotient of $K_0\tilde{\mathscr{C}}$, with extra relations given by (ii). One of the advantages in working with (ii) is the following:

(5.3) LEMMA. *Any element $z \in K_1\mathscr{C}$ equals some $[(M,\alpha)]$.*

Proof. First, express z as $[(S,\sigma)] - [(T,\tau)]$. By (ii), one knows $[(T,1_T)] = 0$, and so, by (ii) again, $-[(T,\tau)] = [(T,\tau^{-1})]$. By (ii) once more, $z = [(S \oplus T, \sigma \oplus \tau^{-1})]$.

Just as in the case of K_0, the K_1–definition is cast in such a way that $K_1\mathscr{C}$ satisfies a certain *universal property*. Namely, given any abelian group G, and any mapping χ from isomorphism classes of $\tilde{\mathscr{C}}$ to G such that

(1) $\chi(M_2,\alpha_2) = \chi(M_1,\alpha_1) + \chi(M_3,\alpha_3)$ for every exact sequence (*),

(2) $\chi(M,\alpha\beta) = \chi(M,\alpha) + \chi(M,\beta)$,

then, there exists a unique homomorphism $\tilde{\chi}\colon K_1\mathscr{C} \to G$ such that $\tilde{\chi}([M,\alpha]) = \chi(M,\alpha)$.

Again, as in the case of K_0, this leads to the fact that an exact functor $f\colon \mathscr{C} \to \mathscr{C}'$ induces an $f_*\colon K_1\mathscr{C} \to K_1\mathscr{C}'$.

Now, if we think of the example where $\mathscr{C}$ is a category of finite dimensional vector spaces over a field, the above equations (1), (2) naturally remind us of the following properties of the *determinant* function on matrices:

$$\det\begin{pmatrix}\alpha_1 & \beta \\ 0 & \alpha_3\end{pmatrix} = (\det\alpha_1)(\det\alpha_3), \ \det(\alpha\beta) = (\det\alpha)(\det\beta).$$

Thus, in the abstract case of a general $\mathscr{C}$, we might say that $K_1\mathscr{C}$ is the recipient group of a 'universal determinant map' for automorphisms in $\mathscr{C}$ (see Section 6 below).

Now, fix a ring R, and form $K_1\mathscr{P}(R)$. It turns out that:

(5.4) THEOREM. *There exists a natural isomorphism $\phi\colon K_1R = GL(R)/E(R) \to K_1\mathscr{P}(R)$ given by $\phi[\alpha] = [(R^n,\alpha)]$, where $\alpha \in GL_n(R) = \mathrm{Aut}(R^n)$.*

Proof. First, ϕ is clearly well-defined on $GL(R)$. Since $K_1\mathscr{P}(R)$ is abelian, ϕ is trivial on $E(R) = [GL(R), GL(R)]$ and hence is well-defined on K_1R. To get

the inverse to ϕ, start with $(P,\alpha)\in\widetilde{\mathscr{P}(R)}$. There exists $Q\in\mathscr{P}(R)$ such that $P\oplus Q\cong R_n$ for some n. The automorphism $\alpha\oplus 1_Q$ on $P\oplus Q$ therefore gives rise to a matrix $\alpha'\in GL_n(R)$. Define $\chi(P,\alpha) = [\alpha']$, the class of α' in K_1R. It can be easily shown that $\chi(P,\alpha)$ depends only on the isomorphism class of (P,α) (and not on the other quantities chosen). Further, χ 'preserves' the relations for $K_1\mathscr{P}(R)$. This involves essentially proving that

$$\left[\begin{pmatrix}\alpha & \gamma\\ 0 & \beta\end{pmatrix}\right] = [\alpha]\cdot[\beta]\in K_1(R)\quad\text{whenever } \alpha\in GL_n(R),\ \beta\in GL_m(R).$$

After assuming $n = m$ which we may, this follows from

$$\left[\begin{pmatrix}\alpha & \gamma\\ 0 & \beta\end{pmatrix}\right]=\left[\begin{pmatrix}1 & \gamma\beta^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right]=\left[\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right]\cdot\left[\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\right]=\left[\begin{pmatrix}\alpha\beta & 0\\ 0 & I\end{pmatrix}\right].$$

6. K_1 and determinant. We have observed above that the group K_1R provides a 'universal determinant theory' for invertible matrices over R. Now, if R is *commutative*, there does exist a 'standard' determinant map, which provides a homomorphism $\det\colon GL(R)\to U(R)$ (= group of units of R). The kernel of this map is denoted by $SL(R) = \bigcup SL_n(R)$. There results a short exact sequence

$$1\to\frac{SL(R)}{E(R)}\to\frac{GL(R)}{E(R)} = K_1(R)\xrightarrow{\det} U(R)\longrightarrow 1,$$

which splits by a map sending $u\in U(R)$ to the class of the 1×1 matrix $(u)\in GL_1(R)$. If we now define $SK_1(R)$ to be the quotient $SL(R)/E(R)$, we obtain $K_1(R)\cong U(R)\times SK_1(R)$, for R commutative. Since $U(R)$ may be assumed known, the computation of $K_1(R)$ boils down to the consideration of $SK_1(R)$. Note that the term $SK_1(R)$ is trivial if and only if, for any $\alpha\in SL_n(R)$, we may transform $\left(\begin{smallmatrix}\alpha & 0\\ 0 & I_m\end{smallmatrix}\right)$ (for suitable m) to I_{n+m} by elementary row and column operations. If this is the case, then $K_1(R)\cong U(R)$, and the 'usual' determinant map on R emerges as the *universal* one.

To illustrate how this can happen, let us examine three classes of examples. In these examples, we shall not only show $SL(R) = E(R)$, but actually show $SL_n(R) = E_n(R)$ for *every* n.

(A) *R an Euclidean domain.* Given $\alpha = (a_{ij})\in SL_n(R)$, let $\varepsilon = \min_j(|a_{1j}|: a_{1j}\neq 0)$. Say $\varepsilon = |a_{1k}|$. After subtracting suitable multiples of the kth column from the other columns, we pass to a new matrix $\beta = (b_{ij})$ with $\min_j(|b_{1j}|: b_{1j}\neq 0) < \varepsilon$. Continuing like this, we obtain a matrix $\gamma = (c_{ij})$ with $\min_j(|c_{1j}|: c_{1j}\neq 0) = 1$, i.e., some c_{1k} is a unit. We may actually assume that c_{11} is a unit (Why?). After further column and row transformations, we may assume the first row and the first column of γ are both zero, aside from the unit c_{11}. Having achieved this, we repeat the entire process above to the minor of c_{11} in γ. Finally, we arrive at a matrix $\delta = \operatorname{diag}(d_1,\cdots,d_n)$, where (necessarily) $d_1\cdots d_n = 1$. It follows easily from (5.1) that $\delta\in E_n(R)$.

From (A), we obtain the following explicit calculations of K_1. For a field F, we have $K_1(F) \cong \dot{F}$ (the multiplicative group of F), and $K_1(F[x]) \cong \dot{F}$. Further, $K_1(\mathbb{Z}) \cong \{\pm 1\}$, $K_1(\mathbb{Z}[i]) \cong \{\pm 1, \pm i\}$, and $K_1(\mathbb{Z}[\sqrt{2}])$, $K_1(\mathbb{Z}[\sqrt{3}])$ are both isomorphic to $\{\pm 1\} \oplus \mathbb{Z}$.

(B) *R a commutative local ring.* Given $\alpha = (a_{ij}) \in SL_n(R)$, we have an equation $b_1 a_{11} + b_2 a_{12} + \cdots + b_n a_{1n} = 1$, where the b_j's are cofactors of a_{1j}. Some a_{1k} must lie outside the maximal ideal of R, hence $a_{1k} \in U(R)$. We may assume that $a_{11} \in U(R)$, and proceed as in (A) to conclude that $\alpha \in E_n(R)$.

(C) *R a commutative finite ring.* Then, again, $SL_n(R) = E_n(R)$ for all n (easy exercise from (B), or otherwise).

In view of (A), (B), (C) above, one may naturally begin to wonder what would be an example of a commutative ring A with $SK_1(A) \neq 0$. One such example is provided by the ring $A = \mathbb{R}[x,y]$ subject to the relation $x^2 + y^2 = 1$. This is the 'coordinate ring' of the real sphere S^1. It turns out that the matrix

$$\sigma = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL_2(A)$$

'survives' in $SK_1(A)$. This fact is rather hard to show just by pure algebra, but Stallings has observed that it can be proved by invoking some topology, as follows We think of each $\alpha = (a_{ij}) \in SL(A)$ as giving a continuous map $\alpha^*: S^1 \to SL(\mathbb{R})$ (namely, $\alpha^*(t) = (a_{ij}(t))$), hence determining a homotopy class $[\alpha^*] \in \pi_1(SL(\mathbb{R}))$. The rule $\alpha \mapsto [\alpha^*]$ clearly gives a homomorphism $f: SL(A) \to \pi_1(SL(\mathbb{R}))$. Since $SL(\mathbb{R})$ is a topological group, its fundamental group is *abelian*. Hence, f is trivial on $E(A)$ $(= [E(A), E(A)])$, and induces $SK_1(A) \to \pi_1(SL(\mathbb{R}))$. For $\sigma = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$, it can be shown topologically that the loop σ^* *cannot* be deformed into the identity in $SL(\mathbb{R})$.* In particular, $[\sigma]$ is nontrivial in $SK_1(A)$. Meanwhile, we have

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \sigma\right)^2 = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}\begin{pmatrix} x & y \\ y & -x \end{pmatrix} = I,$$

so $[\sigma]$ has order 2 in $SK_1(A)$. In fact, it can be shown that $[\sigma]$ *generates* $SK_1(A)$, whence $SK_1(A) \cong \mathbb{Z}/2\mathbb{Z}$. (More examples of $SK_1 \neq 0$ can be found in Section 8.)

Now, how about the *noncommutative* case? For a matrix $\alpha = (\alpha_{ij})$ with noncommuting entries, the job of finding a 'reasonable' expression in $\{\alpha_{ij}\}$ to define 'det(α)' is known to be practically hopeless. Thus, we *do not* have the notion of SK_1 for noncommutative R. Also, keep in mind that the map from $U(R) = GL_1(R)$ to $K_1(R)$ need not be injective. Its kernel contains at least $[U(R), U(R)]$ (since $K_1(R)$ is abelian), and can be larger in general. Thus, if we do hope to get 'det(α)'

*The infinite rotation group $SO(\mathbb{R}) = \bigcup SO(n)$ sits as a deformation retract in $SL(\mathbb{R})$. Thus, $\pi_1(SL(\mathbb{R})) \cong \pi_1(SO(\mathbb{R}))$. The latter group is $\mathbb{Z}/2\mathbb{Z}$, with generator precisely given by the loop σ^* (see Steenrod: *The Topology of Fiber Bundles*, Section 22.9).

of some sort, its value ought to lie in $U(R)/[U(R), U(R)]$, or in one of its quotient groups.

Such 'det' has been successfully constructed only in very special cases. The most notable case is when R is a division ring, i.e., $U(R) = R - \{0\}$. For such R, Dieudonné [11] gave the first definition of det: $GL(R) \to U(R)/[U(R), U(R)]$. He showed that his 'det' is indeed a homomorphism, and further, ker(det) $= E(R)$. This amounts to $K_1(R) \cong U(R)/[U(R), U(R)]$, or, that this 'det' is 'universal'.

Unfortunately, Dieudonné's definition of 'det' is given by induction, so one still does not have a single formula to show how $\det(\alpha_{ij})$ is obtained. To remedy this, let us bypass division rings altogether, and work out a separate example instead. For R, let us take the ring consisting of all 2×2 matrices $\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)$, where a, b, c belong to some field, F. An easy calculation shows that $[U(R), U(R)]$ consists of the matrices $\left(\begin{smallmatrix} 1 & F \\ 0 & 1 \end{smallmatrix}\right)$. This is precisely the kernel of $U(R) \to U(F) \times U(F) = \dot{F} \times \dot{F}$, where the map is induced by $p\colon R \to F \times F$, via $\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) \mapsto (a, b)$. Now, we have:

$$K_1(R) \xrightarrow{p_*} K_1(F \times F) \cong K_1(F) \times K_1(F) \xrightarrow{(\det, \det)} \dot{F} \times \dot{F}$$
$$\cong U(R)/[U(R), U(R)].$$

The composition therefore yields a determinant theory for R, say $\det_R$. The definition of $\det_R$ is in fact, quite simple. If $(\alpha_{ij}) \in GL_n(R)$, where

$$\alpha_{ij} = \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix},$$

then $\det_R(\alpha_{ij})$ is simply

$$\begin{pmatrix} \det(a_{ij}) & 0 \\ 0 & \det(b_{ij}) \end{pmatrix} \quad \text{modulo } [U(R), U(R)].$$

It may be verified that $\det_R$ is actually the 'universal' determinant for R, i.e., $K_1(R) \cong \dot{F} \times \dot{F}$. This results essentially from the fact that $\ker(p) = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ is *nilpotent*, but we shall not give the details here.

PART III. RELATIONS BETWEEN K_0 AND K_1

7. K-theory exact sequences. In Part I and Part II, we have sampled some of the more salient features of the functors K_0 and K_1. We have seen that their definitions are formally alike, but this alone is not sufficient to justify their notations. What remains for us to do now is to address ourselves to the inevitable question: *what is the relationship between K_0 and K_1, and, how useful is this relationship, in theory and in practice?*

The answer to both parts of the question is embodied in the 'K-theory exact

sequences' to be discussed. These sequences exist in several different forms, generally involving five or six terms of the K-groups. What makes them especially significant is the fact that, in all of them, there figures a certain 'connecting homomorphism', which is a map from some K_1 group to some K_0 group. In theory, this builds a bridge between K_1 and K_0, and presages the existence of 'higher' K-groups, which, once defined, hopefully fit into a long exact sequence.* In practice, the exact sequences help transmit information systematically from K_1 to K_0 (sometimes vice versa), and provide an extremely potent tool with which to calculate the K-groups.

In discussing exact sequences, we shall usually skip the drudgery of the exactness proof. It will be more profitable for us to explore instead the *idea* of the connecting homomorphism which highlights the exact sequence in question. We shall begin with the **Mayer-Vietoris sequence,** as it gives the best example of such a connecting homomorphism.

$$\begin{tikzcd} A \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & A_1 \arrow[d, "\eta_1"] \\ A_2 \arrow[r, "\eta_2"'] & A' \end{tikzcd}$$

Take a *Cartesian square* of rings, as shown. By 'Cartesian square', we mean that the square is commutative (ε_i, η_i are ring homomorphisms, $\eta_1\varepsilon_1 = \eta_2\varepsilon_2$), *and* that, given $a_1 \in A_1$, $a_2 \in A_2$, with $\eta_1(a_1) = \eta_2(a_2)$, there exists a unique $a \in A$ such that $\varepsilon_1(a) = a_1$, $\varepsilon_2(a) = a_2$. In this situation, A is called the *fiber product* of A_1 and A_2 over A': it is clearly isomorphic to the subring of $A_1 \times A_2$ consisting of $\{(a_1, a_2): \eta_1(a_1) = \eta_2(a_2)\}$. Thus, by taking arbitrary $A_1 \to A' \leftarrow A_2$ and constructing the fiber product, we get many examples of Cartesian squares.

Given a Cartesian square as above, we may construct

$$K_iA \xrightarrow{(\varepsilon_1, \varepsilon_2)} K_iA_1 \oplus K_iA_2 \xrightarrow{(\eta_1, -\eta_2)} K_iA' \ (\varepsilon_1 \text{ means } \varepsilon_{1*}, \text{etc.})$$

which obviously has *zero* composition for both $i = 0, 1$. The truly remarkable thing is now the following:

(7.1) MILNOR'S THEOREM [13]. *Assume that η_2 is surjective. Then, there exists a natural connecting homomorphism $\partial: K_1A' \to K_0A$ such that the 6-term sequence*

$$K_1A \to K_1A_1 \oplus K_1A_2 \to K_1A' \xrightarrow{\partial} K_0A \to K_0A_1 \oplus K_0A_2 \to K_0A'$$

is exact.

Milnor calls this the *Mayer-Vietoris sequence,* as his inspiration obviously

*K_2 was defined by Milnor about 1967 [13], and subsequently the K_i 's ($i > 2$) were successfully defined by Quillen. The investigation of these 'higher' K 's is now an active field of research. For the most recent literature, see [4].

came from an analogous sequence in algebraic topology which bears the same name.

For our purposes, suffice it to explain the idea behind the construction of ∂. Consider a matrix $\alpha = (a_{ij}) \in GL_n(A')$ which represents a typical element in K_1A'. *How are we going to arrive at some (f.g.) projective module over A?* Let us think of α as an automorphism $f_\alpha: A'^n \to A'^n$; namely, $f_\alpha(z_j) = \Sigma_i a_{ij}z_i$, where $\{z_i\}$ is the standard basis of A'^n. Form the subgroup, $P(\alpha)$, of $A_1^n \times A_2^n$, consisting of $\{(x,y): f_\alpha(\eta_1^n(x)) = \eta_2^n(y)\}$, where $\eta_i^n: A_i^n \to A'^n$ are the obvious maps. $P(\alpha)$ is an A-module by the 'diagonal action' $a\cdot(x,y) = (\varepsilon_1(a)x, \varepsilon_2(a)y)$. Moreover, we have the following beautiful result:

(7.2) LEMMA. (1) *If there exists $(b_{ij}) \in GL_n(A_2)$ which maps to $\alpha = (a_{ij})$, then, $P(\alpha) \cong A^n$.*

(2) *In general, $P(\alpha)$ is a* f.g. *projective A-module (assuming η_2 surjective).*

From (2), it is routine to verify that $\partial[\alpha] = [P(\alpha)] - [A^n] \in K_0A$ gives a well-defined homomorphism from K_1A' to K_0A. Also, the formula for ∂ is such that (1) translates into the statement that $K_1A_2 \xrightarrow{\eta_2} K_1A' \xrightarrow{\partial} K_0A$ has composition zero (by symmetry, one gets $\partial \circ \eta_1 = 0$ as well).

The crux of the matter, therefore, lies in the proof of (7.2). Let us first prove (1). Write $\{x_i\}$, $\{y_i\}$ respectively for the standard basis on A_1^n and A_2^n. Using $(b_{ij}) \in GL_n(A_2)$, we may transform $\{y_i\}$ to a new basis $\{\tilde{y}_j = \Sigma_i b_{ij}y_i\}$. Since

$$\eta_2^n(\tilde{y}_j) = \sum_i a_{ij}z_i = f_\alpha(z_j) = f_\alpha(\eta_1^n(x_j)),$$

we know that $(x_j, \tilde{y}_j) \in P(\alpha)$. For (1), *it suffices to show that $\{(x_j, \tilde{y}_j)\}$ form a free basis for $P(\alpha)$*. To show their independence, let $\Sigma r_j \cdot (x_j, \tilde{y}_j) = 0$, where $r_j \in A$. Then, $\Sigma\varepsilon_1(r_j)x_j = 0 = \Sigma\varepsilon_2(r_j)\tilde{y}_j$. This implies $\varepsilon_1(r_j) = \varepsilon_2(r_j) = 0$, so $r_j = 0$ for all j. Next, given $(x,y) \in P(\alpha)$, express $x = \Sigma u_j x_j$, $y = \Sigma v_j\tilde{y}_j$ ($u_j \in A_1, v_j \in A_2$). The condition that $(x,y) \in P(\alpha)$ is that

$$f_\alpha(\Sigma\eta_1(u_j)z_j) = \Sigma\eta_2(v_j)\eta_2^n(\tilde{y}_j) = f_\alpha(\Sigma\eta_2(v_j)z_j).$$

Since f_α is an automorphism, we have $\eta_1(u_j) = \eta_2(v_j)$, i.e., there exist $r_j \in A$ with $\varepsilon_1(r_j) = u_j$, $\varepsilon_2(r_j) = v_j$. Consequently, $(x,y) = \Sigma r_j \cdot (x_j, \tilde{y}_j) \in P(\alpha)$, proving (7.2)(1).

For (2), we use Whitehead's Lemma (5.1), which says that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}(A').$$

Since we assume here that η_2 is surjective, the induced map $E_{2n}(A_2) \to E_{2n}(A')$ is likewise surjective (check generators for $E_{2n}(A')$!). By (1), we know, therefore, that

$$P\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \cong A^{2n}.$$

Since clearly $P(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}) \cong P(\alpha) \oplus P(\beta)$, we conclude that $P(\alpha)$ is a direct summand of A^{2n}, proving (2).

(7.3) COROLLARY. *Let A be a ring, and I_1, I_2 be two-sided ideals in A such that $I_1 \cap I_2 = 0$. Then, there is an exact sequence:*

$$K_1A \to K_1\left(\frac{A}{I_1}\right) \oplus K_1\left(\frac{A}{I_2}\right) \to K_1\left(\frac{A}{I_1 + I_2}\right) \xrightarrow{\partial} K_0A \to K_0\left(\frac{A}{I_1}\right) \oplus K_0\left(\frac{A}{I_2}\right) \to K_0\left(\frac{A}{I_1 + I_2}\right).$$

Proof. If we set $A_i = A/I_i$, $A' = A/(I_1 + I_2)$, and ε_i, η_i to be the natural projections, the condition $I_1 \cap I_2 = 0$ guarantees that we get a Cartesian square. Here, both η_1, η_2 happen to be surjective. Done by (7.1).

To take a concrete example, let G be a finite group, say of order n. Write $A = \mathbb{Z}G$, its integral group ring. Take I_1 to be the principal ideal in A generated by the central element $\sigma = \Sigma g$ (summed over $g \in G$), and take I_2 to be the 'augmentation ideal' of A, i.e., $I_2 = \{\Sigma m_g \cdot g\colon \Sigma m_g = 0\}$. Clearly, $I_1 \cap I_2 = 0$, so (7.3) applies. Here, of course, $A_2 = A/I_2 \cong \mathbb{Z}$, and $A' = A/(I_1 + I_2) \cong \mathbb{Z}/n\mathbb{Z}$.

To get a good hold on $A_1 = A/I_1$, let us specialize to the case where n is a *prime number* p. Fix a generator x for G. If $t = \eta_1(x)$, the ring A_1 is of the form $\mathbb{Z}[t]$ with a single relation $\Sigma_{i=0}^{p-1} t^i = 0$. Thus, A_1 may be identified with $\mathbb{Z}[\zeta]$ where ζ is a primitive pth root of unity. To illustrate a typical use of (7.3), we shall prove the following interesting result of Rim:

(7.4) THEOREM. *For $|G| = p$, $\varepsilon_1\colon K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}[\zeta])$ is an isomorphism. (In view of (3.3), we get $K_0(\mathbb{Z}G) \cong \mathbb{Z} \oplus J$, where J is the ideal class group of $\mathbb{Q}(\zeta)$.)*

Proof. Write down the Mayer-Vietoris sequence:

$$K_1(\mathbb{Z}[\zeta]) \oplus K_1(\mathbb{Z}) \to K_1(\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\partial} K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}[\zeta]) \oplus K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/p\mathbb{Z}).$$

We claim that $\partial = 0$. This, together with the isomorphism $\eta_2\colon K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/p\mathbb{Z})$, clearly imply (7.4), by exactness. To show $\partial = 0$, it suffices to see that $K_1(\mathbb{Z}[\zeta]) \xrightarrow{\eta_1} K_1(\mathbb{Z}/p\mathbb{Z})$ is *onto.* Let us prove the slightly stronger fact that $U(\mathbb{Z}[\zeta]) \to U(\mathbb{Z}/p\mathbb{Z}) \cong K_1(\mathbb{Z}/p\mathbb{Z})$ is onto. Given a positive integer r prime to p, let $u = 1 + \zeta + \cdots + \zeta^{r-1} \in \mathbb{Z}[\zeta]$. Let $\zeta^r = \xi$, $\xi^s = \zeta (s > 0)$, and $v = 1 + \xi + \cdots + \xi^{s-1} \in \mathbb{Z}[\zeta]$. In $\mathbb{Q}(\zeta)$, we have

$$v = (\xi^s - 1)/(\xi - 1) = (\zeta - 1)/(\zeta^r - 1) = 1/u,$$

so $u \in U(\mathbb{Z}[\zeta])$. This completes the proof as $\eta_1(u)$ clearly equals r.

The above argument, due to Milnor, shows succintly how information on the K_1 level may be passed along to the K_0 groups.

We shall now turn our attention to another important exact sequence in K-theory. This involves the K_i-groups of two rings, R and R', where R' is a 'localization' of R. The main job in setting up this so-called 'Localization Sequence' lies in finding a 'relative term' which hooks up the two homomorphisms $K_i(R) \to K_i(R')$ $(i = 0, 1)$ to give an exact sequence of length 5. Though shorter in length than the Mayer-Vietoris sequence, the localization sequence has proved to be equally important in computations of the K-groups.

To discuss the details, we fix a multiplicatively closed set S containing 1 in the center of a ring, R. For convenience, *we assume, once and for all, that elements of S are not zero divisors in R*. If we form the 'fractions' $\{r/s\colon r \in R,\ s \in S\}$ and manipulate with them in the obvious way, we get a ring R' which is called the *localization of R at S*. The map $r \mapsto r/1$ defines an imbedding $R \subset R'$ (which generalizes the imbedding of an integral domain into its quotient field).

The desired relative term is to be of the form $K_0\mathscr{T}$, where $\mathscr{T}$ is a certain category of left R-modules. The objects M of $\mathscr{T}$ are f.g. (left) R-modules with the following properties:

(T_1) M is 'S-torsion', i.e., there exists $s \in S$ such that $s \cdot M = 0$.

(T_2) There exists a finite $\mathscr{P}(R)$-resolution for M, i.e., an exact sequence

$$0 \to P_m \to \cdots \to P_0 \to M \to 0,$$

where $P_i \in \mathscr{P}(R)$ (and m depending on M).

(7.5) THEOREM. *There exist natural homomorphisms ∂ and δ which give an exact sequence*

$$K_1(R) \to K_1(R') \xrightarrow{\partial} K_0\mathscr{T} \xrightarrow{\delta} K_0(R) \to K_0(R').$$

To apply such a sequence, it is essential to understand the definitions of the two new maps, ∂ and δ. The latter is defined simply by $\delta([M]_{\mathscr{T}}) = \sum_{i=0}^{m}(-1)^i[P_i] \in K_0(R)$, in the notation of ($T_2$). We shall not dwell on the 'well-definition' of δ, which is by no means obvious. Instead, let us proceed to explain the important map ∂—the 'connecting homomorphism'. *Given $\alpha \in GL_n(R')$, how are we going to arrive at some element of $K_0\mathscr{T}$?* First of all, we may find a common denominator $s \in S$ for the entries of α, so that $\beta = s \cdot \alpha$ has entries in R. We claim that $R^n/\beta R^n \in \mathscr{T}$ and $R^n/sR^n \in \mathscr{T}$. The fact that they satisfy (T_2) is obvious, in view of the short resolutions

$$0 \longrightarrow R^n \xrightarrow{\beta} R^n \longrightarrow R^n/\beta R^n \longrightarrow 0, \quad 0 \longrightarrow R^n \xrightarrow{s\cdot 1} R^n \longrightarrow R^n/sR^n \longrightarrow 0.$$

It only remains to show that $R^n/\beta R^n$ satisfies (T_1). For this, take $t \in S$ such that $t \cdot \alpha^{-1} = \gamma$ has entries in R. Then

$$\gamma R^n \subset R^n \Rightarrow tR^n \subset \alpha R^n \Rightarrow stR^n \subset s\alpha R^n = \beta R^n,$$

i.e., $st \in S$ annihilates $R^n/\beta R^n$. We may now define

$$\partial[\alpha] = [R^n/\beta R^n] - [R^n/sR^n] \in K_0\mathscr{T}.$$

Again, we skip the 'well-definition' of ∂ (which turns out to be rather routine.) Suffice it to note that

$$\delta\partial[\alpha] = \delta[R^n/\beta R^n] - \delta[R^n/sR^n] = ([R^n] - [R^n]) - ([R^n] - (R^n]) = 0,$$

so indeed $\delta \circ \partial = 0$. The other compositions in the 5-term sequence are similarly shown to be zero. The *exactness* is much harder; we shall not prove it here.

For a good illustration of (7.5) the reader should take R to be a Dedekind domain and R' its quotient field. In this case $\mathscr{T}$ is precisely the category of all f.g. torsion R-modules. *What does exactness mean at $K_1(R')$, at $K_0\mathscr{T}$, and at $K_0(R)$?*

8. Fundamental Theorem (for regular rings). The main object of study in this section is the K_1 of a 'Laurent polynomial ring', $A[t, t^{-1}]$. Here, A is a ring, t a transcendental element commuting with A, and $A[t, t^{-1}]$ means the ring of all 'Laurent polynomials' $\sum_{i=n}^{m} a_i t^i$, where $n \leqq m \in \mathbb{Z}$ (not necessarily positive). Stated otherwise, $A[t, t^{-1}]$ is the localization of $A[t]$ at $S = \{t^i : i \geqq 0\}$.

Our interest in $K_1(A[t, t^{-1}])$ stems from the striking fact that it turns out to contain, as direct summand, a copy of $K_0(A)$! This phenomenal relationship between K_1 and K_0, regarded by many as the cornerstone of 'classical' algebraic K-theory, is actually part of a much more precise description of $K_1(A[t, t^{-1}])$, now known as the 'Fundamental Theorem' (apparently after [**3**, Ch. 12]):

(8.1) THEOREM. *There exists an abelian group N (depending on A) such that $K_1(A[t, t^{-1}]) \cong K_1(A) \oplus K_0(A) \oplus N \oplus N$. In fact, $N = \ker(j_*\colon K_1(A[t]) \to K_1(A))$, where $j\colon A[t] \to A$, $j(t) = 1$. (Since j splits the inclusion $A \subset A[t]$, we have $K_1(A[t]) \cong K_1(A) \oplus N$.)*

This result has found diverse applications whereby one deduces general theorems about K_0 from general theorems about K_1. For lack of space, we will not exhibit such applications. In what follows, however, we shall try to convey the main ideas used in the proof of (8.1).

To simplify the exposition we shall restrict our attention to a special class of rings, namely, those A which are **left regular:**

(8.2) DEFINITION. *A is called* **left regular** *if A is left noetherian, and every f.g. left A-module has a finite $\mathscr{P}(A)$-resolution.*

The simplest example of a left regular ring is a semisimple ring A with d.c.c. (all left A-modules are projective). The next class of examples is given by Dedekind domains A. Over such domains, any module M generated by r elements has a short $\mathscr{P}(A)$-resolution $0 \to P \to A^r \to M \to 0$.

The term 'regular' seems to have come from algebraic geometry. If A is the local ring of an algebraic variety V (over an algebraically closed field) at a point x, then famous theorems of Zariski and Serre say that *A is regular if and only if x is a non-singular point on V.* Further, if V itself is an affine variety and B is the coordinate ring of V, then *B is regular if and only if V itself is non-singular.* This provides a very rich source of commutative regular rings.

One notable consequence of A being left regular is that the *Cartan homomorphism* (see (4.1)) *$c: K_0(A) \to G_0(A)$ is an isomorphism.* In fact, given $[M] \in G_0(A)$ and a $\mathscr{P}(A)$-resolution $0 \to P_r \to \cdots \to P_0 \to M \to 0$, we have $[M] = \Sigma(-1)^i[P_i] \in \operatorname{Im}(c)$, so c is surjective. It may further be shown that $c'([M]) = \Sigma(-1)^i[P_i]_{\mathscr{P}(A)}$ gives a *well-defined* $c': G_0(A) \to K_0(A)$. Since c, c' are clearly inverse of each other, c is indeed an isomorphism.

Why is it easier to work with A left regular in (8.1)? The explanation lies in the following two properties (quoted here without proof: see [**3**, p. 634, p. 646]).

(8.3) THEOREM. *Let A be left regular. Then* (1) *$A[t]$ is also left regular, and* (2) *$K_1(A) \to K_1(A[t])$ is an isomorphism* (*i.e., $N = 0$ in* (8.1)).

Assuming A left regular below, we shall now prove (8.1), which boils down to $K_1(A[t,t^{-1}]) \cong K_1(A) \oplus K_0(A)$! This will be deduced from the Localization Sequence (7.5), applied to the ring $R = A[t]$, localized at $S = \{t^i : i \geqq 0\}$:

$$(8.4) \qquad K_1(A[t]) \xrightarrow{i} K_1(A[t,t^{-1}]) \xrightarrow{\partial} K_0\mathscr{T} \xrightarrow{\delta} K_0(A[t]) \to K_0(A[t,t^{-1}]).$$

Here $\mathscr{T}$ is precisely the category of all f.g. $A[t]$-modules annihilated by some t^i $(i \geqq 0)$, thanks to (8.3)(1)! To identify the 'relative term', let us consider the map $g: G_0(A) \to K_0\mathscr{T}$ given by viewing an A-module M as $A[t]$-module with *zero* t-action. This g is clearly an *isomorphism*, since we may construct a well-defined inverse g', where $g'([N]) = \Sigma_{i=0}^{\infty}[t^iN/t^{i+1}N]$ for $N \in \mathscr{T}$. We have now a composition of isomorphisms $g \circ c: K_0(A) \to K_0\mathscr{T}$. *One remarkable application of this is that $\delta = 0$!* To show this, it suffices to check that $\delta([P]_{\mathscr{T}}) = 0$, where $P \in \mathscr{P}(A)$ is given zero t-action. To compute δ, consider the short $\mathscr{P}(A[t])$-resolution:

$$0 \to P[t] \xrightarrow{t\cdot 1} P[t] \xrightarrow{t \mapsto 0} P \longrightarrow 0,$$

where $P[t]$ means $A[t] \otimes_A P$. By definition, $\delta[P]_{\mathscr{T}} = [P[t]] - [P[t]] = 0$.

From the exactness of (8.4), $\delta = 0 \Rightarrow \partial$ is surjective. On the other hand, after replacing $K_1(A[t])$ by $K_1(A)$ via (8.3)(2), we see that i in (8.4) is a *split* monomorphism (since $A \subset A[t,t^{-1}]$ splits by 'retracting' t and t^{-1} to 1). It follows that

$$K_1(A[t,t^{-1}]) \cong K_1(A) \oplus K_0\mathscr{T} \cong K_1(A) \oplus K_0(A). \qquad \text{Q.E.D.}$$

We must observe, however, that the above proof differs from the 'traditional' proof of (8.1) in one respect. Namely, in all standard approaches to (8.1) (e.g., [**3**, **6**, **17**]), one achieves the direct decomposition of $K_1(A[t,t^{-1}])$ via an *explicit*

splitting, h, for the map $K_1(A[t,t^{-1}]) \xrightarrow{c'g'\partial} K_0(A)$. Historically, it is the construction of this splitting h which led to the ultimate formulation of (8.1). Let us now explain this famous h, the idea behind which is extremely simple. Given $P \in \mathscr{P}(A)$, we may form $P[t,t^{-1}] = A[t,t^{-1}] \otimes_A P$, which carries an obvious automorphism, namely, multiplication by t! Thus, one sets $h[P] = [P[t,t^{-1}], t \cdot 1] \in K_1\mathscr{P}(A[t,t^{-1}]) \cong K_1(A[t,t^{-1}])$ (by (5.4)). It is routine to show that h *does* split $c'g'\partial$, which is the 'traditional' proof of (8.1).

In the following, we specialize to *commutative* (regular) *domains* A. By factoring out the unit group $U(A[t,t^{-1}])$ from $K_1(A[t,t^{-1}])$, we may translate (8.1) into a theorem about $SK_1(A[t,t^{-1}])$. One first shows, as an exercise, that $U(A[t,t^{-1}]) = U(A) \times \{t^i : i \in \mathbb{Z}\}$. Now $h[A]$ is precisely the 1×1 matrix (t) in $K_1(A[t,t^{-1}])$. Therefore, we conclude that

(8.5) $\quad SK_1(A[t,t^{-1}]) \cong SK_1(A) \oplus \tilde{K}_0(A)$, where $\tilde{K}_0(A) = K_0(A)/\mathbb{Z} \cdot [A]$.

This striking formula enables us to produce commutative domains with *arbitrarily large* SK_1! We only have to recall Claborn's theorem (Section 3), which asserts the existence of Dedekind domains A with $\tilde{K}_0(A)$ arbitrarily prescribed.

From (8.5) we clearly get $SK_1(\mathbb{Z}[t,t^{-1}]) = 0$. More generally, if A is the ring of algebraic integers in a number field F, a famous theorem of Bass-Milnor-Serre says that $SK_1(A) = 0$ (see Section 12). In particular, $|SK_1(A[t,t^{-1}])|$ *equals the class number of* F, by (8.5).

9. An application of the localization sequence. The application we have in mind is to prove the theorem of Auslander-Buchsbaum which states that *a commutative regular local ring must be a UFD* (*Unique Factorization Domain*). The proof to be discussed below, which uses the Localization Sequence in Section 7, was first given by I. Kaplansky. The proof requires a few notions and facts from commutative algebra which will be assumed in this section but will not be needed elsewhere.

We assume it is already known that a commutative regular local ring must be a normal domain (i.e., integrally closed and without zero divisors). Starting with an *arbitrary normal, Noetherian domain* R, with quotient field F, let us first make some general observations. Let $\mathscr{H}$ be the set of all height 1 prime ideals in R (i.e. nonzero prime ideals $\mathfrak{p}$ such that there exist no prime ideals properly between 0 and $\mathfrak{p}$). The *group of divisors*, $D(R)$, is defined to be the free abelian group generated by the symbols $\langle \mathfrak{p} \rangle$, where $\mathfrak{p} \in \mathscr{H}$. Elements of $D(R)$ are called 'divisors'. Since R is normal, the localizations $R_\mathfrak{p}$ are all discrete valuation rings, for $\mathfrak{p} \in \mathscr{H}$. We shall write $v_\mathfrak{p}$ for the corresponding valuation from $\dot{F}$, into $\mathbb{Z}$. To any $z \in \dot{F}$, we attach a divisor:

$$\operatorname{div}(z) = \sum_{\mathfrak{p} \in \mathscr{H}} v_\mathfrak{p}(z) \cdot \langle \mathfrak{p} \rangle \text{ (finite sum),}$$

called the *principal divisor* of z. Since $\operatorname{div}(z \cdot z') = \operatorname{div}(z) + \operatorname{div}(z')$, we have a

subgroup $\operatorname{div}(\dot{F}) \subset D(R)$. The quotient group $D(R)/\operatorname{div}(\dot{F})$ is denoted by $C(R)$, called the **divisor class group** of R. Our interest in $C(R)$ stems from the following elementary fact, stated here without proof:

(9.1) THEOREM (see, e.g. [3, p. 144]). *For R a normal Noetherian domain, $C(R) = 0$ if and only if R is a UFD.*

The formation of $C(R)$ is, of course, slightly reminiscent of the Grothendieck construction in Section 1. It is, therefore, hardly surprising that there exists some K_0-group which maps onto $C(R)$. Indeed, taking $\mathscr{C}$ to be the category of f.g. torsion R-modules, we shall construct a surjection $g: K_0\mathscr{C} \to C(R)$. Given $M \in \mathscr{C}$ and $\mathfrak{p} \in \mathscr{H}$, the localization $M_\mathfrak{p}$ is a f.g. torsion module over the discrete valuation ring $R_\mathfrak{p}$, so $M_\mathfrak{p}$ has a $R_\mathfrak{p}$-composition series. The *length* of this composition series will be denoted by $l_\mathfrak{p}(M_\mathfrak{p})$. We now define:

$$g([M]_\mathscr{C}) = \sum_{\mathfrak{p} \in \mathscr{H}} l_\mathfrak{p}(M_\mathfrak{p}) \cdot \langle \mathfrak{p} \rangle \text{ (finite sum, } \textit{taken in } C(R)).$$

Since $l_\mathfrak{p}$ is additive over exact sequences, g is a *well-defined* homomorphism from $K_0\mathscr{C}$ into $C(R)$. Note that if M is of the special form R/Rx $(0 \neq x \in R)$, then $l_\mathfrak{p}((R/Rx)_\mathfrak{p}) = v_\mathfrak{p}(x)$, so $g([R/Rx]) = \operatorname{div}(x)$. This means that *$g$ vanishes on all expressions $[R/Rx]$*. On the other hand, if M is of the special form $R/\mathfrak{p}$ $(\mathfrak{p} \in \mathscr{H})$, then clearly $g([R/\mathfrak{p}]) = \langle \mathfrak{p} \rangle$. In particular, *$g$ is surjective.*

Now assume, in addition, that R is *regular*. We apply the Localization Sequence (7.5) to $R \subset F$. The category $\mathscr{C}$ above plays the role of the $\mathscr{T}$ in (7.5). Thus, we have an exact sequence:

$$K_1(R) \longrightarrow K_1(F) \cong \dot{F} \xrightarrow{\partial} K_0\mathscr{C} \xrightarrow{\delta} K_0(R) \xrightarrow{j} K_0(F) \cong \mathbb{Z}.$$

If $z = x/y$, where $x, y \in R$ (nonzero), recall that $\partial(z) = [R/Rx] - [R/Ry]$.

Assume that f.g. projective R-modules are stably free. Then, $K_0(R) \cong \mathbb{Z}$ with generator $[R]$. In particular, the map j above is an isomorphism. By exactness, the connecting map ∂ must be surjective. Since $g: K_0\mathscr{C} \to C(R)$ vanishes on $\operatorname{Im}(\partial)$ (as observed above), g must be the zero map. But g is *surjective*. Hence $C(R) = 0$, and by (9.1), R is a UFD! We have now proved:

(9.2) THEOREM. *If R is a regular (normal) domain whose f.g. projective modules are stably free, then R is a UFD.*

Since (f.g.) projectives are free over a local ring, this includes the theorem of Auslander-Buchsbaum.

PART IV. THE STABILITY THEOREMS

10. K_0-Stability Theorem. In its early stage of development, algebraic K-theory owed much to ideas in topology. In fact, much of the inspiration for the subject

actually came from the theory of **vector bundles.** In our treatment above, the emphasis is mainly on pure algebra, so the topological connection has not been made evident. We shall now bridge this gap.

The key observation is that *there exists a close analogy between vector bundles and projective modules.* To make this explicit, let us consider a (connected) compact Hausdorff space, X, and write $\mathscr{B}(X)$ for the category of all *real* vector bundles over X. (We shall assume here that the reader knows the basic terminology in bundle theory.) For $E \in \mathscr{B}(X)$, let $\Gamma(E)$ denote the $\mathbb{R}$-vector space of all global sections of $E \to X$. Then $\Gamma(E)$ is a *module* over $\mathbb{R}(X)$—the ring of continuous real-valued functions on X, by the action:

$$(f \cdot s)(x) = f(x)s(x), \quad f \in \mathbb{R}(X), \; s \in \Gamma(E), \; x \in X.$$

We have now the following two pertinent facts:

(10.1) THEOREM. (1) $\Gamma(E)$ *is a f.g.* **projective** *module over* $\mathbb{R}(X)$.

(2) *If E is an n-bundle (i.e., the fibers of E are n-dimensional $\mathbb{R}$-vector spaces), then $\Gamma(E)$ has constant rank n over $\mathbb{R}(X)$.*

To illustrate the basic ideas involved, let us briefly sketch the proofs of (1), (2). Given $x \in X$, choose local sections $s_i^{(x)}$, $1 \leqq i \leqq r$, which span the fibers over y for all y in a neighborhood U_x of x. After extending the $s_i^{(x)}$, we may assume that they are actually *global* sections. Taking a finite number from among $\{U_x\}$ to cover X (= compact!), we will then have a finite number of global sections, say, $t_1, \cdots, t_m$, which span *all* the fibers. Using the t_i, we obtain a bundle surjection $f: X \times \mathbb{R}^m \to E$ over X. The trivial bundle $X \times \mathbb{R}^m$ decomposes into a direct sum $\cong \ker(f) \oplus E$. Taking global sections we get $\Gamma(E) \oplus \Gamma(\ker f) \cong \Gamma(X \times \mathbb{R}^m) \cong \mathbb{R}(X)^m$, whence $\Gamma(E) \in \mathscr{P}(\mathbb{R}(X))$, proving (1). For the rest, write $A = \mathbb{R}(X)$ and $P = \Gamma(E)$. Let $\max(A)$ denote the maximal ideal spectrum of A. It is well known that $\max(A)$ consists of the $\mathfrak{m}_x = \{f \in \mathbb{R}(X): f(x) = 0\}$ for various $x \in X$. For $\mathfrak{m} = \mathfrak{m}_x$, the localization $P_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module with

$$\mathrm{rank}_{A_\mathfrak{m}} P_\mathfrak{m} = \dim_{A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}} P_\mathfrak{m}/\mathfrak{m}P_\mathfrak{m} = \dim_\mathbb{R} P/\mathfrak{m}P.$$

It is easy to show that $\Gamma(E)/\mathfrak{m}_x \cdot \Gamma(E)$ is canonically isomorphic to E_x (the fiber in E over x), by the evaluation map $s \mapsto s(x)$. Hence, $P \in \mathscr{P}(A)$ has constant rank $= \dim_\mathbb{R} E_x = n$, proving (2).

It turns out further that *any* module in $\mathscr{P}(\mathbb{R}(X))$ is necessarily of the form $\Gamma(E)$, for a suitable $E \in \mathscr{B}(X)$. More precisely, the situation may be summed up as follows.

(10.2) THEOREM. Γ *defines an equivalence of categories from* $\mathscr{B}(X)$ *to* $\mathscr{P}(\mathbb{R}(X))$.

Thus, the 'functor' Γ builds a bridge between algebra and topology. This phenomenon has its first manifestation in Serre's famous paper 'FAC' [**15**], in the context of algebraic geometry. The 'topological rendition' (10.2) was given by Swan [**16**].

Now, how can we profit from (10.2)? Suppose we have a certain topological theorem about vector bundles. Using (10.1) and (10.2) as a 'dictionary', it may be possible to transcribe that theorem into a result about f.g. projective modules over $\mathbb{R}(X)$. One may then ask, could the same result hold true perhaps for f.g. projectives over a *more general* ring? Hopefully, parts of the techniques used to prove the topological result can also be transcribed into its algebraization! If it all works well, one could then get good theorems in algebra which might otherwise never have been found!

To illustrate this principle, let us go through in detail the transcription of a specific topological result. Let X be a connected finite CW-complex, of dimension d. If E is an n-bundle over X, with $n > d$, it is known in topology that we can 'peel off' a trivial line bundle T from E, i.e., $E \cong E' \oplus T$, for a suitable bundle E'. Furthermore, if $n > d + 1$, then the isomorphism type of E' is uniquely determined. These are called the **Stability Theorems** in topology; they result essentially from natural geometric considerations in homotopy theory. The reader does not have to know these theorems beforehand, because our job is only to *transcribe* them — not to *prove* them. First of all, let us restate these Stability Theorems a little more nicely. Let $\mathscr{B}_m(X)$ be the category of m-bundles over X, and let $s_m: \mathscr{B}_m(X) \to \mathscr{B}_{m+1}(X)$ be the functor which sends E to $E \oplus T$ (i.e., adding a trivial line bundle). We have then:

(10.3) THEOREM. (1) *If $m \geqq d$, s_m is surjective on isomorphism classes.* (2) *If $m \geqq d + 1$, s_m is bijective on isomorphism classes.*

To render this into the algebraic form, write $A = \mathbb{R}(X)$, and let $\mathscr{P}_m(A)$ denote the category of f.g. projective A-modules of constant rank m. Note that, under the category equivalence Γ in (10.2), $\mathscr{B}_m(X)$ corresponds to $\mathscr{P}_m(A)$, by (10.1)(2). Further, the trivial line bundle T corresponds to $\Gamma(T) \cong A$. Therefore, we obtain the following 'algebraization' of (10.3):

(10.3)′ THEOREM. *Let X be a connected finite CW-complex of dimension d, and $A = \mathbb{R}(X)$. Let $\sigma_m: \mathscr{P}_m(A) \to \mathscr{P}_{m+1}(A)$ be defined by $P \mapsto P \oplus A$. Then,* (1) *if $m \geqq d$, σ_m is surjective on isomorphism classes*; (2) *if $m \geqq d + 1$, σ_m is bijective on isomorphism classes.*

While (10.3) concerns the *space* X and its *vector bundles*, (10.3)′ deals with the *ring* A and its *projective modules*. The shift of emphasis from the 'topological' to the 'algebraic' is unmistakable. At this juncture, one single obvious question begs to be asked: *for a general commutative ring R, will the algebraic statements* (1), (2) *above still hold for the R-projective modules, for a suitable integer d depending on R?*

Note, however, that the ring A in (10.3)′ has one special property, namely, all $P \in \mathscr{P}(A)$ are of *constant* rank (by (10.2) and (10.1)(2)). For a more general R, of

course, this need not be the case. Thus, we must first readjust the definition of $\mathscr{P}_m$, to allow for the R-projectives of non-constant rank. This is obviously only a small point, because we can simply set

$$\mathscr{P}_m(R) = \{P \in \mathscr{P}(R) \colon \min_{\mathfrak{m}}(\operatorname{rank}_{R_\mathfrak{m}} P_\mathfrak{m}) = m\},$$

where $\mathfrak{m}$ ranges through $\max(R)$. Next, we must search for the right definition of "d" for R. Note that we need not worry about the disappearance of X, because we can comfortably replace it by $\max(R)$. The integer d in (10.3)$'$, however, is in the role of a 'topological dimension', which is lacking in the ring-theoretic context. Here, very fortunately, ideas of algebraic geometry come to our rescue! In algebraic geometry there is a notion of 'combinatorial dimension' which is defined for any topological space. First, call a topological space *irreducible* if it is not the union of two proper closed subsets. The *combinatorial dimension* of a topological space Y, written $\dim Y$, is defined to be the supremum of the integers n, for which there exists a chain $\varnothing \neq Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_n$ of irreducible closed sets $Y_i \subseteq Y$. Now, given a commutative ring R, $\max(R)$ carries the Zariski topology, so, for d, we can use $\dim(\max(R))$ instead (if it is finite). All this being said, we have just about *guessed* (though not *proved*!) a good theorem in algebra, that of Serre and Bass:

(10.4) THEOREM. *Let R be a commutative ring such that* $\max(R)$ *is noetherian,* and* $d = \dim(\max(R)) < \infty$. *Let $\sigma_m \colon \mathscr{P}_m(R) \to \mathscr{P}_{m+1}(R)$ be defined by $P \mapsto P \oplus R$. Then,*

(1) *(Serre) If $m \geqq d$, σ_m is surjective on isomorphism classes.*

(2) *(Bass) If $m \geqq d + 1$, σ_m is bijective on isomorphism classes.*

This theorem applies, for example, to noetherian rings R with finite Krull dimension e. For such rings, it is easy to verify that $\max(R)$ is indeed noetherian, and that $\dim(\max(R)) \leqq e$. Hence, (1) and (2) both apply with d replaced by e. Meanwhile, Bass has obtained far-reaching generalizations of (10.4) to even non-commutative rings and non-projective modules [**2**].

One must observe, however, that (10.4) is *not* supposed to supercede (10.3)$'$. For X as in (10.3)$'$, $\max \mathbb{R}(X)$ is well known to be *homeomorphic* to X itself by $\mathfrak{m}_x \leftrightarrow x$, and hence $\max \mathbb{R}(X)$ *fails* to be noetherian (unless $X = \{\text{point}\}$). Of course, the combinatorial dimension of $\mathbb{R}(X)$ is *zero* since $\max(\mathbb{R}(X))$ is, in fact, Hausdorff. If (10.4) *were* applicable to $R = \mathbb{R}(X)$, it would have yielded totally absurd conclusions about vector bundles, upon translating back to topology via (10.2).**

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*A topological space Y is called *noetherian* if open sets of Y satisfy the ACC (ascending chain condition).

**On the other hand, if we let X be an affine variety (say, over an algebraically closed field k), and consider *algebraic* bundles instead, then X is noetherian in its Zariski topology, and indeed dim max ($k[X]$) equals the variety dimension of X. Here, the analogy is complete, and (10.4) applies to $R = k[X]$, the coordinate ring of X. This is the context originally considered by Serre.

While we shall not attempt to prove the deep result (10.4), we note the following consequences for the K_0-group which the reader can easily verify:

(10.5) COROLLARY. *Let R be as in* (10.4). *Then,*

(1) $K_0(R)$ *is additively generated by* $[P]$, *where* $P \in \bigcup_{i=0}^{d'} \mathscr{P}_i(R)$, $d' = \max(1, d)$.

(2) *If* $Q \in \mathscr{P}(R)$ *and* $P \in \mathscr{P}_m(R)$, *with* $m \geqq d+1$, *then* $[P] = [Q] \Rightarrow P \cong Q$.

(3) *If* P, $Q \in \mathscr{P}(R)$, *then* $[P] = [Q] \Rightarrow r \cdot P \cong r \cdot Q$ *for all sufficiently large* r.

11. K_1-Stability Theorem. We continue to consider, in this section, a commutative ring R whose $\max(R)$ is noetherian, with $\dim(\max(R)) = d < \infty$. The K_0-Stability Theorem says that, in studying projective modules P over R, it suffices to look at those with rank $\leqq d$; further, if P has rank $\geqq d+1$, then the class $[P] \in K_0(R)$ determines the isomorphism type (P). For the functor K_1, it is quite easy to raise analogous stability questions. The *rank* of a projective module is simply to be replaced by the *size* of a matrix. One could then ask, in studying $K_1(R)$, *is it enough to look at matrices of size bounded by some function, say* $f(d)$? *And, if so, would it be possible to pass back information from* $K_1(R)$ *to some finite stage, say* $GL_{f(d)+1}(R)$?

The full answer to these questions constitutes the K_1-Stability Theorem. As for the 'range' $f(d)$ needed for K_1, we shall take $f(d) = d+1$—one bigger than the K_0-range. This could have been predicted if one has the conviction that K_1 is some kind of 'successor' to K_0.

(11.1) K_1-STABILITY THEOREM. *Let R be as above. Then*

(1) $GL_{d+1}(R) \to K_1(R)$ *is surjective.*

(2) *For* $m \geqq d+2$, $E_m(R)$ *is normal in* $GL_m(R)$, *and* $GL_m(R)/E_m(R) \to K_1(R)$ *is an isomorphism.*

In [**2**], Bass proved (1) and conjectured the truth of (2). Some years later, (2) was finally established by Bass-Milnor-Serre [**7**], and by Vasserstein in [**21**]. In these papers, (1), (2) were actually shown to hold not only for R itself, but for any R-algebra which is f.g. as an R-module. The proof of (1) is not too difficult, but that of (2) is both technical and long. We shall refrain from discussing the proofs here; instead, let us examine some special cases of (11.1) to illustrate its content.

To begin with, let R be a commutative *semi-local* ring, i.e., $\max(R)$ is a finite set. Then, $\max(R)$ has dimension 0, and (11.1)(1) says that $U(R) \to K_1(R)$ is onto. We get, therefore, $K_1(R) \cong U(R)$, and (11.1)(2) amounts to $SL_m(R) = E_m(R)$ for all m. These facts can be easily proved, by refining the arguments in Section 6, (B), for the local case.

For R noetherian of dimension 1, all elements of $K_1(R)$ are represented by 2×2 matrices, and $K_1(R)$ is computed by $GL_3(R)/E_3(R)$. This will be useful for the determination of K_1 for algebraic integers in number fields (see Section 12).

Back to an arbitrary d: let us illustrate a typical way of passing information back and forth between $K_1(R)$ and the general linear groups $GL_m(R)$.

(11.2) COROLLARY. *Keep the notations in* (11.1).

(1) *If* $GL_m(R)$ *is f.g. for some* $m \geqq d+1$, *then* $K_1(R)$ *is f.g.*

(2) *If* $K_1(R)$ *is f.g., and* R *is finitely generated as a ring, then* $GL_m(R)$ *is f.g. for every* $m \geqq \max(d+2, 3)$.

Proof. (1) is obvious in view of (11.1)(1). Also, (2) follows from (11.1)(2) if we show that $E_m(R)$ is f.g. for $m \geqq 3$. Let $a_1 = 1, a_2, \cdots, a_r$ generate R as a ring, and let

$$T = \{e_{jk}^{a_i} : 1 \leqq i \leqq r, 1 \leqq j \neq k \leqq m\}.$$

We claim that $E_m(R)$ equals $\langle T \rangle$, the group generated by T. For this, we use the commutator formula $[e_{ij}^a, e_{jk}^b] = e_{ik}^{ab}$ for i, j, and k distinct. Since $m \geqq 3$, $\langle T \rangle$ catches all e_{jk}^{λ} where λ is any monomial in $\{a_i\}$. Finally, since $e_{jk}^a e_{jk}^b = e_{jk}^{a+b}$, $\langle T \rangle$ catches *all* $m \times m$ elementary matrices. Q.E.D.

For an application, let R be the ring of algebraic integers inside a number field. A classical theorem of Hurwitz says that all $SL_m(R)$ are f.g. Since $U(R)$ is also f.g. by the Dirichlet Unit Theorem, we see that all $GL_m(R)$ are f.g. *It follows that* $K_1(R)$ *is f.g.* More generally, the same line of argument can be applied to the *non-commutative case*, to give the following.

(11.3) THEOREM. *If* A *is any ring whose underlying additive group is f.g., then,* $K_1(A)$ *is also f.g.* (compare (3.4), (4.5)).

Finally, let us mention an application of (11.2)(2). For the ring R, we shall take the polynomial ring $\mathbb{Z}[X_1, \cdots, X_{d-1}]$, of Krull dimension d. For this R, one knows that $K_1(\mathbb{Z}) \to K_1(R)$ is an isomorphism (see (8.3)), i.e., $K_1(R) \cong U(R) = \{\pm 1\}$. It follows from (11.2)(2) that:

(11.4) THEOREM. *For* $m \geqq d+2$, $SL_m(\mathbb{Z}[X_1, \cdots, X_{d-1}]) = E_m(\mathbb{Z}[X_1, \cdots, X_{d-1}])$ *can be generated by* $d(m^2 - m)$ *elements. These elements, together with* $\operatorname{diag}(-1, 1, \cdots, 1)$, *generate* $GL_m(\mathbb{Z}[X_1, \cdots, X_{d-1}])$.

Of course, this has made essential use of (8.3), which requires a host of other techniques in algebraic K-theory. However, it does yield a corollary such as (11.4) which is elementary to state, but otherwise quite difficult to prove. In [5], Bass has even singled out (11.4) as the 'first main objective' in his lectures, in order to motivate the various K-theory techniques (such as (8.3), (11.2), $\cdots$) used to prove (11.4).

12. K_1 and arithmetic. Throughout this section, *let* R *be the ring of algebraic integers inside a number field* F. We have already observed, as a consequence of Hurwitz' Theorem on finite generation of $SL_m(R)$, that $SK_1(R)$ is f.g. For those R which happen to be euclidean domains (e.g., $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$, etc.), of course $SK_1(R) = 0$. *For a general* R, *it turns out that* $SK_1(R)$ *is still zero.* This wonderful result is due to Bass, Milnor and Serre [7]. Combined with (11.1)(2), their result yields the beautiful corollary that $SL_m(R) = E_m(R)$ for all $m \geqq 3$. (The concrete meaning of this is, of course, that *any* $n \times n$ *matrix over* R ($n \geqq 3$) *of*

determinant 1 *can be brought to* I_n *by elementary row* (*or column*) *operations, within* R.) This retrieves Hurwitz' Theorem (for $m \geqq 3$), and, in addition, provides a reasonable bound on the number of generators for $SL_m(R)$ (cf. proof of (11.2)(2)).

The proof of $SK_1(R) = 0$ is difficult, because it appeals to rather deep arithmetic properties of R. However, *it is not hard to show that* $SK_1(R)$ *is torsion.* So, let us do that first.

By the Stability Theorem (11.1)(1), we know that *every element of* $SK_1(R)$ *comes from* $SL_2(R)$. Given

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R),$$

let us express the class of σ in $SK_1(R)$ by the symbol $[a, b]$. To justify this notation, we must show that, if $\sigma' = \left(\begin{smallmatrix} a & b \\ c' & d' \end{smallmatrix}\right)$ also belongs to $SL_2(R)$, then σ, σ' are the *same* in $SK_1(R)$. But we have

$$\sigma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ so } \sigma' \cdot \sigma^{-1} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \in E_2(R),$$

proving our claim. Now, the symbols $[a, b]$ have some very remarkable properties. First, by column transformations, we see that $[a, b] = [a + rb, b] = [a, b + ra]$ for all $r \in R$. In particular, $[a, \pm 1] = [0, \pm 1] = 1$. More surprisingly, we get a very pleasant *multiplicative formula*: $[a, b] \cdot [a, c] = [a, bc]$, whenever the first two symbols are defined.* To verify this, let $ay - bx = 1 = at - cz$. Represent

$$[a, b] \text{ by the } 3 \times 3 \text{ matrix } \sigma = \begin{pmatrix} a & b & 0 \\ x & y & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } [a, c] \text{ by the matrix } \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -a \\ 0 & t & -z \end{pmatrix}.$$

Then, $[a, b] \cdot [a, c]$ is represented by

$$\sigma \cdot \tau = \begin{pmatrix} a & bc & -ab \\ x & yc & -ay \\ 0 & t & -z \end{pmatrix}.$$

The advantage of using 3×3 matrices is that we have a and bc in the first row of $\sigma \cdot \tau$! By an easy sequence of row and column operations, we have

$$\sigma \cdot \tau \xmapsto{\text{column}} \begin{bmatrix} a & bc & 0 \\ x & yc & -1 \\ 0 & t & -z \end{bmatrix} \xmapsto{\text{row}} \begin{bmatrix} a & bc & 0 \\ x & yc & -1 \\ -zx & t-zyc & 0 \end{bmatrix} \xmapsto{\text{column}} \begin{bmatrix} a & bc & 0 \\ x & 0 & -1 \\ -zx & t-zyc & 0 \end{bmatrix}$$

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*We have, of course, a similar multiplicative formula in the other variable. This follows, e.g., from the *symmetry* of symbols: $[a,b] = [b,a]$.

From the form of the last matrix, it is immediate that the class of $\sigma \cdot \tau$ coincides with the symbol $[a, bc]$!

It is now easy to show that every symbol $z = [a, b]$ has finite order in $SK_1(R)$. In fact, let $m = |U(R/Ra)|$ (assuming $a \neq 0$). Since b is a unit modulo $R \cdot a$, we have $b^m \equiv 1 (\operatorname{mod} R \cdot a)$. Thus, $z^m = [a, b^m] = [a, 1] = 1$.

How can we show that the symbols are actually *all trivial*? In order to bring out the flavor of the arguments in [7], let us present the proof in a special case, namely, *when F is not totally imaginary* (i.e., there exists at least one imbedding of F into $\mathbb{R}$). One way or another, we have to invoke now some *arithmetic* properties. What we shall use is a somewhat specialized form of Dirichlet's Theorem on primes:

(12.1) *If $Rc + Rd = R$, then there exists $c' = c + rd$ ($r \in R$), such that $R \cdot c'$ is a prime ideal, and such that c' has arbitrarily prescribed signs under the various real imbeddings of F.*

This is proved, for example, in the Appendix of [7]. We shall assume (12.1), and show how to get $SK_1(R) = 0$.

Suppose (for contradiction) that some symbol $z = [a, b]$ has order $t > 1$. First, we would like to rewrite z in some form $[c, d]$, where c, d have nice congruence properties modulo the ideal t^2R. Say

$$\sigma = \begin{pmatrix} a & b \\ x & y \end{pmatrix} \in SL_2(R).$$

Since $\bar{R} = R/t^2R$ is a *semi-local* ring, there exists a sequence of elementary row operations in $\bar{R}$ which brings $\bar{\sigma} \in SL_2(\bar{R})$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If we 'lift' this sequence of operations to R, we will have transformed σ to some $\left(\begin{smallmatrix} c & d \\ * & * \end{smallmatrix}\right)$, where $c \equiv 1 (\operatorname{mod} t^2R)$, $d \equiv 0 (\operatorname{mod} t^2R)$, and $z = [c, d]$.

Now, apply (12.1) to the pair c, d. We may then find $c' = c + rd$ ($r \in R$), such that Rc' is prime, and c' is negative at *precisely one* real imbedding of F (which is assumed to exist!). Let $N = N_{F/Q}$ denote the norm map from F to $\mathbb{Q}$. The sign of $N(c')$ is the product of the signs of c' under the various real imbeddings of F. Hence, $N(c') = -k$, where $k > 0$. Of course, k is an integer, and we easily see that $c' \equiv 1 (\operatorname{mod} t^2R) \Rightarrow -k \equiv 1 (\operatorname{mod} t^2)$ in $\mathbb{Z}$. On the other hand, the field $K = R/Rc'$ has cardinality $= |N(c')| = k$, so its unit group $\dot{K}$ has cardinality $k-1$. Since $z = [c, d] = [c', d]$, we see as before that $z^{k-1} = 1$, so $k \equiv 1 (\operatorname{mod} t)$. The two congruences for k imply that $1 \equiv -1 (\operatorname{mod} t)$; in particular, $t = 2$, and k is odd.

Consider the well-defined homomorphism $g: \dot{K} \to SK_1(R)$, which sends $\delta \in \dot{K}$ to $[c', \delta]$. $\operatorname{Im}(g)$ contains $[c', d]$ of order 2, and $\ker(g)$ contains $\{\pm 1\}$ which are distinct in $\dot{K}$ since $|K|$ is *odd*. It follows that $|\dot{K}|$ is divisible by 4, i.e., $k \equiv 1 (\operatorname{mod} 4)$. This obviously contradicts the earlier congruence $-k \equiv 1$ $(\operatorname{mod} t^2 = 4)$.

HISTORICAL NOTES

In the text above, we have discussed some of the high points in the theory of K_0 and K_1. Through the numerous examples and applications given, we have taken a rather close look at their mutual relationship, as well as their relationship to other existing parts of algebra. While these two functors clearly deserve our attention solely on their own right, we have yet to explain the historical circumstances which brought them into existence, and the guiding forces which helped shape their course of development.

In tracing the origin of K_0, one is inevitably led to [**8**]. This germinal work contained Grothendieck's construction of K_0, which he applied to the category of locally free sheaves on an algebraic variety. Later, this important idea was transplanted into topology by Atiyah-Hirzebruch [**1**], who applied K_0 to the category $\mathscr{B}(X)$ of (say, real) vector bundles over a space X. They defined $K_0(X) = K_0\mathscr{B}(X)$, and defined the higher $K_i(X)$ by using the iterated suspensions of X. This 'cohomology' formalism immediately found many interesting topological applications, and was subsequently called the (topological) K-theory.

Turning now to algebra, the notion of a *projective module* was a crucial part of the homological methods introduced in the late fifties. By the time topological K-theory took shape, the algebraists had also independently formed the 'projective class group'—by applying the Grothendieck construction to (finitely generated) projective modules. Now, according to Serre and Swan, projective modules are the algebraic analogue of vector bundles (see (10.2)). This initiated the pertinent viewpoint that the 'projective class group' ought to be the algebraic counterpart of Atiyah-Hirzebruch's $K_0(X)$. In this light, the projective class group of a ring R should be best denoted by $K_0(R)$.* The challenging question now arises: *is there a general coherent theory of 'higher' $K_i(R)$'s?* Stated otherwise, *does there exist an 'algebraic K-theory'?*

This question was brought sharply into focus by Bass [**2**], who successfully defined $K_1(R)$, and discovered some of the $K_1 - K_0$ exact sequences. Aptly enough, his ideas were topologically inspired. The topological $K_1(X)$ is formed from vector bundles on the suspension of X, and such bundles are given essentially by 'clutching' together two trivial bundles over X. One is thus led to consider *automorphisms* of trivial bundles, or, in algebraic terms, the invertible matrices. In this way, Bass obtained the matrix definition of $K_1(R)$ (with the appropriate generalization to categories), as set forth in Section 5. Quite remarkably, this K_1 is further buttressed by Whitehead's earlier work [**19**] on simple homotopy type, which injects new insight and purpose into the algebraic K_1. Through this fruitful connection, the algebraic techniques used to treat $K_1(R)$ provide feedback into geometric topology. Certain

*Thus, the topological $K_0(X)$ coincides with the algebraic $K_0(\mathbb{R}(X))$, where $\mathbb{R}(X)$ means the ring of continuous $\mathbb{R}$-valued functions on X (see (10.2)).

problems in geometric topology lead to the consideration of various 'obstruction groups' related to K_1 (and K_0); the advent of algebraic K-theory renders a general framework within which to calculate these obstruction groups.

Meanwhile, topological K-theory continues to be a ground of comparison and a source of inspiration for its algebraic counterpart. One major instance is the Fundamental Theorem for the algebraic $K_1(A[t,t^{-1}])$, due to Bass-Heller-Swan [6], which turns out to be an algebraic analogue of the famous Bott Periodicity Theorem for complex vector bundles.

Once the ground work was laid, the study of K_0 and K_1 began to draw attention from researchers in many other fields, such as ring theory, number theory, group representations, quadratic forms, algebraic geometry, and, last but not least, topology. Unfailingly, these patrons bring along with them their bag of favorite things! While looking for applications of K-theory in their own subject, they make diverse contributions to the theory itself. Their concerted effort, in particular, rejuvenated the long-time speculation that there ought to exist a coherent theory of 'higher' algebraic K_i's. The first breakthrough came in 1967, when Milnor introduced the algebraic K_2, based on earlier ideas of Steinberg. This K_2 was quickly embraced by topologists and number theorists, who were the first to discover its applications. The depth and beauty of Milnor's K_2 made the existence of the higher K_i's an even more desirable goal.

In the subsequent period, the search for the higher K_i's reached major proportions. Several different definitions were proposed and tried. Finally, these provisional definitions culminated in Quillen's spectacular discovery of all the 'correct' K_i's! In particular, the subject called Algebraic K-Theory really exists!

References

1. M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. of Symp. in Pure Math., AMS, 3 (1961) 7–38.

2. H. Bass, K-theory and stable algebra, Publ. I.H.E.S., 22 (1964) 489–544.

3. ———, Algebraic K-Theory, Benjamin, New York, 1968.

4. ———, Algebraic K-Theory, I, II, III., Battelle Institute Conference 1972, Springer Lecture Notes in Math. Vol. 341, 342, 343 (1973).

5. ———, Proc. of Conf. on Algebraic K-Theory, Colorado State Univ., 1973.

6. H. Bass, A. Heller and R. Swan, The Whitehead group of a polynomial extension, Publ. I.H.E.S., 22 (1964) 61–79.

7. H. Bass, J. Milnor and J-P. Serre, Solution of the congruence subgroup problem for SL_n ($n \geqq 3$) and Sp_{2n} ($n \geqq 2$), Publ. I.H.E.S., 33 (1967) 421–499.

8. A. Borel and J-P. Serre, Le théorème de Riemann-Roch (d'après A. Grothendieck), Bull. Soc. Math. France, 86 (1958) 97–136.

9. L. Claborn, Every abelian group is a class group, Pac. J. Math., 18 (1966) 219–222.

10. C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.

11. J. Dieudonné, Les déterminants sur un corps non commutatif, Bull. Soc. Math. France, 71 (1943) 27–45.

12. J. Lambek, Lectures on Rings and Modules, Blaisdell, Waltham, Mass., 1966.

13. J. Milnor, Introduction to Algebraic K-Theory, Annals of Math. Studies 72, Princeton Univ. Press, Princeton, 1971.

14. J. Rotman, Notes on Homological Algebra, Van Nostrand, Princeton, N. J., 1970.

15. J.-P. Serre, Faisceaux algébriques cohérents, Ann. Math., 61 (1955) 197–278.

16. R. Swan, Vector bundles and projective modules, Trans. Amer. Math. Soc., 105 (1962) 264–277.

17. ———, Algebraic K-theory, Springer Lecture Notes in Math., 76 (1968).

18. C. T. C. Wall, Finiteness conditions for CW-complexes, Ann. Math., 81 (1965) 56–69.

19. J. H. C. Whitehead, Simple homotopy type, Amer. J. Math., 72 (1950) 1–57.

20. E. F. Whittlesey, Fixed points and antipodal points, this MONTHLY, 70 (1963) 807–821.

21. L. N. Vasserstein, On the stabilization of the general linear group over a ring, Math. USSR Sbornik, 8 (1969) No. 3, 383–400.

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POLYNOMIAL IDENTITIES FOR MATRICES

WILLIAM WATKINS

1. Introduction. In 1946 [7, p. 4], Hermann Weyl stated the following theorem:

THEOREM. (Principle of the irrelevance of algebraic inequalities.) *Let R be an infinite integral domain with n independent indeterminates $x_1, \cdots, x_n$. Let $p \not\equiv 0$ and q be polynomials in $R[x_1, \cdots, x_n]$ such that if $p(r_1, \cdots, r_n) \neq 0$, for some r_i in R, then $q(r_1, \cdots, r_n) = 0$. Then $q \equiv 0$.*

The theorem follows from the fact that the only polynomial in $R[x_1, \cdots, x_n]$ that vanishes on R^n is the zero polynomial.

Weyl used this theorem as follows. If a polynomial equation in the entries of a matrix holds for all non-singular matrices, then the equation holds for the singular matrices too. This follows from Weyl's theorem since the determinant of a matrix is a polynomial in the entries of the matrix which is non-zero if and only if the matrix is non-singular. The determinant plays the part of p in the theorem.

There is another way to use Weyl's theorem. We show that there is a polynomial in the entries of a matrix A that is non-zero if and only if the eigenvalues of A are distinct. So if a polynomial equation in the entries of A holds for all A having distinct eigenvalues, then it holds for all matrices. O. Taussky used an equivalent form of this fact to obtain a factorization of the adjugate of a matrix [3] and a result on commutativity [4]. Here we use it to prove the Cayley-Hamilton theorem and some new theorems.

2. Results. Throughout we assume that $\mathscr{F}$ is an infinite field, $M_n(\mathscr{F})$ is the al-

gebra of n by n matrices with entries in $\mathscr{F}$, I_n is the identity matrix, and $X = (x_{ij})$ is an n by n matrix whose entries are independent indeterminates over $\mathscr{F}$. The next theorem is easy to prove by Weyl's method.

THEOREM 1. *Let A and B be matrices in $M_n(\mathscr{F})$. Then the characteristic polynomials of AB and BA are equal.*

Proof. Let x be another indeterminate over $\mathscr{F}$ and define

$$q(X) = \det(xI_n - XB) - \det(xI_n - BX).$$

If $\det A \neq 0$, then AB is similar to BA, $(A^{-1}(AB)A = BA)$ and so $q(A) = 0$. Thus, $q(A) = 0$, for all A in $M_n(\mathscr{F})$.

We note that if A and B are both singular, then AB is not necessarily similar to BA. For example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The next lemma is needed for the new use of Weyl's theorem.

LEMMA. *Let $f(X, x_1, \cdots, x_n)$ be a polynomial in $\mathscr{F}[x_{ij}, x_t \colon 1 \leqq i, j, t \leqq n]$, which is symmetric in $x_1, \cdots, x_n$. Then there exists a polynomial $g(X)$ in $\mathscr{F}[x_{ij} \colon 1 \leqq i, j \leqq n]$ such that*

$$g(A) = f(A, a_1, \cdots, a_n),$$

for every matrix A in $M_n(\mathscr{F})$ having eigenvalues $a_1, \cdots, a_n$.

Proof. Let $E_t(x_1, \cdots, x_n)$ denote the t-th elementary symmetric polynomial of $x_1, \cdots, x_n$. It is well known [6, p. 79] that every symmetric polynomial is a polynomial in the elementary symmetric polynomials. So there is a polynomial h with coefficients in $\mathscr{F}$ such that

$$h(X, E_1(x_1, \cdots, x_n), \cdots, E_n(x_1, \cdots, x_n)) = f(X, x_1, \cdots, x_n).$$

If $A = (a_{ij})$ has eigenvalues $a_1, \cdots, a_n$, then

$$(x + a_1) \cdots (x + a_n) = \det(xI_n + A). \tag{1}$$

The coefficient of x^{n-t} on the left side of (1) is $E_t(a_1, \cdots, a_n)$ and the coefficient of x^{n-t} on the right side of (1) is a polynomial $S_t(A)$ in the entries a_{ij} of A. (In fact $S_t(A)$ is the sum of all t by t principal subdeterminants of A.) Thus, $E_t(a_1, \cdots, a_n) = S_t(A)$.

Set $g(X) = h(X, S_1(X), \cdots, S_n(X))$. Then

$$\begin{aligned} g(A) &= h(A, S_1(A), \cdots, S_n(A)) \\ &= h(A, E_1(a_1, \cdots, a_n), \cdots, E_n(a_1, \cdots, a_n)) \\ &= f(A, a_1, \cdots, a_n). \end{aligned}$$

As a corollary to the lemma, we obtain that there exists a polynomial $p(X)$ such that

$$(2) \qquad p(A) = \prod_{i \neq j} (a_i - a_j),$$

for every matrix A in $M_n(\mathscr{F})$ having eigenvalues $a_1, \cdots, a_n$. Thus if $p(A) \neq 0$, then the eigenvalues of A are distinct and A is similar to a diagonal matrix [2, p. 171, 172]. From here on, p will denote the polynomial in (2). As an example, we now find a polynomial p in the entries of

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

such that $p(A) = (a_1 - a_2)(a_2 - a_1)$, for every 2 by 2 matrix with eigenvalues a_1, a_2. We compute

$$\begin{aligned} (x_1 - x_2)(x_2 - x_1) &= -(x_1 + x_2)^2 + 4x_1x_2 \\ &= -E_1^2(x_1, x_2) + 4E_2(x_1, x_2). \end{aligned}$$

But $S_1(X) = x_{11} + x_{22}$ and $S_2(X) = x_{11}x_{22} - x_{12}x_{21}$, so

$$\begin{aligned} p(X) &= -S_1^2(X) + 4S_2(X) \\ &= -(x_{11} + x_{22})^2 + 4(x_{11}x_{22} - x_{12}x_{21}). \end{aligned}$$

The rest of the theorems are fairly easy to prove for matrices with distinct eigenvalues because such matrices are similar to a diagonal matrix.

THEOREM 2. (Cayley–Hamilton). *Let A be a matrix in $M_n(\mathscr{F})$ having eigenvalues $a_1, \cdots, a_n$. Then $(A - a_1I_n) \cdots (A - a_nI_n) = 0$.*

Proof. Let $F(X, x_1, \cdots, x_n) = (X - x_1I_n) \cdots (X - x_nI_n)$. Then, $F(X, x_1, \cdots, x_n)$ is symmetric in $x_1, \cdots, x_n$. So the lemma applies to each of the n^2 entries of $F(X, x_1, \cdots, x_n)$ and there is a matrix $H(X)$ of polynomials such that

$$H(A) = F(A, a_1, \cdots, a_n),$$

for every matrix A in $M_n(\mathscr{F})$ having eigenvalues $a_1, \cdots, a_n$. If $p(A) \neq 0$, then A is similar to the diagonal matrix $D = \operatorname{diag}(a_1, \cdots, a_n)$. Also $H(A)$ is similar to $H(D)$. Clearly $H(D) = 0$ and so $H(A) = 0$. We have proved that if $p(A) \neq 0$, then $H(A) = 0$. Thus $H(A) = 0$, for every A in $M_n(\mathscr{F})$. This follows from Weyl's theorem since every entry of $H(A)$ is a polynomial in the entries of A.

The next theorem was proved by Taussky and Wielandt [5] in 1962. Their methods were different than those presented here.

THEOREM 3. *Let A be a matrix in $M_n(\mathscr{F})$ with eigenvalues $a_1, \cdots, a_n$. Define the commutator map $\mathscr{L}: M_n(\mathscr{F}) \to M_n(\mathscr{F})$ by*

$$\mathscr{L}(B) = AB - BA,$$

for every B in $M_n(\mathscr{F})$. Then $\mathscr{L}$ satisfies the polynomial

$$h(x) = x \prod_{i \neq j} (x - (a_i - a_j)).$$

Proof. Suppose $p(A) \neq 0$. Then the eigenvalues of A are distinct and there is a matrix S such that

$$S^{-1}AS = D = \operatorname{diag}(a_1, \cdots, a_n).$$

Define the maps $\mathscr{D}, \mathscr{S}: M_n(\mathscr{F}) \to M_n(\mathscr{F})$ by $\mathscr{D}(B) = DB - BD$, and $\mathscr{S}(B) = S^{-1}BS$, for every B in $M_n(\mathscr{F})$. Then

$$\mathscr{S}^{-1}\mathscr{D}\mathscr{S} = \mathscr{L}. \tag{3}$$

We now prove that $\mathscr{D}$ satisfies h. Let E_{ij} be the matrix in $M_n(\mathscr{F})$ whose only non-zero entry is a 1 in the (i,j) position. The set $\{E_{ij}: 1 \leqq i, j \leqq n\}$ is a basis for $M_n(\mathscr{F})$. We compute

$$\mathscr{D}(E_{ij}) = (a_i - a_j)E_{ij}.$$

Thus $h(\mathscr{D})E_{ij} = 0$, for all i and j and $h(\mathscr{D}) = 0$. By (3) $h(\mathscr{L}) = 0$.

We have proved that $h(\mathscr{L}) = 0$ whenever $p(A) \neq 0$. The entries in any matrix representation of $\mathscr{L}$ are polynomials in the entries of A. By the lemma, the entries in any matrix representation of $h(\mathscr{L})$ are polynomials in the entries of A. Thus $h(\mathscr{L}) = 0$, for every A in $M_n(\mathscr{F})$ having eigenvalues $a_1, \cdots, a_n$.

It is now easy to construct proofs for the next three theorems. Simply prove them for A similar to a diagonal matrix and use Weyl's theorem.

THEOREM 4. *Let $\mathscr{L}: M_n(\mathscr{F}) \to M_n(\mathscr{F})$ be the similarity map defined by $\mathscr{L}(B) = ABA^{-1}$, for all B in $M_n(\mathscr{F})$. Then $\mathscr{L}$ satisfies the polynomial*

$$(x - 1) \prod_{i \neq j} (x - a_i/a_j).$$

THEOREM 5. *Let $\mathscr{L}: M_n(\mathscr{F}) \to M_n(\mathscr{F})$ be the congruence map defined by $\mathscr{L}(B) = ABA^T$, for all B in $M_n(\mathscr{F})$. Then $\mathscr{L}$ satisfies the polynomial*

$$\prod_{i \leqq j} (x - a_i a_j).$$

THEOREM 6. *Let $\mathscr{L}: M_n(\mathscr{F}) \to M_n(\mathscr{F})$ be the Lyapunov map defined by $\mathscr{L}(B) = AB + BA^T$, for all B in $M_n(\mathscr{F})$. Then $\mathscr{L}$ satisfies the polynomial*

$$\prod_{i \leqq j} (x - (a_i + a_j)).$$

In **[1]**, Theorems 5 and 6 are proved by analyzing the elementary divisor structure of $\mathscr{L}$.

REMARKS: We have seen that the matrix properties of non-singularity and distinct eigenvalues are characterized by an algebraic inequality $p(A) \neq 0$ involving the

entries of the matrix A. What other matrix properties can be characterized or inferred from algebraic inequalities? For example: Is there a polynomial $p(X)$ in the entries of X such that A is similar to a diagonal matrix if and only if $p(A) \neq 0$?

R. Merris used the principle of the irrelevance of algebraic inequalities to show that the answer is no. Also it can be shown that there is no polynomial $q(X)$ such that $q(A)$ is the ith coefficient of the minimal polynomial of A, for all matrices A.

References

1. M. Marcus and W. Watkins, Elementary divisors of derivations, Journal of Linear and Multilinear Algebra, 2, June 1974, 65–80.
2. S. Perlis, Theory of Matrices, Reading, Mass., 1958.
3. O. Taussky, The factorization of the adjugate of a finite matrix, Linear Algebra and its Applications, 1 (1968) 39–41.
4. O. Taussky, A generalization of matrix commutativity, Linear Algebra and its Applications, 2 (1969) 349–353.
5. O. Taussky and H. Wielandt, Linear relations between higher additive commutators, Proc. AMS, 13 (1962) 732–735.
6. B. L. van der Waerden, Modern Algebra, Vol. I, Ungar, New York, 1949, English translation.
7. H. Weyl, The Classical Groups, 2nd edition, Princeton, N.J., 1946.

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AN OPTIMAL STRATEGY FOR POT-LIMIT POKER

WILLIAM H. CUTLER

1. Introduction. Poker games yield a wealth of problems for the game theorist. John von Neumann and Oskar Morgenstern analyzed two simple poker games in *Theory of Games and Economic Behavior* and showed that bluffing is a very important part of the game. Later papers generalize some of the results and analyze different games. For a good bibliography, see [1]. The papers may be divided into two groups, the theoretical and the practical. The theoretical deal with extremely simplified games, almost exclusively having only two players (an exception is [4]). This is because the theory of the finite two-person zero-sum game is quite handily taken care of by the Minimax Theorem ([5], p. 153). This tells us that each player has an optimal strategy, in the sense that if both players play these strategies, then neither player can improve his lot, assuming that the other player does not change his strategy. Of course, if one player plays a different, inferior strategy, the other player must usually also change to a non-optimal strategy to get best results. If we remove finite from the hypothesis, the Minimax Theorem does not always hold (see [8]). Many of the poker games analyzed are not finite, but for all of them

the Minimax Theorem does hold. If we allow more than two players, the situation becomes very different, and the theory is much more complicated.

The practical papers deal with some popular poker games, such as draw poker, and deal with many players. Computers are used to analyze the games, and the results are not optimal strategies, but approximations.

This paper is theoretical and involves a game of two players. We shall begin by discovering optimal strategies for several simple games, and shall then solve a game which is complicated enough to be of some interest. We shall conclude by looking at a poker game which can be played with cards, and might be called "deal 'em and bet 'em". Rather than looking at all possible strategies (of which there will be uncountably many), we will search for pairs of optimal strategies. This will be done by making various assumptions about these strategies which will lead to simultaneous equations. Solution of these equations then results in strategies meeting all of the conditions.

For those unfamiliar with poker games, we shall define some terms. At the start, each player puts a certain fixed number of chips into the pot. This is called the **ante.** The cards are then dealt. The first player, hereafter denoted A, can then either **check** (pass the opportunity to bet) or **bet,** whereby he adds additional chips to the pot. It is now the second player's (player B) turn. If A checks, then B can either check, in which case there is a showdown, or bet. If A bets, then B can either **fold** (forfeit his chance at the pot), **call** (in which case he adds the same number of chips to the pot as A bet, and there is a showdown), or **raise,** in which case he adds as many chips to the pot as A bet and some more in addition. If B has bet or raised, then the action reverts to A, who must either fold, call, or raise. If A checks, B bets, and A raises, this is called a **check-raise.** The betting continues until either one player folds or there is a showdown. If a showdown occurs, high hand wins the pot.

In our game, dealing the cards will consist of each player randomly drawing a real number in $[0, 1]$. This game will later be used to approximate optimal strategies in the case where cards are dealt. In [3] the game is analyzed where k raises are allowed, each of exactly n chips. This is quite similar to our game. We shall set the ante at 1 chip, and each bet or raise must be the size of the pot at that time. That is, after the ante, there are 2 chips in the pot, hence the first bet is 2 chips. If the initial bet is made and called, there are then 6 chips in the pot, so the first raise is 6 chips. The second raise is 18, the third 54, etc. This limitation on betting size is rather severe, but is the size of bet recommended by some experts in a no-limit or table stakes game. See [2], p. 59. It will become apparent later that under these conditons the game is easier to analyze than the so-called limit games, where bets and raises are restricted to a certain fixed number of chips.

2. Some simple cases. First of all, let us look at the game analyzed on pp. 211–219 of [5]. In this game, the ante is 1 chip, and player A can either check or bet 2 chips. If A checks, there is a showdown. If A bets, then player B may either call or

fold. This game is obviously in A's favor. Player A has the following strategies: For each $x \in [0, 1]$ he may either bet or check. Hence his pure strategies are characteristic functions on $[0, 1]$ and his mixed strategies are functions $b: [0, 1] \to [0, 1]$ where if A is dealt x, he bets with probability $b(x)$. Similarly, B's mixed strategies are functions $c: [0, 1] \to [0, 1]$, where if B is dealt y and A bets, then B calls with probability $c(y)$.

We shall see that there are pure strategies for both players that are optimal strategies. Let us start by analyzing the game from B's point of view. Suppose he has discovered A's strategy. Then, to decide if he should call A is quite simple. First of all, to call the bet he must risk 2 chips in an attempt to win 4 chips. Hence the pot is giving him 2 to 1 odds. Therefore he should call if he has a $\frac{1}{3}$ or better chance of winning. Specifically, he should call with hand y if

$$u = \frac{\int_0^y b(x)dx}{\int_0^1 b(x)dx} > \frac{1}{3},$$

fold if $u < \frac{1}{3}$, and it doesn't matter if $u = \frac{1}{3}$. Since $f(y) = \int_0^y b(x)dx$ is an increasing function, we note that there is either a single point or an interval where $u = \frac{1}{3}$. Let us suppose, for simplicity, that B chooses the following strategy: There is a fixed number $c \in (0, 1)$, and B calls if and only if $y \geqq c$.

Let us now return to player A. Assuming B plays the previously mentioned strategy, then what are A's expected returns with hand x?

If he checks, his return is

$$E_{\text{check}} = 1(x) + (-1)(1 - x) = 2x - 1. \tag{1}$$

If he bets, his return is

$$\begin{aligned} E_{\text{bet}} &= 1(c) - 3(1 - c) = 4c - 3 && \text{if } c > x, \\ &= 1(c) + 3(x - c) - 3(1 - x) = 6x - 2c - 3 && \text{if } c < x. \end{aligned} \tag{2}$$

Whether A should bet or check depends on which expected return is larger. If $c > \frac{1}{2}$, we get two intervals of betting. The first is $[0, a]$. These are A's **bluffs** (he really has a poor hand, and hoping B folds is the only way he can win). The second is A's legitimate high hand bets, in the interval $[b, 1]$. Since a and b are cut-off points between betting and checking, they must be places where $E_{\text{check}} = E_{\text{bet}}$. Hence, assuming $a < c < b$, we get $2a - 1 = 4c - 3$ or

$$2c - a = 1 \tag{3}$$

and

$$2b - 1 = 6b - 2c - 3 \text{ or}$$

$$2b - c = 1. \tag{4}$$

Finally, since for $y = c$ the probability of B's winning must be $\frac{1}{3}$, we get that the high bets and bluffs must be in the ratio of 2 to 1 or

$$2a + b = 1. \tag{5}$$

Solving (3), (4), and (5) simultaneously we get $a = \frac{1}{9}$, $b = \frac{7}{9}$, and $c = \frac{5}{9}$. Note that all assumed inequalities hold, and hence we have optimal strategies. Hence A must bet if $x < \frac{1}{9}$ or $x > \frac{7}{9}$ and check elsewhere. B must call if $y > \frac{5}{9}$ and fold otherwise.

We shall in general not try to figure out all of the optimal strategies, but for this particular case we might mention that A has only this one optimal strategy, while B has many. (He must always fold if $y < \frac{1}{9}$ and call if $y > \frac{7}{9}$, but if $\frac{1}{9} < y < \frac{7}{9}$ he may vary his strategy.)

Before we continue, it is well to note, as is always the case later, that the ratio of high bets to bluffs is 2 to 1, and the call line falls somewhere in between.

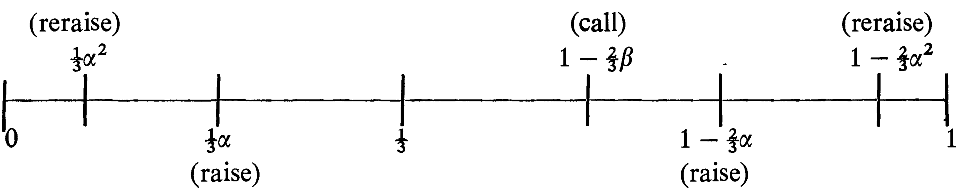
Now let us analyze the game of "force-in." In this game, A is forced to bet, regardless of his hand. B can then fold, call, or raise, and so on. We shall allow for an unlimited number of reraises. The rule that the first player is forced to bet is quite frequently used, and in our study it will help to simplify things.

First we note that since $y = \frac{1}{3}$ is the cut-off point at which B's expectation of winning is $\frac{1}{3}$, then this is the cut-off point between calling and folding. Next we note that B will want to raise on his best hands, say $a \leqq y \leqq 1$. We shall denote the ratio $(1 - a)/(\frac{2}{3})$ by α. Hence $a = 1 - \frac{2}{3}\alpha$. α is the percentage of good hands that B chooses to raise with. Since the ratio of high bets to bluffs must be 2 to 1, he must also raise with half as many bad hands. As we shall see later, it does not matter which bad hands he bluffs, as long as they are hands he would otherwise fold. Let us let him bluff the worst hands, or for $0 \leqq y \leqq \frac{1}{3}\alpha$.

Now if B raises, what does A do? We note a curious thing. A is in exactly the same position as B was before, except that the pot is 3 times as large. A knows that B holds in the range $[0, \frac{1}{3}\alpha]$ or $[1 - \frac{2}{3}\alpha, 1]$, which are in the ratio of 1 to 2. Hence the cut-off point for calling is somewhere between $\frac{1}{3}\alpha$ and $1 - \frac{2}{3}\alpha$, call it b. We denote the ratio $(1 - b)/(\frac{2}{3})$ by β. This is the percentage of good hands that A calls with.

He will also reraise on the α best hands of the collection of good hands that B raised with. Hence A reraises in $[1 - \frac{2}{3}\alpha^2, 1]$. He also bluff-reraises with hands $[0, \frac{1}{3}\alpha^2]$.

Let us graph these points:



We will suppose that the points lie in the relative positions indicated. We will now endeavor to compute α and β.

$1 - \frac{2}{3}\alpha$ is the cut-off between calling and raising (but not calling a reraise), hence the expected returns for each action must be the same.

Therefore

$$E_{\text{call}} = 3(1 - \tfrac{2}{3}\alpha) - 3(\tfrac{2}{3}\alpha) =$$

$$E_{\text{raise}} = -9(\tfrac{1}{3}\alpha^2) + 3((1 - \tfrac{2}{3}\beta) - \tfrac{1}{3}\alpha^2) + 9((1 - \tfrac{2}{3}\alpha) - (1 - \tfrac{2}{3}\beta)) - 9(\tfrac{2}{3}\alpha)$$

which reduces to

$$\alpha^2 + 2\alpha - \beta = 0. \tag{6}$$

Similarly, checking the bluff-raise cut-off point ($\frac{1}{3}\alpha$) we get the equation

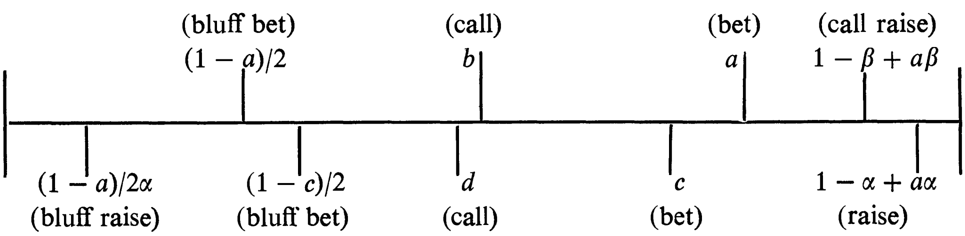
$$-1 = -9(\tfrac{1}{3}\alpha^2) + 3((1 - \tfrac{2}{3}\beta) - \tfrac{1}{3}\alpha^2) - 9(\tfrac{2}{3}\beta) \text{ or } \alpha^2 + 2\beta = 1. \tag{7}$$

Solving (6) and (7) simultaneously, we get $\alpha = \frac{1}{3}\sqrt{7} - \frac{2}{3} \cong 0.215$, and $\beta = \frac{2}{9}\sqrt{7} - \frac{1}{9} \cong 0.477$.

3. The main case. We are now ready to analyze the game with unlimited betting, restricted only by forbidding check-raises. Player A will presumably want to bet on his better hands, say for $a \leqq x \leqq 1$. He will bluff half as often, or for $0 \leqq x \leqq (1 - a)/2$. If Player B is confronted by a check, this indicates that $(1 - a)/2 \leqq x \leqq a$, hence he will bet if $y \geqq c$, where $c < a$. He will also bluff if $y \leqq (1 - c)/2$.

Now if A checks and B bets, then A will call somewhere in $[(1 - c)/2, c]$, call this cut-off point b. If A bets, then B knows that $x \leqq (1 - a)/2$ or $x \geqq a$. Hence his call line lies somewhere in $[(1 - a)/2, a]$, call it d. Also, B will raise with his best hands. But the situation he is in is precisely the force-in situation, so he will raise on the best α good hands, or hands in $[a, 1]$. Hence he raises if $y \geqq 1 - \alpha + a\alpha$. He also bluff-raises if $y \leqq (1 - a)/2\alpha$. Finally, A will call this raise on the best β of his hands in $[a, 1]$, hence for $x \geqq 1 - \beta + a\beta$.

Graphing these points, we get the following picture:



Investigating the bet cut-off at a, we see that

$$3((1 - c)/2) + 1(c - (1 - c)/2) + 3(a - c) - 3(1 - a)$$

$$= -3((1-a)/2\alpha) + (d-(1-a)/2\alpha) + 3(a-d) - 3(1-a)$$

or

$$2\alpha a + 3c - 2d = 1 + 2\alpha. \tag{8}$$

Investigating the point $(1-a)/2$ we get

$$\alpha a + 2d = 1 + \alpha. \tag{9}$$

Finally, the points c and $(1-c)/2$ yield

$$a + b - 2c = 0 \quad \text{and} \tag{10}$$

$$2a - 4b - c = -1. \tag{11}$$

Solving (8), (9), (10), and (11) simultaneously, we get

$$a = 1 - 1/21\sqrt{7} \cong 0.874$$

$$b = d = 5/9 - 1/63\sqrt{7} \cong 0.514$$

$$c = 7/9 - 2/63\sqrt{7} \cong 0.694.$$

We have now completely solved the problem. Cut-off points for bets and reraises are a, $1-(1-a)\alpha$, $1-(1-a)\alpha^2$, $1-(1-a)\alpha^3, \cdots$ or 0.874, 0.973, 0.994 and 0.999. Call lines for raises are $1-(1-a)\beta$, $1-(1-a)\alpha\beta$, $1-(1-a)\alpha^2\beta$, $1-(1-a)\alpha^3\beta, \cdots$ or 0.940, 0.987, 0.997 and 0.999 The number of bluffs to be made are $(1-a)/2\alpha$, $(1-a)/2\alpha^2, \cdots$ or 0.063, 0.014, 0.003 etc.,

Now a word about the bluff-raises. If it is raised back to either player, then some bluff-raises should be made. The bluff-raises should be made with hands which otherwise would have been folded. If the opponent is playing an optimal strategy, then he will never call with a hand worse than any of these, hence it makes no difference which of these hands are bluffed. However, a poor player might call on a weak hand, hence in our strategy we will bluff-raise on the best such hands.

Let us also look at those forbidden check raises. Suppose A is dealt $x = .98$. Our strategy advises him to bet and call a raise. Simple computation will show, however, that if check raises are permitted, he will come out better if he checks, raises if bet into, and folds if reraised. Hence check raises, if permitted, will be used. However, solving the problem with check raises appears to be quite difficult, as the force-in game cannot be applied.

4. Limiting the raises. In any poker game, there is either a limit on the number of raises or a limit on the amount that a given player can bet. This is obviously necessary from a practical standpoint. We shall see, however, that if the game is limited to k raises of pot size, then the game can essentially be played as the game with unlimited raises.

For example, it can easily be seen that the last raise should be made with the

best $\frac{1}{4}$ of those hands that the previous raise was made with. Hence $\alpha = .250$ for this raise. Also, $\beta = .500$.

For the second last raise, $\alpha = .211$ and $\beta = .474$, and for the third last raise, $\alpha = .216$ and $\beta = .477$. Hence, these values converge rather rapidly to the .215 and .477 of the previous section.

5. Other strategies. It can be computed easily that the above mentioned strategy, when played against an optimal strategy, will yield a return of about -0.093 for A. This is the value of the game to A. Hence, as would be expected, the game favors B.

Let us suppose that B is superconservative. Suppose he calls A if $y \geqq .514$, folds otherwise, and never bets or raises. This will give a return to A of about 0.091.

On the other hand, suppose that B is a super-bluffer. That is, he always bets or raises when given the chance. The return to A for this case is also about 0.091.

Hence we see that if A plays the optimal strategy, the return is very small under any of B's possible strategies. We can notice that if B changes his call or bluff lines from the optimal strategy, he usually does not lose anything. For instance, if A has bet, and B's hand falls in the category where he will win if and only if A is bluffing, then his returns for raising, calling, or folding are all the same. Also, if B holds a hand that he is sure is beaten, then it does not matter whether he folds or raises.

Changing the line for legitimate bets or raises will result in a loss to the player, however.

Hence we can get some clue as to the nature of the set of all optimal strategies. High hand raises must be made in precisely the intervals indicated. Calls must be made in the interval between possible bluffs and high hand bets, and must be in the proper ratio with the number of high hand bets. Bluffs may be made almost anywhere, but must be in proper proportion to high hand bets.

6. The discrete game. We will now consider the discrete game, where each player is dealt a 5-card hand from a regular 52-card deck, and the betting proceeds as before. (There is no draw.) There are 2,598,960 different hands, and they are linearly ordered. Straight flushes are best, followed by four-of-a-kind, full house, flush, straight, three-of-a-kind, two pair, one pair, and bust. We shall get our approximate optimal strategies for this game in the obvious way. If the 5-card hand of a player ranks n (from the bottom) out of 2,598,960, we treat his hand as if he were dealt $n/2{,}598{,}960$ in the continuous game.

There are at least 3 objections to this approximation. First of all, some different hands are equal, as the ordering disregards suits. This is a minor objection. Second, the optimal strategies for the discrete case may not be gotten this way from the optimal strategies for the continuous case. This is serious, but according to [5], p. 207, the error is about 1/2,598,960. Finally, the hands are dealt from one deck, not two. That is, what one player holds affects what another player may hold. Milton Parnes pointed out to this author that increasing the rank of a hand does not

necessarily increase its value. In fact, he conjectures that there is a discontinuity at each change of rank. For instance, if you hold a straight flush to the 5, your opponent may hold 31 higher straight flushes or 3 equal ones. However, if you hold four aces with a six kicker, your opponent may only beat you with 27 different straight flushes.

Ignoring all of these difficulties, our approximations for the optimal strategies come out as follows. For the first player, we get:

With	*Procedure*
J9632 or worse	bet, but fold if raised
J9632 — 22J109	check and fold if a bet
22Q43 — KKJ85	check and call
KKJ86 — 9922J	bet and fold if raised
9922Q — 99778	bet, reraise, and fold
997710 — 9 high straight	bet and call a raise
9 high str. — K8743 flush	bet, reraise, and fold
K8752 flush — K10965 flush	bet, 2 reraises and fold
K10972 flush — AAAJJ	bet, reraise, and call
AAAQQ — 88887	bet, 2 reraises, and fold
88889 — 8888Q	bet, 3 reraises, and fold
8888K — 6 high str. flush	bet, 2 reraises, call
7 str. fl. — 10 str. fl.	bet, 3 reraises, fold
J str. fl. — K str. fl.	bet, 3 reraises, call
A str. fl.	bet, 4 reraises

For the second player the strategy is:

With	*Procedure*
QJ763 or worse	fold if bet into, bet otherwise
QJ764 — AKQJ6	fold — check
AKQJ7 — 22J109	Raise if bet into, fold a reraise check if checked to
22Q43 — 77AQ4	call — check
77AQ5 — 222AQ	call — bet
222AK — JJJ87	raise and fold if reraised
JJJ92 — JJJK2	raise twice and fold
JJJK3 — 555JJ	raise and call
555QQ — JJJKK	raise twice and fold
JJJAA — QQQ33	3 raises and fold
QQQ44 — QQQQ9	2 raises and call
QQQQ10 — AAAA4	3 raises and fold
AAAA5	4 raises and fold
AAAA6 — Q str. fl.	3 raises and call
K str. fl.	4 raises and fold
A str. fl.	4 raises and call

Note that if the first player is dealt an eight high straight flush, he is supposed to bet, reraise three times, and fold if raised back. You will not catch this author making that particular play, even though it is undeniably part of the optimal strategy. The reason is two-fold; first, the given strategy is very conservative, and most players

play much more loosely. Second, the author, like practically all poker players, has a tendency to "fall in love with his cards." Nevertheless, inspection of the strategy shows that $\frac{1}{2}$ of all good hands should be folded when reraised.

To continue in this vein, there is a well-known story, told on pp. 65–68 of [**6**], where "Straights" Fowler and Dundee have been raising each other the limit of $25 for awhile. With about $700 in the pot, "Straights" raises one more time, whereby Dundee folds, convinced that his four Queens and Ace kicker will lose to four Kings or better. Irv Roddy, on pp. 165–166 of [7] is quite scornful. Our analysis goes something as follows: Since the pot is offering 28 to 1 odds, "Straights" should be bluffing about 1 time in 29. For the bluff to be a break-even proposition, Dundee should be calling about 28 times out of 29. Hence Mr. Roddy is correct in saying that Dundee should usually call; but there is that one time in 29!

For those who have decided to earn their living playing this paper's strategy, a word of warning. This strategy is very conservative, and playing against most any player, you will find that he quickly tightens up also. The return is very small, if not negligible. One two hour session with a new opponent netted 6 chips.

Finally, playing this game is about as exciting as watching paint dry; in that two hour session, there were two raises!

References

1. Nicholas V. Findler, et. al, Studies on Decision Making Using the Game of Poker, Information Processing, 71-North-Holland Publishing Company, 1972.

2. Oswald Jacoby, Oswald Jacoby on Poker, Doubleday and Co., 1947.

3. Samuel Karlin and Rodrigo Restrepo, Multistage Poker Models, Contribution to the Theory of Games, Vol. 3, Princeton University Press, 1957, pp. 337–364.

4. J. F. Nash and L. S. Shapley, A Simple Three-Person Poker Game, Contribution to the Theory of Games, Vol. 1, Princeton University Press, 1950, pp. 105–116.

5. John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944.

6. *Poker According to Maverick.*

7. I. Roddy, Friday Night Poker, Simon and Schuster, New York, 1961.

8. Maurice Sion and Philip Wolfe, On a Game Without Value, Contribution to the Theory of Games, Vol. 3, Princeton University Press, 1957, pp. 299–306.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

20. A. R. Loch. I would be very interested in hearing about attempts that have been made to improve courses in arithmetic and algebra for those students who have not mastered this material in school. I am particularly interested to learn what steps can be taken to arouse the interest of these students and to encourage them to learn at the collegiate level what they did not learn before.

21. E. M. Mert. Several years ago I came across a very nice exposition of the uses and occurrences of Brianchon's Theorem in geometrical optics. Can somebody supply the reference?

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

A COMMUTATIVITY THEOREM FOR ALGEBRAS

MICHAEL RICH

Let A be a not necessarily associative algebra over a field F. We shall say that A satisfies condition (*) if for each x, y in A there is a scalar $\alpha(x,y)$ in F such that $xy = \alpha(x,y)yx$. The purpose of this note is to prove the following theorem which ties in nicely with results recently obtained on the associativity of algebras [**1, 2**].

THEOREM. *If A is an (not necessarily associative) algebra satisfying* (*), *then A is either commutative or anti-commutative.*

A ring R is called anti-commutative if $ab = -ba$ for all a,b in R. Note that in an anti-commutative ring $x^2 = -x^2$ so that $2x^2 = 0$ for all x in the ring. An example of an associative, anti-commutative ring which is not commutative is provided by the algebra A over a field of characteristic $\neq 2$ with basis $\{a, b, c\}$ and multiplication given by:

$$a^2 = b^2 = c^2 = ac = ca = bc = cb = 0, \ ab = c, \text{ and } ba = -c.$$

Nonassociative anti-commutative rings have been studied extensively in the literature. In particular, Lie algebras [3] are examples of these.

We prove the theorem via the following lemma which is of some interest in its own right.

LEMMA. *Let R be a (not necessarily associative) ring in which $xy = \pm yx$ for all x,y in R. Then R is either commutative or anti-commutative.*

Proof. For each a in R, let $C_a = \{x \in R \mid ax = xa\}$ and $A_a = \{x \in R \mid ax = -xa\}$. By hypothesis, $R = C_a \cup A_a$. If $R \neq C_a$ and $R \neq A_a$, then there is an element $b \in C_a - A_a$ and an element $d \in A_a - C_a$. But $a(b+d) = (b+d)a$ implies that $d \in C_a$ and $a(b+d) = -(b+d)a$ implies that $b \in A_a$. Thus $R = C_a$ or $R = A_a$. Now let $U = \{a \in R \mid C_a = R\}$ and $W = \{a \in R \mid A_a = R\}$. Then $R = U \cup W$. If $R \neq U$ and $R \neq W$, then there are elements $u \in U - W$ and $v \in W - U$. But $u + v \in U$ implies that $v \in U$ and $u + v \in W$ implies that $u \in W$. Therefore $R = U$ or $R = W$ so that R is either commutative or anti-commutative.

Proof of Theorem. We note first that if x, y are elements of A such that $xy \neq yx$, then $x^2 = y^2 = 0$. For by (*) $(x+y)x = \gamma x(x+y)$ and $xy = \alpha yx$ for some α, γ in F. Thus $(1-\gamma)x^2 = (\gamma\alpha - 1)yx$. Now $yx \neq 0$ for otherwise by (*) $xy = 0$. Also $\gamma \neq 1$ for otherwise $\gamma = \alpha = 1$, which is impossible since $xy \neq yx$. Thus, $x^2 = \beta xy$ for some β in F. Similarly $y^2 = \delta xy$ for some δ in F. Now by (*) corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F, there is an η in F such that

$$(\alpha_1 x + \alpha_2 y)(\alpha_3 x + \alpha_4 y) = \eta(\alpha_3 x + \alpha_4 y)(\alpha_1 x + \alpha_2 y)$$

which reduces to

$$(1)\quad (\alpha_1\alpha_3\beta + \alpha_1\alpha_4 + \alpha^{-1}\alpha_2\alpha_3 + \alpha_2\alpha_4\delta)xy = \eta(\alpha_1\alpha_3\beta + \alpha_2\alpha_3 + \alpha^{-1}\alpha_1\alpha_4 + \alpha_2\alpha_4\delta)xy.$$

If in (1) we choose $\alpha_1 = 0$, $\alpha_2 = \alpha_4 = 1$, $\alpha_3 = -\delta$, then the right side of (1) is zero, whereas the left side of (1) is $(1-\alpha^{-1})\delta xy$. Since $xy \neq 0$ and $\alpha \neq 1$, it follows that $\delta = 0$. Thus $y^2 = 0$. Similarly, choose $\alpha_1 = \alpha_3 = 1$, $\alpha_4 = 0$, $\alpha_2 = -\beta$ to derive $\beta = 0 = x^2$.

We can now see that the hypothesis of the lemma applies to A. For if $xy = \alpha yx$, $\alpha \neq 1$, then (1) reduces to

$$(2)\qquad (\alpha_1\alpha_4 + \alpha^{-1}\alpha_2\alpha_3)xy = \eta(\alpha_2\alpha_3 + \alpha^{-1}\alpha_1\alpha_4)xy$$

for any choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F. Choose $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = -\alpha^{-1}$ to get

$1-(\alpha^{-1})^2=0$. Since $\alpha \neq 1$, it follows that $\alpha = \alpha^{-1} = -1$ so that $xy = -yx$. Thus A is either commutative or anti-commutative.

References

1. R. Coughlin and M. Rich, On Scalar Dependent Algebras, Canad. J. Math., 24 (1972) 696–702.

2. R. Coughlin, E. Kleinfeld, and M. Rich, Scalar Dependent Algebras, Proc. Amer. Math. Soc., 39 (1973) 69–73.

3 N. Jacobson, Lie Algebras, Wiley (Interscience), New York, 1962.

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INVARIANCE OF AN INTEGRAL AVERAGE OF A LOGARITHM

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Gauss showed in 1818 that the integral

$$(2/\pi)\int_0^{\pi/2}(x^2\sin^2\theta + y^2\cos^2\theta)^{-1/2}d\theta \tag{1}$$

is unchanged in value if the positive numbers x and y are replaced by $(x+y)/2$ and $\sqrt{xy}$. Repeated use of this invariance allows the elliptic integral to be evaluated numerically by the algorithm of the arithmetic-geometric mean. Gauss proved the invariance by a change of integration variable [**1**], but alternatively it can be deduced as a special case of a quadratic transformation of the hypergeometric function [**3**, eq. 2.1(24)].

We shall prove a similar invariance property of the integral

$$A(x,y) = (2/\pi)\int_0^{\pi/2}\log(x\sin^2\theta + y\cos^2\theta)d\theta, \tag{2}$$

which represents an average of the logarithm over the interval $[x,y]$. The invariance will be used repeatedly to deduce the (previously known) value of the integral. We shall then consider a more general integral and prove a quadratic transformation which yields the invariance of A as a special case. Thus (2) is in several respects a twin of (1).

Assume $x \geqq 0$, $y \geqq 0$, and $x+y>0$. Put $\theta = \phi/2$ in (2) and break the interval of integration into two halves:

$$\begin{aligned}A(x,y) &= \frac{1}{\pi}\int_0^{\pi/2}\log\left(\frac{x+y}{2} - \frac{x-y}{2}\cos\phi\right)d\phi \\ &+ \frac{1}{\pi}\int_{\pi/2}^{\pi}\log\left(\frac{x+y}{2} - \frac{x-y}{2}\cos\phi\right)d\phi.\end{aligned}$$

Replacing ϕ by $\pi - \phi$ in the second integral and recombining, we find

$$A(x,y) = \frac{1}{\pi}\int_0^{\pi/2} \log\left[\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 \cos^2\phi\right] d\phi$$
$$= \frac{1}{\pi}\int_0^{\pi/2} \log\left[\left(\frac{x+y}{2}\right)^2 \sin^2\phi + xy\cos^2\phi\right] d\phi .$$

This proves the invariance property

$$A(x,y) = \tfrac{1}{2} A\left[\left(\frac{x+y}{2}\right)^2, xy\right]. \tag{3}$$

The arguments on the right side are the squares of the arithmetic and geometric means of x and y.

To evaluate the integral if x and y are strictly positive, let

$$\begin{aligned} &x_0 = x, && y_0 = y, \\ &x_{n+1} = \left(\frac{x_n + y_n}{2}\right)^2, && y_{n+1} = x_n y_n, \qquad n \in \mathbb{N}, \end{aligned} \tag{4}$$

where $\mathbb{N}$ is the set of nonnegative integers. Define also

$$\alpha_n = \tfrac{1}{2}(\sqrt{x_n} + \sqrt{y_n}), \quad \beta_n = \tfrac{1}{2}(\sqrt{x_n} - \sqrt{y_n}), \quad \delta_n = \beta_n/\alpha_n, \quad n \in \mathbb{N}. \tag{5}$$

The recurrence relations (4) imply

$$\alpha_{n+1} = \alpha_n^2, \quad \beta_{n+1} = \beta_n^2, \qquad \delta_{n+1} = \delta_n^2. \tag{6}$$

It follows from $\delta_1 = \beta_0^2/\alpha_0^2$ that $0 \leqq \delta_1 < 1$ and hence, as $n \to \infty$,

$$\delta_n \to 0, \quad \frac{y_n}{x_n} = \left(\frac{1-\delta_n}{1+\delta_n}\right)^2 \to 1, \quad \alpha_n^2/x_n \to 1. \tag{7}$$

Successive applications of (3) show that

$$A(x,y) = 2^{-n}A(x_n, y_n) = 2^{-n}\log x_n + 2^{-n}A(1, y_n/x_n), \quad n \in \mathbb{N}. \tag{8}$$

For strictly positive x and y, the integrand of (2) is jointly continuous in x, y, θ, and $A(x,y)$ is continuous in x and y. Since $A(1,1) = 0$ it follows from (8) and (7) that

$$A(x,y) = \lim_{n\to\infty} 2^{-n}\log x_n = 2\lim_{n\to\infty} 2^{-n}\log\alpha_n .$$

Finally, because $2^{-n}\log\alpha_n$ is independent of n by (6), we find

$$A(x,y) = 2\log\alpha_0 = 2\log\frac{\sqrt{x}+\sqrt{y}}{2}. \tag{9}$$

The integral is evaluated by a different method in [4, 322.13]. We leave as an

exercise the simple proof using (3) that (9) is valid if $xy = 0$ and $x + y > 0$. For complex values of x and y, see the end of this note.

We consider next a more general integral. If β and β' are positive numbers, a measure is defined on $[0, \pi/2]$ by

$$(10)\quad d\mu_{(\beta,\beta')}(\theta) = \left[\int_0^{\pi/2} (\sin\theta)^{2\beta-1}(\cos\theta)^{2\beta'-1}\, d\theta\right]^{-1} (\sin\theta)^{2\beta-1}(\cos\theta)^{2\beta'-1}\, d\theta.$$

If $x \geqq 0$, $y \geqq 0$, and $x + y > 0$, we define

$$(11)\qquad L(\beta,\beta'; x, y) = \int_0^{\pi/2} \log(x\sin^2\theta + y\cos^2\theta)d\mu_{(\beta,\beta')}(\theta).$$

The integral in (2) is the special case

$$(12)\qquad A(x,y) = L(\tfrac{1}{2},\tfrac{1}{2}; x, y).$$

THEOREM. *Assume $\beta > 0, x \geqq 0, y \geqq 0$, and $x + y > 0$. Then*

$$(13)\qquad L(\beta,\beta; x,y) = \tfrac{1}{2}L\left[\beta,\tfrac{1}{2};\left(\frac{x+y}{2}\right)^2, xy\right].$$

Proof. Putting $\theta = \phi/2$ we find

$$(14)\qquad L(\beta,\beta;x,y) = \int_0^{\pi}\log\left(\frac{x+y}{2} - \frac{x-y}{2}\cos\phi\right)d\nu_\beta(\phi),$$

where the measure ν_β is defined on $[0,\pi]$ by

$$(15)\qquad d\nu_\beta(\phi) = \left[\int_0^{\pi}(\sin\phi)^{2\beta-1}\,d\phi\right]^{-1}(\sin\phi)^{2\beta-1}d\phi.$$

If the interval of integration in (14) is divided into two halves, we may replace ϕ by $\pi - \phi$ on $[\pi/2, \pi]$ and recombine the integrals to get

$$L(\beta,\beta;(x,y) = \int_0^{\pi/2}\log\left[\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2\cos^2\phi\right]d\nu_\beta(\phi).$$

Since $\nu_\beta = \frac{1}{2}\mu_{(\beta,\frac{1}{2})}$ on $[0,\pi/2]$, we find

$$L(\beta,\beta;x,y) = \tfrac{1}{2}\int_0^{\pi/2}\log\left[\left(\frac{x+y}{2}\right)^2\sin^2\phi + xy\cos^2\phi\right]d\mu_{(\beta,\frac{1}{2})}(\phi).$$

If $\beta = \frac{1}{2}$ the quadratic transformation (13) reduces to (3).

The theorem can be extended to the complex domain without changing the proof if we take the principal branch of the logarithm and assume $(\beta, x, y) \in \mathbb{C}^3_>$, where $\mathbb{C}_> = \{z \in \mathbb{C}: \operatorname{Re} z > 0\}$ is the right half-plane. If $\mathbb{C}_0$ denotes the plane $\mathbb{C}$ cut along the nonpositive real axis, the principal branch of the logarithm is holomorphic on $\mathbb{C}_0$. By putting $\sin^2\theta = u$ in (11), L is seen to be an integral average of the type studied

in [2], and Theorem 8 of that paper asserts that the function

$$(\beta, \beta', x, y) \to L(\beta, \beta'; x, y)/\Gamma(\beta + \beta')$$

has a holomorphic continuation to $\mathbb{C}^2 \times \mathbb{C}_0^2$. By the permanence of functional relations it follows that (13) is valid provided $(x, y) \in \mathbb{C}_>^2$ and $-2\beta \notin \mathbb{N}$. Likewise we conclude that (9) is valid if $(x, y) \in \mathbb{C}_0^2$.

References

1. B. C. Carlson, Algorithms involving arithmetic and geometric means, this MONTHLY, 78 (1971) 496–505.

2. ———, A connection between elementary functions and higher transcendental functions, SIAM J. Appl. Math., 17 (1969) 116–148.

3. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Higher Transcendental Functions, McGraw-Hill, New York, 1953.

4. W. Gröbner and N. Hofreiter, Integraltafel, Zweiter Teil, Bestimmte Integrale, 2nd ed., Springer-Verlag, Vienna, 1958.

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A THEOREM RELATED TO CARTOGRAMS

ASHISH K. SEN

The term cartogram is used by geographers to denote geographical maps which are not drawn in accordance with one of the standard map projections. A particular class of cartograms consists of images of one-to-one mappings $f \equiv \{u(x,y), v(x,y)\}$ of X onto Y with

$$J(f) = u_x v_y - v_x u_y = g(x,y) > 0, \tag{1}$$

where X and Y are bounded portions of the plane and $g(x, y)$ is a prespecified function on X. The value of g at a point may be called the areal scale at the point.

Cartograms of this kind, with a large areal scale near airports and a smaller scale elsewhere, have been drawn for airline pilots. Rand McNally has apparently considered making highway maps with larger real scales around cities in order to show interchanges more clearly. Tobler [3] advocates the use of such cartograms for the empirical verification of several geographical theories. For example, a cartogram with $g(x, y)$ proportional to some measure of buying power would be very useful in a study of the location of economic activities. In another paper, Tobler [4] proposes that cartograms be used in the construction of electoral districts. Several other applications are suggested in [3].

If X and Y are simply connected open sets and g is n times continuously differentiable (i.e., g is C^n) with $\int_X g = \int_Y 1$, then there exists a mapping which obeys condition (1). On applying the Riemann Mapping Theorem, it is readily apparent

that the proof of this assertion reduces to one for the special case where X and Y are unit squares. For this special case, it can be shown that the mapping $f \equiv (u, v)$, where

$$u = \int_0^x \int_0^1 g\,dy\,dx \quad \text{and} \quad v = \frac{\int_0^y g\,dy}{\int_0^1 g\,dy},$$

has the required properties. Furthermore, f is C^n (see Oxtoby and Ulam [2] for a proof when g and f are merely continuous). But this mapping is not unique. However, if $F \equiv (\xi, \eta)$ is any C^{n+1} homeomorphism of the unit square onto itself and g is C^n, then there is a unique C^n homeomorphism $f \equiv (u, v)$ of the unit square onto itself such that (1) holds and v is of the form $\zeta\{\eta(x, y)\}$.

Geographers have been quite concerned with the local properties of cartograms, especially with distortion (often defined in terms of departure from conformality of f). In this context, the following consequence of Tissot's theorem (see, for example, Willmore [5, p. 88]) has proved valuable: For any C^1 mapping

$$df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \mathbf{O}\{\theta(x, y)\} \begin{pmatrix} a(x, y) & 0 \\ 0 & b(x, y) \end{pmatrix} \mathbf{O}\{\phi(x, y)\}, \tag{2}$$

where

$$\mathbf{O}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

At any point at which f is not conformal, θ and ϕ are unique; the functions a and b are always unique.

Given a function g, there is a mapping f and functions a, b, θ, ϕ, such that (2) holds with $ab = g$ (note that $J(f) = ab$). However, such an f is far from unique. Therefore the following theorem—which is the main theorem of this paper—is appealing:

THEOREM. *Let f be a C^1 homeomorphism of X onto Y where X and Y are closed bounded portions of the plane and Y is strictly convex. Also, let a, b, θ, ϕ be as in* (2). *Then f is the only C^1 homeomorphism having these θ, ϕ and $ab = g > 0$.*

REMARK. It has been shown by Lavrent'iev [1] among others that, given functions p and q on X, there exists (under some mild conditions) a mapping f for which (2) holds with $a/b = p$ and $\phi = q$. Moreover, if f_1 and f_2 are two mappings with these properties, the composition $f_1 \circ f_2^{-1}$ must be conformal.

Proof. Assume that f is not unique and that there is another mapping f_1 with

$$df_1 = \mathbf{O}(\theta) \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \mathbf{O}(\phi)$$

and $a_1 b_1 = g$. Let $f_t = tf + (1-t)f_1$. Then

$$J(f_t) = \{ta + (1-t)a_1\}\{tb + (1-t)b_1\}$$
$$= g\{2t^2 + 1 - 2t + (t-t^2)(a/a_1 + a_1/a)\}.$$

But $a/a_1 + a_1/a > 2$ if $a \neq a_1$. Hence $J(f_t) \geqq g > 0$ and

$$\int_X J(f_t) > \int_X g. \tag{3}$$

Next we show that there is a value of t not equal to 0 or 1 such that f_t is one-to-one. When f_t is not one-to-one, the boundary $\mathrm{Bd}\{f_t(X)\}$ is not a simple curve and

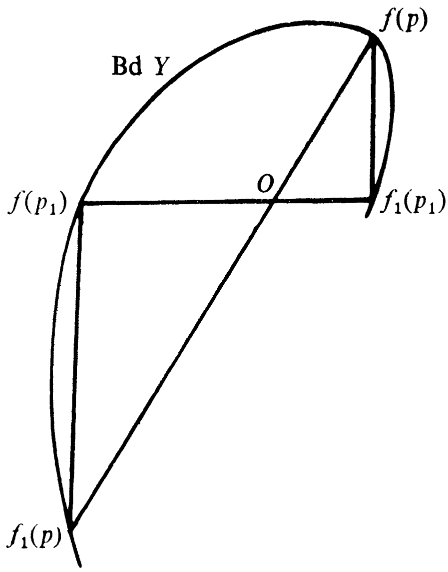

FIG. 1

we have a configuration akin to that shown in Figure 1, where $O \in \mathrm{Bd}\{f_t(X)\}$. On considering similar triangles $f(p)\,f(p_1)\;O$ and $f_1(p_1)f_1(p)\;O$, we see that

(4) length of segment $f(p)f(p_1)$ / length of segment $f_1(p)f_1(p_1) = t(1-t)$.

But owing to the positivity of g, the left side of (4) is bounded below by a number $\delta > 0$. Hence if we choose t in the interval

$$0 < t < \delta(1+\delta)^{-1}, \tag{5}$$

f_t is one-to-one.

Now owing to the convexity of Y, $f_t(X) \subseteq Y$ and if t obeys (5),

$$\int_X J(f_t) \leqq \int_Y 1 = \int_X g. \tag{6}$$

Inequalities (3) and (6) contradict each other. This proves the theorem.

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References

1. M. A. Lavrent'iev, Sur une classe de représentations continues, Rec. Math., 48 (1935) 407–423.

2. J. C. Oxtoby and S. M. Ulam, Measure preserving homeomorphisms and metric transitivity, Ann. Math., 42 (1941) 874–920.

3. W. R. Tobler, Geographical area and map projections, Geographical Review, 53 (1963) 59–78.

4. ———, A continuous transformation useful for districting. Presented at the New York Academy of Sciences Conference on Democratic Representation and Apportionment, November 3, 1972. To appear in the Annal, New York Academy of Sciences.

5. T. J. Willmore, An Introduction to Differential Geometry, Oxford, 1959.

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A REPRESENTATION THEOREM FOR POLYNOMIALS OF TWO VARIABLES

J. C. Molluzzo

The purpose of this note is to prove the following:

Theorem. *Any polynomial of degree* n, $P_n(x,y)$, *over a field of characteristic zero, can be written in the form*

$$P_n(x,y) = \sum_{i=1}^{n} F_i(a_ix + b_iy), \text{ where } \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0$$

if $i \neq j$ *and* F_i, $1 \leqq i \leqq n$, *are polynomials of degree at most* n. *Moreover, there exist polynomials of degree* n *which cannot be represented by fewer than* n *summands.*

We first prove three lemmas. Let $D_n(a_n, a_{n-1}, \cdots, a_0)$ denote the discriminant of the polynomial $f(x) = \sum_{i=0}^{n} a_ix^i$, where $a_n \neq 0$. We consider D_n as a polynomial in the a_i's and assume that all polynomials are taken over a field of characteristic zero.

Lemma 1. *If* $n \geqq 2$, $D_n(0, a_{n-1}, \cdots, a_0) = a_{n-1}^2 D_{n-1}(a_{n-1}, \cdots, a_0)$ *where* D_{n-1} *is the discriminant of an* $n-1$ *degree polynomial.*

Proof. Follows easily from simple manipulations of the determinant form of the representation of D_n.

Note that when $n \geqq 2$, if $D_n(a_n, a_{n-1}, \cdots, a_0) = 0$ and if we let d, $0 \leqq d \leqq n$, be the largest integer such that $a_d \neq 0$, then the polynomial $a_0 + a_1x + \cdots + a_dx^d$ has a multiple root.

LEMMA 2. *If* $n \geqq 2$, $D_n(a_n, a_{n-1}, \cdots, a_0)$ *has no linear factors.*

Proof. Suppose the contrary. Then

$$D_n(a_n, a_{n-1}, \cdots, a_0) = (A_n a_n + \cdots + A_0 a_0) Q_n(a_n, \cdots, a_0),$$

where A_i, $0 \leqq i \leqq n$, are constants and Q_n is a polynomial. First note that at least two of the A_i are non-zero, for if all were zero except A_k, then $D_n(a_n, \cdots, a_{k+1}, 0, a_{k-1}, \cdots, a_0) = 0$ which, by the previous lemma, implies that $x^n + 1$ (if $k \neq 0$ and $k \neq n$) or $x^n + x$ (if $k = 0$) or $x + 1$ (if $k = n$) has a multiple root, but this is not the case. We next observe that $A_0 \neq 0$. If A_0 were zero, at least two other coefficients, A_i and A_j, would be non-zero. Take $a_i = A_j$, $a_j = -A_i$, a_0 arbitrary and all others zero. Then $A_j x^i - A_i x^j + a_0$ would have a multiple root, independently of a_0, but this is not the case. Since $A_0 \neq 0$, then at least another $A_d \neq 0$. Now take $a_d = A_0$, $a_0 = -A_d$, and all others zero. Then $A_0 x^d - A_d$ has a multiple root. This contradiction establishes the lemma.

LEMMA 3. *A polynomial, $P(x,y)$, can be written in the form*

$$P(x,y) = \sum_{i=1}^{r} F_i(a_i x + b_i y) \quad \text{where} \quad \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0$$

for $i \neq j$ and F_i, $1 \leqq i \leqq r$, is a polynomial, if and only if

$$\prod_{i=1}^{r} \left(b_i \frac{\partial}{\partial x} + a_i \frac{\partial}{\partial y} \right) P(x,y) = 0.$$

Proof. The necessity is obvious. Sufficiency follows easily by induction on the number of factors r.

If one takes $r = n = \deg(P)$ in the above lemma, it is easily obtained that the representation is possible with polynomials F_i of degree at most n.

Proof of the theorem. By Lemma 3, $P_n(x,y)$ is of the required form if and only if it is annihilated by the nth order homogeneous partial differential operator

$$T_n = \sum_{i=0}^{n} d_i \frac{\partial^n}{\partial x^i \partial y^{n-i}}$$

where the associated polynomial $d_0 + d_1 z + \cdots + d_n z^n$ has distinct zeros. Now suppose

$$P_n(x,y) = c_0 y^n + c_1 y^{n-1} x + \cdots + c_n x^n + Q_{n-1}(x,y),$$

where $Q_{n-1}(x,y)$ is a polynomial of degree at most $n-1$. Hence,

$$T_n(P_n) = c_0' d_0 + c_1' d_1 + \cdots + c_n' d_n,$$

where $c_k' = (n-k)!k!c_k$, $0 \leqq k \leqq n$. We must then show that $d_0, \cdots, d_n$ can be so chosen that

$$T_n(P_n) = 0 \tag{1}$$

and such that the associated polynomial has distinct zeros. This polynomial has distinct zeros if and only if its discriminant is non-zero. By the factor theorem, the only way its discriminant can be zero for every choice of the d_i that satisfy (1) is that $T_n(P_n)$ be a factor of the discriminant. This is impossible by Lemma 2.

To prove the second statement of the theorem, consider $P_n(x, y) = y^{n-1} + x^2 y^{n-2}$, where $n \geqq 3$. If P_n could be represented by fewer than n summands, then there would exist an operator

$$T_{n-1} = \sum_{i=0}^{n-1} d_i \frac{\partial^{n-1}}{\partial x^i \partial y^{n-1-i}}$$

whose associated polynomial has distinct zeros, and which annihilates P_n. But, $T_{n-1}(P_n) = (n-1)!d_0 + 2(n-2)!d_1 x + 2(n-2)!d_2 y$. Thus, if $T_{n-1}(P_n) = 0$, the associated polynomial would have zero as a multiple zero, contrary to assumption.

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RESEARCH PROBLEMS

Edited by Richard Guy

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

A COMBINATORIAL PROBLEM CONCERNING ORIENTED LINES IN THE PLANE

L. Fejes Tóth

In the Euclidean plane consider a finite number $n > 1$ of oriented lines in a general position, i.e., so that any two have a point in common, but no three do so. The lines decompose the plane into a certain number of convex regions, some of which are well oriented. The term well oriented means that we can travel along the boundary of the region continuously in accordance with the direction of the respective lines. Let w be the number of the well oriented regions and $t = \frac{1}{2}(n^2 + n + 2)$ the total

number of regions. Prove or disprove the conjecture that $w/t \leqq 4/7$ with equality only for three lines bounding a well oriented triangle (Fig. 2).

When I mentioned this problem to H. Hanani he soon proved the inequality

$$\frac{w}{t} \leqq \frac{2[(n^2+n)/3]}{n^2+n+2}$$

which is exact for $n = 2$, 3 and 4 (Figs. 1, 2, 3). It implies $w/t < 2/3$. The proof is

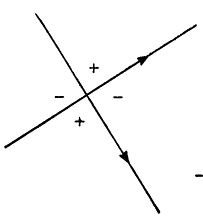

FIG. 1

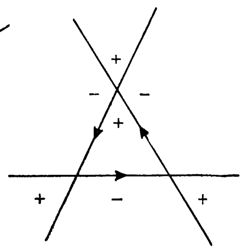

FIG. 2

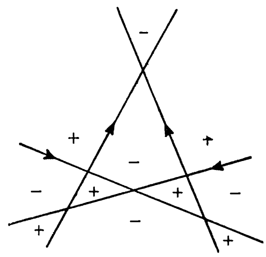

FIG. 3

based on the fact that the unbounded regions have at least one vertex and the bounded regions have at least three vertices. If we knew that there is a sufficiently great number of regions with more than 1 and 3 vertices, respectively, the inequality could be improved.

A simple proof of Hanani's inequality starts from the observation that each edge belongs to at most one well-oriented region. This points to another problem for which the inequality also holds. If the countries of the "map" generated by the n straight lines are colored with two colors, how large a fraction of the total number of countries can have the same color? What if 3 or 4 colors are available, or an unlimited number? This last amounts to asking what fraction of the countries may be chosen as "independent".

N. Sauer observed that in order to attack the above problem it would be useful to analyse the structure of the point-set-union of the bounded regions, a problem which seems to be interesting in itself. Some questions have already been considered. S. Roberts [**4**] claimed to have proved that there are at least $n-3$ triangular regions, but Grünbaum [**2**, p. 26] regards his proof as unconvincing. This has recently been established by R. Shannon [**5**] who has also confirmed Conjectures 2.7 and 2.8 of [**2**]. Martin Gardner [**1**] attributes the question, what is the largest number of non-overlapping triangles that can be produced by n straight line segments, to Kobon Fujimura. For $n = 3, 4, 5, 6$ the answers are 1, 2, 5 and 7, and for $n=7,8,9$ are thought to be 11, 15 and 21, but these have not been proved maximal. Here the lines are not necessarily in general position. A related note is that of Pavlick [3]; though his conjecture, that there is a configuration in the Euclidean plane consisting of n lines meeting in p points, where p is $0, 1$ or any integer satisfying $n-1 \leqq p \leqq \binom{n}{2}$, is false for all $n \geqq 10$.

If one removes the restriction to sets of lines in general position, Grünbaum believes that, provided the lines do not form a pencil (i.e., pass through a single point), $w/t \leqq 5/7$, with equality only for the arrangement of 5 lines shown in Figure 4. One can also consider the problem for arrangements of lines in the projective plane, or for the simple ones (those with the lines in general position) among them.

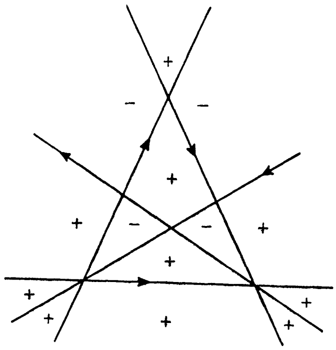

FIG. 4

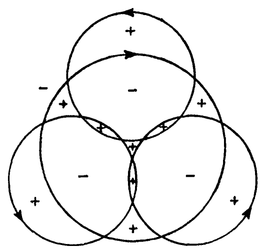

FIG. 5

In another variant the map is generated by n oriented, pairwise intersecting circles which do not form a pencil. If no three circles pass through the same point, the Hanani-type estimate becomes

$$w/t \leqq [2n(n-1)/3]/(n^2-n+2) < 2/3.$$

In the general position case it may be conjectured that $w/t \leqq 5/7$, with equality only in the case of the 4 circles of Figure 5, constructed by Herbert Taylor and Robert Henderson.

I thank Branko Grünbaum for the interesting comments made in the last four paragraphs, and members of staff of the University of Calgary for assistance from their National Research Council of Canada grants.

References

1. Martin Gardner, Mathematical Games, Scientific American, 226 (1972) 102 and 117.

2. Branko Grünbaum, Arrangements and Spreads, Conf. Board Math. Sci. # 10, Amer. Math. Soc., Providence, R. I. 1972.

3. F. Pavlick, If n lines in the Euclidean plane meet in 2 points, then they meet in at least $n-1$ points, Math. Mag., 46 (1973) 221–223.

4. S. Roberts, On the figures formed by the intercepts of a system of straight lines in a plane, and on analogous relations in space of three dimensions, Proc. London Math. Soc., 19 (1889) 405–422.

5. R. Shannon, Ph. D. thesis, University of Washington, 1974.

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THE CAYLEY-HAMILTON THEOREM

CHARLES A. MCCARTHY

In the first course in complex analysis, the power of complex methods is exhibited by an efficient proof of the fundamental theorem of algebra. In the same spirit, one can give a proof of the Cayley-Hamilton theorem. Besides being useful as another easy application of contour integration, this proof has also proved useful in a course of linear algebra for students of applied science who are already familiar with the Cauchy integral formula.

Let A be an $n \times n$ matrix; the entries of A are denoted by $A_{\mu\nu}$, $1 \leqq \mu, \nu \leqq n$. The $n \times n$ identity matrix is denoted by I. Our first step is to show that if $p(z) = c_0 + c_1 z + \cdots + c_r z^r$ is a polynomial, then the matrix $p(A) = c_0 I + c_1 A + \cdots + c_r A^r$ is given by the analogue of Cauchy's formula

$$p(A) = (1/2\pi i) \int_\Gamma p(\zeta)(\zeta I - A)^{-1} d\zeta, \tag{1}$$

where Γ is, say, a circle of sufficiently large radius centered at the origin. Of course this is merely the functional calculus for holomorphic functions of an operator, but for our purposes the following elementary development of (1) suffices. Define, for any $n \times n$ matrix $T = (T_{\mu\nu})$, the norm $\|T\| = \max\{|T_{\mu\nu}| : 1 \leq \mu, \nu \leqq n\}$; note that $\|ST\| \leqq n\|S\| \, \|T\|$, so that $\|A^k\| \leqq n^{k-1}\|A\|$ for $k = 0, 1, 2, \cdots$. It follows that the matrix series $\sum_{k=0}^{\infty} \zeta^{-(k+1)} A^k$ converges in every entry, absolutely and uniformly for $|\zeta| \geqq 2n\|A\|$. For any ζ for which $\sum_{k=0}^{\infty} \zeta^{-(k+1)} A^k$ converges, the product of this series with $\zeta I - A$ is the identity; that is, $(\zeta I - A)^{-1}$ exists at least for $|\zeta| \geqq 2n\|A\|$ and is given there by this series. Term-by-term integration gives the result

$$A^k = \frac{1}{2\pi i} \int_{|\zeta| = 2n\|A\|} \zeta^k (\zeta I - A)^{-1} d\zeta, \quad k = 0, 1, 2, \cdots,$$

from which (1) follows with Γ the circle $|\zeta| = 2n\|A\|$.

Now we know explicitly how to compute the inverse of a matrix: the entries of $(\zeta I - A)^{-1}$ are $[\det(\zeta I - A)]^{-1} M_{\mu\nu}(\zeta)$, where $M_{\mu\nu}(\zeta)$ are the cofactors of $(\zeta I - A)$. Each $M_{\mu\nu}$ is a polynomial (of degree at most $n-1$) in ζ.

Let $c(z) = \det(zI - A)$ denote the characteristic polynomial of A. Formula (1)

then becomes

$$c(A)_{\mu\nu} = \frac{1}{2\pi i} \int_\Gamma \det(\zeta I - A)[\det(\zeta I - A)]^{-1} M_{\mu\nu}(\zeta) d\zeta = \frac{1}{2\pi i} \int_\Gamma M_{\mu\nu}(\zeta) d\zeta .$$

These integrals vanish because $M_{\mu\nu}$ is a polynomial. Thus $c(A)$ is the zero matrix.

Supported in part by NSF GP 28933 and Chalmers Tekniska Högskola, Göteborg,

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FOURIER SERIES OF FUNCTIONS OF BOUNDED VARIATION

C. HOROWITZ

A theorem of Jordan states that a continuous function f of period 2π, and of bounded variation over each period, is the uniform limit of its Fourier series. The theorem is usually proved by using a device called the "second law of the mean." A new and perhaps simpler proof may be based on integration by parts, together with some well-known facts from the theory of Fourier series. Let

$$T_n(x) = \frac{x - \pi}{2} + \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

Then $T_n'(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = D_n(x)$, the nth Dirichlet Kernel. Moreover, it is known that $|T_n(x)| < 2\pi$ for x in $[0, 2\pi)$, and for all n, and that $T_n(x) \to 0$ uniformly for x in any interval $[\delta, 2\pi - \delta]$, $0 < \delta < 2\pi$ (see, e.g., [**1**], p. 42).

Jordan's theorem may now be proved as follows. Let s_n be the nth partial sum of the Fourier series of f. For $x \in (-\infty, \infty)$

$$s_n(x) - f(x) = \frac{1}{\pi} \int_0^{2\pi} [f(x+t) - f(x)] D_n(t) dt.$$

Let $\phi_x(t) = f(x+t) - f(x)$. In particular, $\phi_x(0) = \phi_x(2\pi) = 0$. Thus, integrating by parts,

$$s_n(x) - f(x) = \frac{1}{\pi} \int_0^{2\pi} \phi_x(t) D_n(t) = -\frac{1}{\pi} \int_0^{2\pi} T_n(t) d\phi_x(t)$$

$$= -\frac{1}{\pi} \left[\int_0^{\delta} + \int_{\delta}^{2\pi - \delta} + \int_{2\pi - \delta}^{2\pi} T_n(t) d\phi_x(t) \right].$$

The two outer integrals may be made small *independent of n and x* by choosing δ small. This is true because the T_n are uniformly bounded, and because the total variation of ϕ_x over $[-\delta, \delta]$ tends to zero uniformly in x as $\delta \to 0$ (f is the difference of two uniformly continuous monotone functions). The middle integral is then small independent of x when n is large. Hence, there is uniform convergence.

It should be noted that a simple modification of this proof yields convergence of the Fourier series of a function of bounded variation to the function at each point of continuity. This in turns implies the pointwise convergence of the series to $[f(x+0)+f(x-0)]/2$, on $[0,2\pi)$, as one sees by considering, e.g., the function $f(x)-[f(x_0+0)-f(x_0-0)]\chi_{[x_0,2\pi]}$ (χ = characteristic function), which is of bounded variation, and continuous at x_0.

Reference

1. E. C. Titchmarsh, The Theory of Functions, 2nd edition, Oxford University Press, Amen House, London, 1939.

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

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A SEMI-SEMESTER APPROACH TO INTRODUCTORY MATHEMATICS

FRAN MASAT, ROBERT MEYER, AND WALTER E. MIENTKA

Introduction. Prior to the fall of 1973, Precalculus Mathematics at the University of Nebraska, Lincoln, served an average of 1460 students a year. In the fall semesters the course was taught by instructors who lectured three days per week to large sections of about 120 students. There was also one weekly quiz section, of about 25 students each, supervised by teaching assistants. In the evening, summer, and spring, the course was taught by teaching assistants in small sections of approximately 25 students.

During the fall of 1970, an extensive survey was conducted of the students in Precalculus Mathematics. The survey [3] indicated that this course was serving thirty-one (31) different disciplines from the various colleges of the University. For example, of the fall 1970 class, 22% were in Business Administration, and 19% in Agricultural studies. Although the course had been originally designed as a precalculus course, the enrollment had shifted to include mainly students not continuing into calculus, with only 27% of the students indicating a desire to continue into calculus. This had been strongly suspected from comments by the lecturers as well as from student opinion.

PURSUIT PROBLEMS FOR INDEPENDENT STUDY

W. R. UTZ

1. Introduction. Independent study for undergraduates in a sizeable university is a little different than the same activity at Reed College [**6,7**] or Carleton College [**13, 10, 11**]. At the University of Missouri-Columbia we also have a departmental honors program, Pi Mu Epsilon, occasional summer grants for undergraduate research, etc. These activities are conducted by a capable subset of the department in a programed and orderly manner. However, with thousands of undergraduates in mathematics courses each semester and hundreds of departmental majors, it isn't surprising that from time to time a talented student will show up at an arbitrary staff member's door with a request to participate in independent study. For this reason it is a good idea to have a repertoire of areas and problems that might be suitable for this student.

Over the years I've used pursuit problems as one such topic, and have found that it appeals to about 40% of the students who look to me for independent study. Besides these students, I've had one student who selected this area for an undergraduate thesis and, also, I've used it for projects in courses in nonlinear differential equations. The subject has numerous advantages, the most appealing of which is that it seems to be inexhaustible, and it is for this reason that I'm glad to share it. Another advantage of this topic is that the study assigned may really be independent and so students may be taken on even in very busy semesters at no great cost in time.

2. A Program of Independent Study. Many sophomore level books in differential equations give a pursuit problem as an illustration, and a few books, such as a book of H. T. Davis [**5**], go into these problems systematically. In any case, the undergraduate may already have encountered such problems.

In a series of papers Arthur Bernhart [**1, 2, 3, 4**] has given a summary of many of the classical pursuit problems so that the undergraduate may rapidly discover the variety of possible problems and solutions for many of them.

Other references [**8, 12, 14, 15**] are available and the student may even be exposed to differential games [**9**], but the undergraduates who come to me find this too deep for their available time and background.

A student can give the cited literature a cursory examination in a short time, and by doing so, know what a pursuit problem is. It isn't necessary to give the student every known reference, only enough to start him out. Now, an assignment of devising a pursuit problem, different from any one the student has seen, and then solving it makes a project that can be done independently.

The results are marvelous. Dogs chase bouncing balls, bulls charge up and down hills at varying speeds, polygonal paths are required, dogs capture cats traveling in transcendental paths down dead end alleys, missiles chase fighter planes, velocities (of both the pursued and pursuer) vary, etc. While this process isn't advertised as

undergraduate research, and in no instance have I encouraged a student to expose his work through publication, it does confront the student with many problems common to research.

References

1. Arthur Bernhart, Curves of pursuit, Scripta Math., 20 (1954) 125–141.
2. ———, Curves of pursuit — II, Scripta Math., 23 (1957) 49–65.
3. ———, Polygons of pursuit, Scripta Math., 24 (1959) 23–50.
4. ———, Curves of general pursuit, Scripta Math., 24 (1959) 189–206.
5. H. T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962.
6. F. L. Griffin, Undergraduate mathematical research, this MONTHLY, 49 (1942) 379–385.
7. ———, Further experience with undergraduate mathematical research, this MONTHLY, 58 (1951) 322–325.
8. G. H. Handelman, Aerodynamic pursuit curves for overhead attack, J. of the Franklin Institute, 249 (1949) 205–222.
9. Rufus Isaacs, Differential Games, Wiley, New York, 1967.
10. K. O. May, Undergraduate research in mathematics, this MONTHLY, 65 (1958) 241–246.
11. ———, Undergraduate research: some conclusions, this MONTHLY, 75 (1968) 70–74.
12. C. G. Paradine, Pursuit curves, Math. Gaz., 48 (1964) 437–439.
13. W. C. Ramaley, Independent study for undergraduates, this MONTHLY, 80 (1973) 555–558.
14. I. M. Slade, On algebraic curves of pursuit, Tokohu Math. J., 39 (1934) 111–113.
15. C. E. Wilder, A discussion of a differential equation, this MONTHLY, 38 (1931) 17–25.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before July 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 604* [1944, 46]. *Proposed by H. W. Becker*

Give a combinatorial proof that

$$\sum_{r=0}^{\infty} \frac{1}{(r!)^2}\left(\frac{t}{2}\right)^{2r} \sim \frac{e^t}{\sqrt{2\pi t}} \quad \text{as } t \to \infty.$$

E 2528. *Proposed by L. W. Shapiro, Howard University*

Let R denote the ring of $n \times n$ real matrices with the property that every element not in the first row or on the main diagonal is 0. How many two-sided ideals does R have?

E 2529. *Proposed by S. W. Golomb, University of Southern California*

Let $N_k(n)$ denote the number of "digits" in the k-ary representation of the natural number n. Show that if

$$S_k = \sum_{n=1}^{\infty} \frac{1}{n(N_k(n))^2},$$

then $S_k \sim A \log k$ for some constant A. Find A and estimate the error term.

E 2530*. *Proposed by F. Loupekine, University College of Wales, Aberystwyth*

(a) Show that it is possible to partition the natural numbers into three classes so that if (x, y, z) is a *primitive* Pythagorean triple, then x, y, z are in different classes.

(b)* Can such a partition be made if the above is to hold for *all* Pythagorean triples, not just primitive ones?

E 2531. *Proposed by V. F. Ivanoff, San Carlos, California*

Given points A, B, C, D, E, F in the plane, let ABC denote the *directed* area of triangle ABC, etc. Prove that

$$AEF \cdot DBC + BEF \cdot DCA + CEF \cdot DAB = DEF \cdot ABC.$$

(Remark: The special case $D = F$ was shown by W. L. Williams, *A pentagon theorem*, this MONTHLY 60 (1953), 616–617.)

E 2532. *Proposed by Erwin Just, Bronx Community College*

Solve the following Diophantine equations:

(1) $x^m(x^2 + y) = y^{m+1}$

(2) $x^m(x^2 + y^2) = y^{m+1}$.

Compare Problem E 2464, solution in this issue, p. 403.

E 2533. *Proposed by E. S. Pondiczery, Royal Academy of Poldavia (In Exile)*

Our old friend Professor Euclide Paracelso Bombasto Umbugio, eminent numerologist from Guayazuela, is trying to calculate to an accuracy of 1% the sum of the reciprocals of the 8,877,690 positive integers whose decimal representations contain no repeated digits. Possessing only a desk calculator (Guayazuela is a poor country), he figures it will take him about one year to complete his calculations, performing them at the rate of one reciprocal per second, eight hours a day, six days a week. Help the Professor improve his efficiency.

Editor's comment. Although the Professor is too shy to ask, we know that he would like more accuracy in his sum. Perhaps some of the Professor's many friends who have access to high speed computers could help him out. There is a problem, however, in that the Professor would insist on paying for computer time out of his own pocket; since his salary is quite meager, direct calculation of the sum would be out of the question.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Inequality of Statistical Interest

E 2428 [1973, 807; 1974, 782]. *Proposed by M. S. Klamkin, University of Waterloo, Ontario, Canada*

If a_i $(i = 1, 2, \cdots, n)$ denote real numbers, show that

$$n \min(a_i) \leqq \Sigma a_i - S \leqq \Sigma a_i + S \leqq n \max(a_i),$$

where

$$(n-1)S^2 = \sum_{1 \leqq i < j \leqq n} (a_i - a_j)^2 \qquad (S \geqq 0)$$

and with equality if and only if $a_j =$ constant.

III. *Comment C. L. Mallows, Bell Laboratories, Murray Hill, New Jersey.* The left-hand inequality, $na_1 \leqq \Sigma a_i - S$, will be an equality if and only if $a_1 = a_2 = \cdots = a_{n-1}$ (assuming as in the published solution that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$), contrary to the statement on lines 10–11 of p. 783. This follows by taking $r = 1$ in Corollary 6.1 of my paper (jointly with Donald Richter), *Inequalities of Chebyshev type involving conditional expectations*, Ann. Math. Stat. 40 (1969), 1922–1932. Similarly the dual inequality $\Sigma a_j + S \leqq na_n$ (note the misprint on line 9 of p. 783) is an equality if and only if $a_2 = a_3 = \cdots = a_n$. This follows, too, from my paper by using the inequality dual to that in Corollary 6.1 (i.e., using u_r instead of v_r). Thus equality holds in both if and only if $n = 2$ or all a_i are equal. Certainly $\Sigma a_i - S = \Sigma a_i + S$ if and only if $S = 0$, i.e., if and only if all a_i are equal.

Powers of a Weighted Sequential Sum

E 2434 [1973, 943; 1974, 281]. *Proposed by George O'Brien, York University, Ontario*

Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative numbers such that

$(a_n)^n \to a$ and $(b_n)^n \to b$. Let p and q be nonnegative numbers such that $p + q = 1$. Evaluate $\lim(pa_n + qb_n)^n$. Generalize.

Solution by Joel Pitt, State University College of New York at New Paltz. Assume first that $0 < a, b < \infty$; then

$$\lim_{n\to\infty} (pa_n + qb_n)^n = a^p b^q. \tag{*}$$

To show this, we establish first that if $\{x_n\}$ is a sequence of nonnegative numbers and if $0 < x < \infty$, then $(x_n)^n \to x$ if and only if $n(x_n - 1) \to \log x$. Both conditions obviously imply $x_n \to 1$, so we can assume that $x_n > 0$ and hence $(x_n)^n \to x$ if and only if $n \log x_n \to \log x$. If we define $y_n = 1$ if $x_n = 1$ and

$$y_n = \frac{\log x_n}{x_n - 1} = \frac{\log x_n - \log 1}{x_n - 1} \qquad \text{if } x_n \neq 1,$$

then $n \log x_n = n(x_n - 1)y_n$ and since $y_n \to 1$, the equivalence is shown.

The limiting behavior of $(pa_n + qb_n)^n$ follows immediately: from $(a_n)^n \to a$ and $(b_n)^n \to b$, we conclude that $n(a_n - 1) \to \log a$ and $n(b_n - 1) \to \log b$. Letting $x_n = pa_n + qb_n$, we have

$$\begin{aligned} n(x_n - 1) &= n(pa_n + qb_n - 1) = pn(a_n - 1) + qn(b_n - 1) \\ &\to p \log a + q \log b = \log(a^p b^q), \end{aligned}$$

and thus (*) is established. More generally, the same reasoning yields an analogous result for any finite number of sequences $(a_n)^n \to a$, $(b_n)^n \to b, \cdots, (g_n)^n \to g$.

The limiting behavior of $(pa_n + qb_n)^n$ is slightly more complex if either a or b is 0 or ∞. We can assume that p, q are both positive, for if either is 0, the limit is trivial. With this assumption, (*) is still valid under the obvious interpretations, except in the indeterminate cases $0 \cdot \infty$ and $\infty \cdot 0$. In these cases, any limit is possible.

Also solved by Peter de Buda, O. P. Lossers (Netherlands), Patrick McCray, St. Olaf College Students, Kenneth Schilling, Joel Spencer, Allen Stenger, Temple University Problem Solving Group, and the proposer.

Editor's comment. Most solvers discovered the generalization from 2 to n sequences. More interesting is the case of an infinite family of sequences $\{a_n(i)\}_{n=1}^{\infty}$ for $i = 1, 2, \cdots$ with $(a_n(i))^n \to a(i)$ as $n \to \infty$, and a sequence $\{p_i\}$ of nonnegative numbers with $\sum p_i = 1$. We can again assume without loss of generality that every $p_i > 0$. Let $a = \prod_{i=1}^{\infty} a(i)^{p_i}$ if this product converges and let $a = 0$ otherwise. The proposer shows that if the product diverges or if $a(i) = 0$ for some i, then $(\sum_i p_i a_n(i))^n \to 0$. He also shows that if $a \neq 0$ and if both $a(i) \geqq \varepsilon > 0$ for all i and $(a_n(i))^n \to a(i)$ uniformly in i, then $(\sum_i p_i a_n(i))^n \to a$.

Two Serendipitous Diophantine Equations

E 2464 [1974, 281]. *Proposed by Erwin Just, Bronx Community College*

Solve the following Diophantine equations:

(1) $x^2(x^2 + y) = y^{m+1}$,

(2) $x^2(x^2 + y^2) = y^{m+1}$.

I. *Solution by Chris Freiling, University of San Francisco.* Consider the Diophantine equation

$$x^2(x^2 + y^n) = y^{m+1} \qquad (m, n \geqq 0) \tag{3}$$

of which (1) and (2) are special cases. Ignoring the trivial solution $x = y = 0$ (and $m \geqq 0$, $n \geqq 1$ arbitrary), we see that if either x or y is 0, then the other is also, so suppose nonzero integers x, y satisfy (3). Express the nonzero rational $\alpha = x^2/y^n$ in lowest terms: $\alpha = u/v$, where $u \neq 0$ and $v > 0$. (Equivalently, let $d = (x^2, y^n)$ be taken positive or negative according as y^n is positive or negative, and write $x^2 = du$, $y^n = dv$.) Note that $(u, v) = 1$. Substituting αy^n for x^2 in (3) and simplifying, we obtain

$$u(u + v)y^{2n} = v^2 y^{m+1}.$$

We distinguish two cases:

CASE 1: $m+1<2n$. Then $v^2/u(u + v) = y^{2n-m-1}$ is an integer. Since $(u, v) = 1$, certainly $(u, v^2) = 1$ and because $u \mid v^2$, it follows that $u = \pm 1$. Similarly, since $1 = (u + v, v) = (u + v, v^2)$ and $(u + v) \mid v^2$, we have also that $u + v = \pm 1$. Since $v>0$ we conclude that $v = 2$ and $u = -1$. But then $y^{2n-m-1} = -4$, implying that $y = -4$ (and $2n - m - 1 = 1$). Therefore $x^2 = \alpha y^n = -\frac{1}{2}(-4)^n = (-1)^{n+1}2^{2n-1}$, which cannot be a perfect square. Case 1 is impossible.

CASE 2: $m + 1 \geqq 2n$. Then $u(u+ v)/v^2 = y^{m+1-2n}$ is an integer. Now $(v, u) = (v, u + v) = 1$ and hence v and $u(u +v)$ are relatively prime; since $v \mid u(u + v)$ and $v > 0$, it follows that $v = 1$, and so $y^{m+1-2n} = u(u + 1)$, a product of consecutive integers. Since $y \neq 0$, we must have $m + 1 - 2n = 1$, i.e., $m = 2n$. (This is because if $y^k = u(u + 1)$ with $k > 1$, then since $u(u + 1) = 1$, both u and $u + 1$ would have to be kth powers, an impossibility unless $u = y = 0$.) Thus $y = u(u + 1)$ and $x^2 = uy^n = u^{n+1}(u + 1)^n$. For this to be a perfect square, we must have $u = k^2$ if n is even and $u + 1 = k^2$ if n is odd. The general nontrivial solution is therefore as follows:

(i) If $n = 2r$ is even, then $m = 2n$, $x = k^{2r+1}(k^2 + 1)^r$, and $y = k^2(k^2 + 1)$, where k is any nonzero integer.

(ii) If $n = 2r + 1$ is odd, then $m = 2n$, $x = k^{2r+1}(k^2 - 1)^{r+1}$, and $y = k^2(k^2 - 1)$, where k is any integer other than 0 or ± 1.

(It is easy to verify by substitution in (3) that the above necessary conditions for solution are also sufficient.)

II. *Solution by W. J. Blundon, Memorial University of Newfoundland.* We note first that if $1 + 4y^n$ is a square, then it is an odd square, say $1 + 4y^n = (1 + 2v)^2$ with $v \geqq 0$ implying that $y^n = v(v + 1)$. Since v and $v + 1$ are relatively prime, both must be nth powers, implying that either $v = 0$ or $n = 1$ so that $y = v(v + 1)$.

Consider equation (1): it is equivalent to $(2x^2 + y)^2 = y^2 + 4y^{m+1}$. If $m = 0$, then the right-hand side is $y^2 + 4y = (y + 2)^2 - 4$ which, by the equation, must be a perfect square. Since two nonzero squares cannot differ by 4, necessarily $x = y = 0$ and there is only the trivial solution. Assuming that $m \geqq 1$, the equation can be rewritten as

$$(2x^2 + y)^2 = y^2(1 + 4y^{m-1}),$$

implying that $1 + 4y^{m-1}$ is a square. By the note above, either $x = y = 0$ or $m = 2$ and $y = v(v + 1)$. Ignoring the trivial solution, we have $x^2 = v^2(v + 1)$ so that v is of the form $t^2 - 1$. Thus the nontrivial solutions must be of the form $m = 2$, $x = t^3 - t$, and $y = t^4 - t^2$, where t is any integer other than 0 or ± 1. Substitution shows that these are indeed solutions.

Equation (2) reduces to $(2x^2 + y^2)^2 = y^4 + 4y^{m+1}$. Ignoring trivial solutions again, we see that $m = 0$ is impossible since $x^2(x^2 + y^2) > y$. If $m = 1$, the right-hand side of the reduced equation is $y^2(y^2 + 4)$ and if $m = 2$, it is $y^2\{(y + 2)^2 - 4\}$. Neither case is possible since two nonzero squares cannot differ by 4. Hence $m \geqq 3$ and the equation becomes

$$(2x^2 + y^2)^2 = y^4(1 + 4y^{m-3}).$$

As in the solution of equation (1), this requires $m = 4$ and $y = v(v + 1)$. Then $x^2 = v^3(v + 1)^2$ so that v is of the form t^2. Thus the nontrivial solutions must be of the form $m = 4$, $x = t^5 + t^3$, and $y = t^4 + t^2$, where t is any nonzero integer. Substitution again verifies that these are solutions.

Also solved by J. D. Baildon, Robert Breusch (New Zealand), P. G. de Buda, F. J. Flanigan, M. G. Greening (Australia), G. A. Heuer (Germany), Carl Hurd, Dennis Jespersen, Eleanor Jones, Lew Kowarski, O. P. Lossers (Netherlands), Carolyn MacDonald, L. E. Mattics, J. W. McHutchion, M. R. Murty & V. K. Murty, Walter Read, G. B. Robinson, Michael Shimshoni (Israel), F. B. Strauss, Phil Tracy, and Ken Yocom. Partial solutions by Merrill Barnebey and by Charles Wexler.

Editor's comment. Baildon, de Buda, and Shimshoni all obtain the solution to the more general equation (3) of Solution I above. Flanigan studies the even more general equation $x^\alpha(x^\alpha + y^\gamma) = y^\delta$. Interestingly enough, in one case, solutions to Flanigan's equation depend on solutions to Catalan's equation $m^\nu - n^\mu = 1$. It is still unknown whether there are any nontrivial solutions (i.e., $\nu, \mu \geqq 2$) to this, other than the well-known $3^2 - 2^3 = 1$.

Most solvers solved for x^2 by completing the square (as in Solution II) or by the quadratic formula. Many solutions were incomplete because the solvers did not eliminate the special cases $m = 0$ in (1) and $m = 0, 1, 2$ in (2).

An error occurred in transcribing the original proposal. Because of the interest in the altered problem, your editors decided not to print a correction but to let it stand and to include also the proposer's original problem. See Problem E 2532 in this issue of the MONTHLY.

$d(A) = d(B) = 0$, yet $d(A+B) = 1$

E 2465 [1974, 281]. *Proposed by Claude Anderson, University of California at Berkeley*

If A is a subset of N, the natural numbers, then the density of A, denoted $d(A)$, is defined by the following limit (when it exists):

$$d(A) = \lim_{n \to \infty} \frac{A_n}{n},$$

where A_n is the number of elements of A which do not exceed n. Let $\varepsilon > 0$ be arbitrary. Show that there exist A, B, with $d(A) < \varepsilon$, $d(B) < \varepsilon$, yet $d(A+B) = 1$, where $A+B = \{a+b : a \in A, b \in B\}$. Is it possible to have $d(A+B) = 1$ and $d(A) = d(B) = 0$?

I. *Solution to first part by Bridgewater Problem Solving Group, T. E. Elsner, G. A. Heuer (Germany), Dennis Jesperson, David Singmaster (England), Wolfe Snow, and Dale Worley (independently).* Given $\varepsilon > 0$, choose n with $1/n < \varepsilon$. Let $A = \{1, 2, \cdots, n\}$ and $B = \{n, 2n, 3n, \cdots\}$. Then $d(A) = 0$, $d(B) = 1/n < \varepsilon$, and since $A + B = \{n+1, n+2, \cdots\}$ clearly $d(A+B) = 1$.

II. *Solution to first part by Neal Felsinger, J. T. Gill, J. P. Lambert, L. E. Mattics, William Nuesslein, and the proposer (independently).* Given $\varepsilon > 0$, choose m, n with $1/m < \varepsilon$, $1/n < \varepsilon$ and $(m, n) = 1$. Let $A = \{m, 2m, 3m, \cdots\}$ and $B = \{n, 2n, 3n, \cdots\}$. Then $d(A) = 1/m < \varepsilon$ and $d(B) = 1/n < \varepsilon$. It is known that $A + B$ has finite complement, so that $d(A+B) = 1$.

III. *Solution by L. J. Dickson (Australia), J. T. Gill, O. P. Lossers (Netherlands), William Nuesslein, H. Sherwood, Wolfe Snow, and the Temple University Problem Solving Group (independently).* Let A be the set of natural numbers whose binary representations have nonzero bits only in even positions, and let B be the set of numbers whose binary representations have nonzero bits only in odd positions. Then $d(A) = d(B) = 0$, so that $d(A \cup B) = 0$, and since $A + B = N \setminus (A \cup B)$, it follows that $d(A+B) = 1$.

IV. *Solution by Robert Breusch (New Zealand), the Bridgewater Problem Solving Group, Leonard Carlitz, Michael Doob, J. E. Nyman, T. Šalát (Czechoslovakia), David Singmaster (England), and the proposer (independently).* Let S_1 be the set of squares of the natural numbers, and let $S_2 = S_1 + S_1$ denote the set of natural numbers which are expressible as the sum of exactly two squares. (Some complication is introduced by the fact that the problem stipulates that we must have subsets of N—0 is not allowed.) Define $S_3 = S_2 + S_1$ and $S_4 = S_2 + S_2$ analogously. Certainly $d(S_1) = 0$, and it follows from an estimate of Landau that $d(S_2) = 0$ also. (See L. E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, New York, 1952, p. 254.) Let $A = S_1 \cup S_2$, so that $d(A) = 0$. By the famous "Four Squares Theorem" of Lagrange, $N = S_1 \cup S_2 \cup S_3 \cup S_4$ and since $A + A = S_2 \cup S_3 \cup S_4$ and $d(S_1) = 0$, it is clear that $d(A+A) = 1$.

V. *Solution by Herman Bubbert, Bethel College, Kansas.* Let A be the set of natural numbers which have no digit 9 in their decimal representations. It is easy to see that $d(A) = 0$ and that $A + A = N \setminus \{1\}$ so that $d(A + A) = 1$.

VI. *Solution by Joel Spencer and Scott Forrest (independently).* G. G. Lorentz has shown that given an infinite subset A of N, there exists a subset B of N with $d(B) = 0$ such that $A + B$ has finite complement, and so *a fortiori* $d(A + B) = 1$. (*On a problem of additive number theory*, Proc. Amer. Math. Soc. 5 (1954), 838–841.) Take A to be any infinite set with $d(A) = 0$ (e.g., the primes, the squares). See also H. Halberstam and K.F. Roth, *Sequences*, Oxford, 1966, p. 13.

VII. *Solution by Jerome Minkus, Berkeley, California.* If P denotes the set of odd primes and if $A = \{p, p+1 : p \in P\}$, then $d(P) = 0$ implying $d(A) = 0$. By results of Vinogradov, Van der Korput, and Estermann on the Goldbach Conjecture, it is known that, with the possible exception of a set of density 0, every even integer is the sum of two odd primes; thus $d(A + A) = 1$. (See R. D. James, *Recent progress in the Goldbach problem*, Bull. Amer. Math. Soc. 55 (1949), 246–260.)

Solutions similar to Solution V were submitted by Chris Freiling, V. W. McHutchion, and T. Šalát (Czechoslovakia), and elementary solutions of other types were submitted by Neal Felsinger, G. A. Heuer (Germany), Dennis Jespersen, and L. F. Meyers. Michael Doob took A as in Solution VII and B to be the set of natural numbers divisible by no more than four distinct primes. Then $d(B) = 0$ (see I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, Second Edition, Wiley, New York, 1966, Theorem 11.8, p. 255) but $A + B$ has finite complement (see Wang, *On the representation of a large integer as the sum of a prime and an almost prime*, Sci. Sinica 11 (1962), 1033–1054). K. C. Cheung (Hong Kong) claims that Hua Lo-Keng has shown that all but a finite number of even integers can be expressed as a sum of two primes (cf. Solution VII) but gives no reference. Roy Ryden observes that E 2465 can be found as Problem 6, p. 262, of Niven and Zuckerman, *op. cit.* No solution is offered by them.

$d(A) = d(B) = 0$, yet $d(AB) = 1$

E 2466 [1974, 282]. *Proposed by Claude Anderson, University of California at Berkeley*

With the notation of the previous problem, show that it is possible to have

$$d(A) = d(B) = 0, \text{ and } d(AB) = 1,$$

where $AB = \{ab : a \in A, b \in B\}$.

Solution by Neal Felsinger, Hartford, Connecticut. Let P denote the set of all primes and let Q and R partition P such that the infinite products $\prod_{p \in Q}(1 - p^{-1})$ and $\prod_{p \in R}(1 - p^{-1})$ diverge to 0; this is equivalent to the assumption that the infinite series $\sum_{p \in Q} p^{-1}$ and $\sum_{p \in R} p^{-1}$ both diverge. (For example, we could take Q to consist of 2 and all primes of the form $4k + 1$ and R to consist of all primes of

the form $4k+3$, or we could take Q to consist of $\{p_1, p_3, p_5, \cdots\}$ and R to consist of $\{p_2, p_4, p_6, \cdots\}$ when we list the primes in their natural order: $p_1 < p_2 < p_3 < \cdots$.) Let $A = \{n \in N : p \mid n \text{ only if } p \in Q\}$ and $B = \{n \in N : p \mid n \text{ only if } p \in R\}$; equivalently $n \in A$ if and only if $(n,p) = 1$ for every $p \in R$ and $n \in B$ if and only if $(n,p) = 1$ for every $p \in Q$. Write $R = \{r_1, r_2, \cdots\}$ and for $m = 1, 2, \cdots$ set $A_m = \{n \in N : (n, r_i) = 1$ for $i = 1, 2, \cdots, m\}$. Then $A = \bigcap_{m=1}^{\infty} A_m$ and $d(A_m) = \prod_{i=1}^{m}(1 - r_i^{-1})$. Since $d(A_m) \to 0$ by assumption, we have that $d(A) = 0$. Similarly $d(B) = 0$. But obviously $AB = N$ so that $d(AB) = 1$.

Also solved by Robert Breusch (New Zealand), L. Carlitz, L. J. Dickson (Australia), John Gill, Ellen Hertz, O. P. Lossers (Netherlands), L. E. Mattics, J. W. McHutchion, William Nuesslein, James Nyman, Roy Ryden & Marshall Ruchte, T. Šalát (Czechoslovakia), David Singmaster (England), Wolfe Snow, Joel Spencer, Temple University Problem Solving Group, and the proposer.

Polynomial Approximations to Exponential Functions

E 2467 [1974, 282]. *Proposed by Bruce Reznick and Michael Yoder, Jet Propulsion Laboratory*

Suppose that $a \geqq 3$ and let P_n be an nth degree polynomial with real coefficients. Prove that

$$\max |a^i - P_n(i)| \geqq 1,$$

the maximum being taken over all integers i satisfying $0 \leqq i \leqq n+1$.

I. *Solution by Daniel Gallin, University of San Francisco, and Jacob Sturm, Weizmann Institute of Science, Israel (independently).* We use induction on the degree n of P_n. If $n = 0$, then $P_n(x) = c$, a constant, and if the assertion were to fail we would have both $|1 - c| < 1$ and $|a - c| < 1$. This would imply $|a - 1| \leqq |a - c| + |c - 1| < 1 + 1 = 2$, contradicting the assumption $a \geqq 3$. Now assume the problem is true for all $k \leqq n - 1$ and let $P_n(x)$ be given of degree n. Define

$$Q(x) = \frac{P_n(x+1) - P_n(x)}{a-1}.$$

Then, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} |a^i - Q(i)| &= \frac{|a^{i+1} - a^i - P_n(i+1) + P_n(i)|}{a-1} \\ &\leqq \frac{|a^{i+1} - P_n(i+1)|}{a-1} + \frac{|a^i - P_n(i)|}{a-1} \\ &\leqq \frac{2}{a-1} \max\{|a^i - P_n(i)| : i = 0, 1, \cdots, n+1\} \\ &\leqq \max\{|a^i - P_n(i)| : i = 0, 1, \cdots, n+1\}. \end{aligned}$$

Since the degree of Q is less than the degree of P_n (in fact, $\deg Q = n-1$), the inductive assumption applies and we are done.

II. *Solution by C. P. McCarty, La Salle College.* Let $P_n(x) = a_n x^n + \cdots + a_1 x + a_0$, and suppose to the contrary that $|a^i - P_n(i)| < 1$ for $i = 0, 1, \cdots, n+1$; this is equivalent to the assumption that $-1 < a^i - a_n i^n - \cdots - a_1 i - a_0 < 1$ for $i = 0, 1, \cdots, n+1$. Multiplying the ith of these inequalities by $(-1)^i \binom{n+1}{i}$ and summing, we obtain

$$-\sum_{i=0}^{n+1}\binom{n+1}{i} < \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}a^i - \sum_{k=0}^{n} a_k\left\{\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}i^k\right\}$$
$$< \sum_{i=0}^{n+1}\binom{n+1}{i}.$$

The term in braces vanishes for $k = 0, 1, \cdots, n$ (see W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd Ed., Wiley, New York, 1968, p. 65, or H.W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1959, Formula 1.13). Thus the above inequality reduces to $-2^{n+1} < (1-a)^{n+1} < 2^{n+1}$ or $|a-1| < 2$, contradicting the assumption that $a \geqq 3$.

III. *Solution by D. M. Bloom, Brooklyn College.* Write $P_n(i) = a^i + c_i$ for $i = 0, 1, \cdots, n+1$. If S is the sequence $P_n(0), P_n(1), \cdots, P_n(n+1)$, then the sequence of $(n+1)$st differences of S consists of the single term

$$(*) \qquad (a-1)^{n+1} + \sum_{i=0}^{n+1}(-1)^{n+1-i}\binom{n+1}{i}c_i,$$

which must be 0 since P_n is a polynomial of degree n. (In terms of the classical forward difference operator, (*) is $\Delta^{n+1}P_n(0)$.) It follows that

$$(a-1)^{n+1} = \sum_{i=0}^{n+1}(-1)^{n-i}\binom{n+1}{i}c_i \leqq c\sum_{i=0}^{n+1}\binom{n+1}{i} = 2^{n+1}c,$$

where $c = \max\{|c_i| : i = 0, 1, \cdots, n+1\}$. We have established the stronger result that

$$c \geqq \left(\frac{a-1}{2}\right)^{n+1} \geqq 1.$$

We remark that our proof shows that equality (to 1) is possible only if $a = 3$ and $c_i = (-1)^{n-i}$ for $i = 0, 1, \cdots, n+1$. Conversely, there is a unique polynomial Q of degree not exceeding $n+1$ such that $Q(i) = 3^i + (-1)^{n-i}$ for $i = 0, 1, \cdots, n+1$. For this Q we have $\Delta^{n+1}Q(0) = 0$ implying that the degree of Q does not exceed n; since $\Delta^n Q(0) = 2^{n+1} \neq 0$, it follows that the degree of Q is precisely n and thus for $a = 3$, the lower bound of 1 is best possible.

Also solved by Robert Breusch (New Zealand), J. M. Brown & D. A. Voss, John Christopher, Richard Groeneveld, Harry Lass, O. P. Lossers (Netherlands), L. E. Mattics, William Nuesslein, Kenneth Schilling & Michael Schilling, Michael Skalsky, Wolfe Snow, Michael Steiner (Sweden), F. B. Strauss, and the proposers.

Editor's comment. Let Q_n denote the unique polynomial of degree n for which

$$\max\left\{\left|a^i - Q_n(i)\right| : i = 0,1,\cdots, n+1\right\} = 1,$$

the existence of which was shown in Solution III. Groeneveld and the Schillings have calculated the first few Q_n:

$$Q_0(x) = 2,\ Q_1(x) = 4x,\ Q_2(x) = 4x^2 - 4x + 2,\ Q_3(x) = (8/3)x^3 - 8x^2 + (28/3)x.$$

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before July 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6024. *Proposed by L. Kuipers, Southern Illinois University*

If α is rational and different from 0, and β is irrational, then the sequence $([n\alpha]n\beta)$, $n = 1,2,\cdots$ is uniformly distributed mod 1.

6025. *Proposed by S. F. Wong and B. B. Winter, University of Ottawa*

Let (X,d) be a metric space, T an arbitrary subset of X, and t an arbitrary element of T. As usual, $d(t,A) = \inf\{d(t,a) : a \in A\}$ is $-\infty$ if $A = \phi$; ∂T and T^c are, respectively, the boundary and the complement of T.

(a) Is it always true that $d(t,x) < d(t,\partial T)$ implies $x \in T$?

If not, find a condition on (X,d) which is necessary and sufficient for the validity of this implication.

(b) Is it always true that $d(t,\partial T) = d(t,T^c)$?

If not, find a condition on (X,d) which is necessary and sufficient for the validity of this equality.

6026. *Proposed by Fred Commoner, Cambridge, Massachusetts*

Prove the theorem: Let p be an odd prime. If G is a finite non-abelian group such that p is less than or equal to the least prime dividing $|G|$, then no automorphism of G can send more than $|G|/p$ elements of G to their inverses. There is a non-abelian group G of order p^3 and an automorphism of G sending exactly $|G|/p$ elements of G to their inverses.

6027*. *Proposed by Philip Hanser, Columbia University*

Let f be a continuous real function on R, the reals. Must there exist a strictly increasing function $g: R \to R$ such that $g \circ f$ is everywhere differentiable?

6028*. *Proposed by F. D. Hammer, Stockton State College, New Jersey*

Let Z be the set of all integers. Is there a polynomial in two variables with integral coefficients which is a bijection from Z $\times$ Z onto Z? If so, how many such polynomials are there?

6029. *Proposed by P. P. Carreras, University of Valencia, Spain*

Let $E[t]$ be a linear space provided with a separated locally convex topology t. Show that $E[t]$ is bornological if and only if every absolutely convex bornivorous and algebraically closed subset of $E[t]$ is a t-neighborhood of the origin. (For definitions, see J. Horváth, *Topological Vector Spaces and Distributions* I, Addison-Wesley, 1966, p. 220.)

SOLUTIONS OF ADVANCED PROBLEMS

A Subgroup of Multiplicative Functions

5945 [1973, 1147]. *Proposed by R. Sivaramakrishnan, Engineering College, Trichur, India*

Kesava Menon defines the norm $f^*(n)$ of a multiplicative function $f(n)$ by

$$f^*(n) = \sum_{d|n^2} f(n^2/d)\lambda(d)f(d),$$

in which $\lambda(n) = (-1)^{k(n)}$ where $k(n)$ represents the total number of prime factors of n, each being counted according to its multiplicity.

Characterize the class of multiplicative arithmetic functions $f(n)$ which satisfy $f^*(n) = [1/n]$, $[x]$ being the integral part of x.

Solution by Carl Pomerance, University of Georgia. Let M denote the group of multiplicative functions under convolution product, denoted $f*g$. Hence, the definition of f^* now reads $f^*(n) = (f*(\lambda f))(n^2)$ for every n. Throughout, p will denote a prime and k a natural number. Then, if $f \in M$, we have

$$(1)\qquad \begin{aligned}(f*(\lambda f))(p^{2k-1}) &= \sum_{i=0}^{2k-1} f(p^{2k-1-i})\lambda(p^i)f(p^i)\\ &= \sum_{i=0}^{2k-1} (-1)^i f(p^{2k-1-i})f(p^i) = 0\end{aligned}$$

for all p, k. Let I denote the identity in M, that is $I(n) = [1/n]$. We are interested in characterizing the set $A = \{f \in M : f^* = I\}$. But by definition, $f \in A$ if and only if

$$(2)\qquad (f*(\lambda f))(p^{2k}) = 0$$

for all p, k. Hence (1) and (2) imply that

$$A = \{f \in M\colon f*(\lambda f) = I\} = \{f \in M : f^{-1} = \lambda f\}.$$

Now λ is a totally multiplicative function satisfying $1/\lambda = \lambda$. Hence, if $f, g \in A$, then

$$(f*g^{-1})^{-1} = f^{-1}*g = (\lambda f)*\left(\frac{1}{\lambda}g^{-1}\right) = (\lambda f)*(\lambda g^{-1}) = \lambda(f*g^{-1}),$$

so that $f*g^{-1} \in A$. Hence A is a subgroup of M.

However, from this characterization it is not clear that A contains any functions other than I. We show now that A contains "many" functions. Indeed, define a multiplicative function f by first giving f arbitrary values on prime powers with odd exponents. Then inductively define

$$f(p^{2k}) = \tfrac{1}{2}(-1)^{k+1}f(p^k)^2 + \sum_{i=k+1}^{2k-1} (-1)^{i+1}f(p^i)f(p^{2k-i}) \tag{3}$$

for all p, k. But condition (3) is equivalent to condition (2), so that f is the general member of A.

It is clear from the argument given above that the transformation $f \to f^*$ from M to M is a homomorphism with kernel A. We might ask about the range of this map. But it is easy to show that this map is in fact an epimorphism.

Also solved by the proposer.

Sums of Powers of Roots

5947 [1974, 90]. *Proposed by C. H. Kimberling, University of Evansville*

If $a(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = \prod_{i=1}^n(x-\alpha_i)$ and $s_m = s_m[a(x)] = \sum_{i=1}^n \alpha_i^m$, then (1) find $A(x) = x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$ satisfying $s_m[A(x)] = -s_m[a(x)]$ for $m = 1, 2, \cdots, n$; and (2) find the characteristic polynomial $f(x)$ of

$$V_n = \begin{bmatrix} s_1 & -1 & 0 & \cdots & & 0 \\ s_2 & s_1 & -2 & \cdots & & 0 \\ s_3 & s_2 & s_1 & \cdots & & 0 \\ \vdots & & & & s_1 & 1-n \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1 \end{bmatrix}.$$

Solution by L. Carlitz, Duke University. Put

$$U_r = \begin{bmatrix} s_1 & 1 & 0 & \cdots & & 0 \\ s_2 & s_1 & 2 & \cdots & & 0 \\ s_3 & s_2 & s_1 & \cdots & & 0 \\ & & & & s_1 & r-1 \\ s_r & s_{r-1} & s_{r-2} & \cdots & s_2 & s_1 \end{bmatrix}$$

We shall make use of the following known result (e.g. D. E. Littlewood, *Theory of Group Characters*, Oxford, 1958, p. 83). Then

$$(-1)^r r!\, a_r = d(U_r), \qquad r!\, h_r = d(V_r),$$

where

$$\prod_{i=1}^{n} (1-\alpha_i z)^{-1} = \sum_{k=0}^{\infty} h_k z^k.$$

1. Since $s_m[A(x)] = -s_m[a(x)]$ for $m = 1,2,\cdots,n$, it follows that U_r becomes

$$(-1)^r V_r = \begin{bmatrix} -s_1 & 1 & 0 & \cdots & & 0 \\ -s_2 & -s_1 & 2 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdots & -s_1 & r-1 \\ -s_r & -s_{r-1} & -s_{r-2} & \cdots & -s_2 & -s_1 \end{bmatrix}.$$

Thus $A_r = h_r$ $(r = 1,2,\cdots,m)$, so that

$$A(x) = x^n + h_1 x^{n-1} + \cdots + h_n.$$

2. We shall show that

$$(*) \qquad d(V_n + xI) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} h_k x^{n-k} \qquad (h_0 = 1).$$

Expanding $d(V_n + xI)$ with respect to the last row, we get

$$\begin{aligned} d(V_n + xI) &= (S_1 + x)d(V_{n-1} + xI) + (n-1)S_2 d(V_{n-2} + xI) \\ &\quad + (n-1)(n-2)S_3 d(V_{n-3} + xI) + \cdots + (n-1)!\, S_n \\ &= xd(V_{n-1} + xI) + \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} S_j d(V_{n-j} + xI). \end{aligned}$$

Assume that (*) holds for the values $0,1,\cdots,n-1$. Then

$$\begin{aligned} d(V_n + xI) &= xd(V_{n-1} + xI) + \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} S_j \sum_{r=0}^{n-j} \frac{(n-j)!}{(n-j-r)!} h_r x^{n-j-r} \\ &= xd(V_{n-1} + xI) + \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} x^{n-k} \sum_{j=1}^{k} S_j h_{k-j} \\ &= \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k)!} h_k x^{n-k} + \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} k h_k x^{n-k} \\ &\qquad\qquad \left(\text{since } \textstyle\sum_{j=1}^{k} S_j h_{k-j} = k h_k\right) \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} h_k x^{n-k}. \end{aligned}$$

Since (*) holds for $n = 0$, the proof is complete. It now follows at once from (*) that

$$d(xI - V_n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} h_k x^{n-k}.$$

Also solved by the proposer, who gave the (equivalent) form for A_i:

$$A_i = (-1)^i \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & 1 \\ a_i & a_{i-1} & a_{i-2} & & \cdots & & a_1 \end{bmatrix}.$$

Relatively Prime Additive Sequences

5948 [1974, 90]. *Proposed by A. A. Mullin, U.S. Army Research Office, Arlington, Virginia*

Does there exist an integer-valued sequence g which is both additive and relatively prime? I.e., does there exist a g such that $g(a \cdot b) = g(a) + g(b)$, if $(a,b) = 1$ and $(g(a), g(b)) = 1$, if $a \neq b$, where (a,b) is the greatest common divisor of a and b? Clearly, there exists a nonconstant integer-valued sequence which is both multiplicative and relatively prime.

Solution by William Margolis, College of William and Mary. A simple argument points out the impossibility of g. Consider three distinct primes a, b, c. By hypothesis the sequence

$$g(a), g(b), g(c), g(ab), g(ac), g(bc)$$

is pairwise relatively prime. Thus, at most one term, say $g(c)$, or $g(bc)$ is even.

We then would have $g(a), g(b), g(a) + g(b) = g(ab)$ all odd, which would be very odd indeed.

Also solved by David Bienenfeld (Israel), Ezra Brown, R. E. Dressler, H. M. Edgar, Paul Erdös, W. J. Gorman, Jr., Karl Heuer (W. Germany), Albert Leisinger & Paul Mason, R. B. Levow, O. P. Lossers (Netherlands), L. E. Mattics, J. E. Nyman, J. L. Selfridge, David Singmaster (England), Theresa Vaughan, and the proposer.

Note. Nyman observes that all multiplicative relative prime sequences are restricted to ± 1.

A Well-Poised Hypergeometric Series

5950 [1974, 90]. *Proposed by M. G. Glasser, Battelle Memorial Institute*
Sum the series

$$S = \frac{1}{ab} - \frac{(a+b+2)}{(a+1)(b+1)} + \sum_{n=2}^{\infty} (-1)^n \frac{(a+b+1)\cdots(a+b+n-1)(a+b+2n)}{n!(a+n)(b+n)}.$$

I. *Solution by B. L. R. Shawyer, University of Western Ontario.* Let $S = \sum_{n=0}^{\infty}(-1)^n c_n$. Since $c_n \sim Kn^{a+b-2}$ it is necessary for convergence that $a + b < 2$. For such a and b, c_n is eventually of constant sign and tends monotonically to zero, so that $\sum_{n=0}^{\infty}(-1)^n c_n$ is convergent for $a + b < 2$. Suppose that $a > 0$ and $b > 0$. Then

$$\begin{aligned} B(a,b) &= \int_0^1 (t^{a-1} + t^{b-1})(1+t)^{-a-b}\,dt \\ &= \int_0^1 \sum_{n=0}^{\infty}(-1)^n t^n \binom{a+b+n-1}{a} (t^{a-1} + t^{b-1})dt \\ &= \sum_{n=0}^{\infty}(-1)^n \binom{a+b+n-1}{n}\left(\frac{1}{a+n} + \frac{1}{b+n}\right) = (a+b)S, \end{aligned}$$

whence

$$S = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b+1)}.$$

This formula holds true for all $\operatorname{Re}(a+b) < 2$ by analytic continuation.

II. *Solution by G. E. Andrews, Pennsylvania State University.* The series in question is a well-posed hypergeometric series. If $(a)_n = a(a+1)\cdots(a+n-1)$, then

$$\begin{aligned} S &= \frac{1}{ab} \sum_{n \geq 0} \frac{(a+b)_n(a+b+2n)(a)_n(b)_n(-1)^n}{n!(a+b)(a+1)_n(b+1)_n} \\ &= \frac{1}{ab}\,{}_4F_3\left[\begin{matrix} a+b, \tfrac{1}{2}(a+b)+1, a, & b; & -1 \\ \tfrac{1}{2}(a+b), & b+1, & a+1 \end{matrix}\right] \\ &= \frac{1}{ab}\,\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(1+a+b)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a+b)}. \end{aligned}$$

(by (III. 10), p. 243, Ref. 1). Furthermore we note (Ref. 1, p. 45) that the series converges for $\operatorname{Re}(1-a-b) > -1$.

We note that a q-analog of this sum exists. Namely if

$$[a]_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \text{ then}$$

$$S_q = \frac{1}{(1-q^a)(1-q^b)} - \frac{(1-q^{a+b+2})q}{(1-q^{b+1})(1-q^{a+1})(1-q)}$$
$$+ \sum_{n=2}^{\infty} \frac{(-1)^n q^{(n^2+n)/2}(1-q^{a+b+1})\cdots(1-q^{a+b+n-1})(1-q^{a+b+2n})}{[q]_n(1-q^{a+n})(1-q^{b+n})}$$
$$= \frac{1}{(1-q^a)(1-q^b)} \lim_{d\to\infty} {}_6\phi_5 \left[\begin{matrix} q^{a+b}, -q^{1+\frac{1}{2}(a+b)}, q^{1+\frac{1}{2}(a+b)}, q^a, q^b, d; q, \frac{q}{d}, \\ -q^{\frac{1}{2}(a+b)}, q^{\frac{1}{2}(a+b)}, q^{b+1}, q^{a+1}, q^{a+b+1}, d^{-1} \end{matrix}\right]$$
$$= \prod_{n\geq 1} \frac{(1-q^{a+b+n})(1-q^n)}{(1-q^{a+n})(1-q^{b+n})}$$

(by Eq. (3, 3.1.3), p. 96, Ref. 1).

Reference

1. L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.

Also solved by Richard Askey, L. Carlitz, George Gasper, O. G. Ruehr, David Zeitlin, and the proposer.

On $Q \to Q$ Differentiable Functions

5955. *Proposed by F. D. Hammer, Berkeley, California*

Is there a differentiable function which takes rationals into rationals but whose derivative takes rationals into irrationals?

Solution by William Knight, University of Toronto. Such functions exist and an example is

$$f(x) = \sum_{n=0}^{\infty} g(n!x)/(n!)^2, \tag{1}$$

where $g(y)$ is the periodic function of period one defined on $[-\frac{1}{2}, \frac{1}{2}]$ by

$$g(y) = y(1-4y^2). \tag{2}$$

The function g vanishes at all integers and has a continuous derivative which is unity at all integers.

For any rational x, the series (1) has at most finitely many non-zero terms and they are rational.

The formal derivative of (1) converges uniformly and absolutely and therefore converges to the derivative of f. For any rational x, the derivative of the series (1) is the same as the series for e,

$$(3) \qquad \sum_{n=0}^{\infty} 1/n!,$$

save for at most finitely many terms which are rational. Thus for rational x, the derivative of f is e plus some rational number.

Also solved by Robin Ault, D. Borwein & P. B. Borwein, P. D. Humke, D. J. Newman, Jonathan Rosenberg, J. W. Shaw, Jr., S. J. Sidney, Dan Simchoni (Israel), and Oto Strauch (Czechoslovakia).

Notes. (1) Simchoni proves the following extension of the problem: *Let X stand for either the real line or the complex plane. Let D be a countable subset of X and R any subset of X such that both R and $R\backslash X$ are dense in X. Then, there exists an entire function f (a complex differentiable function defined on all points of the complex plane), satisfying $f: D \rightarrow R, f': D — X\backslash R$. Moreover, if X is the real line then f restricted to X, is real valued.*

(2) Rosenberg refers us to several papers which treat questions of which Problem 5955 is a variation:

(i) K. F. Barth and W. J. Schneider, Entire functions mapping countable dense subsets of the reals onto each other monotonically, J. London Math. Soc. **2** (1970), 620–626.

(ii) ———, Entire functions mapping arbitrary countable dense sets and their complements onto each other, J. London Math. Soc. **4** (1971), 482–488.

(iii) W. D. Maurer, Conformal equivalence of countable dense sets, Proc. Amer. Math. Soc. **18** (1967), 269–270.

(iv) B. H. Neumann and R. Rado, Monotone functions mapping the set of rational numbers onto itself, J. Austral. Math. Soc. **3** (1963), 282–287.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

FILMS

In this and the next few issues of the MONTHLY we present a series of reviews* of the films produced by the College Geometry Project under the direction of Dr. Seymour Schuster of Carleton College.

* Reviews solicited and edited by J. Arthur Seebach, Jr.

The College Geometry Project, funded by the National Science Foundation, was set up at the University of Minnesota to produce written and visual materials for college-level geometry courses for prospective high school teachers. Among the charges of the project was that of experimenting in the production of films. The twelve films produced by the project represent experiments in animation and other special effects, format, pacing and sound. While the primary target audience was the class of prospective high school teachers, the films were presumably designed to be shown to both more and less advanced audiences with only slight accommodation asked of the viewers. (By their very nature, films can generally be shown to, and enjoyed by, viewers outside the primary target audience.)

For each of the films there is a two-page brochure, suitable for posting as promotional material, which gives mathematical background, bibliographical material, and a list of sources of the film.

These brochures do not come with rental films from various university film libraries. Brochures as well as films may be obtained from Mr. James P. Fitzwater, Educational Consultant, International Film Bureau, Inc. 332 South Michigan Avenue, Chicago, Illinois 60604.

These reviews come from various perspectives and taken as a whole furnish a review of the series as well as the individual films.

Caroms: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Chandler Davis. 16mm sound and color; $9\frac{1}{2}$ minutes. Available for rent or purchase from International Film Bureau — sale $145; rental $10. Also available for rent from numerous University Film Libraries.

Here is a uniquely effective film. Its startling simplicity distinguishes it from other mathematical films. At first viewing one could easily feel that the film was hastily put together. This is no doubt because in our search for excellence we are conditioned to look for an authoritative narrator who captivates the viewer with stylistic eloquence. Not so this film! The viewer is cast in the role of a casual passerby who overhears a mathematical conversation. No fanfare accompanies the title, but rather a sudden encounter with soft voices of two geometers discussing a problem and its solution. To enhance this informal setting, a hint of street traffic is heard in the background. There is no music, though (surely facetiously) credits are given for music. The visuals are animations of billiard balls bouncing off walls diagrammed as straight lines, hence the title "Caroms."

The theme of the film is the determination of shortest paths. Four successively more interesting problems in shortest paths are posed and solved neatly by reflections. The final problem is in effect Fagnano's problem of determining a triangle of minimum perimeter inscribed in a given acute-angled triangle.

The approach is heuristic and intuitively convincing, and leads the viewer to seek the technical details in the literature. An excellent set of notes accompanies the

film and provides precise bibliographic information to answer unanswered questions from the film and to provide other methods of solution. Intended for use for both high school and college audiences, this film eminently succeeds in its avowed prime purpose "to stimulate viewers into studying mathematical reflections and their properties."

A. G. FADELL, SUNY at Buffalo

Central Similarities: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Daniel Pedoe. 16mm sound and color; 10 minutes. Available for rent or purchase from International Film Bureau, Inc. — sale $145; rental $10. Also available for rent from numerous University Film Libraries.

The film presents a solution, using central similarities, to the problem of inscribing an ellipse in a given triangle, so that the ellipse will be tangent to the sides of the given triangle at its midpoints.

The film opens with a few quick examples, establishing a visual convention for similarities through the use of animation. It then continues with a tightly organized presentation of the steps leading to the construction of the inscribed ellipse.

To quote the film notes, "the pedagogical appeal of the argument is that it forces the viewer to apply a fairly large number of ideas from his mathematical background: similarity notions, inscription and circumscription of figures, medians, centroids, conics, families of parallel chords, and systems of linear equations." Unfortunately, there are few students who will be able to pull all these concepts from their memory with the speed and dexterity required by the film. The reasoning goes by very quickly, without enough pause to absorb each new concept or argument before the next section builds on it. A student who takes a few seconds to ponder some point will fall behind, and have no chance to catch up, since each idea is stated only once.

Perhaps film is not the medium to present such involved logical arguments, because each viewer must digest them at his own rate. This particular film is probably paced too fast for most people, including the reviewer. It could have been improved by a pause, with a still image on the screen, after each new idea or step. A teacher who plans to use it in a class should preview it ahead of time, and plan to show it at least twice.

Aside from the drawbacks of pacing, the film is technically excellent. The film makes very effective use of animation, with zooms, cross dissolves, and superimpositions.

N. L. MAX, Carnegie-Mellon University

Central Perspectivities: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Seymour Schuster. 16mm sound and color; 13 1/2 minutes. Available for rent or purchase from International Film Bureau, Inc. — sale $185; rental $10. Also available for rent from numerous University Film Libraries.

This film presents the elementary ideas of perspectivity and projectivity. The definitions are given; it is shown that two points and their images determine a unique perspectivity; and it is shown that there exists a projectivity that maps three points of one line onto three points of another. The fundamental theorem of projective geometry is alluded to but not stated. The film is slow paced and easily within the grasp of a beginning geometry student. While the animation technique is well suited to presenting the material, in the reviewer's opinion, the ideas covered are so elementary that they could be presented as effectively on a blackboard.

F. W. STEVENSON, University of Arizona

Orthogonal Projection: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Daniel Pedoe. 16mm sound and color; 13 minutes. Available for rent or purchase from International Film Bureau — sale $185; rental $10. Also available for rent from numerous University Film Libraries.

This little gem of a film opens with the camera looking up at the sun glinting through the leaves of a tree, and then moving down to catch the patterns which the shadows of the leaves make on the ground. The narrator, Daniel Pedoe, suggests that the audience think about casting shadows out of doors, when the sun is directly overhead. The simplicity and unpretentiousness of this opening quickly caught the interest of the class which was viewing the film, and it sets the stage for the whimsical little problem which the narrator proceeds to set up.

Commenting, after some preliminaries, that a triangle casts a triangular shadow, Professor Pedoe asks whether the triangle can be held in such a way that the shadow triangle will be equilateral.

The answer to this problem is generated by inscribing an ellipse which is tangent to the triangle at the midpoints of the sides, then tilting the ellipse until its shadow is a circle. The resulting shadow triangle will be equilateral.

The most important aspect of the film is not the problem, nor even its solution, but the way in which little pieces of information are coordinated to give new information and to suggest "next steps." Noting that circles cast elliptical shadows, it is natural to ask if an ellipse can cast a circular shadow. Noting that triangles can be circumscribed about ellipses, it is natural to ask if an ellipse can be inscribed in a triangle — in particular in a specified way. The most important consequence of this film is to

demonstrate the values of transforming one problem into another, and of raising questions in mathematics.

The film ends by giving examples of problems where a similar "transform approach" pays dividends, and by raising some of the questions needed to fill in the details. For other questions which could be raised, but aren't, a little brochure which accompanies the film is extremely helpful. In addition to pointing out questions which could be raised in a class, the brochure gives references to four geometry texts in which further interests may be pursued. By emphasizing the questioning nature of mathematics, this film stands a good chance of creating those interests.

In addition to everything else, the technical quality of the film is excellent. The photography is sharp, the animation is clear, the background colors are well-chosen, and the pace is judicious. It should suit a variety of audiences: for those with little mathematical interest, the film presents elegant mathematics in an aesthetically pleasing fashion; for those with greater mathematical interest, the film can be used as a springboard for deeper, more detailed mathematical discussion. It is a pleasure to express unqualified enjoyment of this work of art.

J. T. Wood, Colorado College

Geometric Vectors — Addition: A film produced by the College Geometry Project at the University of Minnesota. Mathematicians: William Moser and Seymour Schuster. 16mm sound and color; 17 minutes. Available for rent or purchase from International Film Bureau, Inc. — sale $225; rental $12.50. Also available for rent from numerous University Film Libraries.

The film begins by showing a moving particle. After a series of initial and final positions of the particle are indicated, a pair of such positions are connected by an arrow which denotes displacement from the initial to the final position. Displacement symbols or arrows are considered and differentiated from each other from the point of view of direction and length. The vector is defined as the set of equivalent arrows which is represented by one member of the set. Then the addition of vectors is defined and illustrated by skillful use of varicolored changing diagrams. The commutativity and associativity of vector addition are beautifully done. After presenting the concept of the zero vector and the opposites or additive inverses of vectors, the film reviews these concepts by verifying that it is dealing with a commutative group. It ends by saying that the material covered is the beginning of vector algebra which is important beyond the needs of mechanics. Thus the film is useful to both mathematics and physics students.

This reviewer who has spent not a little time in art museums is gratified by the excellent use of colors and dynamic diagramming to illuminate the concepts. The script is very lucid and just about perfect.

S. Birnbaum, Bronx Community College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(14-17), P, L**. *Fundamentals of Mathematics, 3 Volumes.* Ed: H. Behnke, *et al.* Trans: S.H. Gould. MIT Pr, 1974. *V. I: Foundations of Mathematics, The Real Number System and Algebra,* x + 549 pp; *V. II: Geometry,* xi + 685 pp; *V. III: Analysis,* xiii + 541 pp, $40 set. A masterful, unparalleled exposition of the classical core of mathematics written for college teachers and industrial and government mathematicians. Each topic is treated in sufficient depth to illustrate basic techniques and survey major results, but not in so much detail as to impede the flow of ideas. Initiated under the auspices of the International Commission for Mathematical Instruction, prepared by dozens of authors, and reviewed during several working conferences, the entire work reflects an uncommonly high level of mathematical exposition. Unfortunately, the original writing was done nearly 20 years ago, so the emphasis and scope of the treatises are dated: there is no mention, for instance, of mathematical programming, numerical analysis, statistics or any computer-related mathematics. Nevertheless, as a primer of classical topics, it is without peer. LAS

GENERAL, P. *Mathematics in Today's World.* Science Council of Canada (150 Kent St., 7th Floor, Ottawa, Ont., K1P 5P4), 1974, xxv + 360 pp, (P). Proceedings of three one-day conferences. Part of the Science Council of Canada's Mathematics Study: Mathematics and Policy Planning, Mathematical Statistics and the Environment, Mathematics and Technology. Includes much interesting information and opinions not readily available in other sources. "A book that is meant for browsing." LAS

GENERAL, S(15-18), P. *Was ist Mathematik?* Richard Courant, Herbert Robbins. Springer-Verlag, 1973, xx + 399 pp, $16.30 (P). A third edition, basically unchanged from the second German edition, of the German translation of the famous *What is Mathematics,* Oxford, 1941. JAS

GENERAL, S(13-16), L. *Handbook of Mathematical Formulas.* Hans-Jochen Bartsch. Trans: Herbert Liebscher. Acad Pr, 1974, 524 pp, $9.50. Ranges from arithmetic through advanced calculus, probability, logic and linear optimization. Formulas, explanations and, occasionally, worked examples. No tables of numerical values. LAS

GENERAL, T(15-16: 1, 2). *The Anatomy of Mathematics, Second Edition.* R.B. Kershner, L.R. Wilcox. Ronald Pr, 1974, xiii + 485 pp, $12.50. An introduction to the axiomatic method, number systems, groups, and fields. Only minor changes from the 1950 first edition. FLW

GENERAL, T(13-14: 1). *Man's Mathematical Models.* Bill Williams, Gwen Crotts. Nelson-Hall, 1974, xiv + 370 pp, $11.95. For liberal arts students or teachers. Careful consideration of the language of sets, logic, relations, functions, groups, rings, and fields. FLW

GENERAL, S*, L*. *The Art of Problem-Solving.* Stanley Moses. Transworld Pub, 1974, 183 pp, (P). Written primarily for students, the book analyses the abilities required at various levels of problem-solving, outlines strategies, discusses the applications of observation, intuition and analogy to the solution of problems. Contains a useful section on various methods of proof. TAV

GENERAL, S(13). *Exploring Mathematics on Your Own, V. 1-19.* Donovan A. Johnson, William H. Glenn, M. Scott Norton, C.D.H. Cooper. Distr: Transatlantic Arts. 64 pp, $1.95 @ (P). A series of brief paperback reprints (mostly without copyright information or dates) on high school supplementary topics such as "The World of Statistics", "Topology", and "Computer Programming." The material shows its age, but remains on the whole challenging and interesting. Much of the material is similar to that contained in a single-volume 1972 Dover reprint of the same title by the first two authors (TR, June/July, 1973). The review set was missing six of the 19 volumes. LAS

GENERAL, S. *Creative Constructions.* Dale Seymour, Reuben Schadler. Creat Pub, 1974, 62 pp, $2.50 (P). Illustrations and worksheets for various black and white designs based on circles and regular polygons. Intended for junior high math enrichment. LAS

BASIC, T(13: 1). *Elementary Algebra.* Marshall Fraser. Page Ficklin, 1974, 358 pp, $5.50 (P). Assumes no prior knowledge of algebra. Includes algebraic expressions, linear equations and inequalities, graphing, systems of equations, exponents, factoring, fractions, radicals, and quadratic equations. Contains many interesting applications which appear as problems and exercises. CEC

BASIC, S(13). *Basic Mathematics for the Physical Sciences.* Haym Kruglak, John T. Moore. McGraw, 1963, vii + 354 pp, $5.95 (P). Intended as a reference or review work for students in elementary physical science courses. Chapters, with diagnostic pretests, cover arithmetic, algebra, geometry, trigonometry reasonably well with very brief chapters on calculus (23 pages), measurement, and graphical analysis of experimental data. Appendices. TAV

BASIC, T(13: 1). *Computational Skills with Applications.* Katherine Walker Bell, Reta G. Parrish. Heath, 1975, xv + 475 pp, $6.95 (P). Very elementary mathematics with modules and units for self-paced learning. LLK

BASIC, T(13: 1). *Elements of Algebra, Second Edition.* Francis J. Mueller. P-H, 1975, xii + 436 pp, $9.95. Pre-chapter quizzes are an added feature in this edition (first edition, TR January, 1970). Other revision includes a rewriting of the first chapter, "From Arithmetic to Algebra," and the chapter on functions and relations. LLK

PRECALCULUS, T(13: 1). *Intermediate Algebra.* Marshall Fraser. Freel, 1973, x + 485 pp, $7.50 (P). Includes number systems, operations, equations and inequalities in one and two variables, functions, graphs, logarithms, sequences, counting, and probability. Consists primarily of brief explanations, worked examples, and long lists of drill exercises. There are some splendid word problems selected from ancient sources. CEC

PRECALCULUS, T(13: 1). *Trigonometry: A Unitized Approach.* Reuben W. Farley, *et al.* P-H, 1975, viii + 344 pp, $9.25 (P). Sixteen small units grouped into five blocks. An instructor's manual contains three separate forms for each block test and the final examination. The manual also contains answers to unit tests which are in the text for the students. Audio tapes are available but were not reviewed. LLK

PRECALCULUS, T?(13: 1), S*. *Schaum's Outline of Theory and Problems of Basic Mathematics with Applications to Science and Technology.* Haym Kruglak, John T. Moore. McGraw, 1973, 341 pp, $3.95 (P). In typical format, this volume covers measurement, decimals, percentages, most topics in high school algebra, geometry, and trigonometry. A valuable review source, but an unlikely text for a precalculus course due to the lack of textual material. TAV

PRECALCULUS, T(13: 1). *Modern College Algebra.* Herman R. Hyatt, James N. Hardesty. Scott F, 1975, 374 pp, $9.95. Many examples, supplementary sections, graded exercises (A and B) and a few "C" exercises for use with a computer, where applicable. LLK

EDUCATION, T(15-16: 1), P. *Die Neue Mathematik für Lehrer und Studenten, B. 2.* Heinz Griesel. Hermann Schroedel, 1973, 239 pp, DM 17,80. New mathematics at its worst (or best) for teacher training courses where the language is German. No exercises. JAS

HISTORY, P*, L***. *Einstein, Hilbert, and The Theory of Gravitation: Historical Origins of General Relativity Theory.* Jagdish Mehra. Reidel, 1974, viii + 88 pp, $10 (P). A meticulous account of the intellectual struggles and seminal papers of 1907-1919 in which both Einstein and Hilbert independently and simultaneously arrived at the gravitational equations of general relativity. Although Hilbert's derivation of these equations is widely unrecognized and was dismissed by Einstein as the "pretension of a superman", he did in many ways surpass Einstein in the generality and rigor of his argument. LAS

HISTORY, P, L. *Aristotle, Galileo, and The Tower of Pisa.* Lane Cooper. Kennikat Pr, 1972, 102 pp, $6.75. A critical exegesis of sources concerning the historical basis of the widely held belief that Galileo revolutionized science by overthrowing unfounded Aristotelean claims concerning the behavior of falling bodies. Extensive original passages (mostly in Greek and Latin) comprise half the book as documentation for those who wish to check the translations which are often very subtle.LAS

HISTORY, P, L*. *George Polya: Collected Papers.* Ed: R.P. Boas. MIT Pr, 1974. *V. I: Singularities of Analytic Functions*, xiv + 808 pp, $30; *V. II: Location of Zeros*, x + 444 pp, $22.50. Each volume contains selected reprints with brief paper-by-paper commentaries, together with a complete Polya bibliography. Part of MIT's *Mathematicians of our Time* series. LAS

HISTORY, S**, P, L*. *Evolution of Mathematical Concepts: An Elementary Study.* Raymond L. Wilder. Halsted Pr, 1973, xx + 216 pp, $2.95 (P). Reprint in a popular paperback format of the 1968 edition. An original and noteworthy volume by a master expositor offering a unique cultural anthropology of mathematics for the educated layman and the interested professional. LAS

FOUNDATIONS, T(15-17: 1). *Sets, Logic, and Axiomatic Theories, Second Edition.* Robert R. Stoll. Freeman, 1974, xi + 233 pp, $9.50. A new edition of a highly regarded text. New material includes sections on the axiom of choice, proofs by induction and an appendix on recursive functions. The first edition was published in paperback and was considerably less expensive. CEC

FOUNDATIONS, P**, L*. *Proceedings of the Tarski Symposium.* Ed: Leon Henkin. Proc. of Symp. in Pure Math., V. XXV. AMS, 1974, xx + 498 pp, $40. A stellar collection of survey papers ranging over all areas in which Tarski worked. LAS

FOUNDATIONS, P. *Logic and Arithmetic: Natural Numbers*. David Bostock. Oxford U Pr, 1974, x + 291 pp, $19.25. First part of a two-part work expounding a logicism cleansed of difficulties. Author claims that "the logicist programme, as Frege conceived it, can indeed be completed--but only if we abandon Frege's own conception of numbers as objects" (p. vi). The mechanism for fulfilling the program is the interpretation of the natural numbers as numerical quantifiers and the development of a new logic of the latter. PJC

NUMBER THEORY, P. *Lecture Notes in Mathematics-402: Nombres Transcendants*. Michael Waldschmidt. Springer-Verlag, 1974, viii + 277 pp, $10.30 (P). Notes of an introductory course on transcendental numbers concentrating on the study of the values of the exponential function. The notes are self-contained, clearly written and include several well-conceived exercises. The author discusses the Gelfond-Schneider theorems, the Hermite-Lindemann theorem, various types of transcendence, Gelfond's criterion, zeroes of exponential polynomials, and Baber's generalization of the Gelfond-Schneider theorem. SG

NUMBER THEORY, P. *On the General Rogers-Ramanujan Theorem*. George E. Andrews. Memoirs No. 152. AMS, 1974, 86 pp, $3.10 (P). This paper is devoted to proving a general partition theorem that asserts the identity of two three-parameter partition functions. It gives the strongest known generalization of the Rogers-Ramanujan Identities. CEC

LINEAR ALGEBRA, T(14: 1). *Linear Algebra with Linear Differential Equations*. Franklin Lowenthal. Wiley, 1975, xi + 305 pp, $12.95. Designed as a first course in linear algebra to follow calculus. Thus differential equations can be used as examples where applicable. There is a short final chapter on applications of linear algebra in the solution of differential equations. LLK

ALGEBRA, T(17-18: 1), L. *Commutative Rings, Revised Edition*. Irving Kaplansky. U of Chicago Pr, 1974, ix + 182 pp, $9.75. An edition made new primarily by improvements in the exercises, by the addition of two pages of notes showing where "to appear" articles appeared, and by giving further information about points in the text which were obscure or where recent advances have been made. (First edition reviewed in January, 1972.) JAS

ALGEBRA, P, L. *Finite Rings with Identity*. Bernard R. McDonald. Pure and Appl. Math., V. 28. Dekker, 1974, ix + 429 pp, $27.50. An up-to-date survey of finite rings with identity intended both for experts and for researchers in discrete algebraic modeling (combinatorics, finite geometry, coding, finite linear sequential machines). Among the topics covered are finite fields, finite commutative rings, finite simple rings, matrix rings, modules over a finite ring, polynomial rings, Galois theory, structure of the group of units of a finite ring. A well-written, valuable reference. SG

ALGEBRA, P*. *Boolean Functions and Equations*. Sergiu Rudeanu. North-Holland, 1974, xix + 442 pp, $34.60. The book first handles the algebraic aspects of the theory of Boolean functions. The shorter second part deals with applications to switching theory. Some topics covered: basic properties of Boolean algebras and functions; orthonormal systems and solutions; symmetric systems and solutions; linear Boolean algebras; Boolean transformations; simple and parametric equations; inequalities, injectivity and monotonicity; Boolean calculus; switching equations. A very well-written, self-contained treatise which should become the standard reference in the field. SG

ALGEBRA, T(15-17: 2), L. *Basic Abstract Algebra.* Otto F.G. Schilling, W.Stephen Piper. Allyn, 1975, xiii + 394 pp, $14.95. Intended for introductory undergraduate and graduate courses in abstract algebra: integers, ring theory, linear algebra, polynomial rings, group theory, Sylow theorems, class equation, characters, Galois theory, cyclotomic fields, Hilbert's theorem 90, fundamental theorem of algebra; finite division rings. A well-written book but one which requires maturity on the part of an undergraduate. Not as encyclopedic as many graduate level texts in algebra. SG

ALGEBRA, P*. *Lectures on Linear Groups.* O.T. O'Meara. CBMS Reg. Conf. in Math., No. 22. AMS, 1974, vii + 87 pp, $4.10 (P). A well written and well organized collection of expository lecture notes whose goal is the isomorphism theory of linear groups over integral domains. Assumes a first course in algebra; whatever is needed from projective geometry is developed. Includes a good bibliography. CEC

ALGEBRA, P*. *Algebra of Polynomials.* Hans Lausch, Wilfried Nöbauer. North-Holland, 1973, xi + 322 pp, $19.20. A general approach to polynomials via universal algebra along with a representative sample of results on polynomials over special classes of algebraic structures. A coherent representation of material which previously had been treated to some extent in several papers. Includes an excellent bibliography. CEC

ALGEBRA, P. *Lecture Notes in Mathematics-372: Proceedings of the Second International Conference on The Theory of Groups.* Ed: M.F. Newman. Springer-Verlag, 1974, vii + 740 pp, $19.70 (P). 76 short papers from an August, 1973 conference in Canberra, Australia. Concludes with a brief collection of research problems. LAS

ALGEBRA, T(15: 1, 2), S. *Elements of Modern Abstract Algebra.* Kenneth S. Miller. Krieger, 1975, vii + 188 pp, $10.50. Unaltered reprint of a text originally published by Harper and Row in 1958. Straightforward exposition of groups, rings and ideals, and fields. LAS

ALGEBRA, P. *Einführung in die algebraische Geometrie.* B.L. van der Waerden. Grund. math. Wissenschaften, B. 51. Springer-Verlag, 1973, xi + 280 pp, $17.30. Basically a reprinting of the original 1939 work with the addition of two reprinted papers as appendices, one of which is historical in nature. JAS

ALGEBRA, P. *Introducere in coomologia algebrelor Lie.* Andrei Verona. Editura Academiei Romania, 1974, 284 pp, (P). The fundamental results concerning cohomologies of Lie algebras from an advanced seminar. JAS

FINITE MATHEMATICS, T(13-14: 1, 2). *Finite Mathematics and Calculus with Applications to Business and the Social Sciences.* Robert V. Hogg, *et al.* Cummings, 1974, xi + 402 pp, $10.95. Systems of linear equations, matrix algebra, linear programming, network models, descriptive statistics, probability, game theory, mathematics of finance, and a brief introduction to the calculus. FLW

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics: An Integrated Approach.* Harold D. Shane. Merrill, 1974, viii + 275 pp, $10.95. A nicely organized introduction to linear programming, probability, game theory, Markov chains, matrix algebra, and determinants. FLW

CALCULUS, T(14: 1), S. *Introduction to Vector Analysis, Third Edition.* Harry F. Davis, Arthur David Snider. Allyn, 1975, viii + 328 pp, $12.95. Changes from earlier editions are mainly concerned with optional sections on tensors and a major rewriting of the chapter on advanced topics. LLK

CALCULUS, T(13-14: 2, 3). *Calculus with Analytic Geometry.* Abraham Spitzbart. Scott F, 1975, 770 pp, $14.95. Covers the traditional material for science majors giving extra attention to definitions. Applications are the traditional ones from geometry and physics. It departs from the usual presentation with the introduction of vectors in the first chapter and its discussion of integrals. There is a liberal exposition of analytic geometry. MG

CALCULUS, T(15: 1, 2), L. *Mathematical Methods for the Physical Sciences: An Informal Treatment for Students of Physics and Engineering.* K.F. Riley. Cambridge U Pr, 1974, xv + 533 pp, $26; $8.95 (P). For that "applicable mathematics" course. Reviews calculus, then studies vectors, ordinary differential equations (including Fourier methods), partial differential equations, numerical methods, stationary value problems, matrices and tensors, complex variables. Very carefully written using a heuristic, physical point of view. Nice selection of exercises. DFA

CALCULUS, S(13), P. *Integration Theory: An Outline of a First Course.* J.G. Hocking, Dept. Mathematics, Michigan State U, East Lansing, MI. 47 pp, $1 (P). Notes addressed to calculus teachers developing Riemann integration theory in a manner that permits a proof of integrability of continuous functions without resort to uniform continuity. LAS

CALCULUS, T??(14). *Elements of Differential Calculus for Mathematics, Physics and Engineering Students.* B.S. Fadnis. Asia Pub, 1965, v + 296 pp, $12.50. Intended for students who have taken a course in elementary calculus, the book seems to be little more than a collection of topics that are being left out of many newer "intuitive" calculus texts, e.g., sequences, limits, continuity and series plus a collection of classical topics from analytic geometry such as envelopes and asymptotes. No bargain at this price. TAV

REAL ANALYSIS, T*(15-17: 1, 2). *Elementary Classical Analysis.* Jerrold E. Marsden. Freeman, 1974, xiv + 549 pp, $15. A text book for a one or two semester course in advanced calculus and introductory real analysis. It deals with calculus and Fourier series in Euclidean space. Each chapter is organized to deal with mastering concepts before attempting technical proofs. Lots of exercises of varying degrees of difficulty. An excellent text. CEC

REAL ANALYSIS, T(14-15: 1), S. *Advanced Real Calculus.* Kenneth S. Miller. Krieger, 1975, viii + 185 pp, $10.50. Unaltered reprint of an elementary real variable text first published in 1957 by Harper and Row. LAS

COMPLEX ANALYSIS, P. *Einführung in die Funktionentheorie mehrerer Veränderlicher.* H. Grauert, K. Fritzsche. Springer-Verlag, 1974, vi + 213 pp, $8.10 (P). A German equivalent of Gunning and Rossi with no exercises. JAS

COMPLEX ANALYSIS, T?(15). *Elementary Complex Variables.* W. Allen Smith. Merrill, 1974, x + 274 pp, $12.95. Contains some interesting examples, but cannot be recommended as a text. Differentials are used to prove the derivative formula for log, but are never discussed. $\partial/\partial z$ and $\partial/\partial \bar{z}$ are introduced as bona fide partial derivatives and are defined as operators only "for those bothered by the idea of z (or $\bar{z}$) varying while its conjugate is held constant." Definition of connected set is incorrectly explained. Treatment of curves is especially imprecise. RBK

COMPLEX ANALYSIS, P. *Metode Algebrice in Teoria Globala a Spatilor Complexe.* Constantin Banica, Octavian Stanasila. Editura Academiei Romania, 1974, 348 pp. An exposition of global results obtainable by using sheaf cohomology for complex manifolds. JAS

COMPLEX ANALYSIS, P. *Holomorphic Functions of Finite Order in Several Complex Variables.* Wilhelm Stoll. CBMS Reg. Conf. in Math., No. 21. AMS, 1974, x + 83 pp, $4.10 (P). An exposition of the construction of holomorphic functions with growth estimates to given zero sets. RBK

DIFFERENTIAL EQUATIONS, P. *Uniform Simplification in a Full Neighborhood of a Transition Point.* Yasutaka Sibuya. Memoirs No. 149. AMS, 1974, vi + 106 pp, $3.20 (P). Based on author's own MRC (U. of Wis., Madison) Report No. 1320 (1972-1973), this memoir establishes an asymptotic simplification valid uniformly near a transition point of order ≥ 3 of certain second order linear differential equations with a complex parameter. I-CH

DIFFERENTIAL EQUATIONS, T(18: 2), P*. *Stability of Solutions of Differential Equations in Banach Space.* Ju. L. Daleckii, M.G. Krein. Trans. Math. Mono., V. 43. AMS, 1974, vi + 386 pp, $36.40. A translation of a 1969 Russian book. Deals with differential equations for vector functions with values in infinite dimensional spaces. Requires knowledge of the geometry of Hilbert and Banach spaces and the theory of linear operators acting on them. Includes over 100 exercises and a sizable bibliography. CEC

DIFFERENTIAL EQUATIONS, P. *A Laplace Transform Calculus for Partial Differential Operators.* Thomas Donaldson. Memoirs No. 143. AMS, 1974, 166 pp, $3.60 (P). A general existence theory for a class of hypoelliptic linear partial differential boundary problems which contains the class of operators parabolic in the sense of Petrowski as a special case. No homogeneity restrictions on the polynomial forms associated with the operators. DFA

DIFFERENTIAL EQUATIONS, T(17-18), S, P. *Nonlinear Oscillations.* Nicholas Minorsky. Kreiger, 1974, xviii + 714 pp, $22.50. Unaltered reprint of the 1962 original edition: qualitative (topological) methods, quantitative methods, and applications. LAS

DIFFERENTIAL EQUATIONS, P. *Wybrane dzialy matematyki stosowanej.* Jerzy Gorski, *et al.* PWN, 1973, 316 pp, (P). A series of expository essays, all in Polish, on various topics closely related to differential equations. LAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-395: Numerische Behandlung nichtlinearer Integrodifferential und Differentialgleichungen.* R. Ansorge, W. Törnig. Springer-Verlag, 1974, vii + 313 pp, $11.50 (P). Proceedings of the Oberwolfach conference of December 1973. JAS

DIFFERENTIAL EQUATIONS, P. *Partielle Differentialgleichungen erster Ordnung.* Friedhelm Erwe, Ernst Peschl. Bibliographisches Inst, 1973, 133 pp, (P). Exposition for those with a background in analysis and in ordinary differential and integral equations. JAS

NUMERICAL ANALYSIS, S(17-18), P. *Methods for Solving Systems of Nonlinear Equations.* Werner C. Rheinboldt. CBMS Reg. Conf. in Appl. Math., No. 14. SIAM, 1974, ix + 104 pp, $7.40 (P). The model problems stated in the beginning chapter well motivate the topic: to compute solutions of non-linear equations. This monograph first surveys basic methods (including the old and natural iterative processes, of course), then gets

to some of the modern ones known as update methods or variable metric methods. Links to functional analysis, studies some search algorithms. Concludes with author's view of future outlook. I-CH

NUMERICAL ANALYSIS, T(16-18: 1, 2). *Numerisches Rechnen II.* Ben Noble. Bibliographisches Inst, 1973, 246 pp, (P). Part II of the German translation of the 1964 English original. JAS

FUNCTIONAL ANALYSIS, T(17-18: 1), S, P. *Numerical Solution of Integral Equations.* Ed: L.M. Delves, J. Walsh. Clarendon Pr, 1974, 339 pp, $14.50. A thoroughly edited three-part book based on a Liverpool-Manchester Summer School held in July 1973. Part I: preliminaries; Part II: numerical methods for Fredholm and Volterra equations, eigenvalue problems and integro-differential equations; Part III: applications. With summaries of recent research and a large number of references, this book is an up-to-date introduction to the subject of the title. I-CH

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-399: Functional Analysis and its Applications.* Ed: H.G. Garnir, K.R. Unni, J.H. Williamson. Springer-Verlag, 1974, 586 pp, $18.10 (P). Papers from a major January 1973 international conference held in Madras, India.LAS

FUNCTIONAL ANALYSIS, S(17-18), P. *Funktionalanalysis.* Erika Pflaumann, Heinz Unger. Bibliographisches Inst, 1974, 240 pp, (P). First of several planned volumes based on advanced courses given at the University of Bonn. This one covers basic topological and algebraic structures of function spaces. JAS

FUNCTIONAL ANALYSIS, S(17-18), P. *Analysis V: Funktionalanalysis und Integralgleichungen.* Erich Martensen. Bibliographisches Inst, 1972, 275 pp, (P). Functional analysis and integral equations with an eye to applications. JAS

FUNCTIONAL ANALYSIS, S(17-18), P. *Lecture Notes in Mathematics-394: Iterative Methods for the Solution of a Linear Operator Equation in Hilbert Space--A Survey.* Walter Mead Patterson, 3rd. Springer-Verlag, 1974, 183 pp, $8.20 (P). This expository survey covers the theoretical aspects of 1) iterative methods, 2) successive approximation methods and, 3) gradient methods for solving the linear operator equation $AX = Y$ in a Hilbert space. The ideas of many mathematicians are reproduced, and are appropriately referenced. The numerical or computational characteristics are left for the interested reader. The leisurely pace of the exposition is enjoyable. I-CH

FUNCTIONAL ANALYSIS, T(18: 1, 2). *Lectures in Functional Analysis and Operator Theory.* Sterling K. Berberian. Grad. Texts in Math., V. 15. Springer-Verlag, 1974, ix + 345 pp, $14.80. A graduate text written in an engaging style with clues to the interconnection of ideas. Covers topological groups and vector spaces; convexity; normed, Banach, and Hilbert spaces; category; Banach and C* algebras. Concluding chapter on applications begins with the "punch line" to the proof of Wiener's theorem which was introduced in Chapter 0: "Aperitif." Notes, bibliography, index. RBK

FUNCTIONAL ANALYSIS, P. *Monotonie: Lösbarkeit und Numerik bei Operatorgleichungen.* Erich Bohl. Tracts in Nat. Philo., V. 25. Springer-Verlag, 1974, ix + 255 pp, $26.20. A study of monotone and related topics of operators giving current results concerning existence and actual (approximate) computation of solutions. The first third of the book gives a thorough review of the prerequisites and classical theory. JAS

FUNCTIONAL ANALYSIS, P. *Spatii Liniare Topologice.* Romulus Cristescu. Editura Academiei Romania, 1974, 248 pp. An advanced exposition of the theory of topological vector spaces and operators with a large and very multi-lingual bibliography. JAS

FUNCTIONAL ANALYSIS, P. *Introducere în Analiza Funcţională.* N. Gheorghiu. Editura Academiei Romania, 1974, 231 pp. Basic functional analysis in both complex and real cases through an introduction to spectral theory. Designed mostly as a reference. JAS

OPTIMIZATION, T(14: 1). *Linear Programming in Industry, Theory and Applications: An Introduction, Fourth Revised and Enlarged Edition.* Sven Danø. Springer-Verlag, 1974, xii + 172 pp, $16 (P). A practical approach for a beginning text in operations research. Methods are introduced with examples and an appendix supplies proofs to important theorems. LLK

ANALYSIS, T(15: 2), S*, L. *Mathematical Analysis and Techniques.* A. Page. Oxford U Pr, 1974. *V. I:* viii + 252 pp; *V. II:* ix + 298 pp, $8 (P) each. These volumes contain the content that each undergraduate mathematics major should consider minimal. The presentation leans toward the techniques over theory, but sufficient arguments are included to illuminate the underlying structure. Numerous exercises and examples. Among the topics: Volume I: convergence of series and integrals, mean value theorems and consequences, multivariable calculus methods; Volume II: differential equations (ordinary and partial), Fourier series, matrices and vector fields, complex functions and integration. TAV

ANALYSIS, P. *Lecture Notes in Mathematics-400: A Crash Course on Kleinian Groups.* Ed: Lipman Bers, Irwin Kra. Springer-Verlag, 1974, 130 pp, $7.40 (P). Lectures for non-specialists, given at a special session at the January 1974 AMS meeting, offering an introductory survey of some topics important in the modern theory of Kleinian groups. CEC

ANALYSIS, T(17), P. *Introduction to the Theory and Application of the Laplace Transformation.* Gustav Doetsch. Trans: Walter Nader. Springer-Verlag, 1974, vii + 326 pp, $27.90. Emphasizes theoretical foundations and inserts applications only as they can be adequately handled. Includes the deduction of asymptotic expansions for the image function using the original one, and the deduction the other way around. Careful, readable exposition. Suitable for use by mathematicians, physicists, and engineers. DFA

ANALYSIS, T(18), P. *Treatise on Analysis, V. IV.* J. Dieudonné. Trans: I.G. Macdonald. Pure and Appl. Math., V. 10-IV. Acad Pr, 1974, xv + 444 pp, $34. Chapters XVIII-XX on differential systems, Lie groups and Riemannian geometry. Translated from the 1971 French original. Five more chapters to go. LAS

ANALYSIS, P. *Einige Klassen Singulärer Gleichungen.* Siegfried Prössdorf. Math. Reihe, B. 46. Birkhauser, 1974, xii + 353 pp, $25. A substantial work on integral equations. Possibly of more than average interest because of the author's contact with recent (up to 1972) Russian work. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-410: Séminaire Pierre LeLong (Analyse) Année 1972-1973.* Pierre LeLong. Springer-Verlag, 1974, vi + 181 pp, $7.40 (P).

ALGEBRAIC GEOMETRY, P, *Basic Algebraic Geometry.* I.R. Shafarevich. Trans: K.A. Hirsch. Grund. math. Wissenschaften, B. 213. Springer-Verlag, 1974, xv + 439 pp, $40.20. A coherent, self-contained, well-motivated introduction to modern algebraic geometry. Divided into three parts: algebraic varieties in a projective space; schemes and varieties; algebraic varieties over the complex numbers and complex analytic manifolds. No previous knowledge of algebraic geometry is assumed. Can serve as a text for beginners and as a reference for experts. Hundreds of exercises. A beautiful work. SG

ALGEBRAIC GEOMETRY, P, *Analytic Theory of Abelian Varieties.* H.P.F. Swinnerton-Dyer. London Math. Soc. Lect. Notes, No. 14. Cambridge U Pr, 1974, viii + 90 pp, $6 (P). Presents the analytic theory of abelian varieties necessary for reading Shimura's work; assumes basic complex analysis and Riemann surface theory. The author discusses necessary and sufficient conditions for an n-dimensional complex torus to admit a non-constant meromorphic function and related questions of projective embeddings. He concludes with the study of the endomorphism ring of an abelian manifold, and the duality theory of abelian manifolds. SG

GEOMETRY, T(14: 1), *Euclidean and Non-Euclidean Geometries: Development and History.* Marvin Jay Greenberg. Freeman, 1974, xi + 304 pp, $10.95. Easily readable, almost conversational, and laden with historical accounts, this text uses a modified version of Hilbert's axioms to develop results in neutral and hyperbolic geometry. Many of the numerous exercises develop results used later. Suitable for prospective secondary teachers, liberal arts students and mathematics majors. JNC

GEOMETRY, T*(16-18), S*, P*, L**, *A Comprehensive Introduction to Differential Geometry.* Michael Spivak. Publish or Perish, 1970. *V. 1:* x + 655 pp, $12.50 (P); *V. 2:* vi + 425 pp, $10.50 (P). The "Great American Differential Geometry Book," designed to introduce the fruits of modern thought while maintaining close ties to classical roots. See Extended Review, April 1973. LAS

GEOMETRY, T(16-17: 1, 2), L, *Eine Vorlesung über Differentialgeometrie.* Wilhelm Klingenberg. Springer-Verlag, 1973, x + 135 pp, $6.70 (P). Classical differential geometry of curves and surfaces, "strongly influenced" by the Chicago notes of S.S. Chern in 1954. JAS

GEOMETRY, S(14-15), L, *Grundlagen der Euklidischen Geometrie.* Herbert Meschkowski. Bibliographisches Inst, 1974, 231 pp, (P). Discusses axiom systems, models, order, Dedekind cuts, congruence, similarity, triangulation, polyhedra, area, volume, and projective geometry. Includes exercises after each chapter. RJ

GEOMETRY, P, *Minimal Varieties in Real and Complex Geometry.* H. Blaine Lawson, Jr. Pr U Montreal, 1974, 100 pp, $5 (P). A variety (= manifold) is minimal if the new curvature vector field is zero. These notes are applications of that idea to differential geometry. PJM

GEOMETRY, S(15), P, *The Seven Circles Theorem and Other New Theorems.* C.J.A. Evelyn, G.B. Money-Coutts, J.A. Tyrrell. Stacey International, 1974, viii + 68 pp, £2.80. A beautifully illustrated collection of theorems involving closed chains of six circles and extensions of Pascal's and Brianchon's theorems. JNC

TOPOLOGY, P, *On Closed 3-braids.* Kunio Murasugi. Memoirs No. 151. AMS, 1974, vi + 114 pp, $3.40 (P). The author investigates the relationships between the conjugate classes of 3-braids and the link types of their closure. SG

TOPOLOGY, T***(16-17: 1, 2), S*, L**. *Topology, A First Course.* James R. Munkres. P-H, 1975, xvi + 413 pp, $15.95. A very substantial book and an outstanding text. The claims of flexibility and high quality writing are fully justified. The mathematics is of equal quality and offers, finally, arguments more generally convincing than "to understand analysis" that topology is a subject of broad interest. The standard topics of point set topology including the Nagata-Smirnov metrization theorem and Ascoli's theorem are presented followed by enough homotopy theory to give proofs of the fundamental theorem of algebra and the Jordan curve theorem. Lots of good exercises and good supplementary problem sets introducing new topics illustrate the author's well justified assertion that "open-ended" exercises are essential to real learning. JAS

TOPOLOGY, P. *Studies in Topology.* Ed: Nick M. Stavrakas, Keith R. Allen. Acad Pr, 1975, xxii + 650 pp, $27.50. Proceedings of the annual spring conference on general topology, this 1974 one being held at the University of North Carolina at Charlotte. Special foci: shape theory and infinite dimensional topology. LAS

TOPOLOGY, S(15-16), P, L. *Nearness: A Better Approach to Continuity and Limits.* P. Cameron, J.G. Hocking, S.A. Naimpally, Dept. of Math. Sci., Lakehead U, Thunder Bay, Ontario P7B 5E1, Canada. 1973. *Part I:* 35 pp; *Part II:* 35 pp; $1 each (P). Notes addressed to teachers concerning the pedagogical simplifications of F. Riesz and Kuratowski's nearness concept: x is near a set A if $x \in \bar{A}$. Part I deals with elementary calculus, Part II with advanced courses, especially general topology. LAS

TOPOLOGY, P. *Topological Structures.* Ed: P.C. Baayen. Math. Centre Tracts, No. 52. Math Centrum, 1974, v + 175 pp, Dfl. 19 (P). Proceedings of a 1973 symposium convened to honor J. de Groot (1914-1972). Includes a survey of de Groot's topological work, several papers extending his work, and a collection of 100 open problems in infinite dimensional topology prepared by R.D. Anderson and N.S. Kroonenberg. LAS

TOPOLOGY, P*. *Smoothings of Piecewise Linear Manifolds.* Morris W. Hirsch, Barry Mazur. Annals of Math. Stud., No. 80. Princeton U Pr, 1974, ix + 132 pp, $6.50 (P). An answer to the question: is a given combinatorial manifold the domain of a smooth triangulation of some smooth manifold? The main result is that smoothings are completely classified by linearizations of the tangent bundle. JAS

TOPOLOGY, T(16-18: 1, 2), S, P. *Einführung in die Differentialtopologie.* Theodor Bröcker, Klaus Jänich. Springer-Verlag, 1973, 168 pp, $6.10 (P). Elementary geometric study of manifolds with many exercises and a reasonable index. Assumes only the basics of general topology and analysis. Many pictures and a chapter on dynamical systems. JAS

TOPOLOGY, P. *Classifying Spaces and Fibrations.* J. Peter May. Memoirs No. 155. AMS, 1974, xiii + 98 pp, $3.30 (P). Fibrations are defined by constraining the fibre to live in some fixed category. General results on classifying spaces and homological constructions are then possible, including a large variety of phenomena not previously encompassed by one theory. An excellent book. PJM

TOPOLOGY, P. *The Index Theorem and the Heat Equation.* Peter B. Gilkey. Publish or Perish, 1974, 125 pp, $5 (P). Deals with the proof of the index theorem for the De Rham, Dolbeault, and Signature complexes using heat equation methods for manifolds without boundary. The calculus of pseudo-differential operators and the classical Chern-Gauss-Bonnet formula are discussed in detail. Taken from lecture notes. Includes a bibliography. CEC

PROBABILITY, P, *Lecture Notes in Mathematics-390: Ecole d'Eté de Probabilités de Saint-Flour III-1973*. P.A. Meyer, P. Priouret, F. Spitzer. Springer-Verlag, 1974, vii + 189 pp, $8.20 (P). Three articles: "Transformations of Markov Processes," "Diffusion Processes and Stochastic Differential Equations," and "Introduction to Markov Processes with Parameters in Z_ν". PJM

PROBABILITY, P, *Stochastic Calculus and Stochastic Models*. E.J. McShane. Prob. and Math. Stat., No. 25. Acad Pr, 1974, x + 239 pp, $19.50. A carefully and concisely written development of a calculus based on two new integrals which the author dubs "belated" and "Ito-belated." Their purpose is to provide a computationally easy and theoretically powerful tool for description of stochastic processes which include random noise. TAV

PROBABILITY, P*, *The Theory of Stochastic Processes I*. I.I. Gihman, A. V. Skorohod. Grund. math. Wissenschaften, B. 210. Springer-Verlag, 1974, viii + 570 pp, $52.90. The first of a planned three-volume work intended for mathematicians interested in random processes. Presumes probability, measure theory, functional analysis. Beginning with a general treatment of random processes, this volume is devoted to the mathematical tools of the field, through measurable functions on Hilbert spaces. The treatment is extremely detailed and precise, as it should be at this price!TAV

PROBABILITY, P, *Ergodic Theory, Randomness, and Dynamical Systems*. Donald S. Ornstein. Math. Mono., No. 5. Yale U Pr, 1974, vii + 141 pp, $4.95 (P). A highly technical discussion of the recent developments in the theory of Bernoulli shifts and flows. Contains a complete discussion of the theorem: "Two Bernoulli shifts with the same entropy are isomorphic." TAV

PROBABILITY, P, *Diffusion Processes and their Sample Paths, Second Printing, Corrected*. K. Itô, H.P. McKean, Jr. Grund. math. Wissenschaften, B. 125. Springer-Verlag, 1974, xiv + 321 pp, $27.90. Devoted to stochastic processes associated with diffusion when the assumption of spatial homogeneity is dropped. Includes applications to Brownian motion and nonlinear diffusions. Much is developed from precise treatment of semigroups of transformations. Difficult for all but the experts to assimilate. TAV

STATISTICS, T(15-17), S, P, L**, *Lecture Notes in Economics and Mathematical Systems-100: Cluster Analysis: A Survey*. Benjamin S. Duran, Patrick L. Odell. Springer-Verlag, 1974, vi + 137 pp, $7.40 (P). The cluster problem is to determine an optimal partition of data into subsets (clusters) in such a way that elements assigned to one cluster are similar yet elements from different clusters are not similar. This expository monograph (one-fourth of which is a massive bibliography) surveys and systematizes the many widely varied clustering techniques. LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Marianne Gardner, Carleton; Ih-Ching Hsu, St. Olaf; Richard Jensen, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Pierre J. Malraison, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least four months before publication can take place.

PERSONAL ITEMS

California State University, Fresno: Assistant Professors H. S. Sun and H. B. Haslam have been promoted to Associate Professors.

Dartmouth College: Associate Professor R. A. Groeneveld, Iowa State University, has been appointed Visiting Associate Professor; Assistant Professors K. P. Bogart and S. J. Garland have been promoted to Associate Professors.

Duke University: Dr. Murray Cantor, California State College at Hayward, has been appointed Assistant Professor; Mr. August Lawrence, St. Mary's College, Raleigh, has been appointed Instructor; Professor Francis G. Dressel has retired with the title of Professor Emeritus.

Eastern Nazarene College: Dr. Sheldon Sickler, UCLA, has been appointed Associate Professor; Dr. J. J. McCloy has been appointed Chairman of the Department of Mathematics.

Florida International University: Assistant Professors Pamela A. Geisler, W. T. Kraynek, and A. C. Shershin have been promoted to Associate Professors.

Georgia Institute of Technology: Dr. M. B. Tamburro, UCLA, and Dr. J. C. Wiener, Emory University, have been appointed Assistant Professors; Drs. Catherine C. Aust and A. D. Sloan have been promoted to Assistant Professors.

Illinois State University: Dr. Earl Ecklund, Jr., University of Manitoba, has been appointed Assistant Professor; Assistant Professors Lynn Brown and Arnold Insel have been promoted to Associate Professors.

University of Michigan: Professor F. W. Gehring has been elected to foreign membership in the Finnish Academy of Sciences; Associate Professor H. L. Montgomery has been awarded the 1974 Salem Prize by the French Mathematical Society; Professors Carl H. Fischer and Kenneth B. Leisenring have retired with the title of Professor Emeritus.

Middle Tennessee State University: Associate Professor Joe Evans has been promoted to Professor; Assistant Professors Jim Lea and Samuel Truitt have been promoted to Associate Professors; Mr. William Patrick has been promoted from Instructor to Assistant Professor.

Mississippi University for Women: Professor D. A. King, Chairman of the Department of Mathematics, has been appointed Dean of the School of Arts and Science; Professor Carol Ottinger has been appointed Chairman of the Department of Mathematics.

University of Missouri, St. Louis: Assistant Professors W. L. McDaniel and Alan Schwartz have been promoted to Associate Professors.

Monmouth College: Associate Professor R. A. Kuntz has been appointed Chairman of the Department of Mathematics; Assistant Professor G. B. Swartz has been promoted to Associate Professor.

University of New Orleans: Dr. R. T. Jacob, Emory University, has been appointed Instructor; Assistant Professor Kuang-Ho Chen has been promoted to Associate Professor.

Northeast Missouri State University: Assistant Professors Ronald Knight and J. D. Flowers have been promoted to Associate Professors.

Norwich University: Assistant Professor E. D. True, Johnson State College, has been appointed Assistant Professor; Department Chairman J. J. Heed has been promoted from Associate Professor to Professor; Assistant Professor S. K. Ingram has been promoted to Assistant Dean.

SUNY— College at Brockport: Assistant Professor Norman Bloch has been promoted to Associate Professor; Instructor Dennis Martin has been promoted to Assistant Professor.

SUNY— College at New Paltz: Dr. H. W. Berkowitz, Sun Shipbuilding Company, has been appointed Assistant Professor; Dr. Paul Zuckerman, University of Rochester, has been appointed Assistant Professor; Assistant Professor David Clark has been promoted to Associate Professor.

Texas Tech University: Professor Emmett Hazelwood, Associate Professor Bob Parker, and Professor E. Richard Heineman have retired with the title of Professor Emeritus.

Vancouver Community College: Mr. James A. Moore, Chairman, Mathematics and Science Division, retired on December 31, 1974; Mr. Norman Barton, Chairman, Department of Mathematics, was appointed Chairman, Mathematics and Science Division, on January 1, 1975.

Virginia Polytechnic Institute and State University: Professor David Roselle, Louisiana State University, has been appointed Professor; Assistant Professor Robert Snider, Northwestern University, has been appointed Assistant Professor; Assistant Professor Harold Mick, Northern Illinois University, has been appointed Assistant Professor; Assistant Professor John Burns, Brown University, has been appointed Assistant Professor; Dr. James Thomson, University of North Carolina, has been appointed Assistant Professor.

Daniel H. Wagner, Associates: Drs. W. J. Browning and B. E. Scranton, Purdue University, have been appointed Associates.

Wellesley College: Assistant Professor Alan Shuchat, Mt. Holyoke, has been appointed Assistant Professor; Associate Professor Torsten Norvig has been promoted to Professor.

Wheaton College: Associate Professor Robert Brabenec has been promoted to Professor; Assistant Professors Gregory Dobbins and David Price have been promoted to Associate Professors.

Yale University: Associate Professors Gerald Janusz and Peter Loeb, University of Illinois, have been appointed Visiting Associate Professors.

Dr. E. C. Ackermann, Pennsylvania State University, has been appointed Assistant Professor at Muhlenberg College.

Sister Claire Archambault, Regis College, has been promoted from Assistant Professor to Associate Professor.

Professor Edwin F. Beckenbach, UCLA, has retired with the title of Professor Emeritus.

Assistant Professor Robert Bernhardt, University of North Carolina at Greensboro, has been appointed Assistant Professor at Chicago State University.

Assistant Professor M. L. Berry, West Virginia Wesleyan College, has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics.

Professor W. E. Bleick, Naval Postgraduate School, has retired with the title of Professor Emeritus.

Assistant Professor L. A. Bruckner, University of Maine at Portland-Gorham, has been appointed a Technical Staff Member at Los Alamos Scientific Laboratory.

Associate Professor W. D. Clark, Stephen F. Austin State University, has been promoted to Professor.

Assistant Professor R. S. Cunningham, University of Kansas, has been appointed Assistant Professor at Birmingham-Southern College.

Dr. John Feroe, University of California-San Diego, has been appointed Assistant Professor at Vassar College.

Assistant Professor V. W. Giambalvo, University of Connecticut, has been promoted to Associate Professor.

Professor P. C. Hammer, Pennsylvania State University, has been appointed Professor at Grand Valley State College.

Dr. W. A. Hansen, Northwestern University, has been appointed Assistant Professor at Wilkes College.

Associate Professor M. O. Holoien, Chairman of the Department of Computer Science at Moorhead State College, has been promoted to Professor.

Assistant Professor H. W. Kim, Bucknell University, has been promoted to Associate Professor.

Dr. R. S. King, Dallas Independent School District, has been appointed Assistant Professor of Mathematics and Computer Science at Augusta College.

Instructor Jim Ley, University of Wisconsin — Stout, has been promoted to Assistant Professor.

Mr. J. L. Lowther, University of Iowa, has been appointed Instructor at Michigan Technological University.

Dr. Lewis Lum, University of Tennessee, has been appointed Assistant Professor at Salem College.

Dr. W. A. J. Luxemburg, Professor of Mathematics and Executive Officer for Mathematics at the California Institute of Technology, has been elected a corresponding member of the Royal Academy of Sciences of Amsterdam in the Netherlands.

Dr. S. E. Mosiman, University of Missouri, Columbia, has been appointed Assistant Professor at Loras College.

Associate Professor W. N. Prentice, Denison University, has been promoted to Professor.

Assistant Professor J. F. Ramaley, University of Pittsburgh and On-Line Systems, Inc., has been appointed Manager of Information Services at Ziff-Davis Publishing Company, New York.

Associate Professor T. J. Robertson, University of Iowa, has been promoted to Professor.

Associate Professor R. S. Sabharwal, California State University, Hayward, has been promoted to Professor.

Professor Gerald Samson, Lake Superior State College, received the Distinguished Teaching Award for 1973–74.

Associate Professor J. L. Smith, Muskingum College, has been promoted to Professor.

Assistant Professor C. J. Steib, Southeastern Louisiana University, has been promoted to Associate Professor.

Assistant Professor R. Vaillancourt, University of Ottawa, has been promoted to Associate Professor.

Mr. R. L. Watkins, University of Virginia, has been appointed Visiting Instructor at Virginia Wesleyan College.

Associate Professor C. R. Williams, Midwestern University, has been promoted to Professor.

Associate Professor J. W. Wilson, University of Georgia, has been appointed Program Manager, Materials and Instructional Development Section, Pre-college Division, of the National Science Foundation Education Directorate.

Professor Emeritus I. Albert Barnett, University of Cincinnati, died on September 27, 1974, at the age of 80. He was a Charter Member of the Association.

Dr. Samuel H. Coleman, Georgia Institute of Technology, died on October 16, 1974. He was a member of the Association for nine years.

Dr. Myrtie Collier, Professor Emeritus, Immaculate Heart College, died on June 25, 1974, at the age of 97. Dr. Collier was a Charter Member of the Association.

Mr. H. J. Ereckson, Jacksonville, died on June 1, 1974. He was a member of the Association for five years.

Mr. Felix E. Ginsberg, Rockville, died on May 9, 1974. He was a member of the Association for eighteen years.

Professor Emeritus Cornelius Lanczos, Dublin Institute for Advanced Studies, died on June 24, 1974, at the age of 81. He was a member of the Association for thirty-six years.

Professor Mary Ann Lee, Sweet Briar College, died on September 6, 1974, at the age of 65. She was a member of the Association for thirty-one years.

Professor Robert Lee Moore, University of Texas, Austin, died on October 4, 1974, at the age of 91. He was a Charter Member of the Association.

Professor, Dean, Vice-President Emeritus Webster G. Simon, Case Western Reserve University, died on August 17, 1974, at the age of 82. He was a member of the Association for fifty-seven years.

Professor Emeritus Ralph S. Underwood, Texas Tech University, died on May 18, 1974, at the age of 83. He was a member of the Association for fifty-four years.

FOURTH INTERNATIONAL SYMPOSIUM ON MULTIVARIATE ANALYSIS

The Fourth International Symposium on Multivariate Analysis will be held at Wright State University, Dayton, Ohio, U.S.A., during the period June 16–21, 1975. This symposium, as the earlier ones, is sponsored by the Aerospace Research Laboratories. The symposium will be dedicated to the memory of the late H. Hotelling and P. C. Mahalanobis. It is expected that several distinguished workers in the field will present invited papers in the areas of classification and pattern recognition, contingency tables, design of experiments, distribution theory, econometrics, growth curves, inference, reliability, information and control theory, prediction theory, psychometrics, statistical physics, stochastic problems in hydrology and meteorology, time series and stochastic processes, and various other topics on theory and applications. The list of persons who have already either accepted the invitations or indicated tentative acceptance to present papers include: A. V. Balakrishnan, R. C. Bose, H. Chernoff, S. Das Gupta, D. A. S. Fraser, Y. Fujikoshi, C. W. Helstrom, T. Hida, M. G. Kendall, J. Kiefer, T. Kailath, G. Kallianpur, A. M. Kshirsagar, F. M. Lord, K. V. Marida, D. Middleton, E. Parzen, M. L. Puri, C. R. Rao, M. M. Rao, M. Rosenblatt, Yu. A. Rozanov, M. Siotani, and H. Theil. Several other prominent workers in the field are expected to present invited papers. A tentative and partial list of the chairmen of the sessions include: R. L. Anderson, S. Geisser, H. L. Harter, E. Lukacs, I. Olkin and J. N. Srivastava.

There will be sessions of contributed papers on theoretical aspects as well as *applications*. Persons interested in presenting contributed papers should submit abstracts of their papers, not exceeding 200 words, to P. R. Krishnaiah as soon as possible. Attendance at the symposium is open to anyone interested. The chairman of the local arrangements committee is C. C. Maneri. Further details regarding the symposium may be obtained by contacting P. R. Krishnaiah (Symposium Chairman), ARL/LB, Building 450, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, 45433, U.S.A.

INTERNATIONAL SYMPOSIUM ON INTERVAL MATHEMATICS — UNIVERSITY OF KARLSRUHE

An International Symposium on Interval Mathematics will be held on May 20–24, 1975, at the University of Karlsruhe, on the occasion of its 150th anniversary.

The object of the symposium is to bring together research workers with a common interest in interval mathematics. The program will consist of invited lectures, describing the state of the art, and of contributed lectures. The following scientists have already agreed to give an *invited lecture*: Henrici, Zurich (Switzerland); Moore, Madison, Wisconsin (USA); Ratschek, Düsseldorf (Germany); Urabe, Tokyo (Japan).

The *contributed lectures* will report on original work in all parts of Interval Mathematics.

For further information, please write to: Symposium on Interval Mathematics, Prof. Dr. Karl Nickel, Institut für Praktische Mathematik, Universität Karlsruhe, D-75 Karlsruhe, Federal Republic of Germany.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifty-first annual meeting of the Louisiana-Mississippi Section of the MAA was held at Jackson State College, in Jackson, Mississippi, on February 15–16, 1974.

Dr. R. P. Boas, President of the MAA, gave two invited addresses, "Consequences of Continuity," and "Partial Sums of Infinite Series."

Presiders at the sessions for contributed papers were Dr. David Caveny, Dr. L. S. Haw, and Dr. B. E. Mitchell.

Officers elected for 1974–75 were: Chairman, Russell Whittington, Jr., of Northwestern State University; Vice-Chairman (Mississippi), D. J. Hickman of Jackson State College; and Vice-Chairman (Louisiana), A. J. Hulin of University of New Orleans.

The following papers were presented:

1. *Structure of the Input Monoid Automata*, by W. R. Edwards, Jr., University of Southwestern Louisiana, and Zamir Barel, University of Kansas.

2. *The Additive Group of a Near Integral Domain*, by H. E. Hatherly and Horace Oliver, University of Southwestern Louisiana.

3. *On Tschebyscheff Polynomials for Regular Polygons Inscribed in the Unit Circle of the Complex Plane*, by D. J. Hickman, Jackson State College.

4. *A New Proof of the Vandermonde Determinant Expansion*, by J. J. Johnson, University of Mississippi.

5. *Variations on Hamiltonian Graphs*, by J. M. Kinney, University of Mississippi.

6. *A Simple Finite Difference Construction of a Conformal Mapping*, by C. W. Mastin, Mississippi State University.

7. *Order of Fixed Points of Dendrites Under Finitely Generated Abelian Semigroups of Mappings*, by Patricia Wright, Nicholls State University.

8. *The Reconstruction of Certain Primitive Functions*: *H-Integrals*, by Keith Alford, Alcorn A. & M. College.

9. *Lagrangian Conservation Theorems*, by Margaret M. LaSalle, University of Southwestern Louisiana.

10. *A Short History of Several Complex Variables*, by Yi-Chuan Pan, Jackson State College.

11. *A Sorting Problem*, by D. P. Roselle, Louisiana State University.

12. *An Application of Kantorovich's Inequality*, by G. T. Rizzuto, University of Southwestern Louisiana.

13. *Distribution of the Number of Uniform Variates Exceeding the Sample Mean*, by William Tally, University of Southwestern Louisiana.

14. *Implementation of a Competency-Based Teacher Education Program*, by Rawjwan Thumchai, Jackson State College.

15. *A Comparison of the Bayesian and Non-Bayesian Test of Hypothesis*, by D. M. Bardwell, Nicholls State University.

16. *Path Length Connectivity in Graphs*, by L. D. Strong, University of Mississippi.

17. *The Complete Symmetric Graph Applied to Partitions of the Integers Modulo* $2n$, by K. B. Reid, Louisiana State University.

18. *Group Subsets Having Distinct Differences*, by L. T. Ollmann, Louisiana State University.

19. *An Invariant Subspace Theorem*, by D. A. Hogan, Southern Illinois University.

20. *On the Power Set and Related Problem Solving*, by E. C. Leggette, Jackson State College.

21. *Free Spheres with Mapping Cylinder Neighborhoods*, by B. J. Smith, University of Southwestern Louisiana.

22. *An Example of a Connected Hausdorff Topology on a Countable Set with No Minimal Hausdorff Cotopology*, by Manuel Berri, Louisiana State University in New Orleans.

23. *A Boundedness Theorem for a Nonlinear Differential Equation*, by J. R. Graef and P. W. Spikes, Mississippi State University.

24. *On Nonscillation of Solutions of* $Y''+g(t)\,f(y)=r(t)$, by S. R. Limerick and P. W. Spikes, Mississippi State University.

25. *A Nonscillation Theorem for a Second Order Nonlinear Differential Equation*, by J. R. Graef and P. W. Spikes, Mississippi State University.

26. *What Has Happened to Computer Calculus*, by Paul Ohme, Mississippi College.

27. *A Computer System for Linguistic Studies*, by O. B. Davenport, Jackson State College, and J. R. Oliver, University of Southwestern Louisiana.

28. *The Effects of History of Mathematics on Student Attitudes toward Mathematics: Report of an Experiment*, by C. C. McBride, Louisiana Tech University.

29. *Remedial Mathematics for Sub-15, ACT Composite Score*, by J. F. Reed, Mississippi State University.

30. *Point Divisors in* C^n, by Yi-Chuan Pan, Jackson State College.

31. *A Look at an NSF Education Computing Network*, by J. C. Lewis, Jackson State College.

P. L. FORD, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifty-fourth regular meeting of the Southern California Section of the MAA was held on March 2, 1974, at Harvey Mudd College. The Chairman of the Section, Professor David Outcalt of U.C.S.B., presided. The registered attendance was 147, including 122 members of the Association.

At the business meeting, the results of the election of section officers for 1974–75 were announced as follows: Chairman, Donald Potts, Cal. State at Northridge; 1st Vice-Chairman,

J. M. Hood, Occidental College; 2nd Vice-Chairman, Betty Garrison, Cal. State at San Diego; Program Chairman, Neil Gretsky, U.C. Riverside. The luncheon speaker was Professor Paul Erdös, Hungarian Academy of Sciences, whose topic was "How I became a Mathematician."

The following program was presented:

The irrelevance of algebraic inequalities, by William Watkins, Cal. State at Northridge.

Non-associative algebras and differential equations, by Courtney Coleman, Harvey Mudd College.

The fourth dimension and computer animated geometry, by Thomas Banchoff, Brown University and U.C.L.A.

A simple characterization of commutative rings without maximal ideals, by Melvin Henriksen, Harvey Mudd College.

Construction of new simple groups: challenge for tomorrow, by David Wales, California Institute of Technology.

The use of APL in teaching community college mathematics, by John Clarke, Orange Coast College.

An experiment in student centered education, by Alvin White, Harvey Mudd College.

Hide and seek, data storage, and entropy, by Bob McEliece, Jet Propulsion Laboratories.

How reliable is American industry? by Janet Myhre, Claremont Men's College.

JOHN GREEVER, *Secretary-Treasurer*

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Terms of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms are listed since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January 1977.

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of 1972, and the May issues of 1973 and 1974). Approval for election was given to the following forty-one applicants for academic membership:

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CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN
FLORIDA
ILLINOIS, Rockford College, Rockford, May 9–10, 1975.
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, General Motors Institute, Flint, May 1–3, 1975.
MISSOURI
NEBRASKA
NEW JERSEY
NORTH CENTRAL
NORTHEASTERN, University of Connecticut, Storrs, June 20–21, 1975.
NORTHERN CALIFORNIA
OHIO, Bowling Green State University, Bowling Green, May 2–3, 1975.
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST
PHILADELPHIA, Franklin and Marshall College, Lancaster, November 22, 1975.
ROCKY MOUNTAIN
SEAWAY
SOUTHEASTERN, Central Piedmont Community College, Charlotte, North Carolina ,Spring 1976.
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE
AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.
ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.
ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.
ASSOCIATION FOR WOMEN IN MATHEMATICS
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30–May 2, 1975.
PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.
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THE MATHEMATICAL ASSOCIATION OF AMERICA

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THE FEYNMAN INTEGRAL

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Introduction. In 1922 Norbert Wiener [**1**], treating the Brownian motion of a particle, introduced a measure on the space of continuous real functions, and a corresponding integral. In 1948 Richard Feynman [**2**], studying the quantum mechanics of a particle, introduced a different integral over the same space. He also showed that his integral can be used to represent the solution of the initial value problem for the Schrödinger equation. This suggested that the Wiener integral can likewise be used to represent the solution of the initial value problem for the heat equation, and Mark Kac [**3, 4**] showed this in 1949. Since then function space integrals have been used often in physics and studied extensively in mathematics.

We shall present an introduction to the Feynman integral, beginning with a heuristic definition of it in section 1. Then in section 2 we shall show that it solves the Schrödinger equation, and we shall define it for regions with boundaries in section 3. In section 4 we shall define it precisely. In the remaining sections we shall illustrate its use by evaluating it asymptotically. Our purpose is to show how function space integrals can be used to solve partial differential equations, and also how the application of mathematics has again led to the development of a new branch of mathematics.

Further information about the Feynman and Wiener integrals is contained in references [**5**]-[**8**] and [**9**], [**10**] respectively.

1. A heuristic definition of the Feynman integral. Suppose a particle of mass m is at position y on the x-axis at time $t = 0$, and that it has potential energy $V(y)$. The particle may not remain at position y, but may move to position x at some later time $t > 0$. According to quantum mechanics, this move is not determinate, but it has a probability density $p(x, t)$ of occuring. Furthermore $p = |K(x, t)|^2$, where K is a complex quantity called the probability amplitude. The probability amplitude K satisfies the Schrödinger equation

$$ih \frac{\partial K}{\partial t} = -\frac{h^2}{2m} \frac{\partial^2 K}{\partial x^2} + V(x)K. \tag{1.1}$$

Here h is Planck's constant divided by 2π.

Eq. (1.1) is a linear partial differential equation for K which is of first order in t and of second order in x, so it is like the heat equation, but differs by having a factor i. Because solutions of (1.1) are wave-like, K is also called the wave function of the particle. Since the particle is surely at the position $x = y$ when $t = 0$, K must satisfy the initial condition

$$K(x, 0) = \delta(x - y). \tag{1.2}$$

The function $\delta(x - y)$ in (1.2) is the "delta function," which is supposed to be zero

for $x \neq y$ and to be infinite at $x = y$, in such a way that $\int_{-\infty}^{\infty} \delta(x-y)f(y)dy = f(x)$ for any continuous function f. Since there is no such function, δ must be defined as a generalized function, as a distribution, or as a linear functional characterized by the above identity.

As an application, we shall use K to construct the solution $\psi(x,t)$ of (1.1) which has the initial value $\psi(x,0) = \psi_0(x)$. To do so we write $K = K(x,y,t)$ to emphasize the dependence of K on y. Then we can write ψ in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,y,t)\psi_0(y)dy. \tag{1.3}$$

The fact that K satisfies (1.1) implies that ψ does also, and the initial condition (1.2) implies that $\psi(x,0) = \psi_0(x)$.

There is a unique solution K of (1.1) satisfying (1.2), and it is called the Green's function or the fundamental solution of (1.1). In physics it is also called the propagator, because it describes how the particle travels or propagates from y to x. When the potential is $V(x) = -(m\omega^2/2)x^2$ where ω is a constant, K can be found explicitly to be

$$K(x,t) = \left(\frac{m\omega}{2\pi i h \sin \omega t}\right)^{1/2} \exp\left(\frac{im\omega}{2h\sin\omega t}[(x^2+y^2)\cos\omega t - 2xy]\right). \tag{1.4}$$

We shall now consider the problem of finding K for any potential $V(x)$.

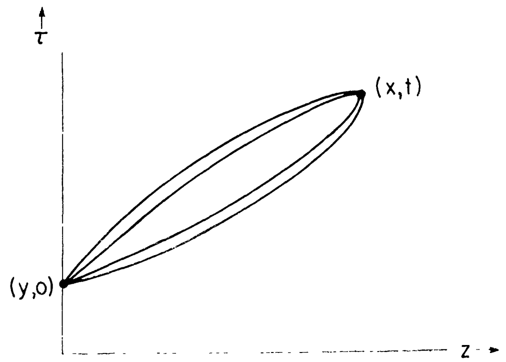

FIG. 1. Four paths $z(\cdot)$ in the set P.

If the particle moves from y at time 0 to x at time t along a path $z(\tau)$, then $z(\tau)$ must be a single valued continuous function of τ with $z(0) = y$ and $z(t) = x$. Let P be the set of such paths. (See Fig. 1.) For each differentiable path $z(\tau)$, we define the "action" $S[z(\cdot), t]$ associated with that path from $\tau = 0$ to $\tau = t$. $S[z(\cdot), t]$ is just the integral of the particle's kinetic energy minus its potential energy:

$$S[z(\cdot),t] = \int_0^t \left\{\frac{m}{2}\left(\frac{dz}{d\tau}\right)^2 - V[z(\tau)]\right\} d\tau. \tag{1.5}$$

Physical considerations [2, 8] suggest that a constant times $\exp\{(i/h)S[z(\cdot),t]\}$ is the propagator associated with the path $z(\cdot)$. Then, in view of the probabilistic interpretation of the propagator, it is natural to represent K as the sum of the propagators associated with all the paths in P. Thus we write

$$K(x,t) = \int_P \exp\left\{\frac{i}{h}S[z(\cdot),t]\right\}Dz(\cdot). \tag{1.6}$$

Eq. (1.5) gives the Feynman path integral representation of the solution K of (1.1) and (1.2). The right side is supposed to be an integral over the set P of continuous paths from $(y,0)$ to (x,t). It is symbolic, because the integrand has been defined only for differentiable paths, and the integral has not been defined at all. Therefore we shall now try to define it.

In order to define $S[z(\cdot),t]$ for any continuous path, we shall represent the derivative dz/dt in (1.5) by a difference quotient. At the same time, we shall replace the integral by a finite sum which approximates it. To do so we divide the interval $(0,t)$ into N equal parts, each of length $\Delta t = t/N$, with division points $t_j = j\Delta t$, $j = 0,1,\cdots,N$. Let $z_j = z(t_j)$. Then we replace the integral (1.5) for S by the sum

$$\sum_{j=1}^{N}\left\{\frac{m(z_{j+1}-z_j)^2}{2\Delta t} - V(z_{j+1})\Delta t\right\}. \tag{1.7}$$

Next we interpret the integration over $Dz(\cdot)$ in (1.6) as integration with respect to $z_1, z_2, \cdots, z_{N-1}$, followed by passage to the limit $N \to \infty$. We do not integrate over $z_0 = y$ nor over $z_N = x$. In taking the limit we must also include a suitable normalization factor to make the limit finite. This leads to the following formula for K:

$$\begin{aligned}(1.8)\quad & K(x,t) \\ &= \lim_{N\to\infty}\left(\frac{mN}{2\pi iht}\right)^{N/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[\frac{i}{h}\sum_{j=1}^{N}\left\{\frac{m(z_{j+1}-z_j)^2}{2\Delta t}-V(z_{j+1})\Delta t\right\}\right]dz_1\cdots dz_{N-1}.\end{aligned}$$

The normalization factor in (1.8) can be found by physical considerations.

The right side of (1.8) is the definition of the Feynman integral or path integral in (1.6). Although (1.6) is an integral over paths, (1.8) does not involve paths. Instead the variables z_j in (1.8) are independent variables of integration. Since (1.8) results from integration over all paths, it is not surprising that no particular path occurs in it.

Let us apply (1.8) to the quadratic potential $V(x) = -m\omega^2x^2/2$. In that case the exponent of the integrand is quadratic, so the integration involves Gaussian functions and it can be done explicitly. Then the limit is exactly the result (1.4). (See [2], [5], [8].) Thus for quadratic potentials, (1.8) is correct. The question whether (1.8) is correct for some larger class of potentials has not yet been decided.

If we set $m = 2hi$, and replace $K(x,t)$ by $W(x,t)$, (1.8) becomes

(1.9) $W(x,t)$

$$= \lim_{N\to\infty}\left(\frac{N}{\pi t}\right)^{N/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\sum_{j=1}^{N}\left\{\frac{(z_{j+1}-z_j)^2}{\Delta t}+\frac{i}{h}V(z_{j+1})\Delta t\right\}\right]dz_1\cdots dz_{N-1}.$$

This is just the conditional Wiener integral of

$$\exp\left\{-\frac{i}{h}\int_0^t V[z(t')]dt'\right\},$$

which is a functional of the path $z(\cdot)$. The integral is called conditional because it involves only paths which satisfy the conditions $z(0) = y$ and $z(t) = x$.

2. The integral satisfies the Schrödinger equation. To show that (1.8) is correct for any $V(x)$, we shall show that the integral satisfies (1.1) and (1.2). Then because that problem has a unique solution K, it will follow that (1.8) is correct. Since our demonstration will be formal, it will be a plausibility argument for (1.8) rather than a proof. For simplicity we shall set $h = 1$ and $m = 1/2$ in this and the next section.

We begin by letting $K_N(x,t)$ denote the expression under the limit in (1.8), so that $K = \lim_{N\to\infty} K_N$. Then we can rewrite K_N as a single integral of K_{N-1} as follows:

(2.1) $K_N(x,t)$

$$= (4\pi i\Delta t)^{-1/2}\int_{-\infty}^{\infty}\exp\left[\frac{i}{h}\left\{\frac{m(x-z_{N-1})}{2\Delta t}-V(x)\Delta t\right\}\right]K_{N-1}(z_{N-1},t_{N-1})dz_{N-1}.$$

In the normalization factor we have set $t/N = \Delta t$. Next we expand K_{N-1} as a finite Taylor series in powers of $x - z_{N-1}$, about x. By using this expansion in (2.1), we obtain

$$\begin{aligned}(2.2)\quad K_N(x,t) = (4\pi i\Delta t)^{-1/2}&\exp[-iV(x)\Delta t]\int_{-\infty}^{\infty}\exp\left[\frac{i(x-z_{N-1})^2}{2\Delta t}\right]\{K_{N-1}(x,t_{N-1})\\ &-(x-z_{N-1})\partial_x K_{N-1}(x,t_{N-1})+\tfrac{1}{2}(x-z_{N-1})^2\partial_x^2 K_{N-1}(x,t_{N-1})\\ &+O[(x-z_{N-1})^3]\}dz_{N-1}.\end{aligned}$$

The notation $O(z^n)$ denotes a function $f(z)$ which satisfies $|f(z)| \leqq A|z^n|$ for some constant A and some neighborhood $|z| < \delta$ of $z = 0$. In (2.2) it refers to the remainder of the Taylor series.

In (2.2), K_N is expressed as the sum of four integrals. The first three are just the first three moments of a Gaussian distribution, so they can be evaluated explicitly. The second integral vanishes because its integrand is odd, and so does the integral of the cubic term in the remainder. Therefore only the first and third integrals contribute, and the order of the remainder is determined by the integral of the fourth degree term. Upon using the values of these integrals in (2.2), we get

$$(2.3)\quad K_N(x,t) = \exp[-iV(x)\Delta t]\{K_{N-1}(x,t_{N-1}) + i\Delta t\partial_x^2 K_{N-1}(x,t_{N-1}) + O[(\Delta t)^2]\}.$$

Now we expand the exponential in powers of Δt and rearrange (2.3) into the form

$$(2.4)\quad \frac{i}{\Delta t}[K_N(x,t) - K_{N-1}(x,t_{N-1})] = -\partial_x^2 K_{N-1}(x,t_{N-1}) + V(x)K_{N-1}(x,t_{N-1}) + O(\Delta t).$$

As $N \to \infty$, both K_{N-1} and K_N tend to K, while t_{N-1} tends to t and Δt tends to zero. Since $t_{N-1} = t - \Delta t$, the left side of (2.4) tends to $i\partial_t K(x,t)$. Thus the limit of (2.4) is just the Schrödinger equation (1.1), so the path integral satisfies this equation.

To show that the path integral also satisfies the initial condition (1.2), we observe that when t tends to zero, $\Delta t = t/N$ also tends to zero. Therefore, in finding the limit of K at $t = 0$, the term $V(z_{j+1})\Delta t$ in the exponent in (1.8) can be omitted. The resulting integral is just the integral of a Gaussian function, which can be evaluated explicitly. Then the limit in (1.8) is found to have exactly the initial value (1.2). (See Buslaev [**11**].) This completes the formal demonstration that (1.8) is correct. More detailed discussions of this demonstration, as well as the construction of (1.8), are given in [**2**], [**7**], [**8**], [**11**] and [**12**].

3. Boundaries and reflection. The Feynman integral representation (1.8) of K can be extended to n dimensions by letting $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}_j$ denote points in n dimensional space, and performing all integrations over that space. In addition an extra normalization factor must be introduced. Although this extension is straightforward, the extension to a region with a boundary is more subtle, as we shall now see.

Let $K_+(\mathbf{x},t)$ satisfy the Schrödinger equation in a domain D_+ exterior to a closed surface B in n dimensional Euclidean space. Then the problem solved by K_+ is the following, if K_+ vanishes on S:

$$(3.1)\qquad ih\frac{\partial K_+}{\partial t} = -\frac{h^2}{2m}\nabla^2 K_+ + V(\mathbf{x})K_+, \quad \mathbf{x} \text{ in } D_+, \quad t > 0,$$

$$(3.2)\qquad \lim_{t\downarrow 0} K_+(\mathbf{x},t,\mathbf{y},0) = \delta(\mathbf{x}-\mathbf{y}),$$

$$(3.3)\qquad K_+(\mathbf{x},t) = 0, \qquad \mathbf{x} \text{ on } B.$$

If instead of $K_+ = 0$ on B, the normal derivative of K_+ vanishes on B, (3.3) must be changed accordingly.

To solve this problem we introduce a copy of D_+ which we call D_-, and we join D_+ to D_- along their common boundary B. In this way, we obtain a two sheeted space D, composed of the sheets D_+ and D_-, which has no boundary. (See Fig. 2.) Let us denote by $K(\mathbf{x},\mathbf{y},t)$ the solution of (3.1) and (3.2) for $\mathbf{x}$ and $\mathbf{y}$ in D. We require K and its normal derivatives to be continuous on B, where the two sheets join. We shall now show that in terms of K, we can write K_+ in the form

$$(3.4)\qquad K_+(\mathbf{x}_+,\mathbf{y}_+,t) = K(\mathbf{x}_+,\mathbf{y}_+,t) - K(\mathbf{x}_+,\mathbf{y}_-,t).$$

Here $\mathbf{x}_+$ and $\mathbf{y}_+$ denote points in D_+ while $\mathbf{x}_-$ and $\mathbf{y}_-$ are the corresponding points

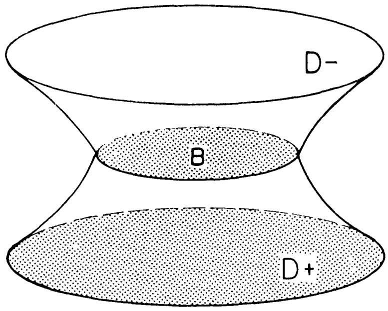

FIG. 2. The double sheeted space D composed of the sheets D_+ and D_- joined along their common boundary B.

in D_-. If the normal derivative of K_+ vanishes on B, (3.4) holds with the sum rather than the difference on the right side.

To show that (3.4) is correct we first observe that K_+ defined by (3.4) satisfies (3.1). This is because both terms on the right satisfy (3.1) in D, and D contains D_+. To see that (3.2) holds we note that the first term on the right tends to $\delta(\mathbf{x}_+ - \mathbf{y}_+)$ as $t \to 0$ while the second term tends to $-\delta(\mathbf{x}_+ - \mathbf{y}_-)$. Therefore, for $\mathbf{x}_+$ in D_+, the second term vanishes as $t \to 0$, so (3.2) is satisfied. Finally we observe that $K_+ = 0$ for $\mathbf{x}_+$ on B because the two terms on the right side of (3.4) are equal when $\mathbf{x}_+$ is on B, and they cancel. Similarly, when the minus is replaced by a plus sign in (3.4), the normal derivative of K_+ vanishes on B by symmetry.

We now represent K by a Feynman integral, which we can do as before since K is defined in D, which has no boundary. In performing the integration over the $\mathbf{z}_j$ in the definition of the integral, we integrate over D, i.e., over D_+ and D_-. In this way we include all paths in D. When we use this integral representation of K in (3.4), we obtain a representation of K_+.

The use of a double sheeted space to solve boundary value problems was introduced by Sommerfeld in 1896 in solving a diffraction problem. Buslaev [11] first used it together with function space integrals to obtain representations of solutions of boundary value problems for parabolic partial differential equations.

4. Rigorous definition of the Feynman integral. The Feynman integral (1.8) can be obtained from the Wiener integral (1.9) by replacing Δt in the Gaussian distribution and in the normalization constant by $-2ih\Delta t/m$. This corresponds to introducing a Gaussian distribution with an imaginary variance. Now the Wiener integral can be defined in terms of a Wiener measure over the space of continuous functions. This measure involves the Gaussian distribution [5]. Therefore it is natural to suppose that the Feynman integral can also be defined in terms of a measure, but a complex measure involving a Gaussian distribution with a complex variance.

In 1960 Cameron [13] showed that no such countably additive measure exists. To date, there is no measure-theoretic definition of the Feynman integral. There are, however, several definitions of it which do not use measure. One of the simplest of these is that of Nelson [14], which we shall now present.

Let $L^2(R)$ denote the Hilbert space of complex valued Lebesgue square integrable functions on the real line R. In this space we seek the solution $\Psi(t)$ of the following initial value problem for the Schrödinger equation:

$$i\frac{d}{dt}\Psi(t) = (H + \hat{V})\Psi(t), \qquad t > 0,$$

$$\Psi(0) = \Psi_0 \in L^2(R).$$

Here $H \equiv -\partial_x^2$ is a second order differential operator and $\hat{V}$ denotes multiplication by the real function $V(x)$.

The operator H is self adjoint if its domain $D(H)$ is properly restricted. Similarly, the operator $\hat{V}$ is self adjoint on the domain $D(\hat{V})$ of all $\Psi \in L^2(R)$ such that $\hat{V}\Psi$ is also in $L^2(R)$. Kato (see [14]) has shown that the operator $H + \hat{V}$ is self adjoint on the domain $D(H)$ if $V(x) \in L^p(R)$ with $p \geqq 2$. Since each of the three operators $(H, \hat{V}, H + \hat{V})$ is self adjoint, the spectral theorem guarantees the existence of the following unitary operators:

$$G^t \equiv \exp(-itH),$$

$$W^t \equiv \exp(-it\hat{V}),$$

$$K^t \equiv \exp(-it[H + \hat{V}]).$$

In terms of K^t the solution $\Psi(t)$ can be written as

$$\Psi(t) = K^t\Psi_0 = \exp[-it(H + \hat{V})]\Psi_0. \tag{4.1}$$

To utilize (4.1), we apply the Trotter-Kato Product Formula [14] to express K^t in terms of G^t and W^t,

$$K^t\Psi_0 = \lim_{N\to\infty} (W^{t/N}G^{t/N})^N\Psi_0, \qquad \forall\Psi_0 \in L^2(R). \tag{4.2}$$

To make (4.2) more explicit, we use the Fourier integral representation of $\Psi_0(x)$ to express $(G^t\Psi_0)(x)$ in the form

$$(G^t\Psi_0)(x) = \left[\exp\left(-it\frac{d^2}{dx^2}\right)\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\hat{\Psi}_0(p)dp\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}e^{ip^2t}\hat{\Psi}_0(p)dp.$$

Here $\hat{\Psi}_0(p)$ is the Fourier transform of $\Psi_0(x)$. Using the definition of $\hat{\Psi}_0(p)$, interchanging the order of integration, which is permissible, and then evaluating the integral with respect to p, we obtain

$$\begin{aligned}(G^t\Psi_0)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_0(x')\int_{-\infty}^{\infty}e^{ip^2t}e^{ip(x'-x)}dp\,dx' \\ &= \int_{-\infty}^{\infty}(4\pi it)^{-1/2}\exp\left[\frac{i}{4t}(x-x')^2\right]\Psi_0(x')dx'\,.\end{aligned}$$

Now we use this expression for $G^t\Psi_0$ in (4.2) and (4.1) to obtain

$$\Psi(x,t) = \lim_{N\to\infty}\prod_{j=0}^{N}\left\{\exp\left(-iV(x_{j+1})\frac{t}{N}\right)\int_{-\infty}^{\infty}\left(\frac{N}{4\pi it}\right)^{1/2}\exp\left[\frac{iN}{4t}(x_{j+1}-x_j)^2\right]\right\} \times\Psi_0(x_0)dx_j\,.$$

We can rewrite this in the equivalent form

$$\Psi(x,t) = \lim_{N\to\infty}\left(\frac{N}{4\pi it}\right)^{(N+1)/2}\int_{-\infty}^{\infty}\Psi_0(z_0)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[i\sum_{j=1}^{N+1}\left\{\frac{N}{4t}(z_j-z_{j-1})^2 - V(z_j)\frac{t}{N}\right\}\right]dz_1\cdots dz_N dz_0, \qquad \forall\Psi_0\in L^2(R), \tag{4.3}$$

where $z_{N+1} = x$.

When we can exchange the order of integration and of taking the limit in (4.3), the coefficient of $\Psi_0(z_0)$ becomes exactly the expression for K in (1.8) with $h = 1$ and $m = 1/2$. To see that this coefficient is equal to the fundamental solution $K(x, z_0, t)$, we just compare (4.3) and (1.3). This comparison yields the Feynman integral representation (1.8) of K.

Equation (4.3) defines the Feynman integral representation of the solution operator K^t as the strong limit of a sequence of linear operators acting in the Hilbert space $L^2(R)$. This definition utilizes the semigroup property of K^t, the same property which is the basis for the formal verification given in Section 2. Other definitions of the Feynman integral, which also depend upon this semi-group property, have been given by Cameron, Babbitt, and Feldman and Nelson (see [**14**]). Recently, C. DeWitt [**16**] has given a definition which is almost measure-theoretical. However, no definition has permitted justification of the interchange of Feynman integration with other integrations, with limits, etc.

5. Asymptotic evaluation of Feynman integrals. We shall now evaluate the Feynman integral (1.8) asymptotically for h small by the method of stationary phase. First we shall explain the method by applying it to the single integral

$$K(h) = \int_a^b \exp\left\{\frac{i}{h}s(z)\right\}dz, \tag{5.1}$$

in which $s(z)$ is a thrice continuously differentiable real valued function. As h tends to zero, the exponential function oscillates more and more rapidly. Consequently contributions to the integral from each subinterval of (a,b) tend to cancel out, and the integral tends to zero. The largest contribution will come from that subinterval in which the oscillation is slowest. That subinterval will be the neighborhood of a point $\bar{z}$ at which $s'(\bar{z}) = 0$. Since $s(z)/h$ is the phase of the integrand, and since s is stationary at $\bar{z}$, $\bar{z}$ is called a point of stationary phase. Let us suppose that there is exactly one such point in (a,b), and that it is an interior point.

We now introduce the new integration variable $\zeta = h^{-1/2}(z-\bar{z})$ so that $z = \bar{z} + h^{1/2}\zeta$. Then $s(z) = s(\bar{z}+h^{1/2}\zeta) = s(\bar{z}) + (h/2)\zeta^2 s''(\bar{z}) + O(h^{3/2})$, since $s'(\bar{z}) = 0$. By using this expansion of s, we can write (5.1) in the form

$$K(h) = h^{1/2}\exp\left\{\frac{i}{h}s(\bar{z})\right\}\int_{h^{-1/2}(a-\bar{z})}^{h^{-1/2}(b-\bar{z})} \exp\left\{\frac{is''(\bar{z})}{2}\zeta^2\right\}[1+O(h^{1/2})]d\zeta. \tag{5.2}$$

As h tends to zero, the limits of integration tend to $\pm\infty$. The integral of a Gaussian function from $-\infty$ to $+\infty$ is known, so (5.2) becomes

$$K(h) = \left[\frac{2\pi h}{s''(\bar{z})}\right]^{1/2} \exp\left\{\frac{is(\bar{z})}{h} + i\frac{\pi}{4}\right\} + O(h). \tag{5.3}$$

Although our calculation is formal, the result (5.3) can be proved to be correct [**17**].

In applying the method of stationary phase to the Feynman integral, we shall use the symbolic form (1.6) because of its similarity to (5.1). The phase S is a functional of the path $z(\tau)$, so its derivative is a functional derivative. Equating it to zero yields a differential equation for the stationary path $\bar{z}(\tau)$, as we shall now show. First we introduce the new variable $\zeta(\tau) = h^{-1/2}[z(\tau)-\bar{z}(\tau)]$, which satisfies $\zeta(0) = \zeta(t) = 0$. Now we set $z(\tau) = \bar{z}(\tau) + h^{1/2}\zeta(\tau)$ in (1.5), and expand $S[z(\cdot),t]$ in powers of $h^{1/2}$, obtaining

$$\begin{aligned} S[\bar{z}(\cdot)+h^{1/2}\zeta(\cdot),t] &= \int_0^t\left\{\frac{m}{2}\left(\frac{d\bar{z}}{d\tau}+h^{1/2}\frac{d\zeta}{d\tau}\right)^2 - V[\bar{z}(\tau)+h^{1/2}\zeta(\tau)]\right\}d\tau \\ &= S[\bar{z}(\cdot),t] + h^{1/2}\int_0^t\left\{m\frac{d\bar{z}}{d\tau}\frac{d\zeta}{d\tau} - V'[\bar{z}(\tau)]\zeta(\tau)\right\}d\tau \\ &\quad + h\int_0^t\left\{\frac{m}{2}\left(\frac{d\zeta}{d\tau}\right)^2 - \frac{1}{2}V''[\bar{z}(\tau)]\zeta^2(\tau)\right\}d\tau + O(h^{3/2}). \end{aligned} \tag{5.4}$$

The $h^{1/2}$ term in (5.4) is linear in ζ, and is called the first variation of S. Equating it to zero, after integrating the first term in it by parts, yields

$$h^{1/2}\int_0^t\left\{m\frac{d^2\bar{z}}{d\tau^2} + V'[\bar{z}(\tau)]\right\}\zeta(\tau)d\tau = 0. \tag{5.5}$$

We want the first variation (5.5) to vanish for every continuous function $\zeta(\tau)$ which

satisfies $\zeta(0) = \zeta(t) = 0$. Therefore the coefficient of $\zeta(\tau)$ in the integrand, which is the functional derivative of $S[\bar{z}(\cdot), \tau]$, must vanish:

$$(5.6) \qquad m\frac{d^2\bar{z}}{d\tau^2} + V'[\bar{z}(\tau)] = 0.$$

This is just Newton's equation of classical mechanics for the stationary path $\bar{z}(\tau)$. In order that z be in P, it must also satisfy the boundary conditions

$$(5.7) \qquad \bar{z}(0) = y, \quad \bar{z}(t) = x.$$

For simplicity we shall assume that (5.6) and (5.7) have a unique solution. It is just the path from $(y, 0)$ to (x, t) given by classical mechanics.

We now use (5.4) and (5.5) in (1.6) to get

$$(5.8) \quad K(x,t) = \exp\left\{\frac{i}{h}S[\bar{z}(\cdot)], t\right\}\int \exp\left[\frac{i}{2}\int_0^t \left\{m\left(\frac{d\zeta}{d\tau}\right)^2 - V''[\bar{z}(\tau)]\zeta^2(\tau)\right\}d\tau\right] \times[1 + O(h^{1/2})]D[h^{1/2}\zeta(\cdot)].$$

The integration is over paths $\zeta(\tau)$ from $\zeta = 0$ at $\tau = 0$ to $\zeta = 0$ at $\tau = t$. The exponent in the integrand of (5.8) is quadratic in ζ and $d\zeta/d\tau$, so the corresponding form (1.8) contains only integrals of Gaussian functions. When the integration is performed and the limit $N \to \infty$ taken, the result is $K(x,t) = K_c(x,t)[1 + O(h^{1/2})]$ where K_c is given by

$$(5.9) \qquad K_c(x,t) = (2\pi i h)^{-1/2}e^{-iM\pi/2}\left|s_{xy}(x,y,t)\right|^{-1/2}\exp\left\{\frac{i}{h}s(x,y,t)\right\}.$$

Here $s(x,y,t) = S[\bar{z}(\tau), t]$ and M is the number of zeros, counted according to order, of $s(x,y,\tau)$ in the open interval $0 < \tau < t$. C. Morette [**12**] was the first to get K_c from the Feynman integral, although Feynman [**2**] indicated the connection. When there are several classical paths $\bar{z}$, K_c is a sum of terms of the form (5.9), one for each path.

The subscript c on K_c stands for classical, because s is just the action associated with the classical path $\bar{z}(\tau)$, and K_c is determined entirely by s. The asymptotic form of K given by (5.9) can also be derived directly from the differential equation (1.1) by the W.K.B. method. It is not valid at the zeros of s_{xy}, which are the caustics of the classical paths through $(y, 0)$. The phase factor $e^{-iM\pi/2}$ represents a phase loss of amount $\pi/2$ for each of the M times the path $\bar{z}(\tau)$ touches a caustic. By using such considerations, Keller [**18**] and Maslov [**19**] introduced the index M for any dimension, and Arnold [**20**] showed that it is the Morse index of a certain multisheeted space on which $s_x(x,y,t)$ is single valued.

When the above method of asymptotic evaluation is applied to the Feynman integral representation of K_+, given by (3.4), it yields (5.9) for the direct classical path from **y** to **x** plus a similar term for the reflected classical path from **y** to the boundary B to **x**. A refined evaluation by Buslaev [**11**], yields another term corre-

sponding to a classical path from (y, 0) which touches B tangentially, proceeds along a geodesic on B, and leaves B tangentially on a classical path to $(\mathbf{x}, t)$. (See Fig. 3.) This path is precisely a surface diffracted ray, which occurs in Keller's [**21**] geometrical theory of diffraction, and the term is exactly the value of the solution associated with that ray.

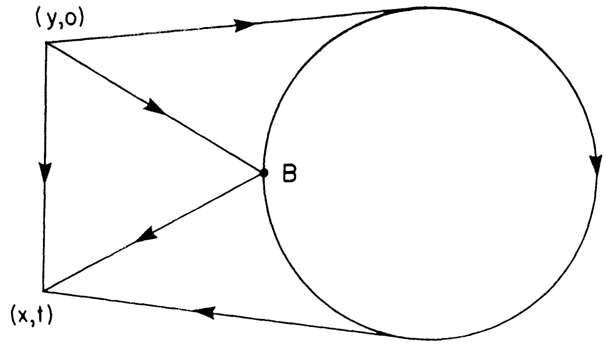

FIG. 3. The three stationary paths from (y,0) to (x,t) are a direct one, one reflected from B and one diffracted by B.

6. The time independent Schrödinger equation. The wave function $\psi(x, t)$ of a particle of energy E is of the form $e^{-iEt/h}u(x)$. It follows from (1.1) that $u(x)$ satisfies the time-independent Schrödinger equation, obtained from (1.1) by replacing $ih\partial/\partial t$ by E. The Green's function $G(x, y, E)$ of this equation is defined as follows:

$$\left[-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - E\right]G(x, y, E) = -\delta(x-y),$$
$$\lim_{x\to\pm\infty}\left|\frac{\partial G}{\partial x} \mp ikG\right| = 0.$$

The last condition is called the radiation condition because it guarantees that G contains only waves radiating to infinity, and none coming in from infinity. In it $k = (2mE/h^2)^{1/2}$. One of the reasons for studying G is that it is the kernel of the resolvent of the Schrödinger operator.

If E is replaced by $E + i\eta$ with $\eta > 0$, then $G(x, y, E + i\eta)$ is readily seen to be the Fourier transform of K:

$$G(x, y, E + i\eta) = \frac{1}{ih}\int_0^\infty \exp\left[\frac{i}{h}(E + i\eta)t\right]K(x, y, t)dt, \qquad \eta > 0.$$

By using the Feynman integral representation of K in the Fourier integral, we obtain the following integral representation of G:

$$(6.1)\quad G(x, y, E) = \frac{1}{ih}\lim_{\eta\downarrow 0}\int_0^\infty\int_P \exp\frac{i}{h}[(E + i\eta)t + S[z(\cdot), t]]Dz(\cdot)dt.$$

We shall now evaluate G asymptotically as $h \to 0$ by applying the method of stationary phase to the integral representation (6.1). Since (6.1) involves integration over paths and over t, we must find a pair $\bar{z}(\tau)$ and $\bar{t}$ which make the phase stationary. By proceeding as we did in the last section, we find that $\bar{z}(\tau)$ must satisfy Newton's equation (5.6) and the boundary conditions (5.7) with t replaced by $\bar{t}$. We also obtain the condition $\partial S[\bar{z}(\cdot),\bar{t}]/\partial\bar{t} = -E$ to make the phase stationary with respect to $\bar{t}$.

This condition, which serves to determine $\bar{t}$, can be written explicitly as

$$\frac{\partial}{\partial \bar{t}} \int_0^{\bar{t}} \left[\frac{m}{2}\left(\frac{d\bar{z}(\tau)}{d\tau}\right)^2 - V[\bar{z}(\tau)]\right] d\tau = -E. \tag{6.2}$$

The derivative of the integrand with respect to $\bar{t}$ in (6.2) vanishes as a consequence of (5.6) and (5.7). Therefore (6.2) yields the following equation with $\tau = \bar{t}$:

$$\frac{m}{2}\left(\frac{d\bar{z}(\tau)}{d\tau}\right)^2 + V[\bar{z}(\cdot)] = E. \tag{6.3}$$

From (5.6) it follows that the left side of (6.3) is independent of τ, so (6.3) holds for $0 \leqq \tau \leqq \bar{t}$.

By using (6.3) at $\tau = 0$ we can determine two possible values of the initial velocity $d\bar{z}(0)/d\tau$. These velocities and the initial condition $\bar{z}(0) = y$ yield two solutions $\bar{z}(\tau)$ of (5.6). If one of these paths $\bar{z}(\tau)$ passes through x, then $\bar{t}$ is determined by $\bar{z}(\bar{t}) = x$ and that path is a stationary path which yields a contribution to the integral (6.1). If both trajectories pass through x they are both stationary paths and the integral is asymptotically equal to the sum of their two contributions. If neither passes through x, then there is no stationary path and the integral is asymptotically equal to zero. If a trajectory passes through x more than once, it makes a contribution to the integral for each such passage, and the integral is asymptotically equal to the sum of all of them. We shall illustrate all these possibilities.

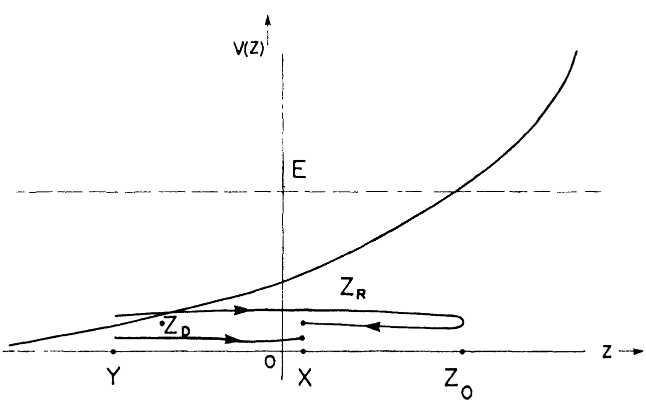

FIG. 4. A monotonic potential $V(x)$ and the two trajectories z_D and z_R from y to x.

Let us first suppose that the potential $V(z)$ is a continuous monotone increasing function of z with $V(-\infty) = 0$ and $V(+\infty) = +\infty$ as in Figure 4. Then for any $E > 0$ there is a unique solution z_0 of the equation $V(z_0) = E$. The point z_0 is called a "turning point," because the velocity of a particle vanishes there and changes sign in its neighborhood, as we see from (6.3). Now let us assume that $y < z_0$. Then if $x > z_0$, no trajectory from y passes through x. However, if $x < z_0$ there are two trajectories from y to x, a direct one $z_D(\tau)$ and a reflected one $z_R(\tau)$ which has been reflected at z_0. (See Figure 4.) We obtain the contributions of these paths to the integral (6.1) by writing the integral as a sum of two integrals, each of which contains just one of these paths. Then we expand the exponent of each integrand to second order about the stationary path, evaluate the resulting Gaussian integral and let η tend to zero.

In this way we find that as h tends to zero, $G(x,y,E) \sim G_c(x,y,E)$ where G_c is defined by $G_c(x,y,E) = 0$, $y < z_0 < x$, and

$$(6.4)\quad G_c(x,y,E) \equiv \frac{m}{h}[p(x)p(y)]^{-1/2}\left[\exp\left(\frac{i}{h}\int_y^x p(z)dz\right) + e^{-i\pi/2}\exp\left(\frac{i}{h}\int_y^{z_0} p(z)dz - \frac{i}{h}\int_{z_0}^{x} p(z)dz\right)\right], \quad y < x < z_0.$$

If $x < y < z_0$, G_c is given by the above expression with x and y interchanged in the first integral. In (6.4) $p(x)$ is the classical particle momentum defined by $p(x) = +[2m\{E - V(x)\}]^{1/2}$. The quantity G_c is the leading term in the "classical," "geometrical optics," or "time-independent W.K.B." approximation to G.

Now we suppose that $V(x)$ is monotone decreasing for $x < 0$ and monotone increasing for $x > 0$ with $V(+\infty) = +\infty$ and $V(0) = 0$. Then for each $E > 0$ there are exactly two turning points z_l and z_r satisfying $V(z) = 0$, with $z_l < z_r$. If y lies in the interval (z_l, z_r) and x lies outside it, there is no trajectory from y passing through x. However, if x also lies in the interval, the two trajectories from y pass through x infinitely many times as they reflect back and forth between z_l and z_r. Thus $G \sim G_c$ where G_c is an infinite series of terms like those in (6.4). Apart from a factor, the series is the geometric series

$$(6.5)\qquad \sum_{n=0}^{\infty} [-e^{i\Phi}]^n = (1 + e^{i\Phi})^{-1}$$

where $\Phi(E)$ is defined by

$$(6.6)\qquad \Phi(E) = \frac{2}{h}\int_{z_l}^{z_r} [2m\{E - V(z)\}]^{1/2} dz.$$

The series (6.5) diverges when $\Phi = (2n + 1)\pi$ where n is an integer. Since G is the kernel of the resolvent operator, its poles as a function of E are the eigenvalues of the Schrödinger equation. Thus the poles of $G_c(x,y,E)$ are asymptotic to the eigenvalues as $h \to 0$. Therefore the asymptotic form of the nth eigenvalue

is the root E_n of the equation

$$(6.11) \qquad \frac{2}{h}\int_{z_l}^{z_r} [2m\{E - V(z)\}]^{1/2} dz = (2n+1)\pi, \qquad n = 0, 1, \cdots.$$

This equation for E_n was found before the discovery of the Schrödinger equation, and it is called the Bohr-Sommerfeld quantum condition. Further details of these calculations are given in [23] and [24].

7. Further developments. We have seen that the Feynman integral can be extended readily to n dimensional space when Cartesian coordinates are employed, but it is more difficult to extend it in other coordinate systems [2]. It has also been extended to complex t by McLaughlin [22], and used by him to obtain both regular and singular asymptotic expansions of solutions of the Schrödinger equation [23].

Function space integral representations have also been found for the solutions of the wave, Klein-Gordon, Pauli, Dirac and quantized field equations ([5], [6]). No asymptotic evaluations of the integrals for the latter three equations have been made because their integrands involve non-commuting matrices. Path integrals employing a Hamiltonian instead of a Lagrangian have also been introduced [24].

Schulman [25] has obtained a path integral representation of the solution of the Pauli equation by using a path integral for the solution of the Schrödinger equation on the group manifold of $SU(2)$. The latter integral reduces exactly to a sum over those paths which satisfy Newton's equations of motion. This is also true for the quadratic potential in Euclidean space. Dowker [26] showed that the same reduction occurs on the group manifold of any compact semi-simple Lie group. Eskin [27] proved the corresponding result for the heat equation on any symmetric space whose group of motions is a complex semi-simple Lie group.

Many other applications of the Feynman integral can be found in the references we have cited and in numerous other papers. In addition the references describe still more developments of the theory from both a physical and a mathematical point of view. Although a great deal has been accomplished, there are many important problems concerning the Feynman integral which are still unsolved.

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References

1. N. Wiener, The average value of a functional, Proc. London Math. Soc., 22 (1924) 454–467.

2. R. P. Feynman, Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys., 20 (1948) 367–387.

3. M. Kac, On distributions of certain Wiener functionals, Trans. Amer. Math. Soc., 65 (1949) 1–13.

4. ———, On some connections between probability theory and differential equations, Proc. of Second Berkeley Symposium on Math. Statistics and Probability, (1951) 189–215.

5. I. M. Gelfand and A. M. Yaglom, Integration in functional spaces and its applications in quantum physics, J. Math. Phys., 1 (1960) 48–69.

6. S. G. Brush, Functional integrals and statistical physics, Rev. Mod. Phys., 33 (1961) 79–92.

7. M. Kac, Probability and Related Topics in Physical Sciences, Lectures in Applied Mathematics Boulder, 1957.

8. R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965.

9. Yu. L. Daletskii, Functional integrals connected with operator evolution equations, Russian Math. Surveys, 17 (1962) 1–107.

10. E. W. Montroll, Markov chains, Wiener integrals, and quantum theory, Comm. Pure Appl. Math., 5 (1952) 415–453.

11. V. S. Buslaev, Continuum integrals and the asymptotic behavior of the solutions of parabolic equations as $t \to 0$. Applications to diffraction, in Topics in Mathematical Physics, ed. Sh. Berman; English Transl., Consultants Bureau, New York, 1967.

12. C. Morette, On the definition and approximation of Feynman's path integral, Phys. Rev., 81 (1951) 848–852.

13. R. H. Cameron, A family of integrals serving to connect the Wiener and Feynman integrals, J. Math. and Phys., 39 (1960) 126–140.

14. E. Nelson, Feynman integrals and the Schroedinger equation, J. Math. Phys., **5** (1964) 332–343.

15. E. J. McShane, Integrals devised for special purposes, Bull. Amer. Math. Society, 69 (1963) 597–627.

16. C. Morette DeWitt, Feynman's path integral. Definition without limiting procedure, Comm. Math. Phys., 28 (1972) 47–67.

17. E. T. Copson, Asymptotic Expansions, Cambridge Tracts in Math. and Math. Phys. 55, Cambridge University Press, 1965.

18. J. B. Keller, Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems, Annals of Physics, 4 (1958) 180–188.

19. V. P. Maslov, Perturbation theory and asymptotic methods (in Russian), Moskov. Gosud. Univ., Moscow 1965; French translation, Dunod, Paris (1972).

20. V. I. Arnold, On a characteristic class entering in the quantization conditions. Functional Analysis and its Applications, 1 (1967) 1–13, (in Russian and also English translation).

21. J. B. Keller, A geometrical theory of diffraction, Amer. Math. Soc., Calculus of Variations and Its Applications; McGraw-Hill, New York, 1958, 27–52.

22. D. W. McLaughlin, Complex time, contour independent path integrals, and barrier penetration, J. Math. Phys., 13 (1972) 1099–1108.

23. ———, Path Integrals, asymptotics, and singular perturbations, J. Math. Phys., 13 (1972) 784–796.

24. M. Gutzwiller, Phase-integral approximation in momentum space and the bound states of an atom, J. Math. Phys., 8 (1967) 1979–2000.

25. L. S. Schulman, A path integral for spin, Phys. Rev., 176 (1968) 1558–1569.

26. J. S. Dowker, Quantum Mechanics on group space and Huygen's principle, Annals of Physics, 62 (1971) 361–382.

27. L. D. Eskin, The heat equation and the Weierstrass transform on certain symmetric Riemannian spaces, A.M.S. Translations, 75 (1968) 239–255.

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THE PROBLEM OF LEARNING TO TEACH

I. THE TEACHING OF PROBLEM SOLVING — BY P. R. HALMOS

The best way to learn is to do; the worst way to teach is to talk.

About the latter: did you ever notice that some of the best teachers of the world are the worst lecturers? (I can prove that, but I'd rather not lose quite so many friends.) And, the other way around, did you ever notice that good lecturers are not necessarily good teachers? A good lecture is usually systematic, complete, precise — and dull; it is a bad teaching instrument. When given by such legendary outstanding speakers as Emil Artin and John von Neumann, even a lecture can be a useful tool — their charisma and enthusiasm come through enough to inspire the listener to go forth and do something — it looks like such fun. For most ordinary mortals, however, who are not so bad at lecturing as Wiener was — nor so stimulating!— and not so good as Artin — and not so dramatic! — the lecture is an instrument of last resort for good teaching.

My test for what makes a good teacher is very simple: it is the pragmatic one of judging the performance by the product. If a teacher of graduate students consistently produces Ph. D.'s who are mathematicians and who create high-quality new mathematics, he is a good teacher. If a teacher of calculus consistently produces seniors who turn into outstanding graduate students of mathematics, or into leading engineers, biologists, or economists, he is a good teacher. If a teacher of third-grade "new math" (or old) consistently produces outstanding calculus students, or grocery store check-out clerks, or carpenters, or automobile mechanics, he is a good teacher.

For a student of mathematics to hear someone talk about mathematics does hardly any more good than for a student of swimming to hear someone talk about swimming. You can't learn swimming technique by having someone tell you where to put your arms and legs; and you can't learn to solve problems by having someone tell you to complete the square or to substitute $\sin u$ for y.

Can one learn mathematics by reading it? I am inclined to say no. Reading has an edge over listening because reading is more active — but not much. Reading with pencil and paper on the side is very much better — it is a big step in the right direction. The very best way to read a book, however, with, to be sure, pencil and paper on the side, is to keep the pencil busy on the paper and throw the book away.

Having stated this extreme position, I'll rescind it immediately. I know that it is extreme, and I don't really mean it — but I wanted to be very emphatic about not going along with the view that learning means going to lectures and reading books. If we had longer lives, and bigger brains, and enough dedicated expert teachers to have a student/teacher ratio of 1/1, I'd stick with the extreme view — but we don't. Books and lectures don't do a good job of transplanting the facts and techniques of

Talks given at the Annual Meeting in San Francisco, January 17, 1974, at a joint AMS-MAA Panel discussion.

the past into the bloodstream of the scientist of the future — but we must put up with a second best job in order to save time and money. But, and this is the text of my sermon today, if we rely on lectures and books only, we are doing our students, and their students, a grave disservice.

What mathematics is really all about is solving concrete problems. Hilbert once said (but I can't remember where) that the best way to understand a theory is to find, and then to study, a prototypal concrete example of that theory, a root example that illustrates everything that can happen. The biggest fault of many students, even good ones, is that although they might be able to spout correct statements of theorems, and remember correct proofs, they cannot give examples, construct counterexamples, and solve special problems. I have seen many students who could state something they called the spectral theorem for Hermitian operators on Hilbert space but who had no idea how to diagonalize a 3×3 real symmetric matrix. That's bad — that's bad learning, probably caused, at least in part, by bad teaching. The full-time professional mathematician and the occasional user of mathematics, and the whole spectrum of the scientific community in between — they all need to solve problems, mathematical problems, and our job is to teach them how to do it, or, rather, to teach their future teachers how to teach them to do it.

I like to start every course I teach with a problem. The last time I taught the introductory course in set theory, my first sentence was the definition of algebraic numbers, and the second was a question: are there any numbers that are not algebraic? The last time I taught the introductory course in real function theory, my first sentence was a question: is there a non-decreasing continuous function that maps the unit interval into the unit interval so that length of its graph is equal to 2? For almost every course one can find a small set of questions such as these — questions that can be stated with the minimum of technical language, that are sufficiently striking to capture interest, that do not have trivial answers, and that manage to embody, in their answers, all the important ideas of the subject. The existence of such questions is what one means when one says that mathematics is really all about solving problems, and my emphasis on problem solving (as opposed to lecture attending and book reading) is motivated by them.

A famous dictum of Pólya's about problem solving is that if you can't solve a problem, then there is an easier problem that you can't solve — find it! If you can teach that dictum to your students, teach it so that they can teach it to theirs, you have solved the problem of creating teachers of problem solving. The hardest part of answering questions is to ask them; our job as teachers and teachers of teachers is to teach how to ask questions. It's easy to teach an engineer to use a differential equations cook book; what's hard is to teach him (and his teacher) what to do when the answer is not in the cook book. In that case, again, the chief problem is likely to be "what is the problem?". Find the right question to ask, and you're a long way toward solving the problem you're working on.

What then is the secret — what is the best way to learn to solve problems? The answer is implied by the sentence I started with: solve problems. The method I advocate is sometimes known as the "Moore method," because R. L. Moore developed and used it at the University of Texas. It is a method of teaching, a method of creating the problem-solving attitude in a student, that is a mixture of what Socrates taught us and the fiercely competitive spirit of the Olympic games.

The way a bad lecturer can be a good teacher, in the sense of producing good students, is the way a grain of sand can produce pearl-manufacturing oysters. A smooth lecture and a book entitled "Freshman algebra for girls" may be pleasant; a good teacher challenges, asks, annoys, irritates, and maintains high standards — all that is generally not pleasant. A good teacher may not be a popular teacher (except perhaps with his *ex*-students), because some students don't like to be challenged, asked, annoyed, and irritated — but he produces pearls (instead of casting them in the proverbial manner).

Let me tell you about the time I taught a course in linear algebra to juniors. The first hour I handed to each student a few sheets of paper on which were dittoed the precise statements of fifty theorems. That's all — just the statements of the theorems. There was no introduction, there were no definitions, there were no explanations, and, certainly, there were no proofs.

The rest of the first hour I told the class a little about the Moore method. I told them to give up reading linear algebra (for that semester only!), and to give up consulting with each other (for that semester only). I told them that the course was in their hands. The course was those fifty theorems; when they understood them, when they could explain them, when they could buttress them with the necessary examples and counterexamples, and, of course, when they could prove them, then they would have finished the course.

They stared at me. They didn't believe me. They thought I was just lazy and trying to get out of work. They were sure that they'd never learn anything that way.

All this didn't take as much as a half hour. I finished the hour by giving them the basic definitions that they needed to understand the first half dozen or so theorems, and, wishing them well, I left them to their own devices.

The second hour, and each succeeding hour, I called on Smith to prove Theorem 1, Kovacs to prove Theorem 2, and so on. I encouraged Kovacs and Herrero and all to watch Smith like hawks, and to pounce on him if he went wrong. I myself listened as carefully as I could, and, while I tried not to be sadistic, I too pounced when I felt I needed to. I pointed out gaps, I kept saying that I didn't understand, I asked questions about side issues, I asked for, and sometimes supplied, counterexamples, I told about the history of the subject when I had a chance, and I pointed out connections with other parts of mathematics. In addition I took five minutes or so of most hours to introduce the new definitions needed. Altogether I probably talked 20 minutes out of each of the 50-minute academic hours that we were together. That's a lot — but it's a lot less than 50 (or 55) out of 50.

It worked like a charm. By the second week they were proving theorems and finding errors in the proofs of others, and obviously taking pleasure in the process. Several of them had the grace to come to me and confess that they were skeptical at first, but they had been converted. Most of them said that they spent more time on that course than on their other courses that semester, and learned more from it.

What I just now described is like the "Moore method" as R. L. Moore used it, but it's a much modified Moore method. I am sure that hundreds of modifications could be devised, to suit the temperaments of different teachers and the needs of different subjects. The details don't matter. What matters is to make students ask and answer questions.

Many times when I've used the Moore method, my colleagues commented to me, perhaps a semester or two later, that they could often recognize those students in their classes who had been exposed to a "Moore class" by those students' attitude and behavior. The distinguishing characteristics were greater mathematical maturity than that of the others (the research attitude), and greater inclination and ability to ask penetrating questions.

The "research attitude" is a tremendous help to all teachers, and students, and creators, and users of mathematics. To illustrate, for instance, how it is a help to me when I teach elementary calculus (to a class that's too large to use the Moore method on), I must first of all boast to you about my wonderful memory. Wonderfully bad, that is. If I don't teach calculus, say, for a semester or two, I forget it. I forget the theorems, the problems, the formulas, the techniques. As a result, when I prepare next week's lecture, which I do by glancing at the prescribed syllabus, or, if there is none, at the table of contents of the text, but never at the text itself, I start almost from scratch — I do research in calculus. The result is that I have more fun than if I had it all by rote, that time after time I am genuinely surprised and pleased by some student's re-discovery of what Leibniz probably knew when he was a teenager, and that my fun, surprise, pleasure, and enthusiasm is felt by the class, and is taken as an accolade by each discoverer.

To teach the research attitude, every teacher should do research and should have had training in doing research. I am not saying that everyone who teaches trigonometry should spend half his time proving abstruse theorems about categorical teratology and joining the publish-or-perish race. What I am saying is that everyone who teaches, even if what he teaches is high-school algebra, would be a better teacher if he thought about the implications of the subject outside the subject, if he read about the connections of the subject with other subjects, if he tried to work out the problems that those implications and connections suggest — if, in other words, he did research in and around high-school algebra. That's the only way to keep the research attitude, the question-asking attitude, alive in himself, and thus to keep it in a condition suitable for transmitting it to others.

Here it is, summed up, in a few nut shells:

The best way to learn is to do — to ask, and to do.

The best way to teach is to make students ask, and do. Don't preach facts — stimulate acts.

The best way to teach teachers is to make them ask and do what they, in turn, will make their students ask and do.

Good luck, and happy teaching, to us all.

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II. THE PROBLEM OF LEARNING TO TEACH — BY E. E. MOISE

It was a real pleasure to listen to Professor Halmos's talk. Seldom have I heard so much to agree with, and so much to applaud. He has given us a beautiful description of our task as teachers. And the description implied — as it had to — a wholesale rejection of the naive empiricism and naive behaviorism which have become an endemic plague in much of the educational world.

In the present state of our knowledge, teaching is an art. In mathematics, at least, attempts to turn it into a science have been retrogressive, in every case that I know of. Even when mathematics is taught poorly or only passably, we take for granted that students will have an opportunity to react to it in different ways, and to learn it at different levels, according to their own talents, temperaments, and motivations. At least, we *used* to take this for granted, until various people found ways to put a stop to it. I believe that the ultimate caricature of good mathematical teaching is linear error-free programming. Under this scheme, instead of taking care to ensure that every student is provided with the most stimulating challenges that he can react to successfully, people use their best efforts to create a situation in which nobody is faced with any challenge at all.

Certain ways of using "modules" have the same vice in a milder form. Some schools are now using a scheme under which courses are split up into small parts (the "modules"), with a standard test for each of them. When a student had passed the test on one module, he is ready to move on to the next. In some schools, at least, the rules prescribe that a student's grade at the end of the year is based on the number of modules that he has completed. Since the tests are of such a sort that almost any student can pass them eventually, the moral conveyed by all this is that an A-student is one who acquires a C-knowledge of mathematics at high speed. I suppose it is possible for a student in such a program to analyze ideas in depth, and to spend lots of time working on hard problems. But to behave in such a way, the student would have to resist the suggestions conveyed to him by the people who are receiving pay on the ground that they are promoting the student's intellectual development.

One of the difficulties with the pseudoscientific "learning theorists" is that they concentrate their attention on those aspects of the learning process that are capable of being meticulously observed and measured. Such a proceeding is not valid, or

even safe; we simply don't know enough about learning processes to do anything predicated on the notion that our knowledge is complete. Curiously, there is empirical evidence against the validity of the empiricists' conception of learning.

In the early 1960's, Dr. Lyn Carlsmith (Harvard Educational Review, vol. 34 (1964), pp. 3–21) found a group of 20 male students, in the Harvard College class of 1964, whose fathers had gone overseas when their sons were no more than six months old, and had not returned until at least two years later. She then took a carefully matched control group of 20 male students whose fathers had not been absent in their early childhood. The SAT test was given to both groups. This test is in two parts, mathematical and verbal. Ordinarily, the difference $M - V$ of the mathematical and verbal scores M and V is positive for boys and negative for girls. The control group conformed to this expectation: in 18 cases out of 20, $M - V$ was positive. But in the "father-absent" group, $M - V$ was positive in only 7 cases out of 20. For a smaller group of 18 doctors' sons, similarly matched, the results were even more striking: in the control group, $M - V$ was positive in 7 cases out of 9, while in the father-absent group, $M - V$ was positive in only *one* case out of 9.

Further study of larger samples confirmed all this. Apparently, $M - V$ diminishes sharply as the duration of the father's absence increases; and the absence of the father in the *first six months* of a boy's life makes a significant difference in the relation between his SAT scores twenty years later.

These results are hard to reconcile with two views now widely held, namely, (1) intellectual capacities that seem to be purely cognitive really are, and (2) these capacities are acquired in ways that are readily accessible to empirical study. I believe that both these notions are not just inexact but very wide of the mark. It would be interesting to know just what it is that fathers teach their baby boys, and how the fathers go about it.

Obviously this study left important questions unanswered. For example, did the absence of the father inhibit the growth of mathematical faculties, or promote the growth of verbal ability, or both? (There is the prior question whether the "mathematical" part of the SAT test measures the sort of ability that produces a mathematician.) The study reminds us, however, of something that we should have known all along, that some of the most important learning processes go on when nobody is looking, and that they go on in ways that are very hard to keep track of. It is simplistic to suppose that people remember what they are told, and understand the things that are explained to them clearly. More commonly, people remember what interests them, and understand the things that they enjoy understanding. Thus intellectual development is linked with development of personality, and the refinement and enlargement of esthetic perceptions is a vital part of intellectual growth. This sort of growth does not lend itself to mechanization.

The processes by which people learn to teach are equally obscure. Some years ago — or so the story goes — a class in Social Relations at Harvard played an elaborate prank on their section man. The section man was in the habit of pacing back

and forth while talking. In a secret caucus, the class agreed on a imaginary line, down the middle of the classroom. When the teacher moved to the left of the line, the class became eager and alert. When he stepped to the right of the line, the class became apathetic. When the class had taught the teacher to stay to the left of the center line, they gradually moved the line, until at the end of two or three weeks they had the teacher boxed into a corner. He had no idea of what was going on.

This was different, in a way, from the usual process under which students teach teachers to teach. But I think that the main difference is the students knew what they were doing. Ordinarily, I believe the process is unconscious for everybody.

This brings us, at last, to the question that I was supposed to be discussing at the outset: granted that teaching is an art, learned by experience, what can we do to help people to learn it? It seems to me that beginning teachers can probably get a great deal of help from policies which could easily be carried out in most departments.

One of the greatest troubles, I believe, in the initial teaching experience, is that the learning of teaching is virtually solitary. At the places that I know about, senior faculty members visit each teaching fellow's classes about once a semester. Even for purposes of evaluation, these procedures are perfunctory, and their value as teacher training is nil. It is hard to think of another art that people are expected to learn in such a way, with no significant help in the form of knowledgeable criticism.

This suggests that we should try to turn the learning of teaching into a group activity. I propose the following scheme. Beginning teachers would be organized in groups of about five, with *identical* teaching assignments, preferably a single course. They would share an office, so that they could conveniently discuss the problems that they all faced. Schedules would be arranged so that they could visit one another's classes. They would all meet, at least once a week, in a sort of "teaching seminar," to discuss what was going on. Each would have full responsibility for his own section, pacing the course to suit himself, subject only to the loose constraints imposed by the place that the course was supposed to fill in the curriculum. Each would make up his own assignments, and write his own hour tests and final examination. If a highly skilled senior faculty member formed part of the group, or met with them as an advisor, this would no doubt be helpful; but I believe that the senior man ought to be an advisor and not a boss. Classroom visits by peers would be much more frequent than visits by the advisor.

I see reason to hope that this sort of consultative effort would improve and vastly accelerate the process by which teachers learn by experience. Some features of it may need further explanation.

(1) I believe that there are such things as pedagogical principles. But even if we agree on what these are, they are hard to demonstrate, or even to convey, by abstract statements, and the art of putting them into practice takes quite a while to learn. I think that discussions of pedagogic questions are of immediate practical utility in proportion to their specificity. Hence an arrangement under which beginning teachers

would discuss not the general problems of education but rather, at a given moment, the problem of teaching a particular topic at a particular stage in a particular course. Under these conditions, I think that general ideas will emerge, in such forms that their meanings will be clear and the extent of their validity will be evident. This is why I think it vital for the teaching assignments to be identical; we need a situation in which the people discussing teaching problems have the same problems on their minds.

(2) If the group has a supervisor who tells everybody exactly what to do, he will almost certainly be telling some of them the wrong things. It is no part of any teacher's job to duplicate the performance of any other teacher however skilled. Teaching is an interpersonal relation, and optimal styles depend on the personalities of individuals. Such styles do and should change in response to class reactions.

(3) Moreover, if all important decisions are made by some higher authority, the beginning teacher will be less likely to come to grips, in his own mind, with the sort of problem that he will have to solve for himself in future years when the boss is gone. Hence the proposal that beginning teachers have full responsibility for their own courses, at a time when they have the benefits of consultation and criticism.

It seems likely that this scheme would amply repay the effort that it would require. Obviously, the only way to find out is to try it. I believe, however, that it involves at least one important hazard and has at least two important limitations.

First, it may be that working under observation, even by peers, will make people over-cautious, in an attempt to avoid the possibility of looking foolish. Probably this danger can be minimized if people are clearly aware of it. It seems especially important for the advisor to be aware of it, and for him to be of a gentle disposition.

Second, the whole scheme, in the form described, deals with fairly traditional teaching, in which the general content and method of the course are taken as given. This means that the skills acquired are only the beginning of professional maturation. The best courses that I know of were of the teacher's own design, and in some cases they were improvisations, whose outcome was not known even to the teacher at the outset. I believe, however, that fairly conventional teaching is a natural first step in professional development. This is a limit to the problems that one man can think about in one semester.

Finally, I don't think that we ought to feel complacent about our present lack of an adequate theory of teaching. If we had such a theory, we would be better off, and I think that one of the tasks of the coming generation is to create one. I have no idea of the form that such a theory might take. Perhaps its most likely inventors are people each of whom has a sophisticated and creative grasp both of mathematics and of psychology.

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III. THE PROMOTION OF PARTICIPATION — BY GEORGE PIRANIAN

I address this to teachers of graduate and undergraduate students, to teachers in junior colleges, and perhaps to high-school teachers. Teachers in elementary schools already know what I have to say.

My colleagues have discussed ways to stimulate classroom participation. Paul Halmos has talked about participation by students, and Ed Moise proposes to inject life into the teachers. I shall try to reinforce their message with a story, and I'll mention a few relevant technicalities.

In 1967–68, The University of Michigan let me teach a section of the honors course in calculus. Because of a long period without freshman contacts, the prospect filled me with fear; but the students were a lovable lot, and we soon developed effective cooperation.

We had a solid book. Unfortunately, the author had taken himself a bit too seriously, and consequently the text was on the dreary side. To compensate, I regularly assigned special problems. For example, I asked the students to prove or disprove that if a real-valued function on the line is continuous at a point, then it is continuous throughout some neighborhood of that point. The subsequent classroom discussion of such a problem could chew up an entire period. But the course ran well, and I was so pleased that at the end I asked one of the girls to grade papers for me during her sophomore year.

In June, Addison-Wesley sent me a copy of Joseph Kitchen's *Calculus of One Variable*. Because the book looked lively, I thought we should try it, and to show my affection for the grader, I wrote to the publisher and requested that he send a copy to her home.

In September, when Lisa came to my office, I asked her opinion, and she said "It's just like the book we used, except that the Piranian problems are already in it." The students bought the book, and I looked forward to a great year.

After one week, I felt apprehensive, and soon I sensed the cold shadow of failure. Despite the excellent text and the bright students, the class sat glued to the runway. And then it happened that Kitchen skipped a point I consider important, and this forced me to devise a special problem for the occasion. The consequence was dramatic. With a roar of the engines and a slight shudder of the fuselage, we took off for the white clouds in the blue sky.

The moral is simple: no matter how sound, complete, and clear my text or lecture notes may be, the students should know that I'm developing the course especially for them, and that I'm turning myself inside-out in their behalf. For example, I must not assign homework by opening the text to page 93 and saying "for next time, try problems 3, 7, 10, 16, 19, and let me see, 21; class dismissed." We'll come back to this in a few minutes.

I must not give the impression of a man hired to teach as many students as possible and wired to do it with maximum industrial efficiency. I must indulge in extensive participation; the best way to achieve this is to recognize that this year's students require a new course, and that regardless of the cost, my section deserves special treatment. You can't teach with the left hand; you can't teach with the right hand. Like playing volley-ball, swimming, or racing a small sailboat, the job requires both hands, both arms, and the muscles of the legs and the torso.

The job takes more. I can preach an eloquent sermon on the gospel according to Darboux and Riemann, or give a spirited performance on Cantor sets, or use both hands and feet in a glorious axiomatic fugue, and yet reap substantial failure. No man can please all the people all the time, and no style of teaching is effective for all students. Therefore, successful teaching requires cooperation from the class.

You and I would know how to live, if we were young again. Meanwhile, multitudes of boys and girls suffer from awkwardness, uncertainty, and hesitation. Ask a dozen of your students with how many of their classmates they are acquainted, and you'll be astonished to learn about the bleakness and academic isolation in which some of them exist. A few years ago, I hit upon an unobtrusive way of sending a bit of mature wisdom across the generation gap. Early each term, I distribute a dittoed sheet listing the Ann Arbor addresses and telephone numbers of the entire class. This may encourage collaboration on homework; but it does not produce the miserable situation in which Archibald copies Merthiolate's paper fifteen minutes before it is due. Half of the class may meet for a great jam session. Leaders emerge, and the strong give guidance to the weak. I should share my salary with four or five students. They do some of my most difficult work, and I receive credit for their success. The kids learn to communicate, and when the homework is done, they may be so full of social steam that they go jogging together. If a few hundred of us were to trot from the Hilton to the Fisherman's Wharf, San Francisco would notice our physical condition and our social cohesion.

I've come back to homework. I do not know how to present mathematical ideas so effectively that students can take possession of them simply by sitting at my feet and smelling my socks. Let me change to a slightly less offensive metaphor: after grazing in my lush pastures, the students must ruminate; they must dedicate substantial time to the chewing of the cud. That's why we need homework.

Suppose now that our calculus text has a set of problems on integration by parts, a set on masses and centroids, a set on cylindrical and spherical coordinates. In each set, the problems range from the trivial to bread-and-butter drill, and they may end with a few important stinkers.

This is a reasonable arrangement of the text. A natural way of running the homework show is to assign problems from Set 23 today, problems from Set 24 tomorrow, and so forth. This is efficient for the teacher, for the students, and for the grader, and it is consistent with the principle of orderly progress. Nevertheless, the practice is a manifestation of pedagogic brutality. The poor boy who can barely manage

Problem 7 never gets the benefit of Problems 10, 16, and 19, except during a discussion that he endures passively because in his inexperienced view it comes too late to be of any use.

A more effective assignment for tomorrow might look like this:

Problem 18 in Set 22,

Problems 15 and 16 in Set 23,

Problems 9 and 12 in Set 24,

Problems 1 and 4 in Set 25.

Under this plan, the difficult problem comes after a week of experience with easier exercises in the same topic. The student profits from repeated exposure, and the teacher has several opportunities for clarifying the basic principles and demonstrating the necessary technique. A tough piece of meat calls for slow cooking, and a difficult idea requires thought on several consecutive days. Use the scheme of staggered assignments, tell the students that you've carefully programmed the homework for maximum effectiveness, and make certain that you're telling the truth.

Staggered homework is a small technicality; but it makes a difference. Also, it illustrates the dictum that genius is the capacity for taking trouble.

I urge the mathematical community to strengthen its pedagogical effort, not by buying new gadgets, not by creating new committees of experts, but by intensification of personal effort. Let each man assume the responsibility for teaching with greater vitality. If this reduces his rate of publication by thirty percent, so much the better. It will be good news for libraries, and it will help save Mathematical Reviews.

In the deliberations among the elders, the first question about a man should be how well he teaches, the second question, how good his publications are — never, how numerous.

I do not say this because we should create more mathematicians; there are enough of us. Nor am I concerned with the problem of generating stronger enrollment in mathematics classes to prevent economic dislocation of superannuated fuddy-duddies. There's one commodity that the world needs above everything else, and for which we'll never develop a satisfactory substitute. We need good men and women. As teachers, we have the desperately urgent task of communicating to the young some of the intellectual values of civilized mankind. We have the task of inspiring students to rise to the highest level of excellence that they can attain. Our survival depends on our collective success. I apologize for ending on such a serious note; but we face a problem of the utmost importance.

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SALOMON BOCHNER ON CHARLES S. PEIRCE

I

CAROLYN EISELE

My remarks at the Conference on the History of American Mathematics, relative to the paper written by and read for Salomon Bochner at Texas Tech University in May 1973, are still applicable to his *Mathematical Reflections* in this MONTHLY, 81 (1974) 827–853. For Professor Bochner's analysis of Charles S. Peirce's mathematical treatment of the continuity concept in the overall framework of Peirce's philosophical system, reflects a lack of acquaintance with a large segment of basic material in that area of Peirce's writings. Those manuscripts lie unpublished in Peirce's handscript in Houghton Library at Harvard University, and are only gradually being made available to the scholarly world in new editions.

I am editing the forthcoming edition of *The New Elements of Mathematics by Charles S. Peirce*. Since this material is not readily accessible, Professor Bochner seems to have relied heavily on the *Collected Papers of Charles Sanders Peirce*, vol. I–VI (1932–1935), probably in the belief that everything of value in Peirce's systematic thinking was included in those volumes. Peircean scholars, however, now speak of that collection as selected portions of selected papers where long mathematical passages of reference have been deleted and where purely mathematical papers have been totally omitted. It is unfortunate that Professor Bochner makes no mention of volumes VII and VIII of the *Collected Papers*, edited by Arthur Burks and published in 1958, long after volumes I–VI.

Professor Bochner also cites as evidence of the validity of his assessment of Peirce as "a real American Tragedy" and "a great philosopher, and an even greater failure," negative conclusions found in parts of Murray Murphey's *The Development of Peirce's Philosophy* (1961), written at a time when much of the unpublished manuscript material was still in a disordered state. Because of different depository arrangements at that time, Peirce papers were to be found in several different places in the Harvard libraries and some of them were restricted in use. It has taken years of dedicated work on the part of several scholars, notably Max and Ruth Fisch, to bring order into the collection. This was necessary before Professor Fisch could embark on the writing of the definitive biography of Peirce. Today the Peirce Collection is all of one piece and runs to some 1650 manuscripts and some 1600 correspondence folders, a vast monument to Peirce's genius.

During the past dozen years incomplete manuscripts have had component parts restored, much research has taken place in them, a vast literature is being created. Professor Bochner writes of the failure of Peirce to produce a comprehensive treatise. Now it happens that Peirce *wrote* several treatises, although opportunities for publication on his terms were denied to him. This was the case with Peirce's books on mathe-

matics, and I dare say that Professor Bochner will be pleased one day to find the "fundamental theorem on prime numbers" in vol. 2, chap. 4, art. 49 as Theorem 28 in the *New Elements*.

In view of the many inquiries that have arisen regarding Peirce's mathematics, as a consequence of Professor Bochner's article, I shall submit to the editor of the MONTHLY, in the near future, a more extended view of Peirce's philosophy and his mathematical competence. For those who would now have a bird's-eye view of Peirce as mathematician and scientist, a biographical statement may be found in the newly published vol. X of the *Dictionary of Scientific Biography*.

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SALOMON BOCHNER ON CHARLES S. PEIRCE

II

MAX H. FISCH

Quite apart from the interest of the other "mathematical reflections" in Salomon Bochner's essay in the MONTHLY for October 1974, his pages on Charles S. Peirce (828, 830–831, 838–852) may become a landmark in the history of mathematics and of philosophy, as being the last essay on Peirce as mathematical philosopher written without benefit of Carolyn Eisele's edition of his mathematical writings.

Since readers familiar with Peirce may otherwise be put off by the criticisms and the doubtful statements of preceding pages, I urge them to read first the positive appreciation of him that runs from page 848 line 14 through page 852 line 2. By itself, it would rank high among short accounts of his importance in the history of science. There is a sentence pointing forward to it on page 841: "Charles Sanders Peirce has a much more sophisticated version [than Leibniz had] of duality between continuity and discontinuity, or at least an anticipation of such a duality, a version of duality which reached into the 20th century and even became a hallmark of it — but, of this, later on."

Readers *not* familiar with Peirce, on the other hand, may welcome a little help at the start toward separating out and putting aside the more questionable parts of Professor Bochner's account. In what follows I shall refer to his "Mathematical Reflections" as MR; to his previous article on "Continuity and Discontinuity in Nature and Knowledge" (*Dictionary of the History of Ideas*, ed. Philip P. Wiener, 1973, vol. I, pp. 492–504) as C & D; and to Peirce's *Collected Papers* as CP (cited by volume and paragraph number).

I begin with minor details and move toward weightier matters.

Professor Bochner speaks of the Greek expression *tò synechés* as "adverbial"

(C & D 493, MR 839). It is rather the neuter singular of the adjective used as a noun, and means "the continuous," "that which is continuous," and hence "continuum."

Bochner regularly uses the Latin phrase *lex continui* as if it were Leibniz's name for his law of continuity (C & D 493, 498, 501, 502; MR 845, 846, 849, 850); but Leibniz called it rather *lex continuitatis* or, in French, *la loi de la continuité*.

Bochner thinks that Peirce's synechism is Leibniz's *lex continui* updated (C & D 501) and called by another name (MR 846). As we shall see, this is not quite so. He says that "Peirce obviously did not want to admit, to himself or to others, his indebtedness to Leibniz in this matter" (MR 846). I believe I have shown, in an article cited by Bochner (MR 846), that Peirce had no such reluctance in this matter or in any other. Bochner finds it "altogether remarkable that in the vast collection of Peirce's papers the name of Leibniz appears extremely rarely" (MR 846). Perhaps less often than those of Aristotle, Kant, and Hegel, I grant, but not far behind them. The evidence, however, is not yet in; the so-called *Collected Papers* are a small sample and not a fair one; they omit nearly all his mathematical and scientific writings and a large part of his logical, philosophical and historical ones; we await Professor Eisele's edition of the mathematical writings, and a much more comprehensive and strictly chronological edition of Peirce's writings as a whole.

In Bochner's opinion, Herbart and Fechner were justified in coining "synechology" because they wanted the adjective "synechological," but Peirce had no such justification for coining "synechism"; it was just "a linguistic affectation" (MR 847). Yet Bochner himself elsewhere uses Peirce's adjective "synechistic" (MR 851), and he ignores the facts that "synectic" was already in mathematical use before Peirce's time (for example, by Cauchy), and in medical use long before the times of Herbart and Fechner. That Peirce knew all this appears from his definitions of "synectic" and of "cause" (1.) in the *Century Dictionary*. He also defined "synechiology" (or "synechology") for the *Century*. If it would have served his purpose, his ethics of terminology would have forbidden him to coin another term.

But Peirce was, among many other things, a lexicographer. When a lexicographer wants an English name for a theory representing something as of prime importance, he takes the Greek name for that something, if it has one, and to the combining form of its Greek name he adds the anglicized Greek suffix-*ism*. Peirce wants a name for a theory representing continuity as of prime importance in framing scientific hypotheses. The Greek name for continuity is *synecheia*. Its combining form is *synech-*. Hence, *synechism*. Neither *lex continuitatis* nor synechology nor any other previously coined term had quite that meaning.

Bochner rightly observes that, analytically, in mathematics as commonly taught, the concept of continuity is subordinate to that of function, but that, historically, "it was the urge to come to grips mathematically with the concept of continuity that was greatly responsible for the gradual emergence of the concept of a mathematical function too." He has the impression, however, that Leibniz understood this better than Peirce did (MR 841). My impression is rather that Peirce was perfectly familiar

both with the historical emergence of function within continuity and with the analytic definition of continuity within function — that is, with the distinction (or distinctions) between the continuous and the discontinuous as coordinate species of function. (See his *Century Dictionary* definitions.) But neither Leibniz nor he at any time conceived continuity as a character shared *only* by continuous functions. The difference was that in the end Peirce conceived and defined a "true continuity" which was *not at all* shared by continuous functions; and what writers on the theory of functions called a continuum, Peirce in the end called a pseudo-continuum (CP 1.185; 6.176; 7.652).

Though he rightly stresses the pervasiveness in Peirce's writings of the categorial triad, Firstness, Secondness, and Thirdness, Bochner is mistaken in finding a parallel triad in Synechism, Tychism, and Agapism (C & D 502, MR 852). Peirce nowhere links the latter three with each other or with the former three. "Agapism" occurs only in a single paper, on theories of evolution, and the only triad to which Peirce assigns it has for its first and second members tychism and anancism (CP 6.302). If there is any triad to which tychism and synechism belong, its second member is pragmatism, which Bochner says he does not understand at all (MR 830). If he really doesn't, then he may not quite understand tychism or synechism either, since Peirce describes synechism as a synthesis of tychism and pragmatism (CP 4.584). All three are regulative principles for the framing of scientific hypotheses. All three belong to what Peirce called the logic of science. (Parenthetically, one thing *I* don't understand is Bochner's phrase on MR 847, "the presence of synechism.")

Bochner says that "Peirce was one of the first of a species of philosophers who, by trend, intent, or circumstances, had been blurring the several demarcations between mathematics, mathematical logic, philosophy of mathematics, and general philosophy" (MR 847). Not by intent, certainly, for I can think of nobody who kept returning so frequently, over so long a time, to the classification of the sciences, or who was more concerned to get the demarcations and the priorities clear. For example, nobody was more insistent that logic did not depend on psychology, and that mathematics did not depend on logic or on any other science.

Bochner follows Morris Cohen and Murray Murphey in ascribing to Peirce the lifelong ambition of creating "a vast philosophic system," the master key to which should be his synechism or theory of continuity (C & D 502, MR 838). And Bochner further follows Murphey in attributing Peirce's failure to his finding no way to utilize the continuum concept effectively (MR 839). But Bochner goes on to argue that what Peirce attempted was impossible (C & D 502, MR 847).

Now it is true that in the five-year period 1890-1894, Peirce drafted several sketches of such a system, published a series of articles outlining some parts of it, and projected a twelve-volume *Principles of Philosophy*. But what Peirce devoted his life to, before 1890 and after 1894 and between them, was a system of logic — more exactly, of logic "considered as semiotic," the general theory of signs. And within that field, his more special study was that of "the logic of science."

In all that Peirce wrote on continuity, the key sentence is: "Synechism is not an ultimate and absolute metaphysical doctrine; it is a regulative principle of logic, prescribing what sort of hypothesis is fit to be entertained and examined" (CP 6.173). And next to that the sentence: "The general motive is to avoid the hypothesis that this or that is inexplicable" (CP 6.171). That is, hypotheses of discontinuity are not to be adopted until hypotheses of continuity have failed. This is not to say that there are no discontinuities — for Peirce held, as Bochner recognizes, that discontinuity is involved in all existence (MR 849) — but only that science does not advance by supposing discontinuities to be absolute.

Which brings me, finally, to the two positive emphases in Bochner's account; namely, on "Peirce's anticipation of the 20th century duality principle of quantum theory" (MR 850) and on his "insight into the fact that *evolution*, in its post-Darwinian explication, straddles and fuses continuity and discreteness both" (849). There have been several studies of Peirce's theory (or theories) of evolution, but none which works this out in detail. A short essay by Charles Hartshorne on "Charles Peirce and Quantum Mechanics" (*Transactions of the Charles S. Peirce Society* 9: 191–201, 1973) has preceded Bochner in finding anticipations of wave-particle duality. But no student of Peirce has begun with his first professional publication, in 1863, on "The Chemical Theory of Interpenetration," and followed on through the fifty-year vein it opened up. Peirce's career was that of a chemist and physicist, and it is high time for such a study. If Bochner's essay, coming so soon after Hartshorne's, gives rise to one or more such studies, it will have done us a great service.

Meanwhile, we may hope to learn from Professor Eisele's edition whether Peirce succeeded in constructing a usable mathematical definition of "true continuity."

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A NEW USE FOR AN OLD COUNTEREXAMPLE

MARY L. BOAS AND R. P. BOAS, JR.

We all learn in calculus courses to expand a function in a power series and to be sure that the remainder tends to zero, so that we know that the series actually converges to the function we got it from. Students do not like to investigate remainders in cases when the series obviously converges, and have to be convinced, by an example, of a function whose convergent power series does not represent it. The usual example is $\exp(-1/x^2)$ (defined to be 0 at 0), where the power series is just $0+0+0+\cdots$, which obviously does not represent this function. Students, and even their teachers, in subjects other than mathematics, are apt to react to this example by saying something like "This is a contrived example and such functions

never occur in applications." Here we discuss the mathematics of a problem from elementary quantum mechanics whose solution inevitably involves $\exp(-1/x^2)$.

Before stating the physical problem, we state the mathematical problem to which it leads. Consider the following differential equation on $r > 0$:

$$r^{-1}(rR')' = \begin{cases} -\alpha^2 R, & 0<r<a, \\ \beta^2 R, & r>a, \end{cases} \tag{1}$$

where $-V_0 < E < 0$,

$$\alpha^2 = C(V_0 - |E|),\ \beta^2 = C\,|\,E\,|, \tag{2}$$

and C is a positive constant. The problem is to decide whether, however close V_0 is to 0, there is always an E such that (1) has a solution which is bounded at the origin and approaches 0 as r tends to infinity.

The physical problem is the quantum mechanical problem of a particle in a two-dimensional potential well. The differential equation for this problem is the Schrödinger equation in two dimensions in polar coordinates, namely

$$\nabla^2\psi + C(E-V)\psi = 0, \quad C = 2m/\hbar^2,$$

where the potential well is defined by $V = -V_0$ for $r<a$ and $V=0$ for $r>a$. To solve the Schrödinger equation one assumes a solution of the form $R(r)\Theta(\theta)$ and finds that $d^2\Theta/d\theta^2 = -n^2\Theta$; with $n = 0$, this then leads to (1) and (2) as the equations determining R.

The classical analogue may be thought of as a golf ball (idealized as a point mass) in a cup of radius a. In classical mechanics, if the golf ball has less kinetic energy than the potential energy it would require to get to the rim of the cup, it will be trapped in the cup. However, in quantum mechanics it is conceivable that when the cup is sufficiently shallow (this means that V_0 in (1) is sufficiently close to 0) there are no values of E that will allow a solution of the required kind. That is, perhaps (1) has no eigenvalues for V_0 near 0; the physicist then says that there is no bound state. It is well known ([3], pp. 87, 88) that in the analogous three-dimensional case (which is harder to visualize) this is just what happens, whereas in the one-dimensional case there is a bound state no matter how small V_0 is. The problem ([3], p. 99, problem 10) is: what happens in the two-dimensional case described by (1) and (2)?

The most satisfying way to see whether (1) always has eigenvalues would be to solve the differential equation explicitly, but unfortunately it cannot be solved in terms of elementary functions. It can, however, be solved in terms of tabulated functions by a technique that is frequently used in physics but is absent from most introductory textbooks on differential equations. In each of the intervals $0 < r < a$, $r > a$, (1) is a version of Bessel's equation (see, e.g., [2], pp. 559–573). For physical reasons we require solutions that are bounded as $r \to 0$ and tend to 0 as $r \to \infty$.

We can find, from reference books that deal with Bessel functions, that any solution of (1) for $0 < r < a$ that is bounded at 0 is a constant multiple of the function denoted by $J_0(\alpha r)$, and any solution for $r > a$ that tends to 0 at ∞ is a multiple of $K_0(\beta r)$. (You should not be frightened by the Bessel functions, since all that you need to know here is easily looked up.) We also know (from the definition of "solution" or from physical requirements) that R and R' must be continuous at $r = a$, so we try to piece our two partial solutions together by using this constraint.

We need only one fact each about J_0 and K_0, namely the identities

$$J_0'(x) = -J_1(x), \qquad K_0'(x) = -K_1(x), \tag{3}$$

(J_1 and K_1 being two other Bessel functions). Now we have $R = AJ_0(\alpha r)$ for $0 < r < a$, $R = BK_0(\beta r)$ for $r > a$, and for continuity of R and R' we need

$$AJ_0(\alpha a) = BK_0(\beta a), \tag{4}$$

$$A\alpha J_0'(\alpha a) = B\beta K_0'(\beta a). \tag{5}$$

By (3) we can write (5) as

$$-\alpha AJ_1(\alpha a) = -\beta BK_1(\beta a). \tag{6}$$

If we divide (4) by (6) we get

$$\frac{J_0(\alpha a)}{\alpha aJ_1(\alpha a)} = \frac{K_0(\beta a)}{\beta aK_1(\beta a)} \tag{7}$$

and from (2) we get

$$\alpha^2 + \beta^2 = CV_0. \tag{8}$$

Our problem now comes down to whether (7) and (8) can be satisfied simultaneously for arbitrarily small V_0. If we put $\alpha a = x$, $\beta a = y$, this reduces to asking whether or not the curve defined by

$$\frac{J_0(x)}{xJ_1(x)} = \frac{K_0(y)}{yK_1(y)}, \tag{9}$$

or for short

$$J(x) = K(y), \tag{10}$$

intersects arbitrarily small circles $x^2 + y^2 = CV_0$ with center at the origin.

We cannot expect to find an explicit solution of (9), so we shall have to resort to approximations. Perhaps the first thought that anybody has nowadays is to turn to the computer; but a numerical solution of (9) for y as a function of x will not really tell us whether a solution of (9) actually passes through the origin, although it may strongly suggest that it does. It is also instructive to see how much can be learned *without* using the computer.

Since we are interested in the possibility of a solution that goes through $(0,0)$, it is rather natural to replace the functions in (9) by their approximate forms for x

and y near 0. These can be looked up, and are ([1], pp. 360 (9.1.10); 375 (9.6.13) (9.6.11))

$$\begin{aligned} J_0(x) &= 1 + O(x^2), \quad J_1(x) = \tfrac{1}{2}x + O(x^3), \\ K_0(y) &= -\log y + O(1), \quad yK_1(y) = 1 + O(y^2 \log y), \end{aligned} \tag{11}$$

where as usual $O(x^2)$ means a function which, after being divided by x^2, is bounded as $x \to 0$ (and similarly for other O-terms). If we retain only the leading term in each formula, (1) reduces to $-\log y = 2/x^2$, i.e.,

$$y = \exp(-2/x^2). \tag{12}$$

Since the curve whose equation is (12) does pass through the origin, we are led to conclude that the curve (10) also passes through the origin and consequently intersects all circles centered at the origin, however small they may be. Therefore the answer to the original problem is that (1) has eigenvalues for all small V_0, so that the two-dimensional problem is like the one-dimensional problem rather than like the three-dimensional problem.

It is interesting to note that we would get the same (correct) conclusion if we used its power series (namely $y = 0$) to "represent" $y = \exp(-1/x^2)$, even though it does not really represent the function at all!

You may well raise the question of whether this intuitively reasonable procedure really works. Have we actually shown that (10) has a solution $y = \phi(x)$ (say) for which $y \to 0$ as $x \to 0$, or have we tacitly assumed the answer to our question by introducing the approximate formulas for the Bessel functions as $x \to 0$ and $y \to 0$?

We can dispose of this objection by considering the remainder terms in the approximate formulas. We shall need the following fact: if $C_1(x) \to 0$ as $x \to 0$ then there is a neighborhood of the origin in which $1/(1 + C_1(x)) = 1 + C_2(x)$, where $|C_2(x)| < 2|C_1(x)|$. To see this, let x be so close to 0 that $|C_1(x)| < \frac{1}{2}$. Expanding $1/(1 + C_1(x))$ by the binomial theorem, we have

$$\begin{aligned} \frac{1}{1 + C_1(x)} - 1 &= -C_1(x) + C_1(x)^2 - \cdots, \\ \left| \frac{1}{1+C_1(x)} - 1 \right| &\leqq |C_1(x)| \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\ &= 2|C_1(x)|. \end{aligned}$$

As $x \to 0$, we have (giving separate names to the O-terms in (11))

$$J(x) \equiv \frac{J_0(x)}{xJ_1(x)} = \frac{1 + A_1(x)x^2}{\frac{1}{2}x^2 + A_2(x)x^4},$$

where $|A_1(x)|$ and $|A_2(x)|$ are bounded in some neighborhood of 0. That is, by our remark above, $J(x) = 2x^{-2}(1 + A_1(x)x^2)(1 + A_3(x)x^2)$, where $A_3(x)$ is another bounded function. Hence

$$J(x) = 2x^{-2}(1 + A_4(x)x^2), \tag{13}$$

where $|A_4(x)|$ is bounded in a neighborhood of $x = 0$.

Similarly, there is a neighborhood of $y = 0$ in which

$$K(y) = \frac{K_0(y)}{yK_1(y)} = (-\log y + B_1(y))(1 + B_2(y)y^2 \log y),$$

where $|B_1(y)|$ and $|B_2(y)|$ are bounded, so that

$$K(y) = -\log y + B_3(y), \tag{14}$$

where $|B_3(y)|$ is bounded.

Now suppose that x is fixed and $y > C_1 \exp(-2/x^2)$, where C_1 is a (perhaps large) positive number. Then $-\log y < -\log C_1 + 2/x^2$. If $0 < x < \frac{1}{2}$ and C_1 is large enough, this makes $K(y) < J(x)$ by (13) and (14). Similarly if $y < C_2 \exp(-2/x^2)$, where C_2 is a small positive number, we have $K(y) > J(x)$. Since $K(y)$ is continuous for small positive y, and $K(y) - J(x)$ is negative for $y > C_1 \exp(-2/x^2)$ and positive for $y < C_2 \exp(-2/x^2)$, we must have $K(y) = J(x)$ for some intermediate value of y. Take this y to be $\phi(x)$; then $K(\phi(x)) = J(x)$. We have then established that (10) has a solution $y = \phi(x)$ for $0 < x < \frac{1}{2}$ (at least), and that $\phi(x)$ is between two constant multiples of $\exp(-2/x^2)$ as $x \to 0$. Hence the curve $y = \phi(x)$ does indeed pass through the origin, since $y = \exp(-2/x^2)$ does so.

Notice that all we have succeeded in doing so far is to show that there is a solution $y = \phi(x)$ of (9) or (10) of the form $y = C(x)\exp(-2/x^2)$, where $C(x)$ is a bounded function. To obtain a closer approximation we need more precise versions of the approximate formulas in (11). The formula for $K_0(y)$ is more accurately

$$K_0(y) = -\log y + \log 2 - \gamma + O(y^2),$$

where γ is Euler's constant $0.5772\cdots$. This leads to

$$K(y) = -\log y + \log 2 - \gamma + O(y^2). \tag{15}$$

Taking one more term in the formulas for $J_0(x)$ and $J_1(x)$, we have

$$J_0(x) = 1 - \tfrac{1}{4}x^2 + O(x^4), \quad J_1(x) = \tfrac{1}{2}x^2 - \frac{1}{16}x^4 + O(x^6).$$

Then

$$\begin{aligned} J(x) \equiv \frac{J_0(x)}{xJ_1(x)} &= \frac{2}{x^2}\,\frac{1 - \frac{1}{4}x^2 + O(x^4)}{1 - \frac{1}{8}x^2 + O(x^4)} \\ &= 2x^{-2}(1 - \tfrac{1}{4}x^2 + O(x^4))(1 + \tfrac{1}{8}x^2 + O(x^4)) \\ &= 2x^{-2}(1 - \tfrac{1}{8}x^2 + O(x^4)) \\ &= 2x^{-2} - \tfrac{1}{4} + O(x^2). \end{aligned}$$

Combining this with (15) we have a better approximate formula for $y = \phi(x)$:

$$-\log y + \log 2 - y = 2x^{-2} - \tfrac{1}{4},$$

and so for a second approximation

$$y = 1.44 \exp(-2/x^2). \tag{16}$$

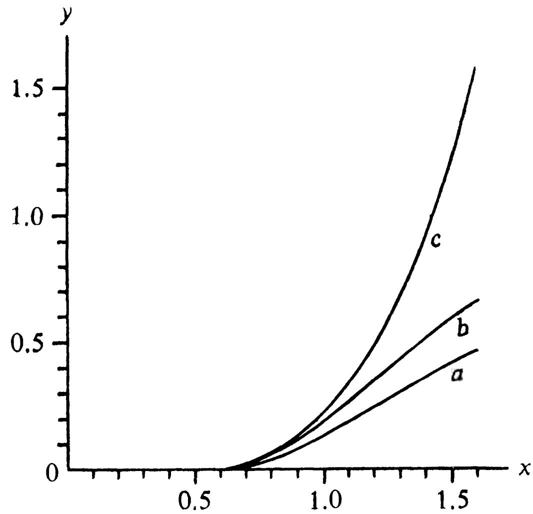

Fig. 1

If we take more terms in (11) we can no longer solve even the approximate equations explicitly. At this point further analytical work hardly seemed worth while and we did in fact solve (9) by computer. We append graphs of the first two approximations (12) (curve a in the figure) and (16) (curve b) and of the computer solution (curve c); these give some idea of how rapidly the approximate solutions deviate from the exact solution as x goes away from 0. They also show that (16) is indeed better than (12).

We are indebted to Leonard Evens for help with the computations and to Joanne Lokay for drawing the graphs.

References

1. M. Abramowitz and I. A. Stegun, Editors, Handbook of Mathematical Functions (National Bureau of Standards, Applied Mathematics Series, 55, Washington, D.C., 1964).
2. M. L. Boas, Mathematical Methods in the Physical Sciences, Wiley, New York, 1966.
3. L. I. Schiff, Quantum Mechanics, 3rd ed., McGraw-Hill, New York, 1968.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

22. Peter A. Lindstrom. I would appreciate receiving references on successful mathematics laboratories not at the remedial two-year college level, but at the level of pre-calculus mathematics, calculus, linear algebra, and differential equations.

23. David E. Kullman. The "utility problem" (also called the "water, gas, and electricity problem") in graph theory is frequently referred to as being "ancient," yet its origins are very obscure. Can anyone supply a reference to the "utility problem' (or an equivalent problem) in the literature prior to 1900?

MATHEMATICAL NOTES

EDITED BY R. A. BRUALDI

Material for this Department should be sent to R. A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

A MEAN ERGODIC THEOREM

RYOTARO SATO

The purpose of this note is to give a brief proof of the following theorem, which is a generalization of a well-known mean ergodic theorem ([1], p. 661).

THEOREM. *Let T be a bounded linear operator on a Banach space X and (k_n) a strictly increasing sequence of positive integers. Suppose the $\|(1/k_n)\ \sum_{i=0}^{k_n-1}T^i\|$ are bounded. If $\lim_n \|(1/k_n)T^{k_n}x\| = 0$ and the set $\{(1/k_n)\ \sum_{i=0}^{k_n-1} T^i x\}$ is weakly sequentially compact, then $(1/k_n)\ \sum_{i=0}^{k_n-1} T^i x$ converges strongly to some $x_0 \in X$ with $Tx_0 = x_0$.*

Proof. Since $(1/k_n)T^{k_n} = (1/k_n)[I + (T - I)\ \sum_{i=0}^{k_n-1} T^i]$, the $\|(1/k_n)T^{k_n}\|$ are

bounded. Hence if we let $Y = \{y \in X;\ \lim_n \|(1/k_n)T^{k_n}y\| = 0\}$, then Y is a closed subspace. Clearly $TY \subset Y$. Let (k_n') be a subsequence of (k_n) such that weak-$\lim(1/k_n') \sum_{i=0}^{k_n'-1} T^i x = x_0$ for some $x_0 \in X$. Then for any bounded linear functional f on X, $f(Tx_0 - x_0) = \lim f((T^{k_n'}x - x)/k_n') = 0$. Thus $Tx_0 = x_0$. Next let f be a bounded linear functional on X such that $f(Ty - y) = 0$ for all $y \in Y$. Then, since $x \in Y$, it follows at once that $f(x) = f(x_0)$. This implies that $x - x_0$ belongs to the closed subspace generated by the set $\{Ty - y;\ y \in Y\}$. Let $\varepsilon > 0$ be given and choose $y \in Y$ such that $\|(x - x_0) - (Ty - y)\| < \varepsilon$. Then

$$\left\| \frac{1}{k_n} \sum_{i=0}^{k_n-1} T^i x - x_0 \right\| \leqq \left\| \frac{1}{k_n} (T^{k_n}y - y) \right\| + \varepsilon \cdot \sup_n \left\| \frac{1}{k_n} \sum_{i=0}^{k_n-1} T^i \right\|,$$

and hence the proof is complete.

Reference

1. N. Dunford and J. T. Schwartz, Linear operators. I: General theory, Pure and Appl. Math. vol. 7, Interscience, New York, 1958.

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NONMEASURABLE INVARIANT SETS

JOHN ROSENTHAL

Amongst the basic theorems of probability are the Zero-One Laws. Intuitively these guarantee that if an experiment is repeated infinitely often, then events whose occurrence is unaffected by modification of finitely many outcomes, occur either with probability 1 or probability 0. So in particular, to guarantee that such an event occurs with probability 1 it suffices merely to show that it occurs with positive probability.

Most important of these are the Kolmogorov [**5**] and the Hewitt-Savage [**4**] Zero-One Laws. In measure theoretic terms the Kolmogorov Zero-One Law says the following: For each $i \in N$ (the positive integers), let (X_i, B_i, μ_i) be a probability space. Let $X = \prod\{X_i \mid i \in N\}$ have the usual product measure $\mu = \prod\{\mu_i \mid i \in N\}$. A subset A of X is called *invariant* (or a tail event) if $\langle a_i \rangle_{i \in N} \in A$, $a_i = b_i$ for all but finitely many i implies $\langle b_i \rangle_{i \in N} \in A$. The Kolmogorov Zero-One Law says that invariant measurable sets have measure 0 or 1. (The Hewitt-Savage Zero-One Law gives the same result in the case where (X_i, B_i, μ_i) are all the same and where the notion of invariance is weakened to requiring A to be left fixed by finite permutations of N.)

These theorems are not only used in probability and statistics to prove that sets have measure 1 by showing they have positive measure, but they are also used for the same purpose in logic. Sacks [**8, 9**], for example, uses this technique in showing

that almost all of certain countable models of set theory give desired independence results.

Particularly in logic the question arises — is the assumption of measurability necessary? Or in other words, are there nonmeasurable invariant sets? The answer to this last question is yes. The rest of this paper will be devoted to constructing an example. Because of the requirements of invariance this construction cannot proceed by a routine axiom of choice argument, as is used in the construction of the usual nonmeasurable sets, but instead needs a more careful transfinite induction argument.

The basic measure theory needed may be found e.g. in [**3**]. The basic set theory needed may be found e.g. in [**4**]. The basic probability, as well as typical uses of the Zero-One Laws, may be found e.g. in [**1, 7**].

We shall deal with the case where each $X_i = \{0,1\}$, $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$. The basic idea in the construction will be to build A so that both, it and its complement (A^{compl}) intersect the complement of every open set of measure < 1. ($X = 2^N$ has the standard product topology.) This will guarantee that A is not measurable by making μ^*A (the outer measure of A) and μ^*A^{compl} both 1. The sets A and A^{compl} will be built to have this property and to be invariant by transfinite induction. Before commencing the construction we review several well-known facts.

Let $|S|$ denote the cardinality of S, also called the power of S.

LEMMA 1. *If $S \subset 2^N$ is measurable and $|S| < 2^{\aleph_0}$, then $\mu S = 0$.*

Proof. (Of course if $2^{\aleph_0} = \aleph_1$, then this is trivial.)

Let $S_0 = S$. It suffices to show there exists a measurable set S_1 such that $|S_1| < 2^{\aleph_0}$ and $\mu S_1 = 2\mu S_0$. For then by repeated use of this fact, if $\mu S_0 > 0$, then there exists an S_m such that $\mu S_m > 1$.

Let $T_0 = \{a - b \mid a, b \in S_0\}$. ($2^N$ is viewed as a group in the standard way.) So $|T_0| < 2^{\aleph_0}$. So there is an $a \in 2^N - T_0$. By construction, $S_0 + a = \{b + a \mid b \in S_0\}$ is disjoint from S_0. As μ is translation invariant, $\mu(S_0 + a) = \mu S_0$. So if

$$S_1 = S_0 \cup (S_0 + a),$$

then S_1 has the desired properties.

LEMMA 2. *There are exactly $2^{\aleph_0}$ open sets in 2^N.*

Proof. For there are only countably many basic open sets and every open set is a countable union of basic open sets.

THEOREM. *2^N contains an invariant, nonmeasurable set. In fact, there is an $S \subset 2^N$ such that S is invariant and $\mu^* S = \mu^* S^{\text{compl}} = 1$.*

Proof. Let β_1 be the first ordinal of the cardinality of 2^N. Let $\langle 0_\alpha \rangle_{\alpha < \beta_1}$ be a list of all open sets of measure < 1.

For any set $T \subset 2^N$, let $T' = \{\langle b_i \rangle_{i \in N} \mid$ there is a $\langle a_i \rangle_{i \in N} \in T$ with $a_i = b_i$ for all but finitely many $i\}$, i.e., T' is the smallest invariant set containing T.

We define S_α, T_α for $\alpha < \beta_1$ by transfinite induction as follows:

$S_0 = T_0 = \varnothing$,

$S_\alpha = \bigcup_{\beta < \alpha} S_\beta$, $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$ for limit ordinals α.

If $\alpha = \beta + 1$, then by construction S_β and T_β both have power $|\beta| \cdot \aleph_0$ and hence both have power $< 2^{\aleph_0}$. On the other hand, as $\mu(0_\beta^{\text{compl}}) > 0$, 0_β^{compl} has power $2^{\aleph_0}$. Hence, there is an $a_\beta \in 0_\beta^{\text{compl}} - (S_\beta \cup T_\beta)$. Let $S_\alpha = (S_\beta \cup \{a_\beta\})'$. In a similar fashion there is a $b_\beta \in 0_\beta^{\text{compl}} - (S_\alpha \cup T_\beta)$. Let $T_\alpha = (T_\beta \cup \{b_\beta\})'$.

By construction the S_α's and T_α's form a pair of increasing, disjoint, invariant sequences of sets and for each α both $S_{\alpha+1} \cap 0_\alpha^{\text{compl}}$ and $T_{\alpha+1} \cap 0_\alpha^{\text{compl}}$ are nonempty.

Let

$$S = \bigcup_{\alpha < \beta_1} S_\alpha, \qquad T = \bigcup_{\alpha < \beta_1} T_\alpha.$$

So S, T are disjoint and invariant and $\mu^*S = \mu^*T = 1$ as neither S nor T is contained in any open set of measure < 1.

Hence, in particular, S has the desired properties.

REMARKS.

1. By a standard modification of the above argument, it is possible to produce $2^{\aleph_0}$ pairwise disjoint, invariant, nonmeasurable sets of outer measure 1. And hence, by taking unions of these, there are $2^{2^{\aleph_0}}$ invariant, non-measurable sets.

Essentially the idea is to produce simultaneously in one induction $\{S^\alpha \mid \alpha < \beta_1\}$ which are pairwise disjoint and invariant, and such that each S^α intersects every open set of measure < 1. The S^α's are defined inductively as the unions of S_γ^α's, $\gamma < \beta_1$, where the S_γ^α's are invariant, form an increasing sequence (for α fixed), have the appropriate disjointness property ($S_\gamma^\alpha \cap S_{\gamma'}^{\alpha'} = \varnothing$ if $\alpha \neq \alpha'$) and such that $S_{\gamma+1}^\alpha$ intersects 0_γ^{compl}. The crucial point is to guarantee that at any stage of the construction fewer than $2^{\aleph_0}$ points have been used, and also to guarantee that, when the β_1 steps are completed, each S_γ^α has been defined. This may be arranged as follows:

Let $p: \beta_1 \times \beta_1 \to \beta_1$ be 1-1, onto such that (*) $\alpha < \alpha'$ implies $p(\alpha, \gamma) < p(\alpha', \gamma)$. Let l, r be the "left" and "right" inverses of p, i.e., l, r: $\beta_1 \to \beta_1$ such that $p(l(\alpha), r(\alpha)) = \alpha$, $l(p(\alpha, \gamma)) = \alpha$, and $r(p(\alpha, \gamma)) = \gamma$. An example of such a "pairing function" and its "inverses" is defined in Gödel [**2**].

At step α, we define $S_{l(\alpha)}^{r(\alpha)}$ as follows: (Condition (*) guarantees that if $\gamma < l(\alpha)$, then $S_\gamma^{r(\alpha)}$ was already defined at an earlier step.)

Case 1: $l(\alpha) = 0$. Let $S_{l(\alpha)}^{r(\alpha)} = \varnothing$.

Case 2: $l(\alpha)$ is a limit ordinal. Let $S_{l(\alpha)}^{r(\alpha)} = \bigcup_{\gamma < l(\alpha)} S_\gamma^{r(\alpha)}$.

Case 3: $l(\alpha)$ is a successor ordinal. Let $a_\alpha \in 0_{l(\alpha)-1}^{\text{compl}} - \bigcup_{\gamma < \alpha} S_{l(\gamma)}^{r(\gamma)}$.

(This exists as the construction guarantees that $\bigcup_{\gamma<\alpha} S_{l(\gamma)}^{r(\gamma)}$ has power $\aleph_0 \cdot |\alpha|$.) Let $S_{l(\alpha)}^{r(\alpha)} = (S_{l(\alpha)-1}^{r(\alpha)} \cup \{a_\alpha\})'$.

2. Also by essentially the same argument one can show there are $2^{\aleph_0}$ disjoint, nonmeasurable subsets of $[0,1]$ (under Lebesgue measure) which are invariant under translation modulo 1 by the rationals.

In this case one must arrange that each set intersects the complement of every open subset of $[0,1]$ of measure < 1. If $T \subset [0,1]$, T' is, in this case, defined to be the smallest set which contains T and which is invariant under translation modulo 1 by the rationals.

3. The above example may be generalized to many other product spaces.

References

1. L. Breiman, Probability, Addison-Wesley, Reading, Massachusetts, 1968.

2. K. Godel, The Consistency of the Continuum Hypothesis, Princeton University Press, Princeton, N. J., 1940.

3. P. R. Halmos, Measure Theory, Van Nostrand, Princeton, N. J. 1950.

4. ———, Naive Set Theory, Van Nostrand, Princeton, N.J. 1960.

5. E. Hewitt and L. J. Savage, Symmetric measures on Cartesian products, Trans. Amer. Math. Soc., 80 (1955) 470–501.

6. A. N. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933.

7. M. Loeve, Probability Theory, 3rd Ed., Van Nostrand, Princeton, N. J. 1963.

8. G. E. Sacks, Degrees of Unsolvability, Annals of Mathematics Studies # 55, Princeton University Press, Princeton, N.J., 1963.

9. ———, Measure-theoretic Uniformity in Recursion Theory and Set Theory, Trans. Amer. Math. Soc., 142 (1969) 381–420.

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NEAR DOMAINS AS GENERALIZATIONS OF D-RINGS

J. A. GRAVES AND J. J. MALONE

The main purpose of this note is to show that the set of D-rings is properly contained in the set of near domains. Near domains, since they can be embedded in near fields, thus constitute a more fitting near ring generalization of integral domains than do D-rings.

DEFINITION. A *near domain* is a (left) near ring satisfying the left Ore condition and the right cancellation law. A near ring D satisfies the left Ore condition if for all $f \neq 0$, $g \neq 0 \in D$, there exist $a \neq 0$, $b \neq 0 \in D$ such that $af = bg$. Near domains were introduced in [3].

The results presented in this note depend upon the following:

Example. Let N be the set of formal power series over the real numbers which

have zero coefficients for the first degree term and for all terms of even degree. Define addition and multiplication on N as termwise addition and composition of functions. We show that $(N, +, \cdot)$ is a near domain.

Cartan [2] in Section 4 of Chapter 1 shows that composition is associative and is left distributive over addition. From this it easily follows that N is a near ring.

Next we establish the left Ore condition. Let $\phi = \Sigma f_i x^i$ and $\gamma = \Sigma g_i x^i$ be arbitrary non-zero elements of N such that the first non-zero coefficient of ϕ is f_s and of γ is g_t.

We must find non-zero α and $\beta \in N$ such that $\alpha \cdot \phi = \beta \cdot \gamma$. Let the first non-zero coefficient of α be a_t and of β be b_s. The first non-zero term of $\alpha \cdot \phi$ and of $\beta \cdot \gamma$ is then the st-th. Equating coefficients of x^{st} we get that $f_s a_t^s = g_t b_s^t$. A non-zero value can be assigned to a_t and b_s can be then computed since the root to be taken is an odd root. Equating coefficients of x^{st+2} we get $s f_s a_t^{s-1} a_{t+2} = t g_t b_s^{t-1} b_{s+2}$. A value can be assigned to a_{t+2} and b_{s+2} can be computed since g_t and b_s are non-zero. Note that, in general, a_{t+k} and b_{s+k} are determined simultaneously by equating coefficients of x^{st+k}; $k = 0, 2, 4, \cdots$. Now assume that $k = 2rt + q$, $0 \leqq q < 2t$, and that all coefficients of α with subscripts less than $t + k$ have been determined. For convenience, $f_s(\Sigma a_i x^i)^s$ is called the s-series, $f_{s+2}(\Sigma a_i x^i)^{s+2}$ is called the $(s + 2)$-series, etc.

In looking at the product of $\alpha \cdot \phi$, we see that for any k the s-series will contain an x^{st+k} term whose coefficient is $f_s a_t^{s-1} a_{t+k}$ plus f_s times a sum of products of a's in which all subscripts are less than $t + k$. If $2t \leqq 2rt$, the $(s + 2)$-series will contribute to the coefficient of x^{st+k}. However, the a's involved in the coefficient of x^{st+k} in the $(s + 2)$-series are exactly those a's involved in the coefficient of x^{st+q} in the s-series, and hence are a's which are already known. In general, the coefficient of $x^{st+2rt+q}$ in $\alpha \cdot \phi$ is the sum of the $x^{st+2rt+q}$ coefficients from the s-series, $(s + 2)$-series, $\cdots$, $(s + 2r)$-series involving, respectively, the following as the highest subscript appearing in each series: $t + 2rt + q$, $t + 2(r - 1)t + q, \cdots, t + q$. Thus, in $\alpha \cdot \phi$ the coefficient of x^{st+k} contains the term $f_s a_t^{s-1} a_{t+k}$ plus a sum of products of f's and a's, all of which are known.

A similar analysis can be made for $\gamma \cdot \beta$. Equating coefficients of x^{st+k}, we obtain

$$f_s a_t^{s-1} a_{t+k} + \cdots = g_t b_s^{t-1} b_{s+k} + \cdots.$$

Then a value may be assigned arbitrarily to a_{t+k} and the value of b_{s+k} computed since g_t and b_s are non-zero. It follows that the left Ore condition holds.

We turn now to the right cancellation law. Let $\alpha, \beta, \gamma \in N$; $\gamma \neq 0$; satisfy $\alpha \cdot \gamma = \beta \cdot \gamma$. If either α or $\beta = 0$, it is easy to show that both are zero and so are equal. Assume then that α and β are non-zero. Note that if a_t is the first non-zero coefficient of α, then b_t must be the first non-zero coefficient of β. If c_s is the first non-zero coefficient of γ, then the first non-zero coefficient of $\alpha \cdot \gamma$ or $\beta \cdot \gamma$ is that of x^{st}. Equating coefficients yields $c_s a_t^s = c_s b_t^s$. Since s is odd, this implies $a_t = b_t$. From

this point on the process of equating coefficients can be carried on, as was done above, to show $\alpha = \beta$.

In passing, it can be noted that N is not commutative since $2x^3 \cdot x^5 \neq x^5 \cdot 2x^3$ and is not right distributive since

$$[x^3 + (x^3 + x^5)] \cdot x^3 \neq x^3 \cdot x^3 + (x^3 + x^5) \cdot x^3$$

and thus is not a ring.

DEFINITION. A *D-ring* is a near ring M such that (i) $ab = 0$, a and $b \in M$, implies $a = 0$ or $b = 0$, and (ii) for every $a \in M$ there are non-zero elements a_l, $a_r \in M$ such that $a_l a$ and aa_r are in the multiplicative center of M.

THEOREM. *The set of D-rings is properly contained in the set of near domains.*

Proof. With Theorem 2.8 of [3] it was established that a D-ring is a near domain. Hence we only need show that there exists a near domain that is not a D-ring. Such a near domain is furnished by the example above, since 0 is the only element in its multiplicative center. Except for the trivial case $M = \{0\}$, a D-ring must necessarily have at least one non-zero element in its multiplicative center.

Let α be an arbitrary element in N. If a_n is the first non-zero coefficient of α, select $b_3 \in R$ such that

$$0 \neq b_3 \neq \sqrt[n-1]{a_n^2}$$

Then $b_3x^3 \cdot \alpha = a_n(b_3x^3)^n + \cdots$, whereas $\alpha \cdot b_3x^3 = b_3(a_nx^n)^3 + \cdots$. Since, by the choice of b_3, $a_nb_3^3 \neq b_3a_n^3$ it follows that only 0 is in the multiplicative center of N.

D-rings were introduced in [1] as a generalization of integral domains and were shown to embed in near fields. It was shown in [3] that near domains embed in near fields and, in fact, that a near domain embeds in its near field of quotients. Thus near domains constitute a more fitting generalization of integral domains than do D-rings.

It was also shown in [3] that a finite near domain is a near field. The example presented above shows that an infinite near domain need not even have an identity element since x is the only possible identity under composition and x is not in N.

Additional results on near domains are given in [4].

References

1. G. Berman and R. J. Silverman, Near-rings, this MONTHLY, 66 (1959) 23–34.
2. H. Cartan, Elementary theory of analytic functions, Addison-Wesley, Reading, Mass., 1963.
3. J. A. Graves and J. J. Malone, Embedding near domains, Bull. Austral. Math. Soc., 9 (1973) 33–42.
4. J. A. Graves and J. J. Malone, Euclidean near domains, submitted for publication.

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LEFT-CONTINUOUS RANDOM WALK AND THE LAGRANGE EXPANSION

J. G. WENDEL

1. Introduction. The purpose of this note is to display a connection between the seemingly unrelated subjects that occur in the title. Before coming to the precise statements, let us review the basic definitions.

Taking the more classical subject first, we recall that Lagrange's expansion arises in the following way. Let f be a function of a complex variable, which is analytic and non-vanishing at the origin. The implicit function theorem then guarantees that in a neighborhood of the origin we can solve the equation

$$y = zf(y); \tag{1}$$

that is, there is a unique function y which is analytic and satisfies $y(0) = 0$, $y(z) = zf(y(z))$. Next, let g be analytic at 0. Then so is the composition $g \circ y$, and it makes sense to ask for its Maclaurin series; the *Lagrange expansion*

$$g(y(z)) = g(0) + \sum_1^\infty \frac{z^n}{n!}[(d/dt)^{n-1}(g'(t)f(t)^n)]\big|_{t=0} \tag{2}$$

provides the answer. (Proofs may be found in many texts, e.g., [**2**] §189.) We note the special cases

$$y(z)^k = \sum_1^\infty \frac{z^n}{n!}[(d/dt)^{n-1}(kt^{k-1}f(t)^n)]\big|_{t=0}, \qquad k = 1,2,3,\cdots, \tag{3}$$

and especially

$$y(z) = \sum_1^\infty \frac{z^n}{n!}[(d/dt)^{n-1}f(t)^n]\big|_{t=0}. \tag{4}$$

We turn now to random walk, which for present purposes we take to be the study of the sequence of partial sums S_n of independent identically distributed *integer-valued* random variables $X_1, X_2, \cdots$. S_n represents the position, in the lattice points of the line, of a particle which starts at $S_0 = 0$ and suffers displacements at times $1, 2, \cdots$ of amounts $X_1, X_2, \cdots$. The random walk is called *left-continuous* (LC) in case its only possible negative displacements are -1, so that $\Pr\{X_n \geqq -1\} = 1$ for $n = 1,2,\cdots$. (Right-continuous random walk is defined dually.) We assume left-continuity from here on.

Much is known about LC random walks, as the interested reader may confirm by consulting [**4**]. They are especially easy to study because they possess the "left-ward" intermediate value property: if $n_1 < n_2$ and $a = S_{n_2} < S_{n_1} = b$, then between the times n_1 and n_2 the walk must visit every lattice point between a and b at least once. LC random walks include classical games of chance, in which at each play the gambler either loses his unit stake, breaks even or wins some number of units; the

position S_n then represents the net increase in the gambler's fortune after n plays of the game. They also overlap the theory of branching processes, under the interpretation that $X_n = -1$ signifies death of the nth individual without issue, while $X_n = k \geqq 0$ means that at death he is replaced by $k+1$ new individuals.

We are especially interested in certain *hitting times* of the random walk. For $k = 1, 2, \cdots$ we let N_k be the first time when S_n reaches $-k$, $N_k = \infty$ if S_n never visits $-k$; by the LC property the latter means that $S_n > -k$ for all n. Thus we explicitly permit the random variables N_k to be improper, i.e., $\Pr\{N_k < \infty\}$ may be less than one. For games of chance the interpretation of N_k is the ruin time for a gambler whose initial fortune equals k, and $N_k = \infty$ means that he may play forever.

Of course $N_k \geqq k$. A deeper fact about these hitting times is contained in the following beautiful theorem due to Kemperman [3]:

THEOREM. *For left-continuous random walk we have*

$$\Pr\{N_k = n\} = (k/n)\Pr\{S_n = -k\}. \tag{5}$$

I.e., the chance that the random walk visits $-k$ for the first time at n is k/n of the chance that without qualification it visits $-k$ at n.

We can now summarize the purposes of the paper. In the next section, we use the Lagrange expansion to deduce the hitting-time relation, that is, we derive (5) from (3). (For $k = 1$ this is indicated in [4], problem 12, pg. 234.) In §3 we give a direct proof of (5) based on an elegant combinatorial theorem of Dwass [1]. Finally in §4 we show that the Lagrange expansion is a *consequence* of the hitting-time relation, and we "close the loop" by deducing Dwass from Lagrange.

2. Generating functions, and a proof of the hitting time relation. We define the generating function for the steps of the random walk by

$$P(z) = E(z^{X_n}) = \sum_{k=-1}^{\infty} p_k z^k, \qquad 0 < |z| < 1,$$

where $p_k = \Pr\{X_n = k\}$. (These quantities do not depend on n, since by assumption the steps X_n are identically distributed.) By independence of the steps, it follows that the partial sum S_n has generating function

$$E(z^{S_n}) = P(z)^n.$$

It is convenient to define $f(z) = zP(z)$, with $f(0) = p_{-1}$, and to note that $f(z)^n = E(z^{n+S_n})$. The function f is analytic at least in the unit disk.

Let $y(z) = E(z^{N_1})$, the generating function for the hitting time of -1. (Although N_1 may be improper, we see that $\Pr\{N_1 < \infty\} = y(1-) = \lim_{z \to 1\ |z|<1} y(z)$.) More generally, let $y_k(z)$ be the generating function for N_k. We now argue that

$$y(z) = p_{-1}z + \sum_{k=0}^{\infty} p_k z y_{k+1}(z) \tag{6}$$

and

$$y_k(z) = y(z)^k, \quad k = 1, 2, \cdots. \tag{7}$$

The first of these follows by considering the situation after the first step X_1 of the random walk. We have

$$y(z) = E(z^{N_1}) = \sum_{-1}^{\infty} p_k E(z^{N_1} \mid X_1 = k).$$

The term $k = -1$ contributes $p_{-1}z$, since $N_1 = 1$ iff $X_1 = -1$. For the remaining terms, one step has been consumed in moving to position k, and in order to reach -1 the particle must eventually move $k+1$ steps leftward. Clearly, the time T_{k+1} to accomplish this is the same function of steps $X_2, X_3, \cdots$ that N_{k+1} is of $X_1, X_2, \cdots$. Hence

$$\begin{aligned} E(z^{N_1} \mid X_1 = k) &= E(z^{1+T_{k+1}} \mid X_1 = k) \\ &= zE(z^{N_{k+1}}) \\ &= zy_{k+1}(z). \end{aligned}$$

To prove (7) we note that $N_{k+1} = N_1 + (N_{k+1} - N_1)$ and observe that the term in parentheses is the time required for the particle to move the k steps leftward from -1 to $-k-1$. It thus has the same probability distribution as N_k; moreover, it is independent of N_1. Therefore

$$y_{k+1}(z) = E(z^{N_{k+1}}) = E(z^{N_1})E(z^{N_k}) = y(z)y_k(z),$$

and the proof is completed by induction.

We now combine (6) and (7) to obtain

$$y(z) = \sum_{-1}^{\infty} p_k z y(z)^{k+1} = zf(y(z)),$$

on recalling the definition of f. Clearly $y(0) = 0$ and y is analytic in the unit disk. Hence y is the unique solution of (1) that meets the side conditions set forth in the introduction. Consequently (3) holds, and by virtue of (7) we have

$$\begin{aligned} \Pr\{N_k = n\} &= \text{coefficient of } z^n \text{ in } y(z)^k \\ &= \frac{1}{n!}(d/dt)^{n-1} kt^{k-1} f(t)^n \big|_{t=0} \\ &= \frac{(n-1)!}{n!} \cdot \text{coefficient of } t^{n-1} \text{ in } kt^{k-1} f(t)^n \\ &= \frac{k}{n} \cdot \text{coefficient of } t^{n-1} \text{ in } E(t^{k-1+n+S_n}) \end{aligned}$$

$$= \frac{k}{n} \Pr\{k-1+n+S_n = n-1\}$$

$$= \frac{k}{n} \Pr\{S_n = -k\},$$

proving (5).

Let us note in passing that the quantity $y(1-)$ may be computed as the least positive root of the equation $q = f(q)$; thus, $\Pr\{N_1 < \infty\} = q$, and further, $\Pr\{N_k < \infty\} = q^k$. We have N_1 proper iff $q = 1$, iff $f'(1) \leqq 1$, iff $E(X_n) \leqq 0$.

3. Dwass' theorem and another proof of the hitting-time relation. Dwass' theorem states an identity for certain functions defined on finite sequences of real numbers and their cyclic permutations. Specifically, let $a_1, a_2, \cdots, a_n$ be real numbers, and examine their partial sums $s_1 = a_1$, $s_2 = a_1 + a_2, \cdots$, $s_n = a_1 + a_2 + \cdots + a_n$. Let m^* be the greatest s_i and define m_1 as the smaller of m^* and 0; thus, $m_1 = \min(0, \max(s_1, \cdots, s_n))$. Of course, m_1 is zero unless all s_i are negative.

We now repeat the construction for each *cyclic* permutation of the a_i and obtain $m_2 = \min(0, \max(a_2, a_2 + a_3, \cdots, a_2 + a_3 + \cdots + a_n + a_1))$, $m_3, \cdots, m_n$. The result of Dwass has the following simple statement:

THEOREM. $m_1 + m_2 + \cdots + m_n = \min(0, a_1 + a_2 + \cdots + a_n)$.

As a special case, suppose that the a_i are integers not less than -1. Then it is clear that the only possible values of the m_i are -1 and 0, and we have the immediate consequence.

COROLLARY 1. *The number of m_i that equal -1 is* $-\min(0, s_n)$.

In view of our interest in the first occurrence of a partial sum with value $-k$, it is natural to single out the sequences $a = (a_1, a_2, \cdots, a_n)$ of integers $\geqq -1$ with sum $-k$, whose earlier partial sums s_m, $m < n$, are all greater than $-k$. For ease of reference, we call such sequences *good.* With this definition we can state

COROLLARY 2. *Exactly k of the cyclic permutations of a are good.*

Proof. We apply Corollary 1 to the sequence $a^* = (a_1^*, \cdots, a_n^*)$ obtained from a by reversal, i.e., by setting $s_i^* = a_{n-i+1}$; we note that reversals of cyclic permutations of a are cyclic permutations of a^*. Setting $s_0 = 0$ and forming the partial sums s_i^* of the a_i^*, we get $s_i^* = s_n - s_{n-i} = -k - s_{n-i}$. Therefore

$$s_i^* < 0 \quad \text{iff} \quad s_{n-i} > -k. \tag{8}$$

Corollary 1 implies that exactly k cyclic permutations of a^* have all negative partial sums. Then it follows from (8) that exactly k cyclic permutations of a have $s_m > -k$ for all $m < n$, which is what we had to prove.

The hitting-time relation (5) is a direct consequence of Corollary 2. In fact, let

$p(a) = p_{a_1}p_{a_2}\cdots p_{a_n} = \Pr\{X_i = a_i \text{ for } i = 1,2,\cdots,n\}$, and set $G(a) = 1$ if a is good, $= 0$ otherwise. Then the probability of first visiting $-k$ at time n is given by

$$\Pr\{N_k = n\} = \Sigma\, p(a)G(a), \tag{9}$$

the sum being taken over all sequences a for which $s_n = -k$. The sum is unchanged if all sequences a are subjected to a cyclic permutation; hence we may replace the right side of (9) by its average over the n possible cyclic permutations. Furthermore, the individual factors $p(a)$ remain invariant. This means that in (9) we may replace $G(a)$ by its average value. Corollary 2, however, may be viewed as stating that this average value is k/n for each a with $s_n = -k$. Therefore

$$\Pr\{N_k = n\} = (k/n)\Sigma\, p(a) = (k/n)\Pr\{S_n = -k\},$$

as we wished to show.

4. Probabilistic proof of the Lagrange expansion; more on Dwass' theorem. Our concern lies solely with the form of the coefficients in (2), and not with matters of convergence, which are covered by general theory. Hence it is sufficient to treat the special cases (3), (4), for the general case then follows by linearity.

The argument in §2 which leads from (3) to (5) is clearly reversible. Taken together with the combinatorial proof of (5) above, this remark already proves (3) when f is a probability generating function (pgf), i.e., when $f(z) = \Sigma_0^\infty f_n z^n$ has non-negative coefficients with sum one. (In our case $f_n = p_{n-1}$.)

Let $y(z) = \Sigma_1^\infty b_m z^m$. Substituting in (1) and solving recursively for the b_m we find that $b_m = P_m(f_0, f_1, \cdots, f_{m-1})$, where P_m is a certain well-defined polynomial in the coefficients of $f(z)$; for example, $b_1 = f_0$, $b_2 = f_0 f_1$, $b_3 = f_0 f_1^2 + f_0^2 f_2$. The coefficient of z^m in (4) is also a well-determined polynomial in the same variables, and the argument above shows that the two sets of polynomials are equal in pairs whenever the f_n are nonnegative and their partial sums are less than one. This implies that the two sets of polynomials are pairwise identical, in other words, that (4) holds in the general case.

To establish (3) we argue similarly. Fix k and write

$$y(z)^k = \sum_{m=k}^{\infty} c_m z^m = \left(\sum_1^\infty b_m z^m\right)^k.$$

Then c_m is a polynomial in $b_1, \cdots, b_{m-k+1}$ and hence in $f_0, \cdots, f_{m-k}$. The form (3) is correct in case f is a pgf, and therefore in general.

We conclude with the observation that the theorem of Dwass may be viewed as a consequence of the Lagrange expansion. Here is a sketch of the argument.

First, deduce (5) from (3). Then reinterpret (5) as asserting the identity of Dwass in the special case when the a_i are integers $\geqq -1$. Extend to arbitrary negative

integers by replacing each $a_i = -k < 0$ by a string of $k - 1$'s. Pass to rational a_i by rescaling. Take limits to get the general case.

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References

1. M. Dwass, Fluctuation theorems for cyclic random variables, Ann. Math. Statist., 33 (1962) 1450–4.
2. Goursat-Hedrick, Mathematical Analysis, 1, Ginn & Company, Boston, 1904.
3. J. H. B. Kemperman, The passage problem for a Markov chain, University of Chicago Press, Chicago, 1961.
4. F. Spitzer, Principles of random walk, Van Nostrand, Princeton NJ, 1964.

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RESEARCH PROBLEMS

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In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

WHERE DO ALL THE TRIANGLES GO?

R. B. EGGLETON

A *tiling* of the euclidean plane may be defined as a covering of the plane by sets (*tiles*) with pairwise disjoint interiors. We are here concerned with tilings by rational triangles (triangles with all three sides of rational length; the area is not necessarily rational).

It was first realized by J. H. Conway [**1**] that the plane can be tiled using exactly one triangle from each congruence class of rational triangles. A constructive demonstration of this result is given in [**2**], where several more restricted classes of rational triangles are also shown to yield tilings of the plane. The basic idea developed in [**2**] is that of tiling an infinite strip, and using such strips to cover the whole plane. However, when two strips are placed side by side, vertices of triangles at the edge of one strip do not necessarily coincide with vertices of triangles in the other. Call

a tiling by triangles *strict* if any point common to the boundaries of two tiles is either a vertex of both or of neither. *Is there a strict tiling of the plane using exactly one triangle from each congruence class of rational triangles?*

A constructive demonstration of such a tiling (and several related ones, including a three-dimensional analog) could be given if the following problem can be resolved affirmatively.

If a, b, c are given rationals, such that $0 < a \leqq b$ and $c \geqq 0$, does the equation

$$x - \frac{a}{x} = y - \frac{b}{y} + c \tag{1}$$

have infinitely many solutions (x, y) in rationals?

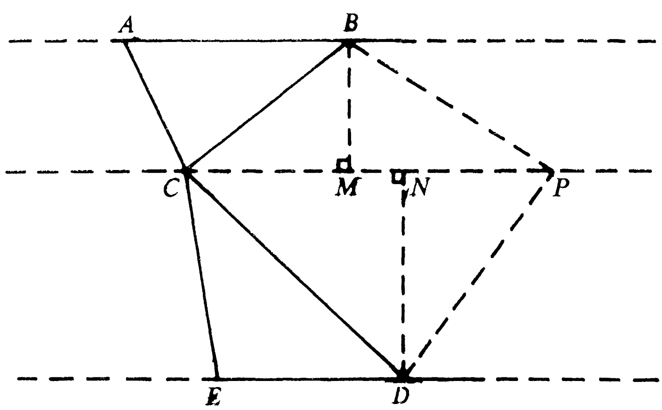

FIG. 1

The connection between the tiling problem and the rational Diophantine equation will now be described. As part of the tiling, suppose two rational triangles ABC and CDE have been located in the plane with AB parallel to DE, and vertices C coinciding (Figure 1). We wish to locate a point P which is at rational distances from B, C, D, and such that CP is parallel to AB. Then BCP and CDP are rational triangles which extend the strict tiling in the two adjacent strips defined by the selected pair of triangles. The point P should be located on the appropriate side of these triangles to avoid interior overlaps, and moreover should be such that neither BCP nor CDP is congruent to any of the finitely many triangles already located elsewhere in the tiling. Let BM and DN be perpendiculars from B and D onto CP. From elementary properties of rational triangles (cf. [2]) it follows that there exist rationals a, b, c such that $BM^2 = 4a$, $DN^2 = 4b$, $MN = c$, and there is no loss of generality in assuming $0 < a \leqq b$ and $c \geqq 0$. With this notation, the existence of a suitable point P corresponds to existence of rationals x, y such that

$BP = x + (a/x)$, $MP = x - (a/x)$, $DP = y + (b/y)$, $NP = y - (b/y)$. Thus (1) is just the consistency condition $MP = MN + NP$.

For each suitable P there are four corresponding rational points on the graph of (1), one in each quadrant. If (x, y) is one of them, the others are $(-a/x, y)$, $(x, -b/y)$, $(-a/x, -b/y)$. In the case $a < b$, $c > 0$ the graph has rational points, since $x = y = (b-a)/c$ satisfies (1). Moreover, if (1) has arisen from an actual pair of rational triangles as described above, the graph contains rational points corresponding to C. (Note that in the degenerate case $a = b$, $c = 0$ the graph contains (t, t) for every rational t.)

Beginning with known rational points on the graph, others can usually be located by tangent and chord processes as described in [3]. In particular, we note two such constructions. A line of positive unit gradient through a rational point (x, y) typically meets the graph in a second point (x', y'), which is also rational. Indeed,

$$x' = \frac{ay(y-x)}{bx - ay}, \quad y' = \frac{bx(y-x)}{bx - ay}.$$

A line through two rational points of the graph, not of positive unit gradient nor parallel to the coordinate axes, typically meets it in a third point which is also rational since the abscissae of the three points are roots of a cubic polynomial with rational coefficients.

If the graph has infinitely many rational points, these constructions are adequate to prove that the set of rational points is unbounded. (It follows that in the tiling construction a suitable point P is available arbitrarily far from C and on the appropriate side of the triangles already located.) However, the crucial question is whether such constructions (or others) can generate an infinite set of rational points on the graph.

References

1. J. H. Conway, Problem 5328, this MONTHLY, 72 (1965) 915.
2. R. B. Eggleton, Tiling the plane with triangles, Discrete Math., 7 (1974) 53–65.
3. L. J. Mordell, Diophantine Equations, Academic Press, London and New York, 1969, chap. 9.

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A SIMPLE CHARACTERIZATION OF COMMUTATIVE RINGS WITHOUT MAXIMAL IDEALS

MELVIN HENRIKSEN

In a course in abstract algebra in which the instructor presents a proof that each ideal in a ring with identity is contained in a maximal ideal, it is customary to give an example of a ring without maximal ideals. The usual example is a zero-ring whose additive group has no maximal subgroups (e.g., the additive group of (dyadic) rational numbers; actually any divisible group will do; see [**1**, p. 67]). This may leave the impression that all such rings are artificial or at least that they abound with divisors of 0.

Below, I give a simple characterization of commutative rings without maximal ideals and a class of examples of such rings, including some without proper divisors of 0. To back up the contention that this can be presented in such a course in abstract algebra, I outline proofs of some known theorems including a few properties of radical rings in the sense of Jacobson.

The *Hausdorff maximal principle* states that every partially ordered set contains a maximal chain (i.e., a maximal linearly ordered subset). It is equivalent to the axiom of choice [**4**, Chapter XI].

Since the union of a maximal chain of proper ideals in a ring with identity is a maximal ideal, and since the union of a maximal chain of linearly independent subsets of a vector space is a maximal linearly independent set, we have:

(1) *Every ideal in a ring with identity is contained in a maximal ideal.*

(2) *Every non-zero vector space has a basis.*

As usual we denote the ring of integers by Z, and for any prime $p \in Z$, we denote by Z_p the ring of integers modulo p, and by Z'_p the zero-ring whose additive group is the same as that of Z_p.

It is not difficult to prove that a commutative ring R has no nonzero proper ideals if and only if either R is a field or R is isomorphic to Z'_p for some prime p. See [**5**, p. 133]. Hence:

(3) *An ideal M of a commutative ring R is maximal if and only if R/M is either a field or is isomorphic to Z'_p for some prime p.*

For any commutative ring R, let $J(R)$ denote the intersection of all the ideals M

of R, such that R/M is a field. If R has no such ideals, let $J(R) = R$. In the latter case we call R a *radical ring*. The knowledgeable reader will recognize $J(R)$ as the Jacobson radical of R. See [2, Chapter 1].

Of the many known properties of radical rings, we need only the following two, the first of which follows immediately.

(4) *A homomorphic image of a (commutative) radical ring is a radical ring.*
(5) *$J(R)$ is a radical ring.*

Proof. If $J(R)$ is not a radical ring, then there is a homomorphism ϕ of $J(R)$ onto a field F with identity element 1. Choose $e \in J(R)$ such that $\phi(e) = 1$, and define $\phi' : R \to F$ by letting $\phi'(a) = \phi(ae)$ for each $a \in R$. If $a, b \in R$, then

$$\phi'(a + b) = \phi((a + b)e) = \phi(ae + be) = \phi(ae) + \phi(be) = \phi'(a) + \phi'(b),$$

$$\text{and } \phi'(ab) = \phi(abe) = \phi(abe)\phi(e) = \phi(aebe) = \phi(ae)\phi(be) = \phi'(a)\phi'(b).$$

Therefore ϕ' is a homomorphism of R onto F, and hence its kernel contains $J(R)$. But $e \in J(R)$ and $\phi'(e) = 1$. This contradiction shows that $J(R)$ is a radical ring.

It follows easily from (1), (3), and (4) that no ring with identity is a radical ring and that every zero-ring is a radical ring.

THEOREM. *A commutative ring R has no maximal ideals if and only if*
(a) *R is a radical ring.*
(b) *$R^2 + pR = R$ for every prime $p \in Z$.*

Proof. Suppose first that (a) and (b) hold. Since R is a radical ring, no homomorphic image of R can be a field, so, by (3) it suffices to show that for any prime $p \in Z$, the zero-ring Z_p' is not a homomorphic image of R. Suppose, on the contrary, that there is a homomorphism ϕ of R onto Z_p' with kernel M. If

$$c = \sum_{i=1}^{n} a_i b_i \in R^2, \text{ then } \phi(c) = \sum_{i=1}^{n} \phi(a_i)\phi(b_i) = 0,$$

so $R^2 \subset M$. Moreover, $\phi(pa) = p\phi(a) = 0$, so $pR \subset M$. Hence $R^2 + pR \subset M \neq R$, so (b) fails. The contradiction shows that R has no maximal ideals.

Suppose next that R has no maximal ideals. By (3) and the definition of $J(R)$, R is a radical ring. Suppose (b) fails for some prime p, let $I = R^2 + pR$, and let ϕ be the natural homomorphism of R onto R/I. If $a, b \in R$, then $0 = \phi(ab) = \phi(a)\phi(b)$, so R/I is a zero-ring, and since $0 = \phi(pa) = p\phi(a) = 0$, R/I has characteristic p and hence is a vector space over Z_p. By (2), since $I \neq R$, R/I has a basis $\{x_\alpha\}_{\alpha \in \Gamma}$ and each $x \in R/I$ may be written uniquely as $x = \sum_{\alpha \in \Gamma} a_\alpha x_\alpha$ with $a_\alpha \in Z_p$ and $a_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$. For any fixed $\alpha_0 \in \Gamma$, the mapping ψ_0 such that $x\psi_0 = a_{\alpha_0}$ is a homomorphism of R/I onto Z_p'. Then $\phi \circ \psi_0$ is a homomorphism of R onto Z_p'. By (3), the kernel of $\phi \circ \psi_0$ is a maximal ideal, contrary to assumption. Hence (a) and (b) hold.

Recall that an abelian group G is *divisible* if $nG = G$ for every $n \in Z$ and note that G is divisible if and only if $pG = G$ for every prime $p \in Z$. It follows from the theorem that a zero-ring whose additive group is divisible has no maximal ideals.

COROLLARY. *Let S be a commutative ring with identity that has a unique maximal ideal R. If $R^2 + pR = R$ for every prime $p \in Z$, then R has no maximal ideals. In particular, if the additive group of S is divisible, then R has no maximal ideals.*

I conclude with some explicit examples:

Examples. (i) For a field F, let $F[x]$ denote the ring of polynomials in an indeterminate x with coefficients in F, and let $F(x)$ denote the field of quotients of $F[x]$. Let

$$S(F) = \left\{ h(x) = \frac{f(x)}{g(x)} \in F(x) : f(x), g(x) \in F[x] \text{ and } g(0) \neq 0 \right\}.$$

It is easy to verify that $S(F)$ is an integral domain whose unique maximal ideal is $R(F) = xS(F)$. If F has characteristic zero, then, by the corollary, $R(F)$ has no maximal ideals. If F has prime characteristic, then, since $[R(F)]^2 = x^2R(F)$, the ring $R(F)$ does have maximal ideals.

(ii) Let G denote the additive semigroup of non-negative dyadic rational numbers, and let $U(F)$ denote the semigroup algebra over G with coefficients in a field F. We may regard each element of $U(F)$ as a polynomial in $x^{(\frac{1}{2})^n}$ for some positive integer n. Let $T(F)$ denote those elements of the quotient field of $U(F)$ whose denominators fail to vanish at 0. It is not difficult to verify that $R^*(F) = \{h \in T(F) : h(0) = 0\}$ is the unique maximal ideal of $T(F)$ and that $[R^*(F)]^2 = R^*(F)$. By the corollary, $R^*(F)$ has no maximal ideals (and no proper divisors of 0).

(iii) Let F_1 be a field of characteristic 0, let F_2 be a field of prime characteristic p, and let R be the direct sum of the ring $R(F_1)$ described in (i) and the ring $R^*(F_2)$ described in (ii). Since each of these latter two rings is a radical ring, so is R. For, otherwise, there would be a homomorphism ϕ of R onto a field F. Then $\phi[R(F_1)]$ and $\phi[R^*(F_2)]$ are ideals of F whose (direct) sum is F, and hence one of them is all of F, contrary to the fact that $R(F_1)$ and $R^*(F_2)$ are radical rings. Also, while $R^2 \neq R$ and $pR \neq R$, it is true that $R^2 + pR = R$, so R has no maximal ideals.

One can create more rings satisfying the hypothesis of the corollary by starting with any commutative ring S with identity and divisible additive group, taking its localization S_M at a maximal ideal M, and letting $R = MS_M$. See [**1**, Chapter 2].

I am grateful to Professor Barbara Beechler for valuable criticisms of earlier drafts of this note.

References

1. L. Fuchs, Abelian Groups, Hungarian Academy of Sciences, Budapest, 1958.

2. I. N. Herstein, Noncommutative rings, MAA — The Carus Mathematical Monograph No. 15, 1971.

3. S. Lang, Algebra, Addison-Wesley, Reading, Mass., 1965.

4. S. Warner, Modern Algebra, Vol. II, Prentice-Hall, Englewood Cliffs, N. J., 1965.

5. O. Zariski and P. Samuel, Commutative Algebra, Vol. I, Van Nostrand, Princeton, N. J., 1958.

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THE DIFFERENTIABILITY OF a^x

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A "from scratch" proof of the differentiability of a^x, $a > 0$, is avoided by essentially all modern-day authors. A slick and popular way of handling the problem is to define a^x as $e^{x \log a}$ its differentiability and other properties following from that of the functions e^x and $\log x$. Unfortunately, the usual definitions of e^x and $\log x$ involve relatively sophisticated ideas (e.g., integration or power series). Furthermore, the student, having heard of e, the natural logarithm base, at an early stage of his development, is hardly enlightened when he is told that e is e^1. He would have a much better feeling for the "naturalness" of e if it were defined as that number a for which $(a^x)' = a^x$.

The purpose of this note is to provide a direct and relatively simple way of getting at the differentiability of a^x. We define $a^x = \lim a^r$ as $r \to x$ through rational values of r from which continuity and other basic properties follow (see e.g., [**1**, p. 63]). The differentiability question obviously reduces to showing that the function $F(x) = (a^x - 1)/x$ has a limit at 0. Since $F(-x) = a^{-x}F(x)$, it suffices to show only that the right-hand limit exists. By a similar observation, we may assume that $a > 1$. As a final reduction, we note that, for $a > 1$, F is bounded below on $(0, \infty)$ and, hence, it is sufficient to show that F is increasing on $(0, \infty)$.

Define $S(x, n) = 1 + x + \cdots + x^{n-1}$ so that $S(x, n)(x-1) = x^n - 1$. Since

$$\begin{aligned} n(a^{1/n} - a^{1/(n+1)}) &= na^{1/(n+1)}(a^{1/n(n+1)} - 1) \\ &> S(a^{1/n(n+1)}, n)(a^{1/n(n+1)} - 1) \\ &= a^{1/(n+1)} - 1, \end{aligned}$$

the sequence $\{F(1/n)\}$ is decreasing. Therefore, for positive rational numbers $m/n < p/q$, we have

$$\begin{aligned} F(m/n) &= F(1/pn)S(a^{1/pn}, pm)/pm \\ &< F(1/qm)S(a^{1/qm}, pm)/pm = F(p/q). \end{aligned}$$

In other words, F is increasing on the positive rationals. By continuity, F is increasing on $(0, \infty)$.

We conclude by noting that $(a^x)'$ is proportional to a^x and that the constant of proportionality can be taken to be 1 if a is chosen suitably, leading to an appealing definition of e (cf. [**2**, p. 41–44]).

References

1. Casper Goffman, Introduction to Real Analysis, Harper & Row, New York, 1966.
2. Edmund Landau, Differential and Integral Calculus, 2nd ed., Chelsea, New York, 1960.

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A NOTE ON A FORMULA OF CHEBYCHEF

Robert E. Dressler and Jan van de Lune

The main results obtained by Chebychef in elementary prime number theory are based upon the identity [**1**, pg. 76]

$$\log[x]! = \sum_{n<x} \Psi(x/n), \tag{1}$$

where $\Psi(x) = \sum_{n\leqq x} \Lambda(n)$ and $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime}, \\ 0 & \text{if } n \neq p^k. \end{cases}$

The proof of (1) given in this note is, in our opinion, less complicated and shorter than the existing proofs. It is based on the simple observation that if f is a decreasing, positive, continuous function defined for $t \geqq 0$ and if f is eventually less than 1, then

$$\sum_{n=1}^{\infty} [f(n)] = \sum_{r<f(0)} [f^{-1}(r)]. \tag{2}$$

First note that if $L(x)$ is the least common multiple of the integers $1, 2, \cdots, [x]$, we have $L(x) = \prod_p p^{[\log x/\log p]}$. Also observe that

$$L(x) = \exp\{\sum_{n\leqq x} \Lambda(n)\} = \exp \Psi(x).$$

Now, in (2), take $f(t) = x/p^t$ $(x > 0,\ p > 1)$ to obtain

$$\sum_{n=1}^{\infty} \left[\frac{x}{p^n}\right] = \sum_{r<x} \left[\frac{\log x/r}{\log p}\right]. \tag{3}$$

Then

$$\begin{aligned} \sum_{n<x} \Psi(x/n) &= \sum_{n<x} \log L(x/n) \\ &= \log \prod_{n<x} L(x/n) = \log \prod_{n<x} \prod_p p^{\alpha} \end{aligned}$$

$$= \log \prod_p p^{\theta} \overset{(3)}{=} \log \prod_p p^{\gamma} = \log[x]!,$$

where $\theta = \Sigma_{n<x}\alpha$, $\alpha = [\log(x/n)/\log p]$, $\gamma = \Sigma_{n=1}^{\infty}[x/p^n]$.

Reference

1. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Band 1, Chelsea, New York, 1953.

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SOME THEOREMS FROM GEOMETRIC FUNCTION THEORY: APPLICATIONS

W. P. Novinger

This note is primarily for those persons who have a complex analysis background that includes the Riemann Mapping Theorem and who wish to pursue some of its extensions and ramifications. Recall that the Riemann Mapping Theorem says that if Ω is a proper, simply connected region in the complex plane, then there is a one-to-one analytic function f from the open unit disc Δ onto Ω. A noted theorem of C. Carathéodory [**1**, p. 96] states that if Ω is a *Jordan* region, then each such f has an extension to a homeomorphism of the closure of Δ onto that of Ω. Carathéodory's theorem has many applications and, of course, is of interest in its own right. However, the topological machinery which is required for a thorough treatment of this result is formidable; indeed, the Jordan Curve Theorem is involved and it can be argued that this is neither a prerequisite for, nor an appropriate part of a first course in complex analysis. An alternative approach, which is more self-contained and suffices for the applications we shall discuss here, is that presented in Rudin [**5**, pp. 279–282].

Definition. A point $\beta \in \partial\Omega$ is called **simple** if to each sequence (α_n) in Ω such that $\alpha_n \to \beta$, there corresponds a continuous function γ: $[0,1] \to \Omega \cup \{\beta\}$ and a strictly increasing sequence (t_n) in $[0,1)$ such that $t_n \to 1$, $\gamma(t_n) = \alpha_n$, and $\gamma(t) \in \Omega$ for $0 \leqq t < 1$.

Carathéodory's theorem as stated earlier is then replaced by the following one.

Theorem 1. (Rudin [**5**, Theorem 14.19, p. 281].) *Let Ω be a bounded simply connected region such that every boundary point is simple. Then each one-to-one analytic map of Δ onto Ω has an extension to a homeomorphism of their respective closures.*

Remark. The proof given in [**5**] requires a preliminary result [**5**, Theorem 14.18(a)] whose proof in turn depends upon Fatou's theorem on the existence

almost everywhere of radial limits for bounded analytic functions on the disc Δ. In keeping with the elementary nature of this note, it seems appropriate to point out that a reasonably short and completely elementary proof of this preliminary result can be given in which all measure theoretic considerations are avoided, see [**4**]. At any rate, let us assume that Theorem 1 is at our disposal. We proceed to the consideration of some of its ramifications.

One of the most familiar applications of the Riemann Mapping Theorem and Theorem 1 is a solution to the Dirichlet problem for all regions satisfying the hypothesis of Theorem 1, given that the problem has a solution for the disc Δ. The line of reasoning is well known and will not be repeated here. A similar but perhaps less well-known application is a proof of the existence of real continuous functions on the closed disc $\bar{\Delta}$ which are harmonic in Δ, but whose harmonic conjugates are not even bounded on Δ. This can be argued as follows: Take $\Omega = \{x + iy\colon 0 < x < 1$ and $-x^2 < y < x^2\}$. Then the function $z \mapsto z$ obviously has a continuous argument u on the closure $\bar{\Omega}$ which is (of course) harmonic in Ω. As Ω clearly satisfies the hypothesis of Theorem 1, there is a homeomorphism f of $\bar{\Delta}$ onto $\bar{\Omega}$ which is analytic on Δ. The composition $u \circ f$ is continuous on $\bar{\Delta}$, harmonic in Δ; but it is easily seen that $u \circ f$ has unbounded harmonic conjugates on Δ because u does on Ω.

As a final application of the Riemann Mapping Theorem and Theorem 1, a somewhat circuitous route to a solution of a problem which appears to have first been solved by G. H. Hardy in 1913 [**3**, p. 157] will be described.

PROBLEM H. *Find a power series* $\Sigma a_n z^n$ *which converges uniformly on* $\bar{\Delta}$ *but not absolutely.*

Hardy indicates in [**3**] that this problem was suggested to him by M. Riesz in 1911 to whom, in turn, it had been suggested by H. Bohr. Hardy gives several specific examples in his paper [**3**]. One can also find examples in [**7**, p. 197]. The method of solution given below is partially suggested by [**2**, Exercise 7, p. 52], where the problem is to find a continuous function on $\bar{\Delta}$ with non-absolutely convergent power series; but first some additional results from geometric function theory must be discussed. As a matter of fact, in a classroom situation, solving Problem H becomes of secondary importance and serves only as one interesting consequence of the prerequisite theory. In particular, this is an opportune place to do an important theorem relating a power series to the area of its image and a Tauberian theorem due to Fejér. Specifically, we are referring to the next two results.

THEOREM 2. *If* $f(z) = \Sigma a_n z^n$ *is a one-to-one analytic function on* Δ, *then the area of the image* $f(\Delta)$ *is* $\pi \Sigma n\,|a_n|^2$.

THEOREM 3. *If* $f(z) = \Sigma a_n z^n$ *is uniformly continuous on* Δ *and if* $\Sigma n\,|a_n|^2 < \infty$, *then* $\Sigma a_n z^n$ *converges uniformly on* $\bar{\Delta}$.

For proofs of Theorems 2 and 3 which are completely elementary (though not exactly easy) see [**6**, pp. 121–125].

When Theorems 1, 2, and 3 are combined the following consequence obtains.

THEOREM 4. *Let Ω be a bounded simply connected region such that each boundary point of Ω is simple. If $f(z) = \Sigma a_n z^n$ is a one-to-one analytic map of Δ onto Ω, then $\Sigma a_n z^n$ converges uniformly on $\bar{\Delta}$.*

The final result which we need for our solution to Problem H is quite easy to establish and appears in [**5**, Exercise 25, p. 350].

THEOREM 5. *Suppose $f(z) = \Sigma a_n z^n$ and $\Sigma |a_n| < \infty$. Then $\lim_{\rho \uparrow 1} \int_0^\rho |f'(rb)|\, dr < \infty$ for every $b \in \partial \Delta$.*

In geometric terms, the conclusion of Theorem 5 is that f maps every radius of Δ onto an arc of finite length.

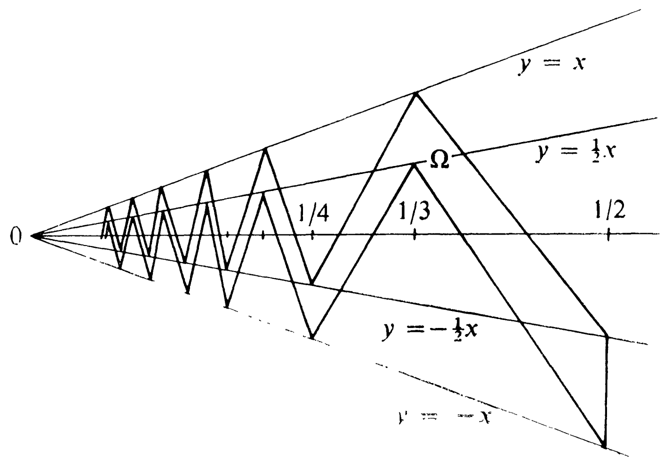

FIG. 1

Consider now the bounded simply connected region Ω which appears in Fig. 1. It is readily seen that each boundary point of Ω is simple. So by Theorem 4 and the Riemann Mapping Theorem there is a homeomorphism $f(z) = \Sigma a_n z^n$ from $\bar{\Delta}$ onto $\bar{\Omega}$ with $\Sigma a_n z^n$ uniformly convergent on $\bar{\Delta}$. Let b be that point in $\partial\Delta$ such that $f(b) = 0$. Then it is quite clear that f maps that radius of Δ which terminates at b onto an arc in $\Omega \cup \{0\}$ which terminates at 0. The length of this image arc is obviously not finite, and so by Theorem 5, $\Sigma |a_n| = \infty$. Hence we have produced a one-to-one power series which converges uniformly but not absolutely on $\bar{\Delta}$.

Added in proof. In a recent article, *Real proofs of complex theorems* (*and vice versa*), this MONTHLY, 81 (1974) 115–137, Lawrence Zalcman uses the same technique as was used above for producing uniformly but non-absolutely convergent power series. He also traces the first example of such a series to Fejér (1910).

References

1. C. Carathéodory, Theory of Functions of a Complex Variable, vol. II, 2nd English ed., Chelsea, New York, 1960.

2. P. L. Duren, Theory of H^p Spaces, Pure and Appl. Math., vol. 38, Academic Press, New York, 1970.

3. G. H. Hardy, A theorem concerning Taylor's series, Quart. Jour. Math., 44 (1913) 147–160.

4. W. P. Novinger, An elementary approach to the problem of extending conformal maps to the boundary, this MONTHLY, 82 (1975) 279–282.

5. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

6. W. Veech, A Second Course in Complex Analysis, Benjamin, New York, 1967.

7. A. Zygmund, Trigonometric Series, vol. I, 2nd ed., Cambridge University Press, London and New York, 1968.

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MATHEMATICAL EDUCATION

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EXPERIMENT WITH AUDIO-TUTORIAL PRECALCULUS

VIVIENNE MAYES, DAVID SCASTA AND PATRICK CONOLEY

Investigators in various teaching fields have designed experiments comparing self-paced methods of instruction with traditional teaching methods. Among these investigators are Sheppard [1970] and Moore [1969] who found that students in introductory psychology achieved more in a self-paced presentation, and Weaver [1969] who found no significant difference in student achievement between self-paced and traditional presentations of freshman biology. The intrinsic natures of the disciplines taught may have influenced these differing evaluations of the effectiveness of self-paced programs. During the summer of 1972 a design was conceived to compare a self-paced, audio-tutorial presentation of precalculus mathematics with a conventional lecture presentation of the subject. The design was implemented at Baylor University during the autumn of that year. This paper describes the design and offers some interpretations of the data obtained from the experiment.

I. The experiment began with the 193 students taking the precalculus course (elementary algebra and trigonometry). It was assumed that each of the students

had met the general university requirements for admission — that each had completed two units or more of high school mathematics, that each had scored at least 17 as a composite score on the *American College Testing Program Assessment*, and that each had finished in the top half of his graduating class.

On the first two class periods of the semester, the students were given the mathematics sections of the Cooperative School and College Ability Test (SCAT) [Educational Testing Service (ETS), 1955] and the Algebra III Test of the Cooperative Mathematics Test Series (ACMT) [ETS, 1963]. In addition, certain other information was obtained from the students so that they could be matched fairly closely according to the following criteria:

A. Performance parameter: same SCAT score.

B. High school parameters:
   1. Same or similar high school as determined by region (in-state or out-of-state) and by size of hometown. (Local students were matched to the same school.)
   2. Same mathematics preparation in terms of number and nature of high school math courses (within one course).
   3. Same or adjacent quarter rank in high school graduating class.

C. Personal parameters:
   1. Same sex.
   2. Same age (within one year).
   3. Same college classification (within one year).
   4. Same non-mathematics major.

Forty-five pairs were then selected from the matched sets of students. The selection was made so that the proportions of high, medium, and low SCAT-scoring pairs reflected the proportions of high, medium, and low SCAT-scoring students in the whole class. Thus, with respect to SCAT scores, the selected pairs represented two identical cross-sections of the whole precalculus population.

Coin tosses assigned one member from each of the forty-five pairs to a lecture section, which served as the control for the experiment. The other forty-five students were returned to a pool of 148 students who received the audio-tutorial presentation. In order to avoid prejudicial behavior, the course instructor did not know which of the 148 audio-tutorial students were involved in the experiment.

After their exposure to the presentations of precalculus, the students were given the Trigonometry Test of the Cooperative Mathematics Test Series (TCMT) [ETS, 1962], and the ACMT [ETS, 1963] was re-administered. The TCMT and pre-and post-ACMT scores served as the data for the comparison of the two sections.

Since there is no universal "Lecture Method," and since self-paced programs of instruction likewise vary considerably, it is important to indicate the points of difference and similarity between the two presentations of precalculus as they were implemented in the experiment. The students in the lecture section met with the pro-

fessor for three one-hour sessions per week. A standard published text was used as a curriculum guide [Robinson, 1970]. Three problem-solving oriented, one-hour quizzes were given during the semester.

In contrast, there were no scheduled meetings for class instruction in the audio-tutorial section. A text, written at Baylor specifically for self-study, and a problem workbook [Hample, 1971] weie used. Four quizzes were prepared for each of the ten units of the course. Most of the quizzes had eight multiple choice questions followed by two problems where the students were required to show all of their steps. A schedule of weekly examination periods was distributed at the beginning of the semester. The students were allowed to spend the entire six-hour testing period taking quizzes over at most two of the units featured for each week. However, the students were allowed to repeat any unit during the four consecutive weeks in which it was featured.

The points of similarity between the two programs were as follows. The same course content was presented to both sections. In both sections homework assignments were collected and graded each week. The same tutoring service was available to all of the students. Both the audio-tutorial section and the lecture sections were under the guidance of the same professor. And the students of both sections were required to complete the course by the end of the semester.

Given the above similarities and differences in method, the experiment was designed to test the hypothesis that students with audio-tutorial presentation differ significantly from students with lecture presentation in achievement as measured by performance on the TCMT and by improvement on the ACMT. The 0.05 level of significance was adopted for all of the statistical tests.

Before statistical measures could be calculated, however, the usual semester's attrition rate had to be taken into account. After 11% of the lecture students and 13% of the experimental, audio-tutorial students had dropped the course, only thirty-three of the original pairs remained.[1] The unpaired audio-tutorial students who were identical (on the basis of the matching criteria defined above) to audio-tutorial students in the experimental pairs served as a pool of replacements. Substitutions from this reserve allowed the resurrection of two pairs in which only the audio-tutorial member had dropped the course. A net total of thirty-five pairs was available for statistical analysis.

II. The two sections were divided into aptitude groups according to SCAT score rank — the seven lowest scorers of each section in group 1, the seven second lowest in group 2, and so on to the seven highest who were assigned to group 5. The histogram (Fig. 1) depicts the differences in means between the pre- and post-

[1] As Born [1971] has found, whereas the lecture students withdrew early in the semester (the usual dropping time when students rearrange their work-study schedules), most of the audio-tutorial students dropped later in the semester (perhaps when it was apparent to them that they were unable to finish the course within the remaining time).

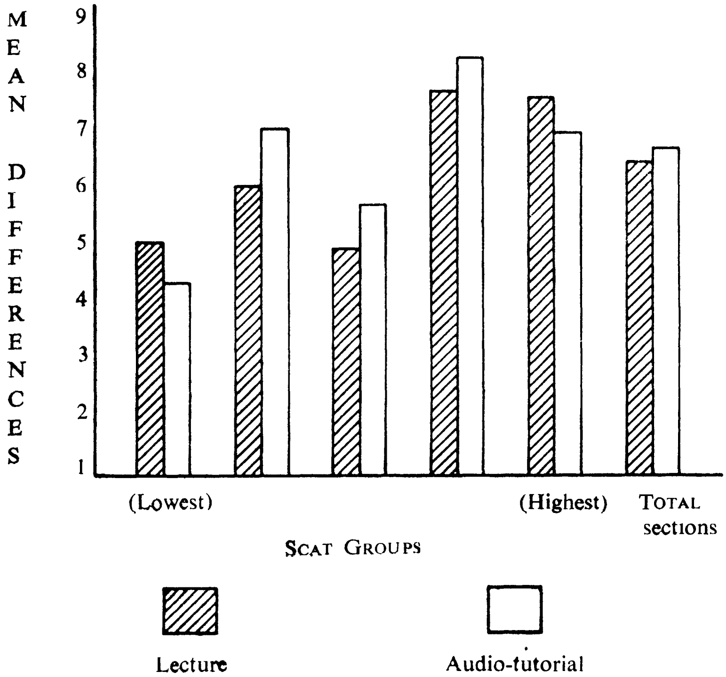

FIG. 1
Differences between means for pre- and post-ACMT scores

ACMT scores for each of the SCAT groups and for the two sections as a whole. The section totals suggest that there is a slight trend for the audio-tutorial section as a whole to make greater improvement during the semester than the lecture section. When the sections are broken down into SCAT groups, this audio-tutorial edge seems to be accounted for by the middle scoring groups (2, 3, 4), while the lowest (1) and the highest (5) SCAT groups tend to improve more in the lecture section. The following statistical analysis indicates that these trends are too weak to be accepted at this time. Nevertheless, they may propose a line for further research.

A Split Plot Factorial 5.2 Analysis of Covariance (SPFAC-5.2) [Kirk, 1968, 482–485] was performed on the ACMT data. This design compensates for initial differences in abilities among the students by using the pre-ACMT scores as a covariate to adjust the post-ACMT scores. The design simultaneously analyzes for differences in final performance which might be related either to the SCAT aptitude levels (Treatment A) or to the two educational methods (Treatment B). Further, the design analyzes the AB-interaction between the various SCAT groups and the educational methods.

III. In summary, no significant difference was detected between the effects of the

audio-tutorial and lecture methods of teaching. (Omitted details available upon request.) Consequently, the null hypothesis cannot be rejected. Although the trend of the data favored the audio-tutorial method, the trend was exceedingly small and is probably attributable to chance. Whereas replication would better clarify this trend, a tentative conclusion is that both methods of presentation are probably equally effective in teaching students pre-calculus mathematics.

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References

David G. Born, Student Withdrawals in Personalized Instruction Courses and in Lecture Courses Symposium. Paper presented at the meeting of the Rocky Mountain Psychological Association, Denver, May 1971.

Educational Testing Service, Cooperative Test Division. Algebra III, Form A, Princeton, N.J., 1963.

———, School and College Ability Test (SCAT) Form 1A, Part II, pp. 4–5, Part IV, pp. 8–9, Princeton, N. J. 1963.

———, Trigonometry, Form B, Princeton, N. J. 1962.

Stephen R. Hample and Larry Knutson, Study Guide to Accompany Algebra and Trigonometry, Harper and Row, New York, 1971.

Roger Kirk, Experimental Design, Brooks/Cole, Monterey, California, 1968, pp. 248–251, 482–485.

J. W. Moore, J. M. Mahan, and C. A. Ritts, Continuous Progress Concept of Instruction with University Students, Psychol. Reports, 23 (1969) 887–892.

T. J. Robinson, Algebra and Trigonometry, Harper and Row, New York, 1970.

W. C. Sheppard and H. G. MacDermott, Design and evaluation of a programmed course in introductory psychology, J. Appl. Behavior Analysis, 3 (Spring 1970) 5–11.

G. M. Weaver, Evaluation and Analysis of Audio-Tutorial Instruction in a General Education Biology Course, Doctoral Dissertation, Commerce, Texas: East Texas State University 1969.

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A PROGRAM IN MATHEMATICAL ANALYSIS FOR THE LIFE AND MANAGEMENT SCIENCES

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Historically, the departments of mathematics at most colleges and universities have served a two-fold purpose: (1) a service department for the physical sciences and engineering, and (2) the training of mathematicians at the undergraduate and graduate level. In addition certain elementary courses have been offered to satisfy the basic mathematics requirements of various departments in non-technical fields.

For example, many schools now offer elementary mathematics courses for the students in business administration, and introductory statistics courses (precalculus) for students in the social sciences. Also, the needs of prospective elementary and secondary teachers have been recognized and considerable progress has been made at many schools in providing course work and guidance relevant to these needs.

During the past two or three decades various disciplines in the life and management sciences have increasingly adapted quantitative methods for their work. Except for some course offerings like those previously mentioned, most mathematics departments have been remiss in performing a service function for those areas in the life and management sciences and the various interdisciplinary studies which intersect with one or more of these areas. Students in these areas desiring or needing training in mathematics beyond the very elementary level have been forced to take courses in calculus and higher level mathematics that were designed for students in mathematics, physical sciences, or engineering.

Texas Tech University has instituted a sequence of six courses which are specifically designed for students in the life and management sciences. These six courses are part of a unified pattern and from the very beginning of the sequence the student has ready access to computing equipment with hands-on operation under the supervision of a staff member. In these six courses, the students learn and use the ideas and techniques of mathematics in their fields of interest. Throughout these courses realistic applications from different fields are introduced. In these applications, care is taken to ensure that the problems are clearly stated and that all assumptions, approximations and idealizations used to obtain a mathematical answer are explicitly discussed. It is our intent to see that the applications are realistic and that their treatment is as complete and intellectually honest as the level of the student's training allows.

The first three courses are each three-hour courses accompanied by a laboratory meeting for one hour each week. These three courses cover probability, linear algebra, and calculus, with some introductory material in difference and differential equations. In designing these courses it has been necessary to borrow and delete from several courses that have been part of the traditional curriculum in mathematics in order to arrive at a course sequence that provides a relevant mathematical background for the life and management sciences.

The one-hour laboratory course provides an opportunity for the student to learn to program and operate a small computer. The computer work necessarily covers some basic problems normally encountered in programming, but more importantly, hands-on use of the computer enables the student to grasp some of the mathematical concepts which are frequently troublesome for the beginner. In addition, it allows students to obtain approximate solutions to "real" problems, rather than concentrating on obtaining closed form solutions to contrived problems.

For our laboratory we have purchased a Wang 2200. It is equipped with an output

writer-plotter, a CRT display, and magnetic tape cassette program storage unit. This machine utilizes BASIC language programming.

The fourth and fifth courses contain more probability, an introduction to the techniques of statistics and optimization theory, and applications. These two courses also discuss some of the basic aspects of the theory and techniques of numerical analysis.

The sixth course is one in mathematical modeling theory. In recent times the disciplines of life and management sciences have devoted a great deal of their time to investigations of those topics in their areas which, they feel, might be quantified. These investigations consist of (1) recognition of a real problem whose mathematical aspects are not clearly defined; asking a series of questions whose answers will shed light on the problem, (2) constructing a mathematical model which will aid in answering these questions, (3) using the ideas and techniques of mathematical analysis to solve the mathematical model, (4) making relevant computations, (5) interpreting the mathematical results in the language of the original problem. This procedure is applied to several examples taken from the life and management sciences.

Specifically, this sequence of courses is being offered for selected students in economics, sociology, psychology, business, pre-med, agricultural sciences, and biology. A student who completes all six courses has a recognized minor in mathematics.

In planning this program we had hoped for two sections with 30 students in each section; however, during the first enrollment period (Fall, 1973) the enrollment exceeded 240 students. The prerequisite for enrolling in this program is a good background in high school mathematics, to include two years of algebra and one year of geometry.

This sequence of courses covers the following mathematical topics. Applications to the life and management sciences are interwoven throughout each course.

COURSE 1. Introduction to linear algebra, finite probability, and differential calculus.

COURSE 2. Properties of derivatives: the integral calculus; trigonometric, logarithmic, and exponential functions; introduction to the calculus of several variables.

COURSE 3. Further topics in linear algebra, probability and the calculus of several variables; introduction to series, difference equations and differential equations.

COURSE 4. Estimation, tests of hypotheses, regression, correlation, analysis of variance.

COURSE 5. Effective algorithms for optimization problems of practical significance; both linear and nonlinear problems are considered.

COURSE 6. The first half of the course is conducted in a lecture style. The ideas of model building are introduced by using examples in different fields. Examples might be: programming models for resource allocation, linear programming models

of pollution control, Markov chain models for learning theory, game theory for models of conflict, etc. As one builds a model, the role of asking the right question and viewing the real world from the right perspective needs to be emphasized. The same problem might be modeled more than once in order to stress that there is not necessarily a best model. Attention is given to evaluating the model against the real world.

The last half of the course is spent in analyzing two or three real problems in which the class fully participates. These examples are in an area in which the instructor can show competence, or in an area where some other faculty member (possibly outside the mathematics department) has some expertise.

Basic References

COURSE 1, 2, and 3

1. Joshua Chover, The Green Book of Calculus, Benjamin, Menlo Park, Cal., 1972.
2. W. E. Baxter and C. W. Sloyer, Calculus with Probability for the Life and Management Sciences, Addison-Wesley, Reading, Mass., 1973.

COURSE 4

1. H. L. Alder and E. B. Roessler, Introduction to Probability and Statistics, 5th ed., Freeman, San Francisco, 1972.
2. Statistics: A Guide to the Unknown, edited by Judith M. Tanur, Frederick Mosteller et al., Holden-Day, San Francisco, 1972.
3. J. K. Wani, Probability and Statistical Inference, Appelton-Century Crofts, New York, 1971.

COURSE 5

1. Arthur M. Geoffrian, Perspectives on Optimization, Addison-Wesley, Reading, Mass., 1972.

COURSE 6

1. R. C. Atkinson, G. H. Bower and E. J. Crothers, An Introduction to Mathematical Learning Theory, Wiley, New York, 1965.
2. J. C. Kemeny and J. L. Snell, Mathematical Models in the Social Sciences, Blaisdell, Waltham, Mass., 1962.
3. R. D. Luce and H. Raiffa, Games and Decisions, Wiley, New York, 1957.
4. Maynard Thompson and D. P. Maki, Mathematical Models and Applications, Prentice-Hall, Englewood Cliffs, N.J., 1973.

Additional References

1. Peter Abell, Model Building in Sociology, Schocken, New York, 1971.
2. R. G. D. Allen, Mathematical Economics, Macmillan, New York, 1959.
3. American Economic Review.
4. American Journal of Sociology.
5. American Sociological Review.
6. R. C. Atkinson, Studies in Mathematical Psychology, Stanford University Press, Stanford, Calif. 1964.
7. Otomar J. Bartos, Simple Models of Group Behavior, Columbia University Press, New York, 1967.
8. Edward Batschelet, Introduction to Mathematics for Life Scientists, Springer Verlag, New York, 1971.

9. William J. Baumol, Economic Theory and Operations Analysis, 3rd ed., Prentice Hall, Englewood Cliffs, N.J. 1972.

10. E. G. Bennion, Elementary Mathematics of Linear Programming and Game Theory, Michigan State University Public Service, East Lansing, 1960.

11. Bibliography of Selected Mathematical Models collected by the participants in the Summer Institute in Model Building and Statistics, Rice University, Houston, Summer 1972, 436 references (to books and journals) briefly annotated and listing the type of application and the mathematical prerequisites.

12. E. K. Bowen, Mathematics with Applications in Management and Economics, 3rd ed., Irwin, Homewood, Illinois, 1972.

13. Grace Bush and John Young, Foundations of Mathematics with Applications to the Social and Management Sciences, 2nd ed., McGraw-Hill, New York, 1973.

14. Frank Carlborg, Introduction to Statistics, Scott, Foresman, Glenview, Illinois, 1968.

15. Alpha C. Chiang, Fundamental Methods of Mathematical Economics, McGraw-Hill, New York, 1967.

16. James S. Coleman, Introduction to Mathematical Sociology, Collier-Macmillan Limited, London, 1964.

17. C. H. Coombs and J. N. Bowen, Additive of Risk in Portfolios, Perception and Psychophysics, 10 (1971) 43–46.

18. A. Crowe and A. Crowe, Mathematics for Biologists, Academic Press, New York, 1969.

19. CUPM, A Basic Library List for Two Year Colleges, 1971.

20. CUPM, Applied Mathematics in the Undergraduate Curriculum, 1972.

21. CUPM, Introductory Statistics without Calculus, 1972.

22. CUPM, Recommendations for an Undergraduate Program in Computational Mathematics, 1971.

23. CUPM, Recommendations for the Undergraduate Mathematics Program for Students in the Life Sciences, 1970.

24. J. G. Defares and I. N. Sneddon, An Introduction to the Mathematics of Medicine and Biology, Year Book Medical Publishers, Chicago, 1961.

25. Jean Draper and Jane Klingman, Mathematical Analysis, 2nd ed., Harper & Row, New York, 1972.

26. P. H. DuBois, An Introduction to Psychological Statistics, Harper & Row, New York, 1965.

27. Econometrics.

28. Economic Journal.

29. Economica.

30. A. I. Forsythe, et al., Computer Science: Basic Language Programming, Wiley, New York, 1970.

31. John Freund, Modern Elementary Statistics, 4th ed., Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

32. Douglas Friedrich, A Primer for Developmental Methodology, Burgess, Minneapolis, 1972.

33. Geographical Analysis.

34. Samuel Goldberg, Introduction to Difference Equations, Wiley, New York, 1958.

35. W. C. Guenther, Concepts of Statistical Inference, 2nd ed., McGraw-Hill, New York, 1973.

36. Josef Hadar, Mathematical Theory of Economic Behavior, Addison-Wesley, Reading, Mass., 1971.

37. Handbook of Mathematical Psychology, Wiley, New York, 1963–1965.

38. W. C. Hoffman, Bibliography for an Undergraduate Course in Applications of Mathematics in the Life Sciences, 1970, available from CUPM.

39. J. E. Howell and D. Teichroew, Mathematical Analysis for Business Decisions, rev. ed., Irwin, Homewood, Illinois, 1971.

40. John B. Johnson, G. Baley Price and Fred Van Vleck, Linear Equations and Matrices, Addison-Wesley, Reading, Mass., 1966.

41. John B. Johnson, G. Baley Price and Fred Van Vleck, Sets, Functions and Probability, Addison-Wesley, Reading, Mass., 1968.

42. Journal of Conflict Resolution.

43. Journal of Mathematical Psychology.

44. Journal of Mathematical Sociology.

45. John G. Kemeny and Thomas E. Kurtz, BASIC Programming, 2nd ed., Wiley, New York, 1971.

46. John G. Kemeny and J. Laurie Snell, Finite Markov Chains, Van Nostrand, Princeton, N. J. 1960.

47. Oscar Kempthorne, An Introduction to Genetic Statistics, The Iowa State University Press, Ames, Iowa, 1969.

48. R. S. Ledley, Use of Computers in Biology and Medicine, McGraw-Hill, New York, 1965.

49. C. C. Lin and L. A. Segel, Mathematics Applied to Deterministic Problems, Macmillan, New York, 1974.

50. Mathematical Thinking in Behavioral Science, Readings from Scientific American, Freeman, San Francisco, 1968.

51. J. McKinsey, Introduction to the Theory of Games, McGraw-Hill, New York, 1952.

52. G. A. Miller, Mathematics and Psychology, Wiley, New York, 1964.

53. Howard M. Nahikian, A Modern Algebra for Biologist, The University of Chicago Press, Chicago, 1964.

54. A Pollatsek and A. Tversky, A Theory of Risk, J. Mathematical Psychology, 7 (1970) 540–553.

55. Proceedings of the Conference on the Applications of Undergraduate Mathematics in the Engineering, Life, Managerial, and Social Sciences, June 13–15, 1973. Available from School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30305.

56. Proceedings Summer Conference for College Teachers on Applied Mathematics at the University of Missouri-Rolla, 1971, available from CUPM.

57. Production of Case Studies and Resource Materials for the Teaching of Applied Mathematics at the Advanced Undergraduate Level, a carefully written sequence of five to ten mathematical models taken from different disciplines. This material will be classroom-tested in the spring of 1974, (to be published by CUPM).

58. Production of Source Book on Applications for Undergraduate Mathematics to the Social Sciences, currently being prepared under auspices of CUPM and MSSB, (to be published).

59. Anatol Rapoport, Fights, Games and Debates, The University of Michigan Press, Ann Arbor, 1960.

60. Anatol Rapoport, Two Person Game Theory, the University of Michigan Press, Ann Arbor, 1966.

61. Nicolas Rashevsky, Mathematical Principles in Biology, Thomas, Springfield, Ill., 1961.

62. Robert Rosen, Optimality Principles in Biology, Plenum Press, New York, 1967.

63. C. A. B. Smith, Biomathematics, vol. 1 and 2, 4th ed., Hafner, New York, 1966.

64. J. M. Smith, Mathematical Ideals in Biology, Cambridge University Press, London, 1968.

65. W. R. Smythe and L. A. Johnson, Introduction to Linear Programming with Applications, Prentice-Hall, Englewood Cliffs, N. J. 1966.

66. G. R. Stibitz, Mathematics in Medicine and the Life Sciences, Year Book Medical Publishers, Chicago, 1966.

67. Jay Strum, Introduction to Linear Programming, Holden-Day, San Francisco, 1972.

68. C. A. Theodore, Applied Mathematics: An Introduction, rev. ed., Irwin, Homewood, Illinois, 1971.

69. W. R. Uttal, Real-Time Computers-Techniques and Applications in the Psychological Sciences, Harper & Row, New York, 1967.

70. K. E. F. Watt, Ecology and Resource Management, McGraw-Hill, New York, 1968.

71. John Davis Williams, The Compleat Strategyst, rev. ed., McGraw-Hill, New York, 1966.

72. S. I. Zukhovitskiy and L. I. Avdeyeva, Linear and Convex Programming, Saunders, Philadelphia, 1966.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before August 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E2534. *Proposed by C. H. Kimberling, University of Evansville*

Consider the array of numbers $a(j,k)$ defined for $j,\ k = 0,1,\cdots$ as follows: $a(j,0) = 1$ for $j = 0,1,\cdots$; $a(0,k) = 2$ for $k = 1,2,\cdots$; $a(j,k) = a(j,k-1)+a(j-1,k)$ for $j,k \geqq 1$. Prove the following:

(i) If p is prime, then $p \mid a(j,p-j+1)$ for $j = 2,3,\cdots,p-1$.

(ii) If $j+2k$ is prime, then it divides $a(j,k)$.

(iii) If $a(j,k)$ is prime, then it divides $a(mj,mk)$ for $m = 1,2,\cdots$.

E2535. *Proposed by M. S. Klamkin, University of Waterloo*

A body is projected in a uniform gravitational field and is subject to a resistance

which is a function of its speed $|\mathbf{v}|$. If the acceleration $\mathbf{a}$ of the body always has a constant direction, no matter what the initial velocity $\mathbf{v}_0$, show that $\mathbf{a} = \mathbf{a}_0 e^{-kt}$ for some constant k.

E2536. *Proposed by Jacob Brandler, Brooklyn College*

If $x^6 = x$ for every element x in the ring R, prove that R is a Boolean ring. Generalize.

E2537. *Proposed by David Shelupsky, City College of C.U.N.Y.*

Find all continuous functions f defined on $(0, \infty)$ for which $f(x_1 y) - f(x_2 y)$ is independent of y.

E2538*. *Proposed by J. Garfunkel, Flushing, New York*

Let ABC be a triangle. If X is a point on side BC, let AX meet the circumcircle of ABC again at X'. Prove or disprove: If XX' has maximum length, then AX lies between the median and the internal angle bisector issuing from A.

E2539*. *Proposed by A. Vince, Woods Hole, Massachusetts*

Let F_n denote the nth Fibonacci number: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove or disprove: If $m^2 \mid F_n$, then $m \mid n$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Hide and Seek in the Unit Disk

E2469 [1974, 405]. *Proposed by David Gale and C. Roger Glassey, University of California, Berkeley*

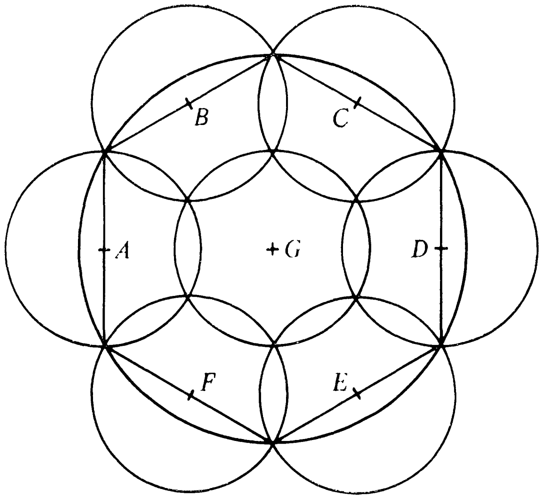
Two players, the Hider and the Seeker, simultaneously choose points in a closed disk of unit radius. The Hider escapes if his point is more than $\frac{1}{2}$ unit from that of the Seeker. Show that if both players play optimally in the sense of game theory, then the Hider will be caught with probability 1/7.

Solution by Ron Evans, University of Wisconsin. As shown in the accompanying figure (drawn by Michael Goldberg), the closed unit disk centered at G (on which the game is played) can be covered by the 7 closed disks of radius $\frac{1}{2}$ centered at $A, B, \cdots, G$, where the six points $A, B, \cdots, F$ are uniformly distributed at a distance $\sqrt{3}/2$ from G.

Let the Seeker (S) adopt the strategy of playing at random on one of the 7 points $A, B, \cdots, G$. Then the probability that S catches the Hider (H) is $\geqq 1/7$, regardless of H's strategy.

Now let H adopt the strategy of playing on G with probability 1/7 and on an arbitrary point on the boundary of the unit disk with probability 6/7. It remains to show that the probability that S catches H is $\leqq 1/7$, regardless of S's strategy.

Let Δ be the closed disk of radius $\frac{1}{2}$ centered at G. In view of H's strategy, the probability that S catches H is positive if and only if S and H either both play in



Δ or both play outside of Δ. Given that S and H both play in Δ, the probability that S catches H is $u = 1$. Given that S and H both play outside of Δ, let t be the probability that S catches H. Then $t \leqq 1/6$, since an arc of the unit circle contained in a disk of radius $\frac{1}{2}$ has length $\leqq 2\pi/6$ (because its chord has length $\leqq 1$). Let p be the probability that S plays in Δ. Then the probability that S catches H is

$$up(1/7) + t(1-p)(6/7) \leqq 1/7.$$

Also solved by Barefoot Carolina Hillbilly, D. J. Bordelon, Michael Goldberg, Robert Kopp, C. F. Larimer, Harry Lass, P. W. Lindstrom, O. P. Lossers (Netherlands), D. J. Newman & M. Slater, William Nuesslein & J. C. Hrbek, S. M. Samuels, George Schillinger, Eugene Tidmore & Danny Turner, and the proposers.

Editor's comment. Bordelon attacks the more general problem where the disk of play has radius D and the Hider escapes if his point is more than R units from that of the Seeker. The solutions are qualitatively different depending on the magnitude of R/D: If $2^{-1/2} \leqq R/D < 1$, then both players have pure strategies and the value of the game is $\pi^{-1} \sin^{-1} (R/D)$. If $\frac{1}{2} \leqq R/D < 2^{-1/2}$, then both players have strategies similar to those described by Evans for the case $R/D = \frac{1}{2}$, and the value of the game is $(1 + \pi/\sin^{-1}(R/D))^{-1}$. If $R/D < \frac{1}{2}$, then the situation is more complicated. Bordelon comments that his results extend those of an unpublished working paper by Samuel Kneale and Russell Johnson, entitled *Probability of hitting an alerted intelligently evasive target* (Operations Research, Inc., Silver Spring, Maryland).

A Simplex Equality Characterizing the Centroid

E2470 [1974, 405]. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let $A_0A_1 \cdots A_n$ be an n-simplex in R^n and let G be its centroid. Let P be any point and let M_i be the intersection of the line A_iG with the hyperplane through P parallel

to the face opposite A_i. Show that

$$\sum_{i=0}^{n} \frac{GM_i}{GA_i} = 0,$$

where GM_i/GA_i denotes the directed ratio of the distances from G to M_i and G to A_i, respectively. Show that this equation characterizes the point G as the centroid; i.e., if the sum is 0 for every point P, then necessarily G is the centroid. (Cf. Problem E2394 [1974, 283].)

Solution by V. T. Norton, Jr., Bowling Green State University. Given arbitrary points P and G, let $(p_0, p_1, \cdots, p_n)$ and $(g_0, g_1, \cdots, g_n)$ be their barycentric coordinates with respect to the simplex $A_0A_1 \cdots A_n$. (We assume $\Sigma p_i = \Sigma g_i = 1$.) Let $\lambda_i = GM_i/GA_i$. Since M_i lies on the hyperplane through P parallel to the face opposite A_i, the ith coordinate of M_i is p_i. Then, since the ith coordinate of A_i is 1,

$$\lambda_i = \frac{p_i - g_i}{1 - g_i}.$$

(We are assuming that G is no vertex of the simplex, so that λ_i is defined.)

If G is the centroid of the simplex, then $g_i = (n+1)^{-1}$ for $i = 0, 1, \cdots, n$ and the denominator of λ_i is $n/(n+1)$. Thus

$$\Sigma\lambda_i = \frac{n+1}{n}\ \Sigma(p_i - g_i) = \frac{n+1}{n}(\Sigma p_i - \Sigma g_i) = 0.$$

On the other hand, if G is not the centroid, then $g_j \neq g_k$ for some $j \neq k$. To be specific, suppose $g_0 \neq g_1$. Putting $p_0 = g_1, p_1 = g_0$ and $p_i = g_i$ for $i = 2, 3, \cdots, n$, we have

$$\Sigma\lambda_i = \lambda_0 + \lambda_1 = \frac{-(g_1 - g_0)^2}{(1-g_0)(1-g_1)} \neq 0.$$

Also solved by M. G. Greening (Australia), I. G. Kastansas (Greece), and the proposer.

Two New Triangle Inequalities

E2471 [1974, 406]. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let a, b, c denote the sides of a triangle ABC and let m_a, w_a, h_a denote the median, angle bisector, and altitude to side a respectively. Show that

$$(1)\quad \frac{(b+c)^2}{4bc} \leqq \frac{m_a}{w_a}, \qquad (2)\quad \frac{b^2+c^2}{2bc} \leqq \frac{m_a}{h_a}.$$

When does equality hold?

Solution composed from those submitted by Leon Bankoff, Los Angeles, California, and Leonard Goldstone, Watervliet, New York. If k_a denotes the symmedian

to side a, then

$$m_a = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},$$

$$w_a = \frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2 - a^2},$$

$$k_a = \frac{bc}{b^2+c^2}\sqrt{2b^2 + 2c^2 - a^2}.$$

By the triangle inequality, $a^2 > (b-c)^2$, so $(b+c)^2 - a^2 < (b+c)^2 - (b-c)^2 = 4bc$, and therefore

$$(*) \qquad \frac{(b-c)^2}{4bc} \leqq \frac{(b-c)^2}{(b+c)^2 - a^2},$$

with equality if and only if $b = c$. By adding unity to each side of (*), we obtain

$$\frac{(b+c)^2}{4bc} \leqq \frac{2b^2 + 2c^2 - a^2}{(b+c)^2 - a^2},$$

which is equivalent to

$$\frac{b+c}{2\sqrt{bc}} \leqq \sqrt{\frac{2b^2 + 2c^2 - a^2}{(b+c)^2 - a^2}},$$

and therefore

$$\frac{(b+c)^2}{4bc} \leqq \frac{b+c}{2\sqrt{bc}}\sqrt{\frac{2b^2 + 2c^2 - a^2}{(b+c)^2 - a^2}} = \frac{m_a}{w_a},$$

which is inequality (1). Equality holds if and only if $b = c$.

For (2), it is evident that

$$\frac{m_a}{h_a} \geqq \frac{m_a}{k_a} = \frac{b^2 + c^2}{2bc}$$

since $h_a \leqq k_a$. Because h_a and k_a divide BC in the ratios $c^2 : b^2$ and $(a^2 + c^2 - b^2) : (a^2 + b^2 - c^2)$ respectively, equality holds in (2) if and only if these ratios are equal, i.e., if and only if $(b^2 - c^2)(a^2 - b^2 - c^2) = 0$. Thus equality holds in (2) if and only if $b = c$ or A is a right angle.

Also solved by David Beran, and by John Lillington (England). Partial solutions (generally with incomplete equality conditions) by Erhart Braune (Austria), R. J. Cormier, J. Garfunkel, M. G. Greening (Australia), I. G. Kastanas (Greece), Lew Kowarski, Carolyn MacDonald, Dewey Moore, Kumar Murty & Ram Murty, Eva Nyulassy (Czechoslovakia), Basilios Papaioannou, St. Olaf College Students, Michael Steiner (Sweden), and the proposer.

Another Binomial Coefficient Summation

E2472 [1974, 406]. *Proposed by David Shelupsky, City College of New York, and H. W. Gould, West Virginia University*

Let n and p be nonnegative integers. Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^{2k}}{\binom{2k}{k}}\left\{\frac{1}{(2k+1)}\frac{3}{(2k+3)}\cdots\frac{2p-1}{(2k+2p-1)}\right\}=\frac{2p-1}{2p-1+2n},$$

where we make the interpretation that when $p = 0$ the "empty product" in the curly brackets is unity. (The case $p = 0$ is Formula (4.12) of H. W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972.)

I. *Solution by P. S. Bruckman, University of Illinois, Chicago Circle.* Call the given sum $S(n,p)$. By suitable rearrangement, we can rewrite $S(n,p)$ as

$$S(n,p)=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{-p-\frac{1}{2}}{k}},$$

and this is the special case $j = 0$, $z = n$, $x = -p-\frac{1}{2}$ of Formula (4.1) of *Combinatorial Identities.*

II. *Solution by O. G. Ruehr, Michigan Technological University.* Rewriting $S(n,p)$ as in Solution I, we recognize it as the hypergeometric function ${}_2F_1(-n,1;p+\frac{1}{2};1)$, which has a closed form expression (Vandermonde's Theorem). See *Higher Transcendental Functions*, Vol. I, Bateman Manuscript Project, McGraw-Hill, 1953, p. 104. The condition on p can be relaxed to $p \neq -\frac{1}{2}, -\frac{3}{2}, \cdots, -n+\frac{1}{2}$.

A direct elementary proof can be based on the familiar beta integral. Write the series as

$$S(n,p)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+p+\tfrac{1}{2})\int_0^1 t^k(1-t)^{p-\frac{1}{2}}\,dt.$$

Interchange the order of integration and summation; then use the Binomial Theorem and the well-known formula $\sum_{k=0}^{n}(-1)^k\binom{n}{k}kt^k = nt(1-t)^{n-1}$ (obtained by differentiating both sides of the binomial formula) to get

$$S(n,p)=-n\int_0^1 t(1-t)^{n+p-\frac{3}{2}}dt+(p+\tfrac{1}{2})\int_0^1(1-t)^{n+p-\frac{1}{2}}dt.$$

The first of these integrals is a beta function and the second is easily evaluated. Simplification yields the desired result.

Also solved by R. G. Buschman, L. Carlitz, Sylvan Greene, O. P. Lossers (Netherlands), Ram Murty & Kumar Murty, F. C. Smith, and David Zeitlin.

Editor's comment. Greene proves the result directly by induction on n by establishing the recurrence formula

$$S(n+1,p) = \frac{2p-1}{2p+1}S(n,p).$$

Zeitlin also uses induction on n, but uses the following recurrence formula:

$$S(n+1,p) = S(n,p) + \frac{2n}{2p+1}S(n-1,p+1) - \frac{2n+2}{2p+1}S(n,p+1).$$

Smith evaluates the sum directly by using the discrete analog of the Fundamental Theorem of Calculus: $\sum_{k=p}^{r} f(k) = F(r+1) - F(p)$ if $f(k) = \Delta F(k)$. Murty and Murty make use of the following well-known inversion theorem: if $f(n) = \sum(-1)^k\binom{n}{k}g(k)$, then $g(n) = \sum(-1)^k\binom{n}{k}f(k)$. (In the notation of the "Umbral Calculus," this takes the form $g^n = (1-f)^n$ if $f^n = (1-g)^n$. — Ed.) Letting $g(n) = S(n,p)$ (and considering p fixed), they establish by induction on n that

$$f(n) = 2^{2n}\binom{2n}{n}^{-1}\left\{\frac{1}{2n+1}\,\frac{3}{2n+3}\cdots\frac{2p-1}{2n+2p-1}\right\},$$

and then apply the inversion. The other solvers use either hypergeometric functions or the beta integral as in Solution II.

Rational Function of a Rational Function of a Polynomial

E 2473 [1974, 406]. *Proposed by Robert Brooks, Harvard University*

Let f and g be irreducible polynomials with coefficients in a field K. Prove that there exists a polynomial which is a rational function of f/g if and only if $f = ag + b$ for some $a, b \in K$.

I. *Solution by Anon, Erewhon-upon-Yarkon.* We assume only that $g(x)$ is nonconstant, and that f/g is in lowest terms, i.e., $f(x)$ and $g(x)$ are relatively prime. Suppose that polynomials $P(t)$ and $Q(t)$, with coefficients in K, exist, not both constant, which are relatively prime and for which

$$\frac{P(f/g)}{Q(f/g)} = h(x),$$

where $h(x) \in K[x]$ is a polynomial. If $Q(t)$ is a constant, which we may take to be 1, then $P(f/g) = h(x)$. Writing $P(t) = a_m t^m + \cdots + a_0$ and clearing fractions leads to

$$a_m f^m + a_{m-1}f^{m-1}g + \cdots + a_0 g^m = g^m h,$$

which implies that $g \mid f^m$, a contradiction.

Assume then that $Q(t)$ is nonconstant, so that $Q(\alpha) = 0$ for some $\alpha \in \bar{K}$, the algebraic closure of K. Either we can solve $f(x) - \alpha g(x) = 0$ or not. If not, then $f(x) - \alpha g(x)$ is a nonzero constant $\beta \in \bar{K}$, and by comparing coefficients, first of

highest terms and then of lowest, we easily show that $\alpha, \beta \in K$ and we are done. If yes, then $f(\gamma) = \alpha g(\gamma)$ for some $\gamma \in \bar{K}$. Clearly $g(\gamma) \neq 0$, for if $g(\gamma) = 0$, then $f(\gamma) = 0$, contradicting the assumption that f and g are relatively prime. Thus $\alpha = f(\gamma)/g(\gamma)$ and from $P(f/g) = h(x)Q(f/g)$ we have $P(\alpha) = h(\gamma)Q(\alpha) = 0$, so that $P(\alpha) = Q(\alpha) = 0$, contradicting the fact that P and Q are relatively prime.

The converse is obvious, for if $f(x) = ag(x) + b$ with $a, b \in K$, let $P(t) = b$, $Q(t) = t - a$. Then $P(f/g)/Q(f/g) = g(x)$.

II. *Solution by Robert Gilmer, La Trobe University, Australia.* Result VI (p. 26) of A. Ostrowski, *Bemerkungen über die Struktur von Ringen, die aus Polynomen in einer Variabel bestehen*, Acta Arith. 1 (1936), 19–42, is the following: *If* $a(x)$, $b(x) \in K[x]$ *with* $\deg a(x) > \deg b(x)$, *and if* $K(t(x)) \cap K[x]$ *properly contains* K *(where* $t(x) = a(x)/b(x)$*), then* $t(x) \in K[x]$ *(implying that* $b(x) \mid a(x)$*) and* $K(t(x)) \cap K[x] = K[t(x)] \equiv \{f(t(x)) : f(x) \in K[x]\}$. [Note that to say $K(t(x)) \cap K[x]$ properly contains K is to say that there exists a nonconstant polynomial which is a rational function of $t(x) = a(x)/b(x)$.—Ed.]

Applying this to our problem, we see that $\deg f(x) = \deg g(x)$, for if not, then either $f(x) \mid g(x)$ or $g(x) \mid f(x)$, both impossible since $f(x)$ and $g(x)$ are irreducible. If $f(x)$ and $g(x)$ are associates, then we are done, so we can assume there is a (unique) nonzero $a \in K$ such that $h(x) = f(x) - ag(x)$ has degree less than that of $f(x)$. Since $K(f/g) = K((f/g) - a) = K(h/g) = K(g/h)$, Result VI implies that $h(x) \mid g(x)$. But $f(x)$ and $g(x)$ are relatively prime so that $h(x)$ and $g(x)$ are also, and this implies that $h(x) = b \in K$. Thus, $f(x) = ag(x) + b$ as was to be shown.

The converse is clear.

We remark that the proposer evidently intended that the field $K(f/g)$ should contain a nonconstant polynomial (otherwise the conclusion is false), and hence b cannot be 0, i.e., $f(x)$ and $g(x)$ cannot be associates. Ostrowski attributes Result VI to Emmy Noether.

Also solved by Ron Evans, Joseph Gruendler, O. P. Lossers (Netherlands), Nan-Shan Shou, Brian Wesselink, and the proposer.

The Maximum of Independent Random Variables

E 2474 [1974, 516]. *Proposed by Paul Abad and David Friedman, San Francisco Unified School District*

Let $X_1, \cdots, X_n$ be independent random variables such that each X_j is uniformly distributed over the interval $[0, a_j]$ with $0 < a_1 < \cdots < a_n$. Let $Y = \max[X_1, \cdots, X_n]$. Find a formula for the probability that $Y = X_i$ in terms of i and the a_j. (Cf. Problem 71–14, S.I.A.M. Review 13 (1971), p. 378.)

Solution by Tim Robertson, University of Iowa. We observe that

$$P[Y = X_i] = P\left(\bigcap_{j=1}^{n} [X_i \geqq X_j]\right)$$

$$= \int_0^{a_i} \int_0^{\min(a_1, x_i)} \cdots \int_0^{\min(a_n, x_i)} (a_1 \cdots a_n)^{-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n dx_i$$

$$= (a_1 \cdots a_n)^{-1} \int_0^{a_i} \prod_{j \neq i} \min(a_j, x_i) dx_i.$$

Now define $a_0 = 0$ and break this integral into parts:

$$P[Y = X_i] = \sum_{k=1}^{i} \int_{a_{k-1}}^{a_k} (a_k a_{k+1} \cdots a_n)^{-1} x_i^{n-k} dx_i$$

$$= \sum_{k=1}^{i} \frac{a_k^{n-k+1} - a_{k-1}^{n-k+1}}{(n-k+1) a_k a_{k+1} \cdots a_n}.$$

Also solved by D. R. Beuerman, Jerry denBroeder, Jay Devore, J. E. Fischer, S. J., W. E. Gould, J. C. Hickman, Dennis Jesperson, R. J. Kulperger, Harry Lass, O. P. Lossers (Netherlands), William Nuesslein, D. K. Pickard (Australia), R. L. Raymond, G. S. Rogers, Thomas Spencer, Selig Starr, and Harold Ziehms.

Editor's comment. Fourteen incorrect or incomplete solutions were received. As the published solution shows, the required probability is $P(\bigcap E_j)$ where E_j is the event $[X_i \geqq X_j]$. Many solvers incorrectly assumed that since the X_j were independent, the E_j were also.

The reference to Problem 71–14 in the S.I.A.M. Review given in the statement of the problem was faulty: the problem appeared [1971, 248] and its solution [1972, 378].

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before August 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6030. *Proposed by David Griffeath, Cornell University*

Let X and Y be jointly distributed real random variables. Consider the conjecture: If X, Y, $X + Y$, and $X - Y$ are all identically distributed, then $X = 0$ almost surely.

Prove or disprove the conjecture in the following cases:

(a) if X is square-integrable;

(b) if X is integrable;

(c) in general.

6031. *Proposed by I. I. Kotlarski, Oklahoma State University*

Let ϕ be a periodic function on $\mathscr{R}$ with period 2π, given by

(1) $$\phi(t) = 1 - \sqrt{\frac{|t|}{\pi}\left(2 - \frac{|t|}{\pi}\right)} \quad t \in [-\pi, \pi].$$

Prove that ϕ is a characteristic function of a real random variable X, and find its probability structure.

Let $X_1, X_2, \cdots, X_n, \cdots$ be independent identically distributed random variables, all distributed according to the characteristic function (1). Denote

$$Y_n = (X_1 + X_2 + \cdots + X_n)/n, \; Z_n = (X_1 + X_2 + \cdots + X_n)/n^2.$$

Show that Y_n does not have a limiting distribution, while the limit distribution of Z_n is the stable distribution with exponent $\frac{1}{2}$.

6032. *Proposed by D. J. Johnson, Air Force Institute of Technology*

Given that L and M are distributive lattices and $[\mathscr{G}, \leqq]$ is the partially ordered set of lattice morphisms from L to M ordered according to the rule $f \leqq g$, if and only if for all x in L, $f(x) \leqq g(x)$ in M (i.e., $f(x) \wedge g(x) = f(x)$), is $[\mathscr{G}, \leqq]$ necessarily a lattice?

6033. *Proposed by S. S. Miller, Cluj, Romania*

Let $w(z)$ be regular in the unit disk D with $w(0) = 0$ and let A be a complex number such that $\operatorname{Re} A \geqq 1$. If $z \in D$, show that

$$|w^2(z) + Aw(z) + zw'(z)| < 1 \text{ implies } |w(z)| < 1.$$

As a special case we get

$$|w^2(z) + w(z) + zw'(z)| < 1 \text{ implies } |w(z)| < 1.$$

6034. *Proposed by Fred Galvin, University of California, Los Angeles*

Suppose the edges of the complete graph on n vertices are colored so that no color is used more than k times. (1) If $n \geqq k + 2$, show that there is a triangle no two of whose edges are the same color. (2) Show that this is not necessarily so if $n = k + 1$.

6035*. *Proposed by Arthur Marshall, Madison, Wisconsin*

For every natural number k let N_k be the kth number in natural order of the sequence consisting solely of primes and the (square-free) products of (two or more) successive primes. Let $\mu(\cdot)$ be the Moebius function: $\mu(N_k) = (-1)^r$, where r is the number of primes dividing N_k. Does the series

$$\sum_{k=1}^{\infty} \frac{\mu(N_k)}{N_k} \ln N_k$$

diverge (positively or negatively), converge, or oscillate?

[*Note*: It is known (Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 2, pp. 585–587) that

$$\sum_{k=1}^{\infty} \frac{\mu(k) \ln k}{k} = -1.$$

Compare also Problem E 2258 [1971, 909]; also Segal, Proc. Amer. Math. Soc., 20 (1969), pp. 287, ff; also Apostol, Proc. Amer. Math. Soc. 40 (1973), pp. 341, ff.]

SOLUTIONS OF ADVANCED PROBLEMS

The Equation $\partial f/\partial x = \partial f/\partial y$

5871 [1972, 780; 1973, 1150]. *Proposed by P. R. Chernoff, University of California, Berkeley*

Let $f(x, y)$ be a real-valued function of two real variables which is separately differentiable. Assume that $\partial f/\partial x = \partial f/\partial y$ everywhere. Must there be a function g of one variable such that $f(x, y) = g(x + y)$? What if we assume *a priori* that f is jointly continuous?

II. *Partial solution by H. F. Royden, Stanford University.* We assume that f is continuous. Changing the sign of one of the variables in the problem, we may restate it as follows: If f is a continuous function in a convex region of the plane which has partial derivatives at each point satisfying

$$(*) \qquad f_x = -f_y,$$

then there is a differentiable function ϕ of one variable such that $f(x, y) = \phi(x-y)$.

It suffices to show that, if f satisfies these conditions on a neighborhood of a (closed) square, then $f(x, y) = \phi(x-y)$ on the square for a suitable ϕ. Without loss of generality we may assume that the square has vertices at $(0, 0)$ and (a, a) with $a > 0$.

The function g defined by

$$g(x, y) = f(x, y) - f(x-y, 0) \text{ for } x-y \geqq 0$$

and

$$g(x, y) = f(x, y) - f(0, y-x) \text{ for } x-y \leqq 0$$

is well-defined, continuous, and has partial derivatives at each point which satisfy $g_x = -g_y$. Moreover, $g(x, 0) = 0$ for $x \geqq 0$ and $g(0, y) = 0$ for $y \geqq 0$. It now suffices to show that $g \equiv 0$ on the square.

Consider the function $h(x, y) = g(x, y) - \varepsilon(x + y)$ for $\varepsilon > 0$. It is continuous and has partial derivatives everywhere on the square. At each point at least one of its partial derivatives is negative. Hence h can assume a maximum on the square only on the bottom or left hand side of the square. Since h is non-positive there, we have $h \leqq 0$ in the square, and hence $g \leqq \varepsilon$ in the square. Since ε was an arbitrary positive number, $g \leqq 0$ in the square. Similarly $g \geqq 0$, whence $g \equiv 0$.

This argument depends strongly on the assumption that our functions are con-

tinuous, so that we know they *assume* maxima on the square. If the hypothesis of continuity is dropped, the problem remains open.

Also solved (partially) by A. M. Gleason.

Note. Several readers, including the solver and the proposer, have pointed out that the original "solution" was incorrect. The application of the chain rule requires that $f(x,y)$ be jointly, not merely separately differentiable.

Cubic Trees

5895 [1973, 208]. *Proposed by Frank Bernhart, Kansas State University*

For $n \geqq 1$ distinguish n points in the interior of a plane circle C and $n+2$ points on the circumference. The $2n+2$ points are to be connected in pairs by $2n+1$ noncrossing arcs within C so that (a) each point on the circumference is the endpoint of one arc, (b) each interior point is the endpoint of three arcs, and (c) each pair of endpoints on the circumference is connected by a path. In graph theory terms, a cubic tree is inscribed in a circle. Case $n = 1$ is illustrated by three radii. If the points on the circumference are labeled in cyclic order $x_1, x_2, \cdots, x_{n+2}$, put $m_i =$ the number of interior points on the unique path between x_i and $x_{i+1} (x_{n+3} = x_1)$.

1. Find a recursive definition for the set of possible sequences

$$M = (m_1, m_2, \cdots, m_{n+2}).$$

2. Start with $n = 1$ and increase the tree by successive steps each consisting of randomly selecting a point x_i, moving it inside C, and joining it to the circumference by two new arcs. Show that for a fixed integer $k \geqq 1$, its fraction of occurrences in the sequences obtained by such constructing is asymptotic to $2^{k-1}/3^k$ as $n \to \infty$.

3. Find a nonrecursive test for determining if a sequence M is a possible sequence.

Solution by D. J. Kleitman, Massachusetts Institute of Technology. (1) Given a finite sequence of integers X define a new one by

(i) inserting a *one* between any two consecutive integers,

(ii) raising by one the value of each of the two consecutive integers now separated by step (i).

Possible sequences are those that can be obtained from $(1,1,1)$ by repeated applications of the steps (i) and (ii). Thus the first application leads to $(1,2,1,2), (2,1,2,1)$. The second application leads to

$$(2,1,3,1,2),\ (1,3,1,2,2),\ (1,2,2,1,3),\ (3,1,2,2,1),\ (2,2,1,3,1);$$

and so on.

(2) Applying the procedure to a particular tree with n circumference points yields n new trees. If a path in the original tree has k interior points, it will appear in two of the new trees (those involving its endpoints) with $k+1$ interior points,

and in the other $n-2$ of the new trees with k interior points. In every case, one new path with one interior point is generated. Thus if $f_n(k)$ is the fraction of occurrences of k in sequences at stage n ($n+2$ circumference points), then

$$\text{for } n>1,\ f_n(1) = \frac{1}{n+2} + \frac{n-1}{n+2} f_{n-1}(1),$$

$$\text{and for } k \geqq 2,\ n>1,\ f_n(k) = \frac{n-1}{n+2} f_{n-1}(k) + \frac{2}{n+2} f_{n-1}(k-1).$$

Let $F(k) = \lim_{n\to\infty} f_n(k) = \lim_{n\to\infty} f_{n-1}(k)$. We then obtain $F(1) = 1/3$, $F(k) = \frac{2}{3}F(k-1)$, whence $F(k) = 2^{k-1}/3^k$.

Also solved by the proposer.

Continued Fraction for $e^{1/z}$

5897 [1973, 209]. *Proposed by I. J. Good, Virginia Polytechnic Institute and State University*

Prove that if z is real or complex and is not zero, then

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z-1+}\,\frac{1}{1\ +}\,\frac{1}{1\ +}\,\frac{1}{3z-1+}\,\frac{1}{1\ +}\,\frac{1}{1\ +}\,\frac{1}{5z-1+}\cdots \\ &= [\overline{1,(2n+1)z-1,1}]_{n=0}^{\infty} \qquad (1) \\ &= [\overline{1,(6n+1)z-\tfrac{1}{2},(24n+12)z,\ (6n+5)z-\tfrac{1}{2},1}]_{n=0}^{\infty} \qquad (2) \end{aligned}$$

Solution by the proposer. Denote the continued fraction (1) by y and think of the part beginning at the third partial quotient as an infinite product of transformations

$$T_3(T_5(T_7(\cdots))),$$

where

$$T_v(w) = 1 + \frac{1}{1\ +}\,\frac{1}{vz-1+}\,\frac{1}{w} = \frac{1}{2} + \frac{1}{4vz-2+4/w}\,.$$

This leads readily to

$$\begin{aligned} y &= 1 + \frac{1}{z-\frac{1}{2}+}\,\frac{1}{12z+}\,\frac{4}{20z+}\,\frac{4}{28z+}\,\frac{4}{36z+}\cdots \\ &= 1 + \frac{2}{2z-1+}\,\frac{1}{6z+}\,\frac{1}{10z+}\,\frac{1}{14z+}\cdots. \end{aligned}$$

The result now follows by straightforward manipulation from Lambert's continued

fraction (for example, *Chrystal's Algebra*, 2, *p.* 522):

$$\tanh x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\,\frac{x^2}{7+}\cdots$$

on putting $x = 1/(2z)$.

Next, denote the continued fraction (2) by η. Then, in a self-explanatory notation, we see that

$$\frac{2}{1+}\,\frac{1}{\eta} = \left[\frac{2}{1+}\,\frac{1}{1+}\,\frac{2}{(2n+2)z-1+}\,\frac{1}{(12n+6)z+}\,\frac{1}{(12n+10)z-1+}\right]_{n=0}^{\infty}.$$

But

$$\frac{2}{1+}\,\frac{1}{1+}\,\frac{2}{vz-1+}\,\frac{1}{w} = 1 + \frac{1}{vz+1/w}.$$

Therefore

$$\frac{2\eta}{1+\eta} = \left[1 + \frac{1}{(2n+2)z+}\,\frac{1}{(12n+6)z+}\,\frac{1}{(12n+10)z+}\right]_{n=0}^{\infty}$$

$$= 1 + \tanh\frac{1}{2z},$$

by Lambert's continued fraction. Therefore $\eta = \exp(1/z)$.

Notes. When z is a positive integer or half a positive integer, (1) and (2) are proved by Perron, *Die Lehre von den Kettenbrüchen* (1929), pp. 132–138.

A slip in the original statement of (2) has been corrected above.

This work was supported in part by the Grant H.E.W. R01 GM 18770–02.

Also solved by M. G. Greening (Australia) whose solution is based on expansions of e^z to be found in A. N. Khovanskii, *The Application of Continued Fractions.*

On Hurwitz Polynomials

5939 [1973, 1067]. *Proposed by J. W. Andrushkiw, Seton Hall University*

Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with real positive coefficients. Show that if for some k, $0 \leqq k \leqq n-3$, the inequality $a_k a_{k+3} \geqq 3a_{k+1}a_{k+2}$ holds, then $f(z)$ cannot have all of its zeros located in the complex left halfplane.

Solution by A. C. Hindmarsh, University of California, Livermore. We shall show that if the polynomial f of degree n has real coefficients a_k and all its zeros lie in the left halfplane $\mathrm{Re}(z) < 0$, then the a_k are positive and

(1) $$a_{k+1}a_{k+2} > a_k a_{k+3}, \quad 0 \leqq k \leqq n-3.$$

The positivity of the a_k is easily shown by factoring f completely over the reals, and noting that each factor (linear of quadratic) has all positive coefficients.

Before proceeding, if one coefficient is positive, we observe that multiplication of (1) by the inequalities in which k is replaced by $k-1$ and $k-2$ in (1) yields, respectively

$$a_{k+1}^2 > a_{k-1}a_{k+3}, \qquad 1 \leqq k \leqq n-3, \tag{2}$$

$$a_{k-1}a_{k+2} > a_{k-2}a_{k+3}, \qquad 2 \leqq k \leqq n-3. \tag{3}$$

Consider first the case where f has all real roots, and write $f(z) = (z-\alpha)g(z)$, where $\alpha < 0$ and g is a polynomial of degree $n-1$ whose (positive) coefficients b_k satisfy (1) to (3). The coefficients of f are then

$$a_k = b_{k-1} - \alpha b_k \qquad (b_{-1} \equiv 0,\ b_n \equiv 0).$$

Substitution yields

$$a_{k+1}a_{k+2} - a_k a_{k+3} = (b_k b_{k+1} - b_{k-1}b_{k+2}) - \alpha(b_{k+1}^2 - b_{k-1}b_{k+3}) + \alpha^2(b_{k+1}b_{k+2} - b_k b_{k+3}).$$

All three coefficients in parentheses are positive for $2 \leqq k \leqq n-3$, by the inductive assumption and by inspection in the special cases for $k = 0$ and $k = 1$. Hence (1) follows, and in turn (2) and (3) also. This reasoning is valid even for $n = 3$, where only the positivity of the b_k is used.

For the next case, we again use induction and suppose f has the pair of nonreal roots $\alpha \pm i\beta$, $\alpha < 0$. Then $f(z) = (z^2 - 2\alpha z + \alpha^2 + \beta^2)g(z)$, where g is of degree $n-2$, and has coefficients b_k satisfying (1) to (3). If $(z-\alpha)^2 g(z)$ has coefficients c_k, then the coefficients of f are

$$a_k = c_k + \beta^2 b_k \qquad (b_{n-1} \equiv 0,\ b_n \equiv 0).$$

Substitution yields

$$\begin{aligned} D_k \equiv a_{k+1}a_{k+2} - a_k a_{k+3} &= (c_{k+1}c_{k+2} - c_k c_{k+3}) \\ &+ \beta^2(b_{k+1}c_{k+2} + b_{k+2}c_{k+1} - b_k c_{k+3} - b_{k+3}c_k) \\ &+ \beta^4(b_{k+1}b_{k+2} - b_k b_{k+3}) = A_0 + A_1\beta^2 + A_2\beta^4. \end{aligned} \tag{4}$$

By the inductive assumptions (and by inspection in special cases), $A_2 \geqq 0$ for $0 \leqq k \leqq n-3$. Substitution of

$$c_k = b_{k-2} - 2\alpha b_{k-1} + \alpha^2 b_k \qquad (b_{-2} \equiv 0,\ b_{-1} \equiv 0)$$

yields

$$A_1 = (b_{k-1}b_{k+2} - b_{k-2}b_{k+3}) - 2\alpha(b_{k+1}^2 - b_{k-1}b_{k+3}) + 2\alpha^2(b_{k+1}b_{k+2} - b_k b_{k+3}).$$

By the inductive assumptions (and by inspection in special cases), each of the three coefficients in parentheses above is nonnegative for $0 \leqq k \leqq n-3$. Hence $A_1 \geqq 0$, and we conclude that D_k in (4) is not less than its value for $\beta = 0$. The same process can be repeated for all other nonreal root pairs of f. Since the value of D_k is positive when all the imaginary parts of the roots are set to 0, we conclude $D_k > 0$, as desired.

Additional remarks. (a) No constant smaller than 1 can be used to bound $a_k a_{k+3}/a_{k+1}a_{k+2}$ for all k and all n.

Proof of (a). For $f(z) = (z+1)^n$, we have $a_k = \binom{n}{k}$, and

$$D_k \equiv Ca_{k+1}a_{k+2} - a_k a_{k+3}$$

$$= C\binom{n}{k+1}\binom{n}{k+2} - \binom{n}{k}\binom{n}{k+3} = \frac{(n!)^2 E_k}{(k+1)!(k+3)!(n-k)!(n-k-2)!},$$

$$E_k = C(n-k)(k+3) - (k+1)(n-k-2).$$

If we let $n = 2k \geqq 6$, then $E_k > 0$ becomes

$$C > \frac{(k+1)(k-2)}{k(k+3)}$$

which requires $C \geqq 1$.

(b) A converse is suggested: If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_k > 0$ satisfying (1), then all the roots of f are in the left halfplane. This statement is true for $n = 3$ but false for all $n > 3$.

Proof. For the case $n = 3$, suppose first that f has real roots x_1, x_2, x_3. We may assume $a_3 = 1$ and write

$$a_0 = -x_1x_2x_3, \quad a_1 = x_1x_2 + x_2x_3 + x_3x_1, \quad a_2 = -(x_1 + x_2 + x_3).$$

If some x_i is positive, then two of them must be, as $a_0 > 0$. Suppose $x_1 < 0, x_2 > 0$, $x_3 > 0$. Then $a_2 > 0$ leads to $x_1 < -(x_2 + x_3)$ while $a_1 > 0$ leads to

$$x_1 > -x_2x_3/(x_2 + x_3).$$

Together these imply $(x_2 + x_3) < x_2x_3/(x_2 + x_3)$, which is clearly false. Hence all $x_i < 0$.

If the roots of f are x_1 and $x_2 \pm iy_2$, $y_2 \neq 0$, then we have (again normalizing by $a_3 = 1$)

$$a_0 = -x_1(x_2^2 + y_2^2), \quad a_1 = 2x_1x_2 + x_2^2 + y_2^2, \quad a_2 = -(x_1 + 2x_2).$$

We find that

$$a_1a_2 - a_0 = -2x_2[(x_1 + x_2)^2 + y_2^2].$$

Since this must be positive, $x_2 < 0$. But $a_0 > 0$ implies $x_1 < 0$.

For $n \geqq 4$, consider the function

$$f(z) = \sum_{k=0}^{n} (1 + \varepsilon k)z^k, \quad \varepsilon > 0.$$

The coefficients $a_k = 1 + \varepsilon k$ are positive and satisfy $a_{k+1}a_{k+2} - a_k a_{k+3} = 2\varepsilon^2 > 0$ regardless of the choice of ε. But for sufficiently small ε, f has at least two roots in $\operatorname{Re}(z) > 0$, because for $n \geqq 4$,

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z-1}$$

does. Hence f disproves the asserted converse.

Also solved by Robert Breusch (New Zealand), O. P. Lossers (Netherlands), and the proposer.

Density of $\sigma(n)/n$

5949 [1974, 90]. *Proposed by C. W. Anderson, University of California, Berkeley*

Let $\Sigma(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n. It is known that $\Sigma: N \to [1, \infty)$ is a dense map. It is also known that there exists p/q, $(p, q) = 1$, $p > q$, such that $\Sigma(n) = p/q$ has no solutions, or $\Sigma(n) = p/q$ has exactly one solution.

(1) Show that there exists a dense infinity of rationals p/q in $[1, \infty)$ for which $\Sigma(n) = p/q$ has no solution. (2) Also show that there exists a dense infinity of such rationals for which $\Sigma(n) = p/q$ has only one solution. For example, $\Sigma(n) = 5/4$ has no solution; $\Sigma(n) = 3/2$ has exactly one solution.

Solution by Neal Felsinger, Hartford, Connecticut. First, note that since σ is multiplicative, Σ is also multiplicative. From this it follows that if m divides n, $\Sigma(m) \leqq \Sigma(n)$ with equality if and only if $m = n$. Next suppose $\sigma(j) \geqq k > j$ and $(k,j) = 1$. Then $\Sigma(n) = k/j$ has no solution or has one solution depending on whether $\sigma(j) > k$ or $\sigma(j) = k$. For, if $\Sigma(n) = k/j$, then $j\sigma(n) = kn$ and so j divides kn. But $(k,j) = 1$, hence j divides n. Then $\Sigma(n) \geqq \Sigma(j) = \sigma(j)/j \geqq k/j$. For equality we must have $n = j$ or $\sigma(j) = k$. The first part now follows:

Let $x \in Q \cap (1, \infty)$, $x = r/s$ where $(r,s) = 1$. Let n be a natural number such that s^2 divides n and $\Sigma(n) > x$. If $(t, n) = 1$ and $k = (tnr/s) - 1$, then $\sigma(tn) = \sigma(t)\sigma(n) > t(nr/s) > k$, hence $\Sigma(m) = k/tn$ has no solutions. Clearly there are arbitrarily large t relatively prime to n so that k/tn can be made arbitrarily close to x.

For the second part, we need the following lemma:

LEMMA. *Let p and q be distinct primes. Then there exists $m = m(p, q)$ such that q divides $\sigma(p^e)$ if and only if $e \equiv -1 \pmod m$.*

Proof. Let s be the least positive integer such that $q \mid 1 + p + \cdots + p^s$. Such an s exists since the number of residues modulo q is finite, so there are $s_1 < s_2$ such that

$$\sum_{i=0}^{s_1} p^i \equiv \sum_{i=0}^{s_2} p^i \pmod q,$$

hence

$$q \mid \sum_{s_1+1}^{s_2} p^i \text{ implying } q \mid \sum_{0}^{s_2-s_1-1} p^i.$$

Let $m = m(p,q) = s + 1$. If $e = km - 1$, then

$$\sigma(p^e) = (1 + p + \cdots + p^{m-1})(1 + p^m + \cdots + p^{(k-1)m}),$$

and so q divides $\sigma(p^e)$. Conversely, if q divides $\sigma(p^e)$, let $e = km + t$, $0 \leqq t < m$. If $e \neq m - 1$, $k \geqq 1$, then q divides $\sigma(p^{km-1})$, and so q divides $(p^{km} + p^{km+1} + \cdots + p^{km+t})$. Thus $q \mid (1 + p + \cdots + p^t)$ implying $t = m - 1$, and so $e \equiv -1 \pmod m$.

For any prime p, define

$$\Sigma(p^\infty) = \lim_{n\to\infty} \Sigma(p^n) = 1 + \frac{1}{p-1}$$

and note that

$$\frac{\Sigma(p^\infty)}{\Sigma(p^n)} = \frac{\sum_{j=0}^{\infty} p^{-j}}{\sum_{j=0}^{n} p^{-j}} = 1 + \frac{\sum_{j=n+1}^{\infty} p^{-j}}{\sum_{j=0}^{n} p^{-j}} < 1 + \frac{2}{p^{n+1}}.$$

We can now show the second part. Let $x \in (1, \infty)$ and $\varepsilon > 0$. We may assume $x - \varepsilon > 1$. Let p_i denote the ith prime. Let I be a finite set of natural numbers such that

$$x - \tfrac{1}{2}\varepsilon < \prod_{i\in I} \Sigma(p_i^\infty) < x.$$

Let n be a natural number such that

$$\prod_{i\in I} (1 + 2/p_i^{(n+1)}) < (x - \tfrac{1}{2}\varepsilon)/(x - \varepsilon)$$

Let M be the lowest common multiple of $\{m(p_i, p_j) \mid i, j \in I, i \neq j\}$. Choose $N > n$ such that M divides N. Define $v = (\prod_{i\in I} p_i)^N$ and $u = \sigma(v)$.

If $i,j \in I$, $i \neq j$, $m(p_i, p_j) \mid N$ so that $p_j \nmid \sigma(p_i^N)$ by the above lemma, then $(u, v) = 1$. Thus $\Sigma(k) = u/v$ has a unique solution. Now

$$u/v = \Sigma(v) = \prod_{i\in I} \Sigma(p_i^N) < \prod_{i\in I} \Sigma(p_i^\infty) < x$$

and

$$u/v = \prod_{i\in I} \Sigma(p_i^N) > \prod_{i\in I} \Sigma(p_i^n) > \prod_{i\in I} \Sigma(p_i^\infty)/ \prod_{i\in I} (1 + 2/p_i^{(n+1)}) > x - \varepsilon,$$

completing the proof.

Also solved by David Bienenfeld (Israel), Paul Erdös (Israel), L. E. Mattics, Carl Pomerance, A. L. Rubin, and the proposer.

Note. The problem of determining the set $\{\Sigma(n)\}$ is, of course, unsolved (and seemingly impossible). Erdös asserts that 73/55 and 74/55 are not in the set. Rubin has shown that if there are infinitely many perfect numbers, then the set of rationals p/q such that $\Sigma(n) = p/q$ has an infinite number of solutions, is dense in $[2, \infty)$.

Neville Robbins, Dean Hickerson and the proposer observe that it is possible to prove that $\Sigma(n)=93/40$ only for $n=80$ and $n=200$. Hence it follows that $93/40 \in A(2) \subset B$ and in turn that $A(2)$ is infinite. It is not known whether $A(2)$ is dense.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

REVISION OF BASIC LIBRARY LIST — AN APPEAL FOR HELP

NSF has approved plans to revise and update the *Basic Library List* for four-year colleges, originally published by CUPM in 1965. Screening of nearly 10,000 books published since 1964 is well under way. Members are invited (1) to suggest ways to improve the style, content, and coverage of the original list and (2) to submit titles (and supporting comments) of their favorite books appropriate for undergraduate libraries, for consideration by the committee. Write by June 30 to MAA Special Projects Office, Department of Mathematics, California State University, Hayward, CA 94542.

FILMS

(For general information about this series of films, see the introduction on page 416 of this MONTHLY, Vol. 82, Number 4.)

Projective Generation of Conics: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Seymour Schuster. 16mm sound and color; 16 minutes. Available for rent or purchase from International Film Bureau, Inc. — sale $225; rental $12.50. Also available for rent from numerous University Film Libraries.

The purpose of this film is to show several different methods of constructing a conic using only points and lines. This goal is admirable since the usual metric construction of a conic is rather awkward. The ideas presented are very appealing and the animation technique is ideal for this approach. The first example of a construction, due to Pascal, is carefully presented. The succeeding methods of generating conics are more detailed and, unfortunately, the pace of the film picks up. The final method, due to Poncelet, is done very quickly and would take many replays for a student to appreciate it. The film simply tries to cover sixty minutes of material in sixteen minutes. Nevertheless, the subject is attractive and the film medium provides a pedagogical dimension that cannot be simulated by the classroom teacher.

F. W. STEVENSON, University of Arizona

Curves of Constant Width: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: J. D. E. Konhauser. 16mm sound and color; 16 minutes. Available for rent or purchase from International Film Bureau — sale \$225; rental \$12.50. Also available for rent from numerous University Film Libraries.

A convex curve has constant width if the distance between parallel pairs of supporting lines is the same for all directions. Employing excellent animation and careful narration, this film introduces the viewer to some basic intuitive notions about convex curves and the intriguing properties of curves of constant width. One sees a Reuleaux triangle generated by animated lines and arcs, followed by a colorful variety of more general Reuleaux polygons and other curves of constant width, culminating in a smooth curve of constant width obtained as an orthogonal trajectory of the tangents to a 3-cusped curve. One lovely sequence shows a platform moving evenly on rollers whose cross sections are noncircular curves of constant width. Later, a harried mime gives an amusing demonstration of why curves of constant width make good manhole covers. The visual demonstration of Barbier's theorem, that any two curves of the same constant width have the same perimeter, may appear somewhat abrupt. An instructor showing this to a class might ask them if they detected a manifestation of the theorem elsewhere in the film.

This is, as advertised, a motivational film, with no mathematical prerequisites, suitable for audiences at any level. This reviewer highly recommends it as a relaxing diversion for a geometry course touching on topics in convexity. Many points hinted at in the film could form the basis for deeper classroom discussion. In this regard, the four page résumé provided with the film contains helpful suggestions for further exploration.

G. D. CHAKERIAN, University of California at Davis

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print.*

GENERAL, S**, P*, L***. *Mathematical Recreations and Essays, Twelfth Edition.* W.W. Rouse Ball, H.S.M. Coxeter. U of Toronto Pr, 1974, xvii + 428 pp, $5 (P). Revision of the 1939 eleventh edition; the first edition of this timeless classic was published in 1892! This edition has current information on primes, a new chapter on combinatorial theory, and even Knuth's dragon designs. LAS

GENERAL. *Rebels Who Count.* Ten 28-minute, black and white video tapes. Mathematician: Kenneth O. May. Distributed by OECA, 2180 Yonge Street, Toronto, Canada. These tapes, written and presented by Professor May, were produced by the Ontario Education Communications Authority in 1970 for TV broadcast on educational television channels. The mathematicians discussed are Galileo, E. Galois, C.F. Gauss, J. Bolyai, W.R. Hamilton, G. Cantor, Lewis Carroll, C.S. Peirce, S. Ramanujan, and Bertrand Russell. Pictures and models are freely used, and most programs involve guest participants. SS

GENERAL, S*(9-13), L*. *The Golden Section and Related Curiosa.* Garth E. Runion. Scott F, 1972, 150 pp, $1.68 (P). Good high-school supplementary material (e.g., for math club). Most suitable for students with some geometry, as it emphasizes constructions and their "validations" (proofs). Exercises, solutions, extensive bibliography. PJC

GENERAL, S*, L*. *Surreal Numbers.* D.E. Knuth. A-W, 1974, 119 pp, $3.95 (P). "In the beginning everything was void and J.H.W.H. Conway began to create numbers. Conway said, 'Let there be two rules...'" A sometimes strained allegory in which a new two-step algorithm is used to create numbers: integers, rationals, irrationals, infinitesimals, infinite cardinals, even some "pseudo-numbers." Absolutely unique, and a lot of fun. LAS

BASIC, T(13: 1). *Beginning Algebra.* Alfonse Gobran. Prindle, 1975, viii + 296 pp, $9.95. Large numbers of word problems highlight this as an exceptional elementary algebra text. LLK

BASIC, T(13). *Basic Mathematics: An Individualized Approach, 6 Units.* Joanna S. Burris. Prindle, 1974. *Unit 1: Introduction and Whole Numbers,* x + 86 pp; *Unit 2: Fractions,* 76 pp; *Unit 3: Decimals,* 44 pp; *Unit 4: Ratio and Proportion,* 45 pp; *Unit 5: Percent,* 61 pp; *Unit 6: Practical Geometry,* 53 pp, $2 each (P). Programmed workbooks for developmental arithmetic. For $500 an adopting school will receive 35 cassette tapes and a complete testing program. LAS

PRECALCULUS, T(13: 1). *Essentials of Trigonometry, Second Revised Edition with Tables.* Joseph B. Rosenbach, Edwin A. Whitman, David Moskovitz. Xerox, 1974, xi + 340 pp, $8.95. Traditional text in trigonometry. LLK

PRECALCULUS, T(13: 1). *Elementary Algebra.* Thomas M. Green. Macmillan, 1975, ix + 541 pp, $10.95. A classical (no functions) algebra book with some interesting highlights ($\Sigma i = n(n + 1)/2$ is the "summation property"). Lots of exercises. PJM

PRECALCULUS, T(13: 1). *Precalculus Mathematics Series.* Charles C. Carico. Wadsworth, 1974, (P). A set of 12 volumes covering precalculus topics. Each volume is in 2-4 units. Each unit is in sections consisting of a brief definition, examples and exercises, followed by a summary review of the unit. LLK

EDUCATION, S, P, L. *Elementary Mathematics Activities, Part A.* Hassler Whitney. Inst for Adv Study, 1974, ix + 121 pp, $3.50 (P). Games, explorations and home-made tools for a practical program in first grade mathematics. A refreshing, down-to-earth approach by one of this century's outstanding theoretical mathematicians. LAS

EDUCATION, S(15). *The Overhead Projector in the Mathematics Classroom.* George Lenchner. NCTM, 1974, ii + 30 pp, $1.10 (P). Includes basic information for making overhead transparencies. Suggests a variety of uses for overhead projector. Numerous illustrations and sample visuals. References and sources of materials furnished. For methods classes. PSJ

EDUCATION, T(14-16: 1), S. *Grundbegriffe der neuen Schulmathematik.* Wilfried Dierks, Ulrich Löttgen. Hermann Schroedel, 1974, 296 pp, DM 21,80. For teachers and interested parents, an exposition of what it's all about. Logic in its semantic sense is used as a device to avoid the confusing formalism often associated with dogmatic set theory at this level. JAS

HISTORY, P, L*. *Stanislaw Ulam: Sets, Numbers, and Universes, Selected Works.* Ed: W.A. Beyer, J. Mycielski, G.-C. Rota. MIT Pr, 1974, xxiii + 709 pp, $25. Selected papers on pure and applied mathematics, supplemented by various commentaries on their significance and consequences. Includes a complete Ulam bibliography, and a complete reprinting of the 1960 *Collection of Mathematical Problems.* LAS

HISTORY, P, L. *Oeuvres Choisies, Tome II.* Waclaw Sierpiński. PWN, 1975, 780 pp, $35. Nearly 150 papers on set theory and its applications (general topology, measure, real variable theory) introduced by five survey essays by A. Mostowski, S. Hartman and E. Marczewski. LAS

HISTORY, P, L. *Travaux de Topologie et ses Applications.* Stefan Mazurkiewicz. PWN, 1969, 380 pp. 62 collected papers on topology, a complete bibliography of Mazurkiewicz' 141 scientific papers, and a professional biography by Casimir Kuratowski. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-405: The Souslin Problem.* Keith J. Devlin, Hårvard Johnsbråten. Springer-Verlag, 1974, viii + 132 pp, $7.40 (P). The Souslin conjecture--stated in the first (1920) volume of *Fundamenta Mathematicae*--is that every complete dense linearly ordered set that has no end points and contains no uncountable collection of pairwise disjoint open intervals is isomorphic to the real numbers. Ronald Jensen showed that this conjecture is independent of ZF set theory axioms, even in the presence of GCH or its negation. These notes offer the first major exposition of Jensen's results. LAS

FOUNDATIONS, P, L. *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic.* Jaakko Hintikka. Oxford U Pr, 1973, x + 291 pp, $18.50. "This book is addressed to two interrelated questions: what is the relevance of formal logic to those numerous activities in which

language is used for some non-linguistic purpose? What kind of information (if any) can deduction (logical inference) give to us?... My answers to both the questions just mentioned turn out to be closely connected...with some of the main ideas of Kant's philosophy of logic and mathematics..." Based on John Locke Lectures, Oxford University, 1964; all essays have appeared in print before but occur here in substantial revision. Particularly valuable is "An Analysis of Analyticity." PJC

FOUNDATIONS, P*, L*. *Truth, Probability and Paradox: Studies in Philosophical Logic.* J.L. Mackie. Oxford U Pr, 1973, xii + 305 pp, $14.50. Six philosophical essays, each developed through careful survey and exposition of rival views: on philosophical analysis, truth, conditional statements, dispositions and powers, concepts of probability, and logical paradoxes. Accompanying appendix catalogs prominent logical paradoxes. Concludes Mackie: "We need a philosophical solution for all the paradoxes of this group, not merely an exclusion device..." (p. 295).PJC

FOUNDATIONS, T(18), P. *The Theory of Ultrafilters.* W.W. Comfort, S. Negrepontis. Grund. math. Wissenschaften, B. 211. Springer-Verlag, 1974, x + 482 pp, $40.20. A definitive treatise that crossbreeds topological and logical aspects of ultrafilters to produce a major tool for unifying diverse aspects of mathematics. Emphasizes ultrafilters defined by various extremal properties. Each chapter is carefully tied to the extensive bibliography by concise references. LAS

FOUNDATIONS, P, L. *Probabilistic Metaphysics.* Patrick Suppes. Philosophical Studies (Uppsala), 1974. *V. 1:* 74 pp; *V. 2:* 85 pp, $14 (P) set. Six lectures refuting the "new theology" of post-Kantian philosophy, e.g., that every event has a determinate cause and that, in principle, scientific knowledge and the grounds for rational belief can be made complete. Suppes argues that these beliefs are demonstrably false and that the appropriate framework for thinking about the world is fundamentally probabilistic rather than logical. LAS

COMBINATORICS, P. *Lecture Notes in Mathematics-403: Combinatorial Mathematics.* Ed: D.A. Holton. Springer-Verlaq, 1974, 148 pp, $7.40 (P). Proceedings of the 1973 second Australian conference on combinatorial mathematics. LAS

COMBINATORICS, P. *Combinatorics.* Ed: T.P. McDonough, V.C.Mavron. London Math. Soc. Lect. Notes, No. 13. Cambridge U Pr, 1974, v + 204 pp, $9.95 (P). Proceedings of a July 1973 conference held at Aberystwyth, Wales. LAS

COMBINATORICS, P. *Lecture Notes in Mathematics-406: Graphs and Combinatorics.* Ed: Ruth A. Bari, Frank Harary. Springer-Verlag, 1974, viii + 355 pp, $12.30 (P). Proceedings of a June, 1973 conference at George Washington University containing 10 expository and survey papers, plus shorter contributed ones. LAS

NUMBER THEORY, P. *The Brauer Group of Commutative Rings.* Morris Orzech, Charles Small. Lect. Notes in Pure and Appl. Math., V. 11. Dekker, 1975, ix + 183 pp, $14.50 (P). A useful summary of the major results concerning the Brauer group. Reviews classical results on Brauer group of a field; discusses Brauer groups of complete local rings, regular domains, domains of dimension ≤ 2; Galois cohomology; Tsen's theorem, cancellation; the torsionness of the Brauer group. Includes an illuminating proof that 2 is a prime number. SG

LINEAR ALGEBRA, T(13: 1). *The Mathematics of Matrices: A First Book of Matrix Theory and Linear Algebra, Second Edition.* Philip J. Davis. Xerox, 1973, xiii + 348 pp, $10.95. Same distinctive flavor as the first (1965) edition. No changes--only correction of errors. LLK

LINEAR ALGEBRA, T(14: 2). *Computational Linear Algebra with Models.* Gareth Williams. Allyn, 1975, xiv + 384 pp, $13.95. A new look in linear algebra texts. Contains an appendix on computer programs, and includes not only eigenvalues and eigenvectors but also Leontief input-output models in economics. LLK

LINEAR ALGEBRA, T(14: 1). *Introduction to Linear Algebra.* Philip Gillett. HM, 1975, xii + 523 pp, $12.95. A very thorough treatment of topics in linear algebra. Organization is unique--row echelon form in Chapter 7, even though matrices are introduced in Chapter 4; inner products in Chapter 10, although vectors are introduced geometrically in Chapter 1. The spectral theorem is the climax. LLK

LINEAR ALGEBRA, T(14-16: 1, 2), S. *An Introduction to the Theory of Linear Spaces.* Georgi E. Shilov. Trans: Richard A. Silverman. Dover, 1974, ix + 310 pp, $4 (P). A relatively sophisticated presentation of the theory including some work with limits and operators on infinite dimensional spaces. Includes chapters on bilinear and quadratic forms and quadric surfaces. JAS

ALGEBRA, T*(14-16: 1), S, L. *Introduction to Modern Algebra, Third Edition.* Neal H. McCoy. Allyn, 1975, xii + 271 pp, $13.95. A highly regarded undergraduate text. In this edition the chapter on linear algebra has been deleted and chapters on the logical structure of mathematics, finite abelian groups, and factorization in integral domains have been added. There are also some additional exercises and a short bibliography which are new. CEC

ALGEBRA, T(15-16: 1). *Introduction to Abstract Algebra.* J.T. Moore. Acad Pr, 1975, xi + 291 pp, $11.95. An interesting first course: groups, rings and fields (as a "type of ring"), a chapter on quotient or factor systems, and a concluding chapter on polynomial rings. Lots of good examples and exercises. May be a little light on fields. No references to further reading. PJM

ALGEBRA, T(15-16: 1), S*, L. *Notes on Applied Modern Algebra.* Larry Dornhoff. Stipes, 1974, iv + 189 pp, $6 (P). An introduction to those parts of modern algebra (e.g., morphisms, finite state machines, Boolean algebra, semigroups, codes, Polya enumeration theory) that are important in computer science. An informal alternative to the well-known texts by Stone and by Birkhoff and Bartee. LAS

ALGEBRA, T*(16-18: 1, 2), S, L. *The Structure of Fields.* David Winter. Grad. Texts in Math., V. 16. Springer-Verlag, 1974, xii + 205 pp, $12.80. Assumes only linear algebra and considerable mathematical maturity. Presents a thorough treatment of classical Galois theory and some of its modern generalizations. Lots of exercises, good index, and several appendices containing useful folklore on tensors, coalgebras and such. JAS

ALGEBRA, S(14-15). *Groups.* D.A.R. Wallace. Prob. Solvers, No. 16. Crane, Russak, 1974, 104 pp, $5.25 (P); $3.75. A nice elementary introduction to groups including sections on abelian groups, permutation groups, Sylow theorems, and solvability. Each chapter consists primarily of definitions, solved problems, and a collection of exercises. Recommended for someone learning on his own. Hard to imagine as a text. CEC

ALGEBRA, P. *Lecture Notes in Mathematics-421: Groupes Discrets.* Valentin Poénaru. Springer-Verlag, 1974, 216 pp, $9.40 (P). An exposition of Stallings' work on the theory of ends, with applications to 3-manifolds. Almost self-contained (need some background in groups presented by relations), and very readable. PJM

ALGEBRA. *Lectures on Boolean Algebras.* Paul R. Halmos. Springer-Verlag, 1974, 147 pp, $3.95 (P). Reprint of 1963 original, based on a 1959 course at Chicago. LAS

ALGEBRA, P. *Quadratische Formen und orthogonale Gruppen, Zweite Auflage.* Martin Eichler. Grund. math. Wissenschaften, B. 63. Springer-Verlag, 1974, xii + 222 pp, $27. Few changes from the 1952 edition: geometry and number theory related to vector spaces (over arbitrary fields) in which a metric is given by a quadratic form. JAS

ALGEBRA, P. *Finite Groups Whose 2-Subgroups are Generated by at Most 4 Elements.* Daniel Gorenstein, Koichiro Harada. Memoirs No. 147. AMS, 1974, vii + 464 pp, $6.40 (P). Determination of all finite simple groups described in the title and classification of all finite simple groups whose Sylow 2-subgroups do not possess an elementary abelian normal subgroup of order 8. SG

ALGEBRA, P. *The Discrete Series of GL_n Over a Finite Field.* George Lusztig. Annals of Math. Stud., No. 81. Princeton U Pr, 1974, 99 pp, $6 (P). The study of a certain representation of the group $Gl_n(F_q)$ where F_q is a field with q elements. This representation, which is a free module over the ring of Witt vectors over F_q, arises from a certain homology group. The author discusses its character and uses it to construct other representation of $Gl_n(F_q)$. SG

ALGEBRA, P. *Finitely Presented Infinite Simple Groups.* Graham Higman. Notes on Pure Math., No. 8. Australian Natl U, 1974, vii + 82 pp, $8.75 (P). An infinite family of finitely presented infinite simple groups is described. The construction is based on work done following a report of R. Thompson's work by B. Jónsson. There is no bibliography. CEC

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics with Algebra.* Daniel D. Benice. Saunders, 1975, 356 pp, $10.95. Topics in finite mathematics and algebra which could precede a short course in calculus. Includes probability, statistics, Markov chains, matrices, and linear programming. LLK

FINITE MATHEMATICS, T(13-14: 1). *Games and Programs: Mathematics for Modeling.* Robert R. Singleton, William F. Tyndall. Freeman, 1974, xxiii + 304 pp, $11.95. Excellent introductory-level book on decision models, theory of games and linear programming. Of interest to students of economics, business administration, psychology, etc. Extensive discussion of decision making. LLK

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics with Applications, Second Edition.* A.W. Goodman, J.S. Ratti. Macmillan, 1975, xiv + 541 pp, $12.95. This second edition preserves the clarity and thoroughness of the first edition (ER, January 1973). Additions and corrections include more elementary problems and an expanded answer section. LLK

CALCULUS, T(13: 2, 3). *Calculus.* Howard E. Campbell, Paul F. Dierker. Prindle, 1975, xii + 752 pp, $16.95. A good "middle of the road" calculus text. Less than 700 pages, includes numerical methods and vector calculus through Green's and Stokes' Theorems. Clear and concise but lacking in applications. LLK

CALCULUS, T**(13: 1). *First Year Calculus.* Ethan D. Bolker, Joseph W. Kitchen, Jr. A-W, 1974, xii + 906 pp, $11.95 (P). An excellent new calculus text with a few *minor* flaws. Limits are only used to calculate derivatives, and remain intuitive. The chain rule is proved right without any tricks, after the derivative is exhibited as a best linear (affine) approximation. Lots of supplementary material in the numerous exercises. Really well motivated. The flaws: Photocopy of a typescript is visually unappealing. Index is a little confusing: e.g., to find the chain rule I had to look up derivative of a composite function. Assumes trigonometric functions and a fair amount of analytic geometry. Three parts: the Derivative, the Integral, and Transcendental Methods. No infinite series, but Taylor's theorem is there. Mean Value theorem relegated to an appendix. PJM

CALCULUS, T(13: 1). *Calculus for Management, Economics and the Life Sciences.* Kenneth Loewen. Prindle, 1975, xii + 388 pp, $12.50. Early introduction to partial derivatives, level curves and implicit functions seems appropriate. Many good examples. The first eight chapters constitute the main thrust of the text with three optional chapters on trigonometric functions, differential equations, and a more formal approach to integration. LLK

CALCULUS, T(13: 1, 2). *Topics in Calculus, Second Edition.* Morton Lowengrub, Joseph G. Stampfli. Xerox, 1975, xiii + 442 pp, $11.95. Many examples of applications in the biological and social sciences. Emphasizes intuitive and geometric understanding. Both differentiation and integration considered in the first five chapters. This new edition has a section on the Riemann integral and many new exercises. (First edition TR, August-September 1970.) FLW

CALCULUS, T(13: 3). *Calculus: One and Several Variables, Second Edition.* Saturnino L. Salas, Einar Hille. Xerox, 1974. *Part 1:* xi + 620 pp, $12.95; *Part 2:* x + 412 pp, $11.95. This second edition has the same format as the first (TR, June 1971); there are some additions and in almost every chapter some rewriting. LLK

DIFFERENTIAL EQUATIONS, T(14: 1). *Introduction to Ordinary Differential Equations, Second Edition.* Shepley L. Ross. Xerox, 1974, ix + 432 pp, $12.50. Second edition of a good basic text in ordinary differential equations. Many sections are unchanged, but Chapter 7 is a complete rewriting of "Systems of Linear Differential Equations." LLK

DIFFERENTIAL EQUATIONS, T(14: 1). *Elementary Differential Equations with Linear Algebra, Second Edition.* Albert L. Rabenstein. Acad Pr, 1975, x + 374 pp, $12.95. A major revision of the first edition (ER, November 1971), it now includes reduction to row echelon form as well as additional exercises and examples. Fewer topics are presented and the level is more elementary. LLK

DIFFERENTIAL EQUATIONS, P. *Dynamical Systems.* Wiesław Szlenk. Aarhus U, 1974, ii + 198 pp, $3 (P). Lecture notes from a 1971/72 course covering selected topics in classical theory and modern transversality theory of dynamical systems. Investigates behavior of a particle on its whole trajectory including behavior at infinity. Includes differential equations, algebraic topology, differential geometry and functional analysis. I-CH

NUMERICAL ANALYSIS, P. *Numerische Methoden bei Optimierungsaufgaben, Band 2.* L. Collatz, W. Wetterling. Birkhauser, 1974, 165 pp, $13. Proceedings of the conference (the fourth in a series) at Oberwolfach, November 1973. JAS

NUMERICAL ANALYSIS, T(16-18), S*, P, L. *The Chebyshev Polynomials.* Theodore J. Rivlin. Wiley, 1974, vi + 186 pp, $15.95. A well written and well organized book. Leads readers quickly to the illuminative characteristics of the Chebyshev polynomial: the definition is simple, the applications are penetrating--in interpolation theory, approximation theory, numerical integration, functional analysis, etc. Enlightening, stimulating and informative; a good book indeed. I-CH

NUMERICAL ANALYSIS, P. *Numerische Behandlung von Eigenwertaufgaben.* L. Collatz, K.P. Hadeler. Birkhauser, 1974, 142 pp, $12. Proceedings of the November 1972 conference at Oberwolfach. JAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-401: Elliptic Operators and Compact Groups.* Michael Francis Atiyah. Springer-Verlag, 1974, 93 pp, $7.40 (P). Notes from a course given at the IAS in 1971, extending the Atiyah-Singer index theorem from finite groups to compact groups. PJM

FUNCTIONAL ANALYSIS, P. *Derivations and Automorphisms of Banach Algebras of Power Series.* Sandy Grabiner. Memoirs No. 146. AMS, 1974, iv + 124 pp, $3.30 (P). Studies derivations, automorphisms, and endomorphisms of certain Banach algebras which are continuously embedded in the space of complex formal power series in one indeterminate. Applications to weighted shifts, spaces of analytic functions, and quasi-nilpotent operators. DFA

FUNCTIONAL ANALYSIS, T(18: 1), P. *Integral Equations Via Imbedding Methods.* Harriet H. Kagiwada, Robert Kalaba. Appl. Math. and Comp., No. 6. A-W, 1974, xviii + 382 pp, $19.50; $12.50 (P). First derives initial value problems corresponding to linear Fredholm integral equations with various forms of kernel, then studies parameter imbedding (for linear and nonlinear equations), radiative transfer, and (briefly) dual integral equations. Discusses numerical schemes, provides examples, computations, programs. Exercises throughout. DFA

FUNCTIONAL ANALYSIS, T(17-18: 1), S, P. *Polynomial Approximation.* Robert P. Feinerman, Donald J. Newman. Williams & Wilkins, 1974, viii + 148 pp, $13. The theme is simple: to study how well polynomials approximate. The coverage is broad, ranging from classical results (e.g., Weierstrass theorem, Bernstein's inverse theorem for periodic functions) to some of the authors' own work on rational approximation and the general Müntz-Jackson theorem. A good book for readers with knowledge of functional analysis. I-CH

FUNCTIONAL ANALYSIS, P. *Linear Lattices.* Hidegorô Nakano. Wayne St U Pr, 1966, 157 pp, $4.95 (P). A reprint of that portion of the 1950 original, *Modular Semi-Ordered Linear Spaces*, which deals with the general theory of linear lattices, together with the material on projectors, spectral representation theory, and reflexive and normed spaces. JAS

FUNCTIONAL ANALYSIS, S(18), P. *Funktionalanalysis II.* Erika Pflaumann, Heinz Unger. Bibliographisches Inst, 1974, 338 pp, (P). An expository work covering mapping spaces, uniform structures, differentiation and integration with values in normed spaces, Banach algebras, and spectral theory. JAS

OPTIMIZATION, T(16-18: 1), P, L. *Introduction to Optimization Methods.* P.R. Adby, M.A.H. Dempster. Halsted Pr, 1974, x + 204 pp, $7.95 (P). This book covers gradient and search methods (both fundamental and advanced) for non-linear optimization, constrained and unconstrained. Treats the Kuhn-Tucker conditions, and the classical and generalized

Lagrangian methods. Includes many references to current research literature. Although many examples involve only simple and well-behaved functions, the authors claim that the illustrated techniques should yield insight into suitability of various optimization methods when applied to practical problems. I-CH

COMPUTER SCIENCE, S*(16), P, L. *Logic Design: A Review of Theory and Practice.* Glen G. Langdon, Jr. Acad Pr, 1974, ix + 179 pp, $14.50. A good guide rather than a text. Historical, technological development. Assumes prior knowledge of Boolean algebra, switching theory and electronics. Discusses inter-relationships between theory and practice. Emphasizes timing considerations. Good bibliography. RWN

COMPUTER SCIENCE, T*(13-14), L. *Fundamentals of Fortran Programming.* Robert C. Nickerson. Winthrop Pub, 1975, xi + 300 pp, $6.95 (P). One of the clearest texts on FORTRAN aimed at non-mathematically oriented students. Includes many simple examples and flowcharts, but no solutions to the problems. Small paperback size. RB

COMPUTER SCIENCE, T(13-14), L. *A Short Course in Basic Fortran IV Programming: Based on the IBM System/360 and System 370.* Robert M. Lee. McGraw, 1972, viii + 239 pp, $6.95 (P). This text puts a heavy emphasis on FORTRAN fundamentals. Most of the book covers branching, looping and input/output. The short sections on subroutines and disk I/O are very good. The complete answer section contains especially nice analyses of questions and problems. A refreshing new FORTRAN manual. RB

COMPUTER SCIENCE, T(15-16: 2), S, P, L. *An Introduction to American National Standard COBOL.* Tate F. Lindahl. Cummings, 1973, 330 pp, $8.95 (P). The best book on COBOL I have seen. Many simple and helpful examples and very clearly written. First part of book covers the DIVISIONS and the basic rules. Last part covers the PROCEDURE DIVISION statements and advanced concepts. RB

COMPUTER SCIENCES, T(13-14). *An Introduction to BASIC, A Time-Sharing Language.* Tate F. Lindahl. Cummings, 1971, 221 pp, $5.95 (P). A manual written for a particular system, namely IBM's ITF. Some of the material is specific to that installation, but most of the general material is common to all BASIC languages and is covered quite well. Many examples, but no flowcharts. Limited solutions to problems. RB

COMPUTER SCIENCE, S*(13-16), L*. *Programming Proverbs.* Henry F. Ledgard. Hayden, 1975, 134 pp, $5.65 (P). "Think first, program later." "Never assume the computer assumes anything." Refreshing bits of programming wisdom, supported by ALGOL 60 and PL/I illustrations. A prophecy of the coming "breakthrough" in programming based on taking style seriously as an essential discipline of the programmer's mental process. LAS

COMPUTER SCIENCE, S*(13-16), L*. *Programming Proverbs for Fortran Programmers.* Henry F. Ledgard. Hayden, 1975, 130 pp, $5.65 (P). A minor variation on the preceding book with FORTRAN illustrations. LAS

COMPUTER SCIENCE, P. *Complexity of Computation.* Ed: Richard M. Karp. SIAM-AMS Proc., V. VII. AMS, 1974, viii + 166 pp, $14.20. Nine papers from an April 1973 symposium held in New York City on one of the "most active research areas in the mathematical theory of computation." LAS

COMPUTER SCIENCE, P. *Computer Aided Geometric Design.* Ed: Robert E. Barnhill, Richard F. Riesenfeld. Acad Pr, 1974, x + 326 pp, $18.50. Papers from a March 1974 conference at the University of Utah, including many concerned with various types of splines. LAS

COMPUTER SCIENCE, P. *Universal Theory of Automata: A Categorical Approach.* H. Ehrig. Teubner, Stuttgart, 1974, 240 pp, DM 22,80 (P). In simplest terms an automaton is a set of states, a set of inputs, a set of outputs and relations between these sets. The relations may be functions, continuous maps, linear maps, etc., and the sets may have additional structure (linear, topological, etc.). Automata in a category provides a way of talking about all of these things at once. That is what this book does. More for the mathematician than the computer scientist, but an excellent book. PJM

COMPUTER SCIENCE, P. *Einführung in die Codierungstheorie I.* Tiko Kameda, Klaus Weihrauch. Skripten zur Informatik, Band 7. Bibliographisches Inst, 1973, 215 pp, (P). Developed from expository lectures given in 1970. Covers two aspects: algebraic and communications coding. JAS

COMPUTER SCIENCE, P. *Funktioneller Aufbau digitaler Rechenanlagen.* Heinz Schecher. Springer-Verlag, 1973, xii + 261 pp, $6.90 (P). Hardware from the point of view of what it does for the user. JAS

COMPUTER SCIENCE, T(15-17: 2), S, P, L. *Mathematical Theory of Computation.* Zohar Manna. McGraw, 1974, x + 448 pp, $19.50. Treats the practical and the theoretical aspects of the theory of computation in a pleasant, informal style. Attempts to formalize the understanding of computation and to make the art of verifying computer programs (the famous debugging technique) into a science. Includes material that the author feels every computer scientist should know. I-CH

COMPUTER SCIENCE, P. *Minicomputer Systems: Structure, Implementation and Application.* Cay Weitzman. P-H, 1974, xii + 367 pp, $15.95. "A complete overview of minicomputer systems, their design, implementation and operation." The emphasis is on hardware, and design configurations using a variety of hardware alternatives. TAV

COMPUTER SCIENCE, P. *Informations-strukturen.* Peter Heyderhoff, Theodor Hildebrand. Skripten zur Informatik, Band 6. Bibliographisches Inst, 1973, 218 pp, (P). A systematic introduction in response to the needs of programmers and users of recently developed software. JAS

COMPUTER SCIENCE, T(15-16: 4), S, L. *Digital Computer Programming: Principles, Techniques, and Applications.* Bing Hou-yi Lieu. Dover, 1974, xiii + 228 pp, $4 (P). An excellent book to use for teaching an elementary course covering a simple accumulator-centered machine language, number systems, FORTRAN and applications. There is also a fine chapter on debugging. FORTRAN is used exclusively after the second chapter. Many examples, flowcharts and selected answers to problems. RB

COMPUTER SCIENCE, T(14-15: 2-4), L. *Computer Data Processing, Second Edition.* Gordon B. Davis. McGraw, 1973, x + 662 pp, $12.95. This book has a number of very nice features, a good glossary, many references, nice sections on higher level languages and assembly level programming. Well suited for a textbook with a fair number of exercises and examples in each chapter. No answer section. RB

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-12: Fachtagung Prozessrechner 1974.* Gerhard Krüger, Rüdiger Friehmelt. Springer-Verlag, 1974, xi + 620 pp, $17.20 (P). Karlsruhe, June 1974. A wide variety of papers concerning both hardware and software. JAS

COMPUTER SCIENCE, T, S, P. *An Introduction to Digital Logic.* A. Potton. Hayden, 1973, x + 144 pp, $8.50. Aims to introduce reader with no previous knowledge of general electronics to the principles and techniques

of understanding and designing electronic digital systems (e.g., a binary adder). As such it may be useful in introductory courses in electronics, electrical engineering, or computer science. The mathematics of Boolean algebra is developed quickly and cursorily, probably too briefly for beginners. Except for the Karnaugh map technique for simplifying logical expressions, the rest of the book is devoted to implementation of counters, sequential logic systems, and arithmetic operators, plus practical design considerations. PJC

APPLICATIONS (BIOLOGY), P. *Mathematical Models in Biology and Medicine.* Ed: N.T.J. Bailey, Bl. Sendov, R. Tsanev. North-Holland, 1974, ix + 152 pp, $15.40. "Mathematical modelling is a universal method for studying nature." Twelve papers plus panel discussion from an IFIP working congress in Varna, Bulgaria in September 1972. LAS

APPLICATIONS (BIOLOGY), P. *Lecture Notes in Biomathematics-2: Mathematical Problems in Biology.* Ed: Pauline van den Driessche. Springer-Verlag, 1974, vi + 280 pp, $11.50 (P). Papers and abstracts from a May 1973 conference at the University of Victoria. LAS

APPLICATIONS (BIOLOGY), P, L. *Ecosystem Analysis and Prediction.* Ed: Simon A. Levin. SIAM, 1974, viii + 337 pp, $14.25 (P). Proceedings of a July 1974 conference at Alta, Utah, sponsored by the SIAM Institute for Mathematics and Society. 32 papers, mostly brief case-studies. LAS

APPLICATIONS (BIOLOGY), P, L. *Some Mathematical Questions in Biology, VI.* Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 7. AMS, 1974, vi + 232 pp, $20 (P). Proceedings of the eighth annual symposium on mathematical biology. Contains two papers on selection vs. chance in evolution, a major paper by E.C. Zeeman applying Thom's catastrophe theory to developmental biology, a differential geometry model for color theory by H.L. Resnikoff and more. An incredible price for 200 pages of photo-copied material: apart from copyright, one could xerox the entire book for one-fourth its cost! LAS

APPLICATIONS (CHEMISTRY), P. *Mathematical Aspects of Chemical and Biochemical Problems and Quantum Chemistry.* Ed: Donald S. Cohen. SIAM-AMS Proc., V. VIII. AMS, 1974, vi + 153 pp, $13.80. 8 papers from an April 1974 symposium in New York City. LAS

APPLICATIONS (CHEMISTRY), P. *Topics in Current Chemistry-49: Symmetry and Chirality.* C. Alden Mead. Springer-Verlag, 1974, 88 pp, $12.30. A molecule is *chiral* if it is different from its mirror image. This monograph deals with symmetry and chirality via representation theory of the symmetric and hyperoctahedral groups. An exposition of the theory set forth by Ruch and Schönhofer in 1970. LAS

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-97: Uber die Stabilität des einfachen Bedienungskanals.* G. Schmidt. Springer-Verlag, 1974, vii + 147 pp, $6.20 (P). Applications of multi-dimensional Markov processes to single-server queuing theory. JAS

APPLICATIONS (ECONOMICS), S(13), L. *Common Globe or Global Commons: Population Regulation and Income Distribution.* John C.G. Boot. Dekker, 1974, xi + 139 pp, $9.75. A pithy retelling of the now well-known causes of "Limits to Growth", culminating in a fairly detailed proposal for system of certificates to regulate population growth. Intended as a supplement or alternative to classical introductory economics texts. Very little mathematics. LAS

APPLICATIONS (ECONOMICS), P. *Programe Liniare cu Mai Multi Indici.* Cerchez Mihu. Editura Academiei Romania, 1974, 169 pp, Lei 6,25 (P). Transportation programs with three and four indices. Intended for engineers, economists, and to help with "achievement of the (Rumanian) State Program." JAS

APPLICATIONS (ECONOMICS), S(17-18), P. *Core and Equilibria of a Large Economy.* Werner Hildenbrand. Princeton U Pr, 1974, viii + 251 pp, $12.50. A mathematically detailed affirmative answer to the basic economic question of whether the cooperative equilibrium of a centralized economy can be achieved by a non-cooperative, decentralized economy governed only by a suitable system of prices. Uses (and introduces) the topology of correspondences (set-valued functions), fixed point theorems and abstract measure theory. LAS

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-99: Production Theory.* Ed: W. Eichhorn, *et al.* Springer-Verlag, 1974, viii + 386 pp, $13.20 (P). Proceedings of an international seminar held at Karlsruhe, May-July 1973. Twenty papers, all in English, on various mathematical models of production. LAS

APPLICATIONS (INFORMATION THEORY), P. *Lecture Notes in Mathematics-398: Théories de l'Information.* Ed: J. Kampé de Fériet, C.F. Picard. Springer-Verlag, 1974, 201 pp, $9.50 (P). Proceedings of the conference at Luminy, June 1973. JAS

APPLICATIONS (MECHANICS), P. *The Foundations of Mechanics and Thermodynamics.* W. Noll. Springer-Verlag, 1974, x + 324 pp, $20.10. Reprints selected by C.A. Truesdell, mostly from the *Archive for Rational Mechanics and Analysis* of Noll's influential papers on the thermodynamics of deformable bodies. LAS

APPLICATIONS (PHYSICS), P. *Symmetry Principles in Solid State and Molecular Physics.* Melvin Lax. Wiley, 1974, xi + 499 pp, $19.50. Point groups and space groups applied to solid state physics. PJM

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-30: Polarization Nuclear Physics.* Ed: D. Fick. Springer-Verlag, 1974, 292 pp, $9.90 (P). Proceedings of an October 1973 meeting at Ebermannstadt. LAS

APPLICATIONS (PHYSICS), T(17-18), P*, L*. *Group Theory and Quantum Mechanics.* B.L. van der Waerden. Grund. math. Wissenschaften, B. 214. Springer-Verlag, 1974, viii + 211 pp, $23. A completely rewritten translation of the 1932 German original. The major change is that chapter 1 now provides a rigorous exposition of quantum mechanics based on the theory of self-adjoint linear operators; the remaining chapters have been revised to reflect this new beginning. LAS

APPLICATIONS (PHYSICS), P. *Combinatorics and Renormalization in Quantum Field Theory.* E.R. Cainaniello. Benjamin, 1973, xv + 121 pp, $9.50 (P); $16. An outgrowth of lectures by the author at Imperial College in London. Definitions and results (without proofs) from advanced linear algebra with applications to modern theoretical physics. Intended as a concise survey with references to the literature for details. JAS

APPLICATIONS (PHYSICS), T?(2, 3), S?. *Mathematik für Physiker.* Dr. Herbert Meschkowski. Bibliographisches Inst. *Band 1: Zahlen,* 1970, 174 pp; *Band 2: Funktionen,* 1970, 179 pp; *Band 3: Elementare Wahrscheinlichkeits-rechnung und Statistik,* 1972, 188 pp, (P). Instant mathematics! Series and sequences, calculus and usable statistics. Moderately advanced but awfully short. JAS

APPLICATIONS (PHYSICS), T(18), L. *Mathematical Methods of Electromagnetic Theory.* K.O. Friedrichs. New York U, 1974, iv + 265 pp, $6.75 (P). Discourse on electromagnetic theory from a mathematical point of view, using an "axiomatic" approach to electric forces. The methods studied include those of proper formulation of physical laws and of justifying simplifying assumptions in those laws. 1972-73 lecture notes from the Courant Institute. DFA

APPLICATIONS (PHYSICS), P. *Equilibrium States on Thin Energy Shells.* Richard Leslie Thompson. Memoirs No. 150. AMS, 1974, vi + 110 pp, $3.30 (P). Probability measures on configurations of particles in a finite lattice obtained by restricting the grand canonical ensemble to an energy shell, or a set of particle configurations which share a common total energy with respect to vector potentials. The weak limits of these measures define a class of states on Z^n which are shown to be Gibbs' states. CEC

APPLICATIONS (PHYSICS), T(15-17: 1, 2). *Classical Groups for Physicists.* Brian G. Wybourne. Wiley, 1974, xvi + 415 pp, $19.95. An extensive introduction to Lie groups and algebras for physicists. Assumes knowledge of quantum mechanics and elementary notions of finite groups. Some proofs are omitted or sketched. The final three chapters give case studies of applications of groups to physics. Includes exercises and a good bibliography. CEC

APPLICATIONS (PHYSICS), P. *Quelques problèmes mathématiques en physique statistique.* Mark Kac. Pr U Montreal, 1974, 79 pp, $7. The first half, on phase-transitions for a one-dimensional model, is from the author's Gibbs lectures at the 1967 AMS annual meeting. The rest concerns gaseous networks and spherical models, and the problem of disordered chains. Aimed at the mathematician, to interest him/her in statistical mechanics. DFA

APPLICATIONS (PHYSICS), P. *The $P(\phi)_2$ (Quantum) Field Theory.* Barry Simon. Princeton U Pr, 1974, xx + 392 pp, $20; $7.50 (P). Post 1970 results emphasizing probabilistic Euclidean strategy instead of the formerly dominant Hamiltonian strategy. An exposition based on lectures in Zurich in 1973. Lots of references but no index for one who looses his way. JAS

APPLICATIONS (SOCIAL SCIENCE), T(15-17: 1), S, P, L*. *Urban Systems Models.* Walter Helly. Acad Pr, 1975, x + 185 pp, $16.50. An introductory text designed to bridge the gap between useful mathematics (calculus, probability and linear algebra) and the specialized research literature of urban analysis. Includes some detailed, simplified case studies, many general theoretical models, extensive references to the applied literature, and generally good (if often artificial) exercises. An excellent source for a seminar in modern applied mathematics. LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.

Florida State University: Dr. S. F. Bellenot, Claremont Graduate School, has been appointed Assistant Professor; Associate Professors J. L. Bryant and T. G. Hallam have been promoted to Professors; Assistant Professors Chiu-Yeung Chan and Phillip Novinger have been promoted to Associate Professors.

University of Nebraska-Lincoln: Drs. George Corliss, Michigan State University, Jeffrey Dawson, Rutgers University, Janina Spears, Kansas State University, have been appointed Assistant Professors; Assistant Professor Melvin Thornton has been promoted to Associate Professor; Instructor Sylvia Wiegand has been promoted to Assistant Professor; Associate Professor David Skoug was awarded a Research Council leave of absence to do research at the University of Minnesota, Minneapolis; Professor Edwin Halfar has been appointed Interim Chairman of the Department of Mathematics and Statistics for the academic year 1974–75.

Northern Michigan University: Assistant Professors Jane Swafford and Theodore Eisenberg have been promoted to Associate Professors.

Pomona College: Assistant Professor Sandy Grabiner, Claremont Graduate School, has been appointed Associate Professor; Associate Professor P. B. Yale has been promoted to Professor; Professor Hugh J. Hamilton has retired with the title of Professor Emeritus.

University of Rhode Island: Dr. Charles Groetsch, University of Cincinnati, has been appointed Visiting Assistant Professor; Dr. Oved Shisha, University of Texas at Austin, has been appointed Visiting Professor.

Rice University: Dr. D. G. Aronson, University of Minnesota, has been appointed Visiting Professor; Associate Professor R. O. Wells, Jr., has been promoted to Professor.

Southern Illinois University, Carbondale: Dr. Melvin Lax, RPI, has been appointed Lecturer; Dr. Russell Hendel, MIT, has been appointed Lecturer; Assistant Professors Worthen Hunsaker, Richard Millman, and David Kammler have been promoted to Associate Professors; Instructor Imogene Beckemeyer has been promoted to Assistant Professor; Assistant Professor Melvin Nathanson has been invited to the Institute for Advanced Study, Princeton, for the 1974-75 academic year.

Southern Methodist University: Associate Professor D. W. Matula, Washington University, has been appointed Professor and Department Head, Department of Computer Science and Operations Research; Mr. W. H. E. Day, Washington University, has been appointed Instructor.

University of Southwestern Louisiana: Assistant Professor R. D. Sidman, University of Connecticut, has been appointed Associate Professor; Assistant Professors Duane Blumberg and Victor Schneider have been promoted to Associate Professors.

Virginia Military Institute: Dr. E. J. McShane, University of Virginia, has been appointed Visiting Professor for the spring semester of the 1974–75 academic year; he will occupy the Mary Moody Northern Eminent Scholars Chair; Associate Professor H. G. Williams, Jr., has succeeded Dr. A. L. Deal, III, as Chairman of the Mathematics Department.

Instructor Elayne A. Idowu, University of Cincinnati, has been appointed Assistant Professor at the University of Pittsburgh.

Associate Professor R. E. Powell, Kent State University, has been promoted to Professor.

INTERNATIONAL CONFERENCE ON ALGOL 68
OKLAHOMA STATE UNIVERSITY

An International Conference on the implementation and use of the algorithmic language ALGOL 68 will be held at Oklahoma State University, June 10–12, 1975. Those who have used this language as a research tool are invited to present a paper describing their work at this Conference. For additional information contact: G. E. Hedrick, Department of Computing and Information Sciences, Oklahoma State University, Stillwater, Oklahoma 74074.

UNIVERSITY OF CHICAGO —
A STUDENT SCIENCE TRAINING PROGRAM IN MATHEMATICS FOR SUMMER 1975

With partial support from the National Science Foundation, the Department of Mathematics of the University of Chicago will conduct an eight-weeks residential STUDENT SCIENCE TRAINING PROGRAM IN MATHEMATICS from June 23 to August 15. Participants are to be 50 talented high school students who have just finished their junior year. The aim of the program is to rouse and strengthen the participants' mathematical powers through a deep experience in mathematical thought and effort. A basic course in number theory and two or three other mathematics courses will be provided, together with a supporting scheme of problem-seminars. Members of Chicago's Department of Mathematics will teach. The basic number theory course will be given by Professor A. E. Ross of the Department of Mathematics at Ohio State University. The Chicago program can be regarded as continuing the highly successful program conducted by Professor Ross at Ohio State for the last decade.

The Department of Mathematics at Chicago invites inquiries about this program from eligible high school students. The Department also invites high school and college teachers to nominate for participation in the program any highly motivated (though possibly mathematically inexperienced) high school junior who shows significant interest, initiative and power in mathematics. And since up to ten undergraduates will serve in the program as counselors, the Department invites application for the position from any undergraduate who has gone through Professor Ross's Summer Science Training program at Columbus.

Communications should be addressed to Professor A. L. Putnam, Director (Summer Science Training Program, Department of Mathematics, University of Chicago, Chicago, IL 60637).

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE WISCONSIN SECTION

The Forty-second Annual Meeting of the Wisconsin Section of the MAA was held at Marquette University, Milwaukee, on Friday and Saturday, May 3–4, 1974, with 108 registrants including 81 members.

The program featured an invited address by Professor M. D. Thompson, Indiana University, entitled "Mathematical Models for the Spread of Epidemics."

The following papers were presented:

1. *Using finite vector spaces in teaching linear algebra*, by Tom Renfrow, Beloit College.
2. *A Constructive proof for the minimum cut — maximum flow theorem*, by J. D. Wine, UW-LaCrosse.
3. *Parenthesizings: from the generalized associative law to the Catalan numbers*, by Gary Klatt, UW-Whitewater.
4. *Statistics in cancer research*, by J. Van Ryzin, UW-Madison.
5. *An application of analysis and group theory in the modular coloring of Pascal's triangle*, by Martin Bulgerin, (student); UW-Eau Claire.
6. *Introduction to non-associative algebras including Cayley numbers*, by F. G. Florey, UW-Superior.
7. *Video cassettes mean freedom in the classroom*, by Alvin Rolland, UW-Eau Claire.
8. *Statistical analysis of air pollution data*, by G. C. Tiao, UW-Madison.
9. *Task scheduling algorithms*, by Fobert Wilson, UW-Madison.
10. *Rational sequences which converge to the square root of integers*, by Wayne Wild, UW-Stevens Point.
11. *Applications of computer graphics to the teaching of undergraduate mathematics*, by Jerry Caldwell, UW-River Falls.
12. *Structurally stable predator-prey systems*, by P. D. Straffin, Jr., Beloit College.
13. *Isolating roots of polynomial equations*, by C. W. Schelin, UW-LaCrosse.
14. *Quadratic words in a free group*, by Norman Frisch, UW-Oshkosh.

At the annual business meeting with Professor Phillip Bender, Chairman, presiding, the governor's report was presented by Ed Wilde and the MAA Contest Committee report was made by Carl Vanderlin, indicating that 20,653 students participated in the preliminary high school contest and 1077 participated in the final contest, a drop of about 20% below the previous year. A motion was passed to urge appropriate MAA officers to follow an aggressive policy of seeking endowments for MAA. Another motion was passed to charge a nominal registration fee of $2.00 to members and guests at meetings except for students. Professor C. H. Johnson, UW-Stevens Point, was elected chairman and Professor R. L. Hall, UW-Milwaukee, was elected vice-chairman.

R. D. WAGNER, *Secretary-Treasurer*

NOVEMBER MEETING OF THE OHIO SECTION

The Ohio Section of the MAA held its annual Autumn Meeting at the University of Cincinnati, November 1 and 2, 1974. Two hundred and ten people attended the meeting. Chairman Louis Green presided; Richard Little was Program Chairman.

The following invited addresses were presented:

Introducing operations research in the undergraduate curriculum: Effect learning with compact video cassettes, by Judah Rosenblatt, Case Western Reserve University.

Operations research — The state of the art, by G. L. Thompson, Graduate School of Industrial Administration, Carnegie-Mellon University.

How to make statistics interesting: Accomplishments and future plans of the Committee on Statistics of the American Statistical Association and NCTM, by Gottfried Noether, University of Connecticut.

A panel discussion was held on *Implications of Operations Research and Statistics for the Undergraduate Mathematics Curriculum.* Participants on the panel were Leonard Sweet, University of Akron, and Samuel Mantel, University of Cincinnati. A question and answer session followed the presentation by the panelists.

The program also included the following swap sessions: Operations Research and the Undergraduate Curriculum, led by James Murtha, Marietta College; and Mathematics in the Two-Year College, led by Michael McSwigan, University of Cincinnati.

Also on the agenda were a meeting of the Executive Board of the Section, and members of the *ad hoc* committees: Committee on Teacher Training and Certification, Committee on Curriculum and Committee on Cooperation between Colleges and Universities.

R. H. ROLWING, *Secretary-Treasurer*

NOVEMBER MEETING OF THE SEAWAY SECTION

The Fall Meeting of the Seaway Section of the MAA was held at St. John Fisher College, Rochester, N. Y., on November 2, 1974, with a registered attendance of 145 people, including 126 members of the Association.

Professor D. O. McKay of the University of Western Ontario, Chariman of the Section, presided at both morning and afternoon sessions.

Greetings from the College were extended by Rev. C. J. Lavery, C. S. B., President of St. John Fisher College.

Professor C. A. Lathan, Monroe Community College, talked on "The Two-Year College Mathematics Journal: The Newest Kid on the Block."

As a variation from the format of previous meetings, this fall meeting had talks on a single theme, the theme being Linear Algebra. Dealing with different aspects of that topic, the following invited talks were presented:

Historical Highlights of Linear Algebra in the Last Fifty Years, by M. F. Smiley, State University of New York at Albany.

A First Course in Linear Algebra, by L. A. Trivieri, Mohawk Valley Community College, and R. J. Wernick, State University College at Oswego.

Computational Aspects of Linear Algebra, by Patricia J. Eberlein, State University of New York at Buffalo.

Singular Values, by Chandler Davis, University of Toronto.

Spectrum Localization of Banach Spaces, by Zdislav Kovarik, McMaster University.

Some Applications of Markov Chains to the Social Sciences, by Christopher Nevison, Colgate University.

Linear Inequalities: Old and New Methods of Solution, by R. E. Reed, State University College at Oneonta.

EMMET STOPHER, *Secretary-Treasurer*

THE FIFTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The Fifty-eighth Annual Meeting of the Mathematical Association of America was held at the Shoreham Hotel, Washington, D.C. from Saturday to Monday, January 25–27, 1975, in conjunction with meetings of the American Mathematical Society, the National Council of Teachers of Mathematics, the Association for Symbolic Logic, and the Association for Women in Mathematics. The sessions of the Association on Saturday and Sunday mornings were joint meetings with the National Council of Teachers of Mathematics; the session on Sunday evening was a joint meeting with the American Mathematical Society. There were registered 3622 persons including 2075 members of the Association.

Sessions were held on Saturday morning, Sunday morning and evening in the Regency Ballroom and Monday morning and afternoon in the Palladian Room of the Shoreham Hotel. Presiding officers were Professor Ronald M. Davis at the first lecture on Saturday morning, Professor Frank Fitzgerald for the remainder of the Saturday morning session, Dr. A. B. Willcox for the panel discussion on Sunday morning, Professor D. I. Schneider for the lecture on Sunday morning, Professor J. H. Wells for the panel discussion on Sunday evening, Professor William Swyter for the Monday morning session, and Professor Dorothy L. Bernstein for the Monday afternoon session.

The Program Committee consisted of Choy-tak Taam, Chairman; Dorothy L. Bernstein, Ronald M. Davis, Ruth M. Davis, Frank Fitzgerald, S. B. Jackson, D. I. Schneider, and William Swyter.

FIRST SESSION OF THE ASSOCIATION

Joint Session with the National Council of Teachers of Mathematics

A Personalized System of Instruction, by Dr. B. A. Green, Jr., Center for Personalized Instruction, Georgetown University

PSI is a form of individualized instruction by which a class of 100 can be taught as if each were in a class of one. Students work individually in consultation with a proctor and the instructor to accomplish well-defined objectives, proceeding at their own paces and working always to the point of mastery of the material. Research shows that in comparison with lecture-recitation classes, students in PSI learn more and like it better, although they have to work harder. On the other hand, PSI courses require far more planning and preparation by the teacher when the course is given for the first time. Success is not automatic.

The First U.S. Participation in the International Mathematical Olympiad, by Professor S. L. Greitzer, Rutgers University, Newark

The participation of a U.S.A. team in the International Mathematical Olympiad for the first time involved diplomatic as well as educational problems. The surprising results (USSR — 256, USA - 243, Hungary — 237, GDR — 236) strongly suggest the necessity on the part of mathematics educators of rethinking in the areas of the secondary school mathematics curriculum, the training of teachers of mathematics, the selection and preparation of gifted mathematics students, and, for some mathematics educators, a change in their opinion on the effects of our existing curricula on our students. Our experiences in these areas were presented for consideration.

Ordering of Fields and Prime Ideals of Witt Rings, by Professor Alex Rosenberg, Cornell University

The notion of ordering of the real numbers was recalled and extended to arbitrary formally real fields. The notion of a homomorphism of a ring was recalled. Quadratic forms were introduced next, and it was shown how equivalence classes of these can be made into a commutative ring. It was then shown that the orderings of fields correspond biuniquely to homomorphisms of the ring arising from quadratic forms to the integers. Applications of this correspondence were given.

SECOND SESSION OF THE ASSOCIATION

Joint Session with the National Council of Teachers of Mathematics

Panel Discussion: Teaching Mathematics to the Beginning Undergraduate

A panel discussion with Professors C. A. Lathan, Monroe Community College, Rochester, New York, Andrew Sterrett, Jr., Denison University, Benjamin Volker, Bucks County Community College, Newton, Pennsylvania, and M. H. Protter, University of California, Berkeley, moderated by Dr. A. B. Willcox, Executive Director of the MAA.

Professor Lathan spoke on "Remedial Mathematics and the Mathematics Curriculum in an Urban Community College." The wide ranges of ability, interest, and age provide a unique instructional and curricular problem for the urban community college. The typical urban community college must provide mathematics from Basic Arithmetic to Linear Algebra and Differential Equations. Both course content and teaching mode have been influenced by the impact of the returning veterans, adults looking for a career or trying to change careers, and normal pressure due to open admission.

Professor Sterrett spoke on "Trends in the Introductory Mathematics Curriculum in Four-Year Colleges". A questionnaire was sent to chairmen of mathematics departments of small colleges in the midwest. Returns indicate that enrollments in computer science, elementary statistics, and finite mathematics have increased in the last five years, largely because of increased emphasis on quantitative methods in the social and life sciences. Chairmen also report that many students are less well prepared for calculus now than they were five years ago, especially in trigonometry and in their ability to perform algebraic manipulations. The number of mathematics majors has declined in many small schools. A more complete summary of the findings is available upon request.

Professor Volker spoke on "The Challenge of Diversity: A Community College Reacts to the Ever-Widening Range of Preparation and Ability in Entering Students." He emphasized that from a community college point of view the beginning undergraduate does not, except, in rare cases, represent a mathematics major. As a consequence, he pointed out, it is necessary to design a curriculum with a wide range of entry points and to match as closely as possible the characteristics of an individual student with the proper beginning course. By necessity, community colleges have developed innovative course materials, experimented with the use of educational technology and made a serious effort to individualize instruction. Professor Volker described such efforts undertaken to date by Bucks County Community College, Newtown, Pennsylvania.

Professor Protter spoke on "A Self-Paced Beginning Course in Calculus and Physics." A first year college physics course frequently contains topics in mechanics and electromagnetic theory which require a substantial amount of calculus. A difficulty in articulation between mathematics and physics arises since the appropriate mathematics is often taught after the need for it arises in the physics course. To solve this problem an experimental course combining first year physics and calculus was initiated at Berkeley during the year 1972–73. The Keller plan was used with the material for the year divided into 60 units, half in mathematics and half in physics. Each student was expected to complete two units per week. The results with about 100 students were generally positive.

Annual Business Meeting of the Association; the Association's Fourteenth Award for Distinguished Service to Mathematics; Presentation of the 1974 Lester R. Ford Awards; Award of the 1975 Chauvenet Prize; Presentation of a Resolution of Appreciation to a Member of the MAA.

The Open University in Great Britain — A Progress Report, by Dr. R. J. Wilson, Lecturer in Mathematics, The Open University, Great Britain.

The Open University offers degree courses to students studying part-time at home, and the number of students has now reached 50,000. Each course is an integrated package consisting of correspondence texts, television and radio programs, a summer school, and some local face-to-face tutoring.

At the Las Vegas Annual Meeting in January 1972, the Mathematics Foundation Course was discussed in some detail (see this MONTHLY, May 1972). In the present lecture, the subsequent development of the mathematical curriculum was described, future plans were outlined, and some conclusions drawn.

THIRD SESSION OF THE ASSOCIATION

Joint Session with the American Mathematical Society

A Panel Discussion Sponsored by the Joint Committee on the Training of Graduate Students to Teach: Schemes for Training Graduate Students to Teach: Four Reports.

A panel discussion with Professor D. I. Schneider, University of Maryland, Mr. A. L. Peressini, University of Illinois, Urbana, Professor G. T. Sallee, University of California, Davis, and Mr. Philip Treisman, University of California, Berkeley, moderated by Professor J. H. Wells, Chairman of the Joint Committee, University of Kentucky.

Professor Schneider reported that teaching assistants at the University of Maryland are prepared for their teaching duties by a three-day program prior to the start of the academic year. After a day of orientation, discussion of the role of the teacher, and analysis of basic classroom technique, the teaching assistants work with faculty members in small groups presenting material which is videotaped, discussed by others, and then replayed for the benefit of the presenter.

Mr. Peressini noted that almost all of the undergraduate courses in mathematics at the University of Illinois are taught in small sections of twenty to thirty students each. As a result, new teaching assistants must be in a position to handle all aspects of teaching one or two freshmen level courses. In order to assist them with their new responsibilities as teachers and as graduate students, the Department of Mathematics conducts an orientation program for new teaching assistants during registration week. New teaching assistants are also supervised by regular faculty members during their first year of teaching. A procedure for evaluating the work of teaching assistants on the basis of input provided by a student survey and supervisor ratings has also been developed. The panel presentation was devoted to a discussion of these orientation, supervision and evaluation programs.

Professor Sallee observed that, like most large campuses, the University of California at Davis is faced with the problem of training large numbers of teaching assistants to aid in the instruction of lower-division courses. The current program of the mathematics department, which seems quite successful with both students and teaching assistants, begins with two days of discussions and practice teaching before the quarter begins. During their first quarter of instruction, teaching assistants are then subject to close faculty supervision including detailed conferences after each visit. A complete report is made on the student's development as a teacher at the end of the quarter including recommendations for the degree of future supervision.

Mr. Treisman described an approach to helping graduate students experiment with a wide repertoire on teaching strategies in a manner that promotes the development of basic teaching competence. The core of the approach consists of integrated programs of microteaching, interaction analysis (involving a structured analysis of classroom videotapes), individually designed projects, and reciprocal class visits. Examples of teaching goals amenable to this approach such as reducing the use of "programmed" answers and extending "wait-time" were discussed. Factors inhibiting change were identified, and the importance of the faculty supervisor carrying out the training program in his own classroom was stressed.

FOURTH SESSION OF THE ASSOCIATION

The Challenges to a Discipline in an Era of Interdisciplinary Emphasis, by Dr. L. J. Paige, Assistant Director for Education, National Science Foundation.

There is little doubt but that the nation's problems in energy, resource development, health care

delivery, etc. are growing more complex. As a consequence, there is an increasing need for interdisciplinary cooperation and many authors have argued that the traditional departmental structure of colleges and universities hampers effective interdisciplinary activities. The speaker takes issue with this point of view and points out that organizational structures are not the critical issue. The critical issue is the attitude of individuals and the recognition on the part of departments that there are challenges which they can respond to quite effectively.

The National Institute of Education, by Dr. J. M. Mays, Science Adviser, National Institute of Education, Department of Health, Education and Welfare.

NIE is a new Federal agency charged with helping to improve American education through research, development and dissemination. Areas of emphasis are:

Basic Skills — improving instruction in reading and other basic skills needed for further education and for functioning in society.

Education and Work — providing better knowledge, information, and capabilities for choosing and pursuing careers.

Educational Equity — lowering barriers to education arising from home language and culture, ethnicity, sex, or economic status.

Finance, Productivity, and Management — improving the ability of the educational system to maintain or improve quality and meet new demands.

Dissemination — increasing utilization of results of educational R & D.

What is an Automorphic Form?, by Professor L. J. Goldstein, University of Maryland.

Automorphic forms for subgroups of the modular group were defined, and it was indicated that the Eisenstein series, special cases of which occur in the differential equation of the theta-function, are automorphic forms. Next, the theta functions were introduced and it was mentioned that they are also automorphic forms. Finally, it was shown how the finite-dimensionality of various spaces of automorphic forms yields identities among automorphic forms, which imply deep facts about Diophantine equations.

FIFTH SESSION OF THE ASSOCIATION

On Measuring Things, Eudoxus Revisited, by Professor A. M. Gleason, Harvard University.

Eudoxus of Cnidus (408?–355? B.C.) was, apparently, the first mathematician to give rigorous infinitary arguments. We are told by Archimedes that he was the first to find proofs of the theorems in Euc id Book XII. The basic theory of proportion, as given in Euclid Book V, is also attributed to him. In this presentation, Eudoxus' work was examined in the light of modern concepts. The concept of Eudoxian semi-group was introduced and then applied to the problem of length, area, and volume. This modern treatment was critically compared to the treatment of these topics in Euclid.

The Early Days of Probability Theory, by Professor Mark Kac, Rockefeller University.

The speaker presented reminiscences of the early days of Probability Theory in Poland, especially. those of the speaker's collaboration with the late Professor Steinhaus. General remarks of historical, pedagogical, and scientific nature were also included.

Intuitive Geometry is Alive and Well, by Professor Branko Grünbaum, University of Washington

Intuitive geometry is an easily accessible and rapidly growing area of knowledge, related to combinatorics somewhat like algebraic or differential geometries are to algebra and analysis. Some of its branches (packings and coverings, convex polytopes, etc.) are fairly familiar to many. Several

new and less widely known branches were discussed and illustrated, leading to many exciting open problems. The topics of the talk included arrangements of lines in the plane (in particular, the simplicial arrangements), the process of shelling, and a new point of view on graphs, configurations, maps and polytopes, as well as their often surprising ramifications.

SPECIAL SESSION OF THE ASSOCIATION

Film showings were held in the Empire Room of the Shoreham Hotel on Friday and Saturday evenings. The following films were shown:

Friday

7:00—7:25 P.M. GAUSS-BONNET THEOREM — A Lecture by Carl B. Allendoerfer (in color and with animation)

7:26—7:48 P.M. CYCLOIDAL CURVES OR TALES FROM THE WANKLENBURG WOODS. An Allendoerfer Film (in color)

Films of the Topology Films Project (in color and with sound narration)

8:00—8:05 P.M. A preliminary part from HOW TO TURN A SPHERE INSIDE OUT

8:08—8:22 P.M. REGULAR HOMOTOPIES IN THE PLANE, PART I

8:25—8:50 P.M. TOPOLOGY—a B.B.C. Broadcast as part of The Open University Foundation Course in Mathematics (in b & w)

8:55—9:10 P.M. A Film produced by the Moody Institute of Science (in color): MATHEMATICS OF THE HONEYCOMB

9:15—9:45 P.M. NUMERICAL ANALYSIS—Excerpts from a Videotaped Course Produced by Ben Noble at Oberlin College (in b & w)

9:55—10:20 P.M. STATISTICS AT A GLANCE (in color)

Saturday,

7:00—8:15 P.M. AN APPLICATION OF COMPUTER GRAPHICS TO TEACHING MATHEMATICS: EXHIBITION OF COMPUTER-ANIMATED SUPER-8 MOVIES AND SLIDE SEQUENCES ON CALCULUS, STATISTICS, PURE AND APPLIED ANALYSIS, by Professors R. B. Kirchner and R. W. Nau of Carleton College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Friday at 9:00 A.M. in the Diplomat Room of the Shoreham Hotel, with 43 members present.

The Board approved the appointment by President Boas of the following Nominating Committee for 1975: Victor Klee, Chairman; Deborah T. Haimo, and June P. Wood.

The Board elected Professor Betty J. Hinman of the Downtown College of the University of Houston, Second Vice-President for the period 1975–76.

Professor D. P. Roselle has requested to be relieved of his position as Associate Editor of this MONTHLY for Mathematical and Classroom Notes, effective March 1, 1975, since he will become Secretary of the MAA after the Washington, D. C. meeting. The Board elected Professor R. A. Brualdi of the University of Wisconsin, Madison, to fill the unexpired part of his term extending through 1978.

Professor E. S. Langford has requested to be relieved of his position as Associate Editor of the Problems Section of this MONTHLY, heading the University of Maine Problems Group, which also asked to be relieved of its responsibility. The Board elected Professor D. Ž. Djoković of the University of Waterloo to fill the unexpired part of the term of Professor Langford, effective February 1, 1975, and extending through 1978. Professor Djoković will head a Problems Group for the Elementary Problems at the University of Waterloo.

Having elected by mail ballot Professors J. A. Seebach, Jr. and L. A. Steen, both of St. Olaf College, as Co-Editors of the MATHEMATICS MAGAZINE for the period 1976–80, the Board, in accordance with their nominations, elected the following as Associate Editors of the MAGAZINE for the same period: T. F. Banchoff of Brown University, D. J. Eustice of Ohio State University, Raoul Hailpern of SUNY at Buffalo, R. A. Honsberger of the University of Waterloo, L. M. Kelly of Michigan State University, and P. J. Malraison, Jr. of Carleton College.

The following resolution was approved by the Board:

"The Board of Governors wishes to go on record as recognizing the significant contributions of CUPM to mathematics and the work of the MAA. It believes that projects of the type CUPM has undertaken are important and vital and every effort should be made to investigate avenues of possible permanent financial support."

The Board voted to make the TWO-YEAR COLLEGE MATHEMATICS JOURNAL (TYCMJ) an official journal of the MAA beginning January 1, 1976, with the understanding that the membership of the MAA shall be given the option, effective January 1, 1976, to receive either this MONTHLY or the TYCMJ as an official journal, and obtain the other journals at reduced prices. A form will be sent with the program for the 1975 summer meeting (which also includes a request for payment of dues during the summer), requesting members to list which journals they wish to receive in 1976. Since it is expected that the majority of members will wish to continue receiving the same journals as at present, the form will contain a statement to the effect that, unless the Association's Washington office is otherwise informed by September 1, it will assume that the member wishes to continue receiving the same journals as at present. This notice will also include the dues for the various journal options, as they were approved by the Board, so that those paying dues during the summer will know the amount to pay.

The Finance Committee reported receipt, with deepest appreciation, of the following gifts to the Association:

1. $20,000 from Professor Mary P. Dolciani, which will be used to finance the DOLCIANI MATHEMATICAL EXPOSITIONS.
2. $1,000 from Professor H. M. Gehman.
3. $3,000 from an anonymous donor.

4. Approximately $50,000 to be received as a bequest from Professor C. B. Allendoerfer.

The Board voted to receive, with an expression of gratitude, the following grants made to the Association:

1. $3,200 from IBM to defray the expenses of the 1974 USA Mathematical Olympiad Awards Ceremony.

2. $5,000 from the Spencer Foundation for partial support of travel expenses for the U.S. team to participate in the International Mathematical Olympiad in Erfurt, D.D.R.

3. $5,000 from IBM to defray expenses of the 1975 USA Mathematical Olympiad Awards Ceremony.

In accordance with a recommendation of the Finance Committee, the Board voted to instruct the Secretary-Elect to submit to the membership at the business meeting on August 19, 1975, at Western Michigan University, an amendment to the By-Laws adding the Past-President to the membership of the Finance Committee for a period of two years, after expiration of his term as President, thereby increasing the membership of the Committee from five to six; in addition, this amendment provides that the Past-President presides at a meeting of the Finance Committee where the President is unable to preside, and that in the absence of the Past-President, the senior member (in terms of length of service on the Committee) of the two elected members of the Finance Committee shall preside.

The Board voted to increase substantially the privileges of academic membership effective January 1, 1976, specifically that, for the basic annual dues of $60, an academic member shall receive the following as privileges of academic membership:

(a) One subscription to each MAA journal (worth $45 at regular rates);

(b) One copy of the COMBINED MEMBERSHIP LIST (worth $10 at regular rates);

(c) The privilege of naming one institutional nominee (student or faculty) (worth at least $18 at the regular rate);

(d) The privilege of purchasing additional copies of MAA books, *for student awards only*, at members' discount prices (a saving of up to $5 per book);

(e) The privilege of naming additional institutional *student* nominees for a payment of $10 for each student.

The Board voted to negotiate for the purchase by the MAA of the New Mathematical Library (NML) from Random House, Inc. and that the sale of these books be promoted vigorously. The Board voted that individual volumes in the NML series be priced at $4.00 for nonmembers and $3.00 for members.

In accordance with a recommendation made by the Joint Committee on Relations Between the AMS and MAA, the Board voted to endorse the present version of the proposal whereby the AMS, MAA, and SIAM establish a Joint Projects Committee for Mathematics (JPCM) which is to be an autonomous body under the auspices of the Conference Board of the Mathematical Sciences to deal with those matters of primary concern to the three parent organizations.

The Executive Director reported the membership of the Association as of December 31, 1974, as 18,762, an increase of 132 from a year earlier. He also reported 375 academic members, 41 of which were added in 1974, compared to 15 new academic members in 1973.

At their meeting on January 16, 1974, in San Francisco, the Board had considered possible changes in the By-Laws concerning the number of Governors-at-Large (at present there are six) and their election. The matter had been referred to the Executive and Finance Committees for a recommendation. Upon the recommendation of these Committees, the Board voted to instruct the Secretary-Elect to submit to the membership at the business meeting on August 19, 1975, at Western Michigan University, an amendment providing that the Governors-at-Large henceforth be elected by the Board upon nominations made by the Executive Committee.

In addition, the Board voted to instruct the Executive Committee to make these nominations from specifically designated categories of mathematicians, which it shall determine to assure appropriate representation of all constituencies on the Board.

The Board voted to authorize 64 pages for the February 1975 issue of the TYCMJ and 48 pages for each of the three other issues of 1975.

The Board authorized the eight winners in each year's USA Mathematical Olympiad to be designated "Olympiad Fellows."

The Board voted to approve the following resolution:

WHEREAS (a) the Board is deeply concerned by the substantial deficits incurred by the MAA in recent years, in particular, the deficit of $77,207 in 1973, and

(b) the Board recognizes that avoidance of further budget deficits may involve curtailment of various MAA activities, and thus engender counter-pressure from various members of MAA, and

(c) the Board wishes to strengthen the hands of its officers against those anticipated counter-pressures (even if they should come from members of the Board!) and wishes to encourage the MAA's officers, members of the Executive and Finance Committees, editors, executive director, and committee chairmen to take whatever action may be necessary to avoid further overall budget deficits,

THEREFORE BE IT RESOLVED THAT the above-named officials of the MAA should consider the avoidance of further overall budget deficits in the next few years as a goal at least as important as any other goal of the MAA.

The Board considered the suggestion of Professor Victor Klee that the practice of printing abstracts of lectures delivered at MAA meetings (both national and sectional) be discontinued. Abstracts of lectures delivered at sectional meetings were discontinued several years ago. A straw vote on whether abstracts of lectures delivered at national meetings should also be discontinued showed that 23 members believe they should continue to be printed, while 7 felt that they should not.

LUNCHEON IN HONOR OF THE RETIRING SECRETARY

A luncheon in honor of the Retiring Secretary was held on Saturday at noon in the Diplomat Room of the Shoreham Hotel. President Boas, as master of ceremonies, paid tribute to the services of the Secretary during the past fifteen years and called attention to the citation distributed with the program for the luncheon. (This citation appeared in the February issue of this MONTHLY, pages 110–112.) President Boas read letters of congratulations received from several members, including one from Professor H. M. Gehman, the previous Secretary of the Association, who wrote: "Please tell those present that of all my activities on behalf of the Association, I am most proud of my success in training Henry Alder to take over the duties of the secretaryship. He was an apt pupil!"

President Boas also read the following tribute sent and signed by Professor C. B. Allendoerfer on September 24, 1974, five days before his death:

"The Secretary's job is to keep the President out of trouble. Most of the time Henry played this traditional role and repeatedly 'saved my bacon'. Every succeeding President has benefited from his advice and counsel."

The following mailgram was received from Professor V. O. McBrien: "Even though I cannot be with you at the luncheon to honor Professor Alder, I wish all of you to know that I share your great respect for his tireless efforts. He commands the respect of the entire mathematical community, and I wish him every happiness."

Professor Everett Pitcher, Secretary of the AMS, presented greetings from the AMS. Dr. J. D. Gates, Executive Secretary of the NCTM, conveyed greetings from NCTM and

presented to Professor Alder a Certificate of Merit "in recognition of exemplary contributions to the improvement of mathematics education."

President Boas expressed appreciation to Professor Alder's secretary, Mrs. Trudy Walker, and presented a gift of a silver flower bowl on behalf of the MAA. The Retiring Secretary was then presented with a gift of a Minolta camera, with attachments, and inscribed: "From Your Friends in the MAA, January, 1975."

The Retiring Secretary addressed the audience as follows:

"To say that I am overwhelmed by all this would be an understatement. I greatly appreciate all the kind things which have been said about me. Actually it is I who wish to express thanks to you for having elected me three times to the position of Secretary of the Association and, thereby, given me the opportunity to be of service to the mathematical community.

I can say with complete honesty and sincerity that I have greatly enjoyed working in this position and, in particular, thereby to have had the pleasure of meeting the many dedicated leaders of the mathematical community, who have made the MAA the thriving organization which it is today.

When I accepted the office of Secretary in 1959, I set myself two goals, namely

1. to see to it that the Association continues to carry out with all possible vigor the noble goals for which its founders organized it, and

2. to relinquish the position before there are any hints from anybody that it is time to do so.

Whether or not I have achieved the first goal is not for me to judge, but I feel confident that I have achieved the second goal. I am particularly glad to be able to step out of this position as the youngest retiring Secretary ever of the Association and especially in a city where voluntary retirement from a responsible position after 15 years of service has become a rather rare phenomenon.

If I have been able to be of some service to the mathematical community, it clearly would not have been possible without the great help I have received from so many. Time does not permit me to list all the many individuals to whom I am deeply indebted. I feel compelled, however, to mention five: To be able to carry out the responsibilities of the position of Secretary, you need a good teacher to help you learn the job. No one could have had a better teacher than my predecessor in this position, Professor H. M. Gehman, whose help proved truly superb. I regret very much that he could not be here today.

Next, no one can carry out such a position without an understanding and supportive wife. In this respect, I have been most fortunate indeed. Not only did she have to face my frequent absences at meetings, but she also has assisted me cheerfully with advice on how to deal with many situations which have arisen in the Association. Quite regularly we discussed these during the dinner hour, in fact so much so that the first reaction of my ten-year old son when he heard that I would no longer be Secretary was: "Daddy, what are we going to talk about then at dinner?" I would like to take this opportunity, therefore, to express my profound appreciation to my wife, Benne.

I have been equally fortunate in having had for the past 9 years the help of the best secretary anyone could wish for. I would like to take this opportunity to express my deep gratitude to Mrs. Trudy Walker.

Next, our organization would not be what it is today if it did not have such a superb Executive Director, who is the one who really sees to it that, as his title implies, everything is executed properly, efficiently, and promptly. He has been of tremendous help to me personally, and I would like to take this opportunity to acknowledge this publicly today to our Executive Director, Dr. Alfred B. Willcox.

Last, but certainly not least, you are well aware that one of the main responsibilities of our organization is the publication of journals, books, and other materials. That all these publications always come out in such good form, right on time, and practically free of errors is obviously no accident. For example, in what other organization can you be certain that the issue for January comes out in January? The credit belongs to a person who is rarely seen in public, but who, behind the scenes, makes one of the most substantial contributions to the work of the Association, namely our Editorial Director, Dr. Raoul Hailpern.

It is a source of great satisfaction to me personally that my successor as Secretary of the Association brings such unusual qualifications to this position. First of all, he is young, which I feel is particularly important, believing, as I do, that mathematics is a game for the young and that our Association should continue to appeal particularly to the young. He has already served the Association in several positions with distinction, in particular as the Associate Editor of the MONTHLY for Mathematical and Classroom Notes. I am convinced that the Association will have a bright future with Professor David Roselle as its Secretary, especially since, as I am certain, he will receive the same cooperation and help from you, the membership, as it has been my privilege to receive.

Again, thank you very much."

The audience gave the Retiring Secretary a standing ovation.

The luncheon concluded with remarks by Professor Harley Flanders, in which he recalled that as a member of the Executive Committee in 1959, he had suggested Professor Alder as Secretary of the Association. He reminisced about the days in the late 1950's when he had worked with him on various Association projects in the Northern California Section, in particular, the visiting lectureship program to secondary schools.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting was held on Sunday, January 26, 1975, in the Regency Ballroom of the Shoreham Hotel, with President Boas presiding. He announced with the deepest regret the death of Professor C. B. Allendoerfer on September 29, 1974, and called on Professor B. W. Jones, who addressed the meeting as follows:

"This is neither the time nor the place to try to describe in detail all that Professor Allendoerfer did or all that he was involved in for the MAA and mathematics in general. His many services to the mathematical community are listed in the citation in the February 1972 issue of the MONTHLY, the year when the Award for Distinguished Service to Mathematics was presented to him. Of course, he was President, Editor of the MONTHLY, and involved in every phase of the Association's programs. In recent years, he was primarily concerned with the Committee on Educational Media, producing films and television programs not only for universities and colleges, but also for schools.

It can now be told — he wished it to remain anonymous during his lifetime — that over a period of years close to $25,000 has been contributed by Professor Allendoerfer principally to the Greenwood Fund, which has been used for the production of films by the Association.

I last saw Carl during the first part of September last year, when my wife and I stopped in Seattle for a visit on the way back from Vancouver. He was very pitiful physically, but through his eyes and his labored breath, the old Carl was there. He was concerned even to the last moment with the affairs of the Association.

It is gratifying to find that the presence of Professor Allendoerfer is felt at this meeting— two nights ago when the Allendoerfer films were shown, there was standing room only. Yesterday at Professor Alder's retirement luncheon, the name of Professor Allendoerfer was mentioned on several occasions.

It is my special honor to announce that Professor Allendoerfer has provided in his will for a bequest of approximately $50,000 to the Association to be used in the most appropriate way. I hope that this bequest will be regarded not only as money to be used, but also as a symbol of Carl's devotion to the affairs of mathematics and to the Association, and that it will be an inspiration to us all."

At the conclusion of Professor Jones' remarks, President Boas asked the audience to stand in memory of Professor Carl B. Allendoerfer.

The Association's Fourteenth Award for Distinguished Service to Mathematics was made to Professor Saunders Mac Lane of the University of Chicago. The citation (which appears on pages 107–108 of the February issue of this MONTHLY) was prepared and read by President Boas. Professor Mac Lane, in accepting the Award, stated that he felt greatly honored and deeply touched by the citation which his old friend had prepared. He continued: "We all feel that the Association has been a tremendous force to hold collegiate mathematics together and to encourage better teaching; all of us have worked together for that end." He expressed his great pleasure in working with President Boas, both at Harvard and more recently when both were Presidents of their societies. He then introduced Professor Garrett Birkhoff, who had jointly authored with him the textbook *Survey of Modern Algebra.*

Professor Mac Lane suggested that there were still many things to be done. He expressed his appreciation to the Association and its members for the honor which had come his way.

A specially-bound copy of the citation was presented to Mrs. Mac Lane as a token of appreciation for her continuous support of her husband's multitude of activities.

The tenth set of Lester R. Ford Awards was presented by President Boas to authors of expository articles published in the MONTHLY and MATHEMATICS MAGAZINE in 1973. The Awards, in the amount of $100 each, were presented for six articles (for further details on these Awards, see the February issue of this MONTHLY, pages 208–209).

The Chauvenet Prize for 1975 was awarded to Professor Martin Davis of the Courant Institute of Mathematical Sciences and Professor Reuben Hersh of the University of New Mexico for their joint paper *Hilbert's 10th Problem*, which appeared in the SCIENTIFIC AMERICAN, 229, No. 5 (November, 1973), 84–91. Further details concerning this Prize and its recipients appear on pages 108–109 of the February issue of this MONTHLY.

Professor Davis, speaking in behalf of both recipients, expressed himself as being rather overwhelmed by the occasion. He had been fortunate enough to receive a LeRoy P. Steele Award at the AMS meeting a few days ago, and now a Lester R. Ford Award and shared the Chauvenet Prize for 1975, which made him feel humble and gave him a great deal of pleasure. Professor Davis recalled two of his teachers from the time he was an undergraduate at New York University: Professor Bennington Gill, whose enthusiasm for mathematics was so apparent that he inspired many, and the teacher who had the greatest influence upon him, Professor Emil Post, an unusual and remarkable mathematician, who was not only an inspiring and gifted teacher and mathematician of the first rank, but who, in spite of the conditions under which he worked (he regularly had a sixteen-hour teaching load, no office in which to work), nevertheless managed to produce research. In addition, he had to suffer the rewards of one whose ideas are just a little bit ahead of his time.

Professor Davis continued: "It is particularly important in these rather difficult times for mathematics that we always remember that the enthusiasm and passion for mathematics among talented young people is a precious resource for our subject, and we should all try in our contacts with young people to encourage these feelings as much as it is in our power to do so."

President Boas then read a resolution of appreciation, approved by the Board of Governors, for Professor Howard Eves' long and devoted service as Problem Editor of

this MONTHLY. Since Professor Eves was unable to be present, the specially-bound copy of this resolution will be sent to him.

President Boas reported that the Secretary, Professor H. L. Alder, had been honored at a luncheon the previous day upon his retirement from fifteen years' service in that position. He added that during this period he had kept 7-1/2 presidents of the Association out of trouble by remembering everything or producing the necessary information from his files. He then announced that the Board of Governors had voted to make Professor Alder an Honorary Life Member of the Association and elected him Secretary-Emeritus. He presented him with a plaque certifying these two actions of the Board.

The Secretary expressed his deepest appreciation to the Board for this unexpected and most welcome honor.

He then presented his report as Secretary, announcing first the results of the balloting for Governors in which 1163 votes were cast. Professors Charles Hatfield of the University of Missouri at Rolla and Mary B. Williams of Ohio State University were elected Governors for the three-year term 1975–77.

The Secretary reported on some of the actions taken by the Board of Governors. He expressed the hope that the great expansion in privileges of academic members, with only a nominal increase in dues, will make it possible for essentially all institutions with mathematics departments to become academic members.

The summer meeting of the Association this year will be held August 18–20, 1975, at Western Michigan University in Kalamazoo. The next annual meeting will be held on January 24–26, 1976, in San Antonio, Texas. This meeting will be devoted, at least in part, to the celebration of the Bicentennial of the United States. The program will be somewhat special and is already in the active planning stage. The following annual meeting will be held in St. Louis, Missouri, January 29–31, 1977.

The Secretary took the opportunity to express the Association's deepest appreciation to the local Commitee on Arrangements for their dedicated work in planning and preparing for this meeting so energetically and thoughtfully. He expressed special gratitude to Professor Hewitt Kenyon, Chairman of the Committee, and Professor Choy-tak Taam, Publicity Director.

The Secretary then moved an amendment to the By-Laws providing for an Editor of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL to be added to the membership of the Board of Governors by replacing in Article III, Section 2 (listing the membership of the Board) "the Editor of its publication entitled MATHEMATICS MAGAZINE" by "an Editor of each of its two publications entitled TWO-YEAR COLLEGE MATHEMATICS JOURNAL and MATHEMATICS MAGAZINE," and by adding in Article IV, Section 1(c), after "Monthly": "an Editor of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL." The motion carried.

The Secretary then moved to change all references in the By-Laws from "ordinary" members to "individual" members. (This occurs in Article II, Sections 1, 2, and 4, Article VII, Section 5, and Article VIII, Sections 1 and 4.) The motion carried.

MEETING OF SECTION OFFICERS

The representatives of the Sections met on Saturday evening in the Tudor Room of the Shoreham Hotel. Professor L. E. Mehlenbacher, Chairman of the Committee on Sections, presided for the first part of the meeting. Fifty-seven persons were present representing twenty-six of the twenty-eight Sections.

President Boas welcomed the officers and representatives of the Sections, emphasizing that, to a large extent, the Sections are the Association. He observed that the Section Governors constitute the majority of the Board of Governors and that they do what the Sec-

tions tell them. They, thereby, are the mechanism through which the wishes of the membership are made known. He urged the Sections to avail themselves of this opportunity. He suggested that if the Sections wish to increase the input into the Association from two-year colleges, applied mathematics, etc., it is pretty much up to them to do so.

President-Elect Pollak also welcomed those in attendance, and expressed a wish to visit every Section, if possible, in the next two years, observing that he had already visited several. It was his desire to find out at these Section meetings what the interests and concerns of the Sections are.

Dr. A. B. Willcox, Executive Director, reported on three developments designed to improve the services the MAA performs for its members and the collegiate mathematical community, namely making the TWO-YEAR COLLEGE MATHEMATICS JOURNAL an official journal of the MAA, effective January 1, 1976, increasing the tangible benefits of academic membership, and taking over the publication of the NEW MATHEMATICAL LIBRARY, the series of expository books started by SMSG and more recently published by Random House, Inc.

The Executive Director expressed the hope that the Section Officers will continue their excellent efforts to interest two-year college teachers of mathematics in Section activities, inviting those not already members to attend Section meetings and to join the MAA.

Professor H. L. Alder, the retiring Secretary, singled out as the most important thing he had learned during his fifteen years as Secretary, that the Sections are the lifeblood of the Association, that is, the overall vitality of the Sections determines the effectiveness of the MAA as an organization. For this reason, he always attached special importance to these meetings of Section Officers and found particularly valuable his visits to the Sections. He regretted that he had not been able to carry out his intention to visit all Sections, but he had visited enough Sections to realize the great benefit of these contacts to the officers and hopefully vice versa.

He recalled his own involvement in the affairs of the Association began after he had attended a particularly fruitful meeting of Section Officers as Chairman of the Northern California Section, when many worthwhile activities were discussed of which he had never heard before. Most of these were adopted in that Section and have been carried on ever since.

After becoming a national officer, he found that one of the most valuable sources for ideas for national activities was the set of suggestions made at meetings of Section Officers. He gave numerous examples of ideas, first made at these meetings, which have been implemented. He hoped that the Section Officers would continue to let the national officers have the benefit of their advice and, at the same time, that the Section Officers would implement whatever worthwhile suggestions they hear at these meetings. Only in this way will the Association continue to be able to play so effectively the vital role in American collegiate education which it does today.

Professor Alex Rosenberg, Editor of this MONTHLY, reported that the first six issues of 1975 will carry Main Articles dealing with Computer Image Enhancing, History of Mathematics, Mathematical Physics, Logic, and more standard mathematics. Among articles already accepted or being revised are some dealing with Neurobiology, Control Theory, Mathematical Psychology, Operations Research, and Mathematical Political Science. Thus, Professor Flanders' efforts for broad coverage are being continued.

He reported some slight cosmetic changes designed to save money will gradually come into effect during 1975. For the same reason, beginning with the 1975 volume, authors are being charged for reprints. About eighty per cent of the material that comes to the Editor for Main Articles is rejected, and the rejection rate for other sections is at least this high. Even so, the backlog for all sections runs from a year and a half to two years.

Professor D. W. Bushaw, Chairman of the Committee on the Undergraduate Program

in Mathematics (CUPM), briefly discussed present and projected CUPM projects. Among the former are several collections of materials (fresh and realistic applications of precollege mathematics; applications of undergraduate mathematics in the social sciences; and instructional materials in applied mathematics for advanced undergraduates). A CUPM group has just been formed to bring up to date the *Basic Library List.*

Possible future projects now in various stages of consideration include: activities related to continuing education (especially for elementary teachers and for scientists and engineers in industry); a compilation of applications of undergraduate mathematics in biology; and a curriculum in the mathematics of energy systems.

The forthcoming appearance of a *Compendium* of earlier CUPM documents was announced.

The Section Officers were reminded that questions, suggestions, and comments about CUPM activities in general or in particular are always welcome.

Professor M. D. Thompson reported that the CUPM project to prepare instructional materials for use in teaching applied mathematics at the undergraduate level is nearing completion. The objectives of the project were to prepare materials which illustrate the entire process of applied mathematics and to test these materials in both the traditional lecture format and in a very open-ended instructional setting. The materials consist of nine modules and an Introduction. Eight of the ten units were tested during the second half of the 1973–74 academic year by seven instructors. Most instructors tested two or three units. In June 1974, a conference was held to review the results of the tests. The materials, together with a commentary on the experiences of the preliminary tests, will be published by CUPM in late 1975.

Dr. H. M. Cox, Executive Director of the Annual High School Mathematics Examination, announced that the date for the 1975 examination is Tuesday, March 11. He thanked the sixty contest chairmen for continuing to do such a fine job. He reported that the 1974 examination was designed to be somewhat easier than other recent examinations and that the scoring formula was simplified. As a result, there were some 850 contestants listed on the Honor Roll, a larger number than in any previous examination. A report on the 1974 Examination was sent out last August. Additional copies are available upon request. It was possible to prepare from the 1974 Examination an anectodal/statistical study on honor roll students. This report will be presented at the Denver meeting of NCTM in April.

Dr. Cox announced that, as of January 24, 1975, 331,000 copies of the 1975 Examination had been shipped.

Professor E. F. Beckenbach, Chairman of the Committee on Publications, observed that this Committee is striving to produce usable materials for all segments of the membership. He reviewed the journals, serial publications and miscellaneous publications now published by the MAA. He sought the cooperation of the Sections in the distribution of these publications and in the formulation of their content. He urged that Section Officers write him if they feel that the publication program should be redirected. He suggested that the entire set of MAA publications be available in each Section and someone be put in charge of displaying this set at each Section meeting.

Professor Mehlenbacher then announced that his term as Chairman of the Committee on Sections expires at the end of this meeting and introduced Dean L. H. Lange as his successor. Dean Lange presided for the remainder of the meeting.

Professor C. A. Cable, Chairman of the Allegheny Mountain Section, reported on "An Alternative Program for Undergraduates at Sectional Meetings." Such a program was designed primarily by and for undergraduates for the May, 1974 Section meetings at Allegheny College. This consisted of two main sessions and a section for fifteen-minute student papers for the Friday night part of the program. On Saturday morning the students attended the regular sessions of the program along with the faculty members of the MAA.

Their first session was a panel discussion which was entitled "Job Opportunities for Mathematics and Computer Science Majors." The panel participants were an applied mathematician from Westinghouse Electric Corporation in Pittsburgh, an accountant discussing recent computer uses in bank accounting from the Mellon Bank in Pittsburgh, and a person in charge of research in IBM. The panel participants were most enthusiastic and did an extremely good job. This session was concluded by a very spirited question and answer period. The only cost for these speakers was room and lodging from one night.

The second event of the student schedule was a panel discussion entitled "Graduate School Programs in the Allegheny Mountain Section." Each university having a graduate school program within the sectional boundaries was invited to send a representative to participate on this panel. Each of these panelists discussed items from a questionnaire which they had received about two weeks earlier. This session also concluded with a stimulating question and answer period.

In addition to planning and organizing the student schedule part of the program, the student committee also accepted the responsibility for arranging for dormitory accommodations for all students who desired them. Students from other institutions desiring accommodations were asked to notify this committee a couple of weeks before the meetings and to bring their own sleeping bags.

All in all, the students have done a very commendable job in organizing and carrying out a very worthwhile alternative program, and they are being encouraged to continue.

Professor I. H. Rose, Chairman of the Metropolitan New York Section, reported on three special projects of that Section:

1. The Speakers' Bureau, now in its fifteenth year of supplying speakers (mainly to high school student and teacher groups). For many years successful, the program is now in trouble. Economic and social changes have cut deeply into both the supply of willing lecturers and the demand for their services. Attempts are now being made to revive interest and support by placing greater emphasis on talks having to do with applications and careers.

2. The Math Fair, now in its seventh year. The Fair awards medals for mathematical papers presented by high school students in two all-day sessions. In all its aspects, this project is highly successful.

3. The Math Services Committee, just getting off the ground (with the help of a $300 grant from the MAA). The Committee plans to act as a clearinghouse and information service for the dissemination of information regarding remedial and other types of collegiate mathematics programs. Specific projects will include a periodic newsletter and a local visiting lecturer program for colleges. Preliminary meetings, attended by representatives of more than twenty junior and senior colleges, have been held.

Professor H. K. Stumpff, Secretary-Treasurer of the Missouri Section, reported on that Section's program of visiting lecturers to secondary schools. This activity is coordinated by Professor C. J. Stuth of Stephens College in Columbia, Missouri. At the beginning of the fall semester each year, he contacts professors in all colleges and universities in Missouri requesting volunteers to serve as possible lecturers. At the time they volunteer, each professor is asked to list the titles of the talks he or she is willing to give. A brochure is then prepared and mailed to all Missouri high schools. The coordinator receives all requests for speakers (by name and title of talk) and subsequently screens these requests according to the distance to be traveled and the frequency with which a lecturer has been requested. (Hopefully, no lecturer will find it necessary to respond more than twice in any one semester.) Schools are asked to submit second choices for lecturers along with several alternate dates to facilitate planning. Since the lecturer and his institution pay the travel expenses, high schools are asked to request speakers from nearby institutions. The high schools pay no fees of

any kind other than providing a meal or other appropriate hospitality. The length of the visit may vary from a short one hour talk to a full day visit, which may include lectures and discussions with various groups in the school system.

The entire program operates on a meager budget provided by the Missouri Section through its accumulated State Contest funds. Although still in its infancy, the program is growing. During the two semesters of operations, a total of 18 lecturers have filled 21 of the 29 requests received. The program seems successful enough to warrant continued support by the Missouri Section.

Professor R. H. Rolwing, Secretary-Treasurer of the Ohio Section, reported on the two-week conferences for faculty members held in that Section last summer at the Ohio State University and Kent State University. They were variously referred to as "short courses," "conferences," "workshops," and "seminars." Both were well received and highly successful. A total of some sixty persons attended one or the other of the sessions with the greater number, some 40 participants, at Ohio State. The range of participants was from high school teachers (two) to Ph.D. granting institutions. Most, however, were from 2 and 4 year colleges. One person from the Detroit area which is outside the Section attended both conferences.

The conferences were planned in response to a number of requests from Ohio Section members for the Committee on Cooperation between Colleges and Universities to encourage and/or organize this type of conference. In January of 1974, the OHIO SECTION NEWSLETTER included a questionnaire designed to determine what kinds of conferences or workshops would best meet the needs of the members. There were 36 replies, including suggestions for a seminar on applications of elementary mathematics and a two-week algebra or analysis course, with problem seminars, at the master's degree level.

The first conference on Combinatorics was held June 10–14 at Ohio State University and the second, June 17–26 on Numerical Analysis at Kent State University. The purpose of these seminars was to provide undergraduate teachers with an introduction to the basic ideas of combinatorial theory and numerical analysis. No previous experience was assumed. The only expense to the participants was for room, board, and travel. Dormitory space was made available by the two universities at $5–6 per day per person, double occupancy.

Professor Norman Levine of Ohio State and Professor Donald Koehler of Miami University were in charge of arrangements. The lectures on Combinatorics were given by Professors D. K. Ray-Chaudhuri, Richard Wilson, Thomas Dowling, and Neil Robertson, all of Ohio State, while the entire conference on Numerical Analysis was conducted by Professor James Dailey of Kent State. These Section members generously donated their time and expertise and a vote of thanks is certainly in order. There were many fine complimentary remarks by the participants about the quality of the presentation and the anticipated value for undergraduate courses.

The Committee on Cooperation between Colleges and Universities carried out an evaluation of the conferences to aid in planning future conferences. Among the goals of the program which were successfully achieved was an increased cooperation between large schools and small schools.

The success of last summer's conferences encouraged the Committee on Cooperation between Colleges and Universities to plan a similar workshop for the coming summer. It will be held in northeast Ohio in June. The topic will be Operations Research/Mathematical Programming.

Professor Theodore White, Chairman of the Pacific Northwest Section, reported on the two-year college program, started in that Section in 1966 with special sessions of interest to community college teachers. In 1967, a Vice-Chairman for two-year colleges was appointed. His prime responsibility is the two-year college program at Section meetings.

These meetings last for two days, one of which is devoted to four-year college interests, the other to community college interests. Two-year college participation has increased since 1966. There is a good representation of four-year college persons at the two-year sessions. The Section financially supports the two-year college sessions at the Northwest Mathematics Conference held each fall. This is done because much of the curriculum of the community colleges is elementary and secondary in nature, and this conference is primarily designed for such people.

In response to a question by Professor D. J. Albers regarding what the impact of the TYCMJ had been on two-year college teachers in the Pacific Northwest, Professor White replied that several colleagues who were not members of the MAA had stated their intent to join if the motion to make the TYCMJ an official journal of the MAA were approved by the Board of Governors the previous day.

Professor J. W. P. Mayer, Chairman of the Philadelphia Section, distributed the results of a survey he had made after the 1973 meeting of the Section, which had put more than usual weight on the applied side of mathematics. 68% found the talks stimulating, 48% favored a second annual meeting, 36% did not. However, if the spring activity were a workshop, 72% would attend. For this workshop, 67% of those planning to attend favored one on Mathematics in the Social Sciences, 50% one on Computers, 40% one on Operations Research. A surprising 76% were willing to serve as resource teams in these and other areas.

Professor P. A. Lindstrom, Second Vice-Chairman of the Seaway Section, reported on two recent activities of that Section. The fall 1974 Section meeting dealt with a single topic, Linear Algebra, including its history, research in the area, applications, the computer, and a first course in linear algebra. Although other Sections have tried the one topic theme for a Section meeting, this was the Seaway Section's first such meeting. He also reported on the results of two questionnaires he had sent out in 1974. One was sent to the two-year college members of the MAA in the Seaway Section asking them what they would like to contribute to the Seaway Section and the MAA and what they would like the Seaway Section and the MAA to do for them. The other questionnaire was sent to at least one officer of all Sections asking in what ways their Section has helped the two-year college people and what support their Section had received from the two-year college people.

Professor R. H. Owens, Governor of the Maryland-D.C.-Virginia Section, felt that announcements of Section meetings in the MONTHLY are unnecessarily succinct, at least for their Section. Normally they meet in April and November, but details rarely reach the membership early enough. Often these details are not worked out until a few weeks prior to the meeting for a variety of very good reasons. However, meeting notification does not require details (national meetings are always known long before the program is distributed!). Consequently, the MONTHLY could be of genuine service to the Sections by including items similar to the following:

> Va.-Md.-D.C.: Meetings are held normally on the last Saturday in April and the Saturday before Thanksgiving. Deadline for contributed papers — three weeks before the meeting. Time and place of meeting to be announced.

For many educators, the Section meetings are as important as the national meetings. For young college teachers and even graduate students, these meetings often provide the only forum for presenting a paper on their current interests, and they need a long lead time for preparing a paper. Detailed information, which reaches the recipients two or even one week prior to the meeting, simply does not permit the necessary planning.

The Secretary thought Professor Owens' suggestion an excellent one and urged all Sections to inform Dr. Raoul Hailpern, Editorial Director, as soon as possible when their

meetings are normally held, so that he can use this information in the MONTHLY when he has no specific date for the next Section meeting available.

Professor E. K. McLachlin, Secretary-Treasurer of the Oklahoma-Arkansas Section, reported that his Section plans meetings three years in advance and suggested that other Sections may wish to consider planning meetings for more than one year in advance.

Professor Yousef Alavi, Chairman of the Michigan Section, reported on that Section's visiting high school lecturers' program, which now involves 20 lecturers and last year reached 1200 students. The Section was selected to undertake an Oxford seminar type workshop to acquaint mathematicians and mathematics students with problems from industry. This workshop will be held at General Motors Corporation in Flint, Michigan. A summer seminar was held at Lake Superior, with fifty people participating. This will become an annual event. The idea for this seminar originated from the Ohio Section summer conferences. Lecture notes will be available.

The Section has introduced a small undergraduate activity at Section meetings. They have found MATH CLIPS very informative and now have a MICHIGAN SECTION NEWSLETTER.

The Secretary urged that the officers and the central office of the MAA be placed on the mailing list for Section newsletters. Professor Leonard Gillman, Treasurer of the MAA, suggested that these newsletters be sent also to all other Section Officers. This will be particularly helpful now that Section Officers' meetings are held only every eighteen months.

Professor Mayer then presented a series of suggestions to the MAA, in particular that

1) the Nominating Committee actively solicit suggestions for nominations;
2) more non-university mathematicians be on all national committees;
3) more attention be given to applied mathematics (computer science, operations research, etc.) when planning national meetings;
4) the national meetings of the MAA be held alone, or together with ORSA, SIAM, AMS, etc. in rotation;
5) the MONTHLY more clearly reflect the needs of the undergraduate teacher of mathematics by having many more articles on applied mathematics, bibliographies of new areas of mathematics, etc.
6) the MONTHLY periodically publish a list of all the reports to the various bodies of the MAA, together with the addresses of the authors.

President Boas responded by noting that the membership of the Nominating Committee is always made known, and anyone can make suggestions to that Committee.

Professor Albers expressed the belief that the MONTHLY had made enormous strides in serving the membership of the MAA.

The Executive Director called attention to the MATH CLIPS, which are produced by the Washington office. They are designed to get information to Section Officers, Governors, Officers of the MAA, chairmen of standing committees, a total of about 250 people. These clips come from newspapers, Section newsletters, and other publications which cross the desk of the Executive Director. He requested Section Officers to send him items of particular interest to Sections for inclusion in MATH CLIPS.

Professor D. J. Albers, Chairman of the Northern California Section, reported his Section plans to sponsor a week-long meeting for two-year college faculty, June 16–20, 1975. An arrangement has been worked out with the Continuing Education Office of the University of Santa Clara to have the meeting on the campus of the University and to award two quarter units of credit through the Continuing Education Office. The formal part of the week's activities will include two short courses, each course meeting daily for one hour during the five mornings of the conference. One of these courses, taught by Professor K. R. Rebman of California State University, Hayward, will be on the applications of mathematics

in the biological and social sciences. The second, taught by Professor G. L. Alexanderson of the University of Santa Clara, will cover special problems drawn from number theory, geometry, and combinatorics. In addition to these short courses, there will be special lectures, a minimum of four, in the afternoon by prominent mathematicians, including Professor G. Polya, on a variety of topics.

Professor D. E. Edmondson of the Texas Section reported that his Section was undertaking a study to see what is happening to mathematics enrollments in four-year colleges. He noted that his institution had been losing mathematics majors at a rate of ten to fifteen per cent a year for the past four years, which, however, had been counterbalanced by a growth in enrollments of business and biological science students.

Professor Rosenberg cited the report of the Office of Education which came out in April and reported a reduction of only two to three per cent in the number of mathematics majors.

Professor Mayer reported on the results of his survey which showed that one out of five institutions reported a decline in enrollment in mathematics majors. He suggested a survey by the MAA. Professor G. B. Price agreed that such a survey would be very useful. After further discussion, the Executive Director suggested that the survey be incorporated into the questionnaire sent to departments for the purpose of gathering information for the GUIDEBOOK TO DEPARTMENTS IN THE MATHEMATICAL SCIENCES. Since a new edition has just been prepared, the next opportunity to gather this information will not arise until two years hence; he promised, however, to convey the suggestion for such a survey to the Committee on Advisement and Personnel.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Thursday, January 23, to Sunday, January 26. The forty-ninth Josiah Willard Gibbs Lecture was delivered by Professor Fritz John of the Courant Institute of Mathematical Sciences, New York University, on "A Priori Bounds, Geometrical Effects, and Asymptotic Behavior" on Thursday at 8:30 P.M. in the Regency Ballroom. The Retiring Presidential Address was given by Professor Saunders MacLane of the University of Chicago on "Topology and Logic as Sources of Algebraic Ideas" on Friday at 2:00 P.M. in the Regency Ballroom.

A set of Colloquium Lectures was given by Professor H. J. Keisler of the University of Wisconsin, Madison, entitled "New Directions in Model Theory" on Thursday, Friday, Saturday, and Sunday at 1:00 P.M. in the Regency Ballroom.

The Committee on Employment and Educational Policy presented two panel discussions. The first, on "Seeking Employment Outside Academia: Views From Some Who Have Recently Succeeded," was held on Friday at 8:30 P.M. in the Regency Ballroom. It was moderated by Professor Martha Kathleen Smith of the University of Texas, Austin. The second panel discussion on "Public Science Policy" was held on Saturday at 8:30 P.M. in the Regency Ballroom. It featured Dr. H. Guyford Stever, Director of the National Science Foundation.

The AMS presented a two-day Short Course on Mathematics in Operations Research on Tuesday and Wednesday, January 21 and 22, in the Palladian Room. The program was under the direction of Dr. A. J. Goldman, Chief, Operations Research of the National Bureau of Standards. The speakers and their topics were: Dr. Gordon Raisbeck, Vice-President (and Head of Physical Systems Research), Arthur D. Little, Inc., "Mathematicians in the Practice of Operations Research"; Dr. Christoph Witzgall, Senior Mathematician, National Bureau of Standards, "Linear Programming and Flows in Networks"; Dr. R. E. Gomory, Vice-President and Director of Research, IBM, T. J. Watson Research Center, "Stock Cutting and Its Ramifications: Mathematical Operations Research in Industry";

Professor Frank Proschan, Florida State University, "Mathematical Theory of Reliability, with Applications;" Professor A. F. Veinott, Jr., Stanford University, "Lattice Programming and Inventory Theory"; and Dr. C. M. Harris, Program Director, RMC Research Corporation, Bethesda, Maryland, (Professor on leave from George Washington University), "Queueing Theory and Applications: Some Mathematical Frontiers."

AMS invited addresses were given as follows, all in the Regency Ballroom:

Lie Groups and Differential Equations, Professor Sigurdur Helgason, Massachusetts Institute of Technology, Thursday, 9:00 A.M.

Representations of Semisimple Lie Groups, Professor Wilfried Schmid, Columbia University, Thursday, 10:15 A.M.

Riemann Surfaces and the Moduli Problem, Professor Linda Keen, Graduate School and University Center, City University of New York, Thursday, 2:15 P.M.

Some Recent Discoveries in the Isomorphic Theory of Banach Spaces, Professor H. P. Rosenthal, University of Illinois, Urbana-Champaign, Thursday, 3:30 P.M.

A Survey of Current Non-Quantum General Relativity, Professor R. K. Sachs, University of California, Berkeley, Friday, 9:00 A.M.

A Representation Theoretic Proof of a Formula of Max Noether, Professor N. R. Wallach, Rutgers University, New Brunswick, Friday, 10:15 A.M.

Applications of Homological Algebra to Topology, Professor D. W. Anderson, University of California, San Diego, Saturday, 2:15 P.M.

Martingale Methods in Analysis, Professor D. L. Burkholder, University of Illinois, Urbana-Champaign, Sunday, 2:15 P.M.

The Association for Symbolic Logic met on Thursday and Friday, January 23 and 24. Invited addresses were given by Professor R. A. Shore of Cornell University on *α-Recursion Theory* on Thursday at 10:30 A.M., Professor Alexander Kechris of the California Institute of Technology on *Descriptive Set Theory* on Thursday at 3:45 P.M., and Professor G. L. Cherlin of the Massachusetts Institute of Technology on *Model Theoretic Algebra* on Friday at 10:30 A.M., all in the Tudor Room.

The Conference Board of the Mathematical Sciences sponsored a panel discussion on "Wide-Ranging Applications of Statistics" on Saturday at 2:00 P.M. in the Palladian Room. The session was planned under the direction of Dr. Joan R. Rosenblatt, Chief, Statistical Engineering Laboratory, National Bureau of Standards. Participants in the panel and their topics were Dr. J. B. Kruskal, Bell Telephone Laboratories, "You Have to Really *Like* Applications to Do Them Well"; Dr. M. E. Muller, Director, Computing Activities, World Bank, "Multi-Dimensional Economic Time Series," and Dr. Marvin Zelen, Director, Statistical Laboratory and Professor of Statistical Science, SUNY at Buffalo, "Statistical Science and Biomedical Applications."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of Hewitt Kenyon, Chairman; H. L. Alder, Ruth A. Bari, J. A. Donaldson, I. I. Glick, W. H. Gottschalk, J. C. Owings, William Swyter, Choy-Tak Taam, Juanita S. Tolson, G. L. Walker.

Registration headquarters were located in the upper lobby of the Shoreham Hotel. The Mathematical Sciences Employment Register was maintained from 9:00 A.M. to 4:00 P.M. on Friday, January 24, with interviews scheduled from 9:00 A.M. to 5:40 P.M. on Saturday, Sunday, and Monday. The Register was located in the Empire Room of the Shoreham Hotel. Books and educational media exhibits were displayed in the Ambassador Room of the Shoreham Hotel from noon to 5:00 P.M. on Thursday and from 9:00 A.M. to 5:00 P.M. on Friday, Saturday, and Sunday.

HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 wks. bef. metg.

FLORIDA, early March. Deadline for paper titles 2 wks. bef. mtg.

ILLINOIS, second Friday/Saturday in May.

INDIANA

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI

NEBRASKA, April.

NEW JERSEY

NORTH CENTRAL

NORTHEASTERN, University of Connecticut, Storrs, June 20–21, 1975.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST

PHILADELPHIA, Franklin and Marshall College, Lancaster, November 22, 1975.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers 8 wks. bef. mtg.

SEAWAY

SOUTHEASTERN, Central Piedmont Community College, Charlotte, North Carolina, Spring 1976.

SOUTHERN CALIFORNIA

SOUTHWESTERN, usually in April. Deadline for papers 2 wks. bef. mtg.

TEXAS

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION, California State University, San Francisco, October 18, 1975.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, MGM Grand Hotel, Las Vegas, November 17–19, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Rensselaer Polytechnic Institute, Troy, New York, June 9–11, 1975.

Successful leaves from our tree

Programming and Probability Models in Operations Research by Donald P. Gaver, Naval Postgraduate School, and Gerald L. Thompson, Carnegie-Mellon University, emphasizes the construction of mathematical models and their manipulation to accomplish desired goals. Full of examples, exercises, and problems illustrating various models and methods. 690 pages. Cloth.

The Analysis and Solution of Partial Differential Equations by Robert L. Street, Stanford University, gives your students a clear perspective to relations between analytic and numerical methods, and between qualitative and quantitative methods and results. Over 500 carefully chosen problems stress solutions of physically reasonable systems. 480 pages. Cloth.

Explorations in Number Theory by Jeanne LeCaine Agnew, Oklahoma State University, covers number theoretic functions in a style that *involves* the student. 288 pages. Cloth.

An Introduction to Abstract Algebra by A. Richard Mitchell and Roger W. Mitchell, University of Texas, Arlington, emphasizes algebraic structure and the natural aspects of the subject, helping to develop intuition. 302 pages. Cloth.

Modules: A Primer of Structure Theorems by Tom Head, University of Alaska, Fairbanks, and University of Texas, El Paso, is ideally suited to a follow-up course in abstract algebra. 192 pages. Cloth.

Differential Equations and Related Topics for Science and Engineering by Robert W. Hunt, California State College, Bakersfield, provides the student with understandable and usable methods for grasping primary ideas involved in solving problems with applications in science and engineering. 256 pages. Cloth.

Number Theory: An Introduction to Algebra by Fred Richman, New Mexico State University, solves problems with algebraic tools, and then interprets the answers back into number theory. 188 pages. Cloth.

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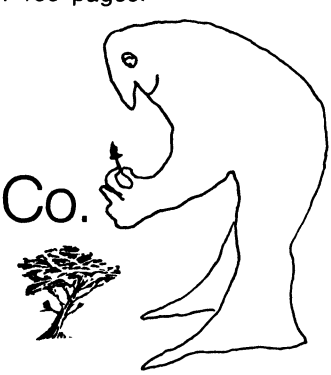

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(*Continued on inside cover*)

JUNE–JULY 1975

NOTICE TO AUTHORS

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SIGNED DIGRAPHS AND THE ENERGY CRISIS

FRED S. ROBERTS AND THOMAS A. BROWN

1. Introduction. Recently, the seriousness of the "energy crisis" has been increasingly revealed. Attempts to understand the patterns of energy use and the effects of various energy conservation strategies require the understanding of an extremely complex system. Such a system involves many variables interacting with each other, reacting to changes in each other, and so on. In attempting to model such a complex system, one faces a tradeoff between the accuracy of the model's predictions and the ability to obtain the detailed information needed to build the model. In this paper, we describe one way to model complex systems, such as those underlying the energy crisis, which is based on a minimal amount of information about the system.

It is useful to divide methodologies for analyzing complex systems into two types, the arithmetic and the geometric. (Cf. Kane [3], Kane, Vertinsky, and Thompson [5], and Coady, Johnson and Johnson [1].) Arithmetic methodologies tend to be numerical and precise, and usually aim at the optimization of a few parameters. They tend to be present-oriented and relatively sensitive to change or modification of the basic parameters. Geometric methodologies tend to be rather non-numerical, and take account of variables which are not readily quantifiable. Their aim is an analysis of structure and shape, and especially of changing patterns of structure which may have different ramifications for the future. The models we shall discuss are geometric in nature and make use of signed and weighted directed graphs.

In Sections 2 and 3 we describe the models in general terms. Section 4 introduces a specific rule for change of value of the variables. Sections 5 and 6 give criteria for a system to be stable under external pressures. Finally, the results are applied to specific energy questions in Section 7.

2. Signed digraphs. We begin by recalling some elementary definitions of graph theory. A directed graph or **digraph** D consists of a set V called the **vertices** and a subset A of $V \times V$ called the set of **arcs**. Our digraphs may have **loops**, that is, arcs of the form (x, x). A **signed digraph** consists of a digraph together with an assignment of a sign $+$ or $-$ to each arc. Following Harary, Norman, and Cartwright [2] we say that a **sequence** in a digraph D is a sequence of vertices $x_1, x_2, \cdots, x_t$ so that for all i, (x_i, x_{i+1}) is an arc. The **sign** of a sequence is the product of the signs of its arcs, and the **length** of a sequence is the number of arcs in it. Finally, a **cycle** is a sequence $x_1, x_2, \cdots, x_{t-1}, x_t$ with $x_1, x_2, \cdots, x_{t-1}$ distinct and $x_t = x_1$.

The idea of studying energy demand and other environmental problems by means of signed digraphs was introduced in Roberts [9]. In such applications, the vertices of a digraph are taken to be variables relevant to the problem being studied (e.g., population, energy capacity in a given region, energy price, etc.). There is an arc from vertex x to vertex y if a change in x has a significant effect on y. This

arc is assigned a + sign if the effect is augmenting, i.e., if, all other things being equal, an increase (decrease) in x leads to an increase (decrease) in y; and a − sign if the effect is inhibiting, i.e., if, all other things being equal, an increase (decrease) in x leads to a decrease (increase) in y.

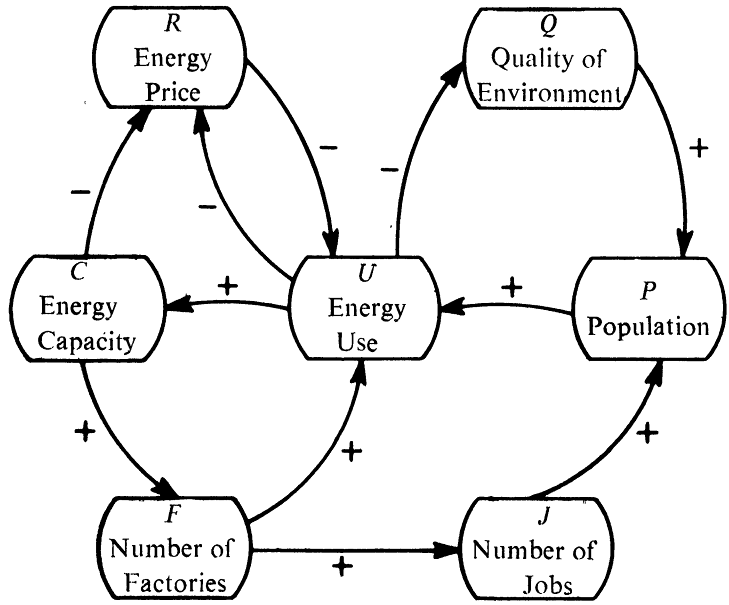

FIG. 1. Signed digraph for energy demand in electrical power.
From Roberts [9].

Fig. 1 shows a sample signed digraph for energy demand in electrical power usage in a given area, constructed for talking purposes. (As is conventional, the vertices are represented by ovals, and we draw an arrow from vertex x to vertex y if and only if (x, y) is an arc.) For example, there is an arc from population P to energy use U with a + sign because as population goes up, energy use goes up. There is a negative arc from energy use U to quality of (physical) environment Q because as energy use goes up, the quality of the (physical) environment goes down, as the result of increased smog, thermal pollution, etc.

The change of sign of an arc has an interesting interpretation. Consider for example the arc from energy use U to energy price R. According to the present system, this arc is −, because according to the present rate structure, the more you use, the less you pay (per kilowatt hour). It has been suggested that the rate structure should be inverted, and that large users should pay more rather than less. This strategy, known as **inverting the rate structure,** corresponds to changing the sign of arc (U, R) from − to +. In the same way, other changes in the signed digraph, in particular other changes of sign, might correspond to potential strategies for modifying the energy use system. We shall be interested in evaluating these strategies.

Often, a signed digraph is the most detailed mathematical model of a complex system attainable. This is true in particular if some of the variables cannot be measured, as for example the variable "environmental quality" in the signed digraph of Fig. 1. Difficulty of measurement is a property of many of the variables arising in societal problems. Even with an oversimplified model such as a signed digraph, there are still some precise conclusions which can be reached. For example, if Fig. 1 is an accurate model of the signs of effects in an energy demand system, then one can pinpoint certain augmenting subsystems. The cycle C, F, U, C corresponds to such a system. An increase in energy capacity C leads, through this subsystem, to an increase in the number of factories F, which in turn leads to more energy use U, which finally leads to a further increase in energy capacity C. An augmenting or positive feedback subsystem often contributes to instability, especially if there are many such subsystems present. (Sometimes inhibiting or negative feedback subsystems can create instability of another type, by contributing increasing oscillations.) The directed cycle C, R, U, C is another augmenting subsystem. For an increase in energy capacity leads, via this subsystem, to a decrease in price, which leads to an increase in use and hence to a further increase in capacity. It is easy to see that, in this figure, all subsystems containing the energy capacity vertex C and corresponding to (simple) cycles such as C, F, U, C are augmenting or unstable. This observation already makes precise, from a structural point of view, why the energy capacity system is so unstable. The signed digraph model is especially good for making such structural observations, for digraph theory has concerned itself over the years with just such notions of structure. (Indeed, one book about digraph theory, Harary, Norman, and Cartwright [2], is called *Structural Models*.)

3. Weighted digraphs. The signed digraph model has in it many oversimplifications. For example, some effects of variables on others are stronger than other effects. Thus, in Fig. 1, it seems clear that the effect of an increase in population P on energy use U is very strong compared to the effect of a decrease in quality of the environment Q on population P. The signed digraph model, however, assumes that all effects are equally strong, by placing unit ($+1$ or -1) weights on each arc. It might be more reasonable to place a different weight $w(x,y)$ on each arc (x,y) of a given digraph, thus yielding a **weighted digraph.** The weight is interpreted as the relative strength of the effect, and can be positive or negative. If each weight is an integer, we shall call the weighted digraph **integer-weighted.** Even more realistic than assigning a weight to each arc is to assume that the strength of an effect corresponding to the arc (x,y) changes depending on the levels of the variables x and y. But even weights are hard to estimate in practice, especially if the variables themselves are hard to measure or define.

Another omission in the signed digraph model is the time lag involved before a change in x has an effect on y. For example, an increase in population P will lead almost immediately to an increase in energy use U, while there is a time lag after an

increase in the number of jobs J before that attracts more population P to an area. The signed digraph model assumes that all effects take place in one unit time. Thus, a more realistic model would introduce a time lag corresponding to each effect. As with weights, time lags are hard to estimate, and there is a tradeoff between the generality of the model and the possibility of estimating its parameters (weights, time lags) in a realistic way. Also, there is a mathematical problem with the introduction of time lags. Specifically, analysis of the dynamic models — which we shall introduce later — becomes quite difficult with the introduction of time lags. For this reason, we shall discuss weighted digraphs, but we shall disregard time lags.

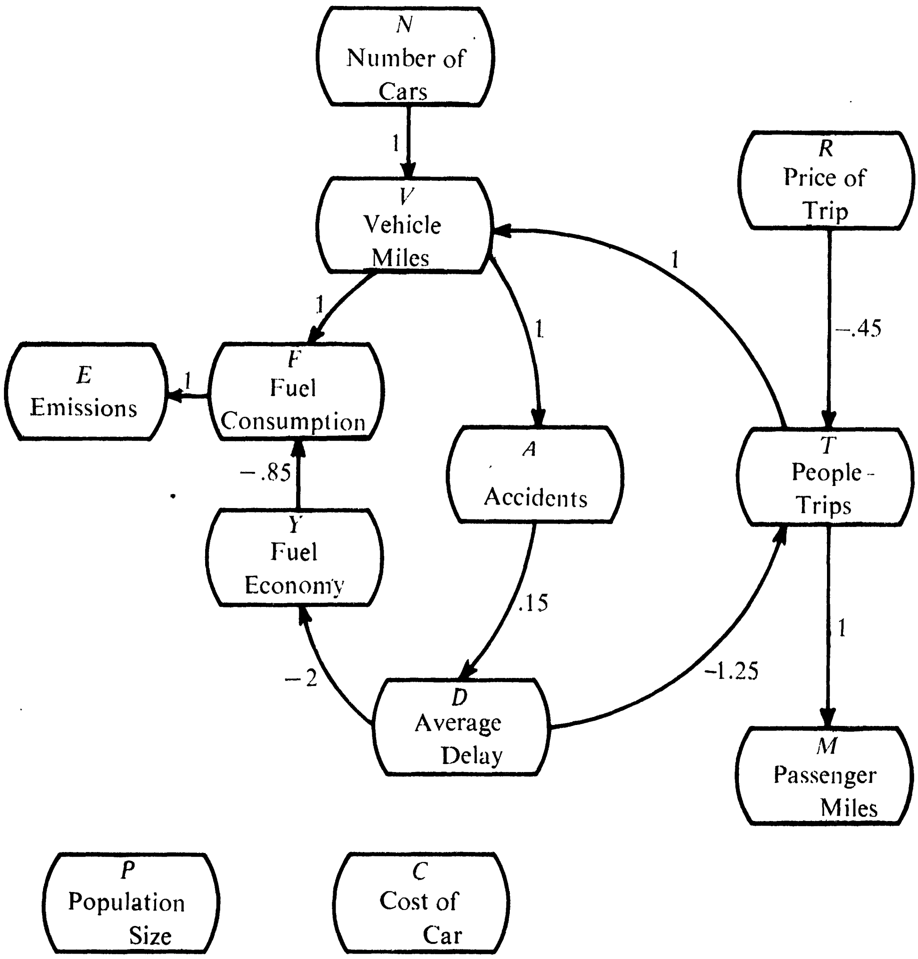

FIG. 2. Weighted digraph for energy use and air pollution produced by the transportation system of San Diego. Short term effects shown only. From Roberts [10].

Fig. 2 shows an example of a weighted digraph, which we shall use to discuss energy use in transportation. This digraph was constructed by a panel of experts

to represent the variables relating to energy use and air pollution resulting from the transportation system of San Diego County, California. Only short-term effects are shown. The experts who built this model had been studying San Diego, under an Environmental Protection Agency contract, to assess strategies for meeting requirements of the Clean Air Act, and used their knowledge to construct the weighted digraph. (We shall discuss below the precise meaning of the weights.) In San Diego, 97% of all trips are made by automobile. In recent years, San Diego has exhibited steadily increasing levels of fuel consumption and air pollution. We shall return to these observations later, and see if they are reflected in the properties of the digraph. (For details on how this and related digraphs were constructed, see Roberts [**10**].)

4. A dynamic model. Some rather interesting conclusions can be reached if we introduce a simple dynamic model for the propagation of changes through the vertices of a signed or weighted digraph. Let us begin with a signed digraph, and list its vertices as $x_1, x_2, \cdots, x_n$. We suppose that each vertex x_i attains a value $v_i(t)$ at each discrete time $t = 0, 1, 2, \cdots$. The succeeding value $v_i(t+1)$ is determined from $v_i(t)$, from an outside pulse $p_i^0(t+1)$ introduced at vertex x_i at time $t+1$, and from information about whether other vertices x_j adjacent to x_i went up or down at the last time period. Specifically, we assume that if there is an arc from x_j to x_i and x_j goes up by a units at time t, then as a result x_i goes up at time $t+1$ by an amount equal to a times the sign of arc (x_j, x_i). Moreover, x_i must increase by an amount equal to any external change $p_i^0(t+1)$ introduced at x_i at time $t+1$. To make all this precise, we define

$$v_i(t+1) = v_i(t) + p_i^0(t+1) + \Sigma_j \operatorname{sgn}(x_j, x_i) p_j(t), \tag{1}$$

where

$$\operatorname{sgn}(x_j, x_i) = \begin{cases} +1 & \text{if } (x_j, x_i) \text{ is } + \\ -1 & \text{if } (x_j, x_i) \text{ is } - \\ 0 & \text{if there is no arc } (x_j, x_i), \end{cases}$$

and

$$p_j(t) = \begin{cases} v_j(t) - v_j(t-1) & \text{if } t > 0 \\ p_j^0(0) & \text{if } t = 0. \end{cases}$$

The quantity $p_j(t)$ will be called the **pulse** at vertex x_j at time t. A **pulse process*** on a signed digraph D is defined by the rule (1), by an initial vector of values

$$V(0) = (v_1(0), v_2(0), \cdots, v_n(0)),$$

and by vectors giving the outside pulse introduced at each vertex at each time period.

* Pulse processes were introduced in Roberts [**9**].

We shall denote these vectors by

$$P^0(t) = (p_1^0(t), p_2^0(t), \cdots, p_n^0(t)).$$

We shall also use the **pulse vector** $P(t) = (p_1(t), \cdots, p_n(t))$.

In applications, one usually determines $V(0)$ as follows. Suppose we know the starting value $v_i(\text{start})$ at each vertex x_i. Then $v_i(0)$ is defined by

$$v_i(0) = v_i(\text{start}) + p_i^0(0),$$

i.e., $v_i(0)$ is the starting value at vertex x_i plus the initial pulse introduced at vertex x_i. Thus, we usually define a pulse process by giving the vector

$$V(\text{start}) = (v_1(\text{start}), v_2(\text{start}), \cdots, v_n(\text{start}))$$

rather than the vector $V(0)$.

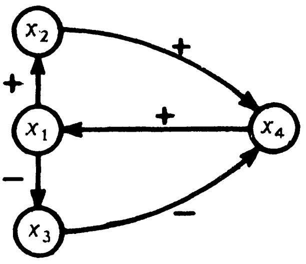

FIG. 3. Signed digraph.

To give an example of how a pulse process works, let us consider a very simple signed digraph, that of Fig. 3. We assume that $P^0(0) = (1,0,0,0)$, that $P^0(t) = 0$, all $t > 0$, and that $V(\text{start}) = (0,0,0,0)$. Thus, $V(0) = (1,0,0,0)$. At time 0, vertex x_1 increased by 1 unit, so at time 1, vertices x_2 and x_3 change, vertex x_2 going up by 1, vertex x_3 going down by 1. Thus, $V(1) = (1,1,-1,0)$ and so $P(1) = (0,1,-1,0)$. Since at time 1, vertex x_2 went up 1, this leads to an increase of 1 unit in vertex x_4 at time 2. But vertex x_3 went down 1 at time 1, so this leads (since $\text{arc}(x_3, x_4)$ is $-$) to a further increase in x_4 by 1 unit at time 2. We conclude that $V(2) = (1,1,-1,2)$, and $P(2) = (0,0,0,2)$. The increase in x_4 of 2 units at time 2 leads in turn to an increase in 2 units in x_1 at time 3. Thus, $V(3) = (3,1,-1,2)$, and so on.

The rule (1) for a pulse process on a signed digraph generalizes, in an obvious way, to a rule for a pulse process on a weighted digraph:

$$v_i(t+1) = v_i(t) + p_i^0(t+1) + \Sigma_j w(x_j, x_i) p_j(t). \tag{2}$$

Eq. (2) is really a system of finite difference equations, with parameters $w(x_j, x_i)$. For it can be rewritten as follows:

$$p_i(t+1) = p_i^0(t+1) + \Sigma_j w(x_j, x_i) p_j(t).$$

With this notion of pulse process, the weights in a weighted digraph have a specific

interpretation. For example, in Fig. 2, the weight $-.45$ on the arc from price of trip to people-trips suggests that as the price of a trip goes up by 1 unit, the annual number of people-trips will go down by .45 units. (Here, a unit was taken to be 10% of a base case level.)

If in a pulse process we have $P^0(t) = 0$ for $t > 0$, the process is called **autonomous.** An autonomous pulse process for which $P^0(0)$ is the vector $(0, 0, \cdots, 1, 0, \cdots, 0)$ with a 1 in the ith place, is called a **simple pulse process starting at vertex** x_i. In a simple pulse process starting at vertex x_i of a signed digraph, the quantities $p_j(t)$ and $v_j(t)$ are related to the **signed number of sequences** of length t from x_i to x_j, i.e., the difference between the number of positive sequences from x_i to x_j of length t and the number of negative sequences from x_i to x_j of length t. Specifically, it is easy to prove the following theorem:

THEOREM 1. *In a simple pulse process starting at vertex x_i of a signed digraph D, the quantity $p_j(t)$ is given by the signed number of sequences from x_i to x_j of length equal to t and the quantity $v_j(t)$ is given by $v_j(start) + p_j^0(0) +$ the signed number of sequences from x_i to x_j of length less than or equal to t.*

The **adjacency matrix** of the weighted digraph D is the matrix $A = (a_{ij})$ with $a_{ij} = w(x_i, x_j)$. The following theorem is easy to prove for signed digraphs using Theorem 1. The proof generalizes readily to the case of weighted digraphs.

THEOREM 2. *Suppose D is a weighted digraph with adjacency matrix A. In a simple pulse process on D starting at vertex x_i, $p_j(t)$ is given by the i,j entry of A^t, while $v_j(t)$ is given by $v_j(start)$ plus the i,j entry of $I + A + A^2 + \cdots + A^t$.*

It is fruitful to restate Theorem 2 in vector notation, in which case we see easily how to generalize it to autonomous pulse processes, obtaining

THEOREM 3. *In an autonomous pulse process on a weighted digraph, $P(t) = P(0)A^t$.*

5. Stability. The main qualitative property of a complex system which we shall study is stability. There are various notions of stability and we shall study only two of them here. We say that a vertex x_j of a weighted digraph D is **pulse stable** under a pulse process if the sequence

$$\{|p_j(t)|: t = 0, 1, 2, \cdots\}$$

is bounded, and **value stable** if the sequence

$$\{|v_j(t)|: t = 0, 1, 2, \cdots\}$$

is bounded. The weighted digraph D is **pulse (value) stable** under the pulse process if each vertex is. To give an example, in exponential growth such as $v_j(t) = 2^t$, variable x_j is both pulse and value unstable, as $p_j(t) = 2^{t-1}$. However, in linear growth such as $v_j(t) = 2t + 5$, variable x is still value unstable but it is now pulse stable.

Under any pulse process, value stability (at x_j) implies pulse stability (at x_j), since

$$|p_j(t)| = |v_j(t) - v_j(t-1)| \leqq |v_j(t)| + |v_j(t-1)|.$$

On the other hand, pulse stability does not imply value stability: consider, for example, the signed digraph of Fig. 4. Unsurprisingly, one can relate stability to the eigenvalues of D, i.e., those of D's adjacency matrix A. We shall do so in this section. Unfortunately, theorems relating stability to the eigenvalues of D are not always useful because they do not relate stability to the structure of the digraph.

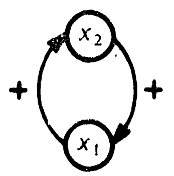

FIG. 4. Under the simple pulse process starting at vertex x_1, vertex x_2 is pulse stable but not value stable.

In the next section, we shall give a sample of a result which does relate stability to structure, and which can be exploited to choose stabilizing strategies. Finally, in Sec. 7, we shall use these results to study the examples of Figures 1 and 2. To begin with, we state a necessary condition for pulse stability.

THEOREM 4. *If a weighted digraph D has an eigenvalue greater than unity in magnitude, then D is pulse unstable under some simple pulse process.*

Proof. Let λ be an eigenvalue with $|\lambda| > 1$ and let U be an eigenvector corresponding to λ such that $\|U\| = 1$.* Write $U = \sum_{i=1}^{n} \alpha_i E_i$, where E_i is the vector with 1 in the ith component and 0 elsewhere. Then, for any integer $t > 0$,

$$|\lambda|^t = \|UA^t\| = \left\| \sum_{i=1}^{n} \alpha_i E_i A^t \right\| \leqq \sum_{i=1}^{n} |\alpha_i| \, \|E_i A^t\|.$$

Since $\|U\| = 1$, each $|\alpha_i| \leqq 1$. Thus

$$|\lambda|^t \leqq \sum_{i=1}^{n} \|E_i A^t\|.$$

It follows that for every $t > 0$, there is some i such that $\|E_i A^t\| \geqq 1/n\,|\lambda|^t$. Since there are only a finite number of E_i, we conclude that for at least one of them, $\|E_i A^t\| \geqq 1/n\,|\lambda|^t$ for arbitrarily large t. Pick $P(0) = E_i$. Then for arbitrarily large t, $\|P(t)\| = \|P(0)A^t\|$ is at least $1/n\,|\lambda|^t$. Since $|\lambda| > 1$ and n is fixed, we conclude that with $P(0) = E_i$, $\|P(t)\|$ gets arbitrarily large as $t \to \infty$, and hence the weighted digraph is pulse unstable under the simple pulse process starting at vertex x_i. Q.E.D.

* Throughout, if $M = (m_{ij})$, we use $\|M\| = \sqrt{\sum_{i,j} m_{ij}^2}$.

COROLLARY. *If an integer-weighted digraph D is pulse stable under all simple pulse processes, then each nonzero eigenvalue has magnitude equal to unity.*

Proof. By Theorem 4 each nonzero eigenvalue has magnitude at most unity. Let $\sum_{i=0}^{n} a_i\lambda^i$ be the characteristic polynomial of A. If i is the least integer such that $a_i \neq 0$, then the product of all the nonzero eigenvalues of D is $(\pm)$ times a_i. Since all entries of A are integers, a_i is an integer. Thus, each nonzero eigenvalue must have magnitude unity. Q.E.D.

Theorem 4 gives a necessary condition for pulse stability. This condition is also sufficient if D has no multiple eigenvalues (except possibly 0). This will follow from Theorem 5 below. To handle the case of multiple eigenvalues, let us consider the Jordan Canonical Form J corresponding to the matrix A. J can be written in the form

$$J = \begin{bmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_r} \end{bmatrix}, \tag{3}$$

where each B_j is an $(e_j + 1) \times (e_j + 1)$ diagonal block and B_j has the form

$$\begin{bmatrix} \lambda & \delta_j & & & & 0 \\ & \lambda & \delta_j & & & \\ & & \lambda & \ddots & & \\ & & & \ddots & \delta_j & \\ & 0 & & & \lambda & \delta_j \\ & & & & & \lambda \end{bmatrix},$$

where λ is an eigenvalue of A and δ_j is 0 or 1, 0 if $e_j + 1 = 1$ and 1 otherwise. (Note that λ may appear in several of the B_j's.)

If J has the form (3), then J^t has the form

$$\begin{bmatrix} \boxed{B_1^t} & & & 0 \\ & \boxed{B_2^t} & & \\ & & \ddots & \\ 0 & & & \boxed{B_r^t} \end{bmatrix}.$$

Now if $\delta_j = 0$, then $B_j = (\lambda)$ and

$$B_j^t = (\lambda^t). \tag{4}$$

If $\delta_j = 1$, then it is easy to prove by induction on t that $b_{k,l}^{j,t}$, the k, l entry of B_j^t, is given by

$$b_{k,l}^{j,t} = \begin{cases} 0 & \text{if } k > l \\ \binom{t}{l-k}\lambda^{t-l+k} & \text{if } k \leqq l. \end{cases} \tag{5}$$

To state our necessary and sufficient condition for pulse stability, let us say that an eigenvalue λ of D is **linked in** J if there is an off-diagonal entry of 1 in some row of J in which λ appears as the diagonal element. Equivalently, λ is linked in J if it appears on the diagonal of some B_j in which $\delta_j = 1$.

THEOREM 5. *Suppose D is a weighted digraph and J is its Jordan Canonical Form. Then the following are equivalent:*

(a) *D is pulse stable under all autonomous pulse processes.*

(b) *D is pulse stable under all simple pulse processes.*

(c) *Every eigenvalue of D has magnitude less than or equal to unity and every eigenvalue of D which is linked in J has magnitude less than unity.*

The proof of Theorem 5 begins with two lemmas.

LEMMA 1. *If A is an $n \times n$ matrix and J is its Jordan Canonical Form, then*

$$\{\|A^t\| : t = 0, 1, 2, \cdots\} \tag{6}$$

is bounded if and only if

$$\{\|J^t\| : t = 0, 1, 2, \cdots\} \tag{7}$$

is bounded.

Proof. The proof uses a topological argument. The matrices A and J are related by a similarity transformation. Now such a transformation is bicontinuous under the topology induced by the norm $\|\cdot\|$, so it follows that $\{A^t\}$ is contained in a sphere if and only if $\{J^t\}$ is contained in a sphere.* Q.E.D.

LEMMA 2. *If a weighted digraph D is pulse stable under all simple pulse processes, then*

$$\{\|J^t\| : t = 0, 1, 2, \cdots\} \tag{7}$$

is bounded. If the sequence (7) is bounded, then D is pulse stable under all autonomous pulse processes.

* Our thanks to Garrett Birkhoff for suggesting this argument.

Proof. By Theorem 3,

$$P(t) = P(0)A^t.$$

Thus, pulse stability under all simple pulse processes implies that the sequence

$$\{\|A^t\|:\ t = 0,1,2,\cdots\} \tag{6}$$

is bounded and hence, by Lemma 1, we conclude that the sequence (7) is bounded as well. Conversely, if the sequence (7) is bounded, then by Lemma 1 the sequence (6) must be bounded. Hence by Theorem 3, D must be pulse stable under all autonomous pulse processes. Q.E.D.

To prove Theorem 5, let us observe that clearly (a) implies (b). We shall prove (b) implies (c) and (c) implies (a). We have already shown that if D is pulse stable under all simple pulse processes, then every eigenvalue of D has magnitude less than or equal to unity. To complete the proof of (b) implies (c), let us show that every linked eigenvalue in J has magnitude less than unity. By Lemma 2, we know from pulse stability that $\{\|J^t\|\}$ is bounded. But now suppose λ is an eigenvalue of D which is linked in J and has magnitude greater than or equal to unity. If λ appears on the diagonal of block B_j, then $b_{1,2}^{j,t} = t\lambda^{t-1}$, by Eq. (5). Since $|\lambda| \geqq 1$, $|b_{1,2}^{j,t}|$ gets arbitrarily large as t approaches ∞, and so $\|J^t\|$ becomes arbitrarily large as well. This completes the proof that (b) implies (c).

To prove that (c) implies (a), it is sufficient by Lemma 2 to show that under assumption (c), $\{\|J^t\|\}$ is bounded. To prove that $\{\|J^t\|\}$ is bounded, it is sufficient to prove that for each j, $\{\|B_j^t\|\}$ is bounded. Let λ be the diagonal entry of B_j^t. By hypothesis, $|\lambda| \leqq 1$. If $\delta_j = 0$, then B_j^t has the form (4) and so $\|B_j^t\|$ is bounded since $|\lambda|^t$ is bounded. If $\delta_j = 1$, then by hypothesis, $|\lambda| < 1$. We show that for $k \leqq l$, $\{|b_{k,l}^{j,t}|:\ t = 0,1,2,\cdots\}$ is bounded, where $b_{k,l}^{j,t}$ is given by Eq. (5). Let $e = e_j$, i.e., e is one less than the dimension of B_j. Observe that $l-k \leqq e$. Thus, for $t > 2e$, we have

$$\binom{t}{l-k} \leqq \binom{t}{e} = \frac{t(t-1)\cdots(t-e+1)(t-e)!}{e!(t-e)!} \leqq \frac{t^e}{e!}$$

and

$$|\lambda^{t-l+k}| = |\lambda|^{t-l+k} \leqq |\lambda|^{t-e},$$

since $|\lambda| < 1$. It follows that

$$|b_{k,l}^{j,t}| \leqq \frac{t^e}{e!}|\lambda|^{t-e}.$$

Since $t^e|\lambda|^{t-e}$ approaches 0 as t approaches ∞, we conclude that $\{|b_{k,l}^{j,t}|\}$ is bounded. Q.E.D.

COROLLARY. *Suppose D is an integer-weighted digraph and J is its Jordan Canonical Form. Then the following are equivalent:*

(a) *D is pulse stable under all autonomous pulse processes.*

(b) *D is pulse stable under all simple pulse processes.*

(c) *Every eigenvalue of D has magnitude less than or equal to unity and no nonzero eigenvalue of D is linked in J.*

(d) *Every nonzero eigenvalue of D has magnitude equal to unity and no nonzero eigenvalue of D is linked in J.*

Proof. Obviously, (d) implies (c). By Theorem (5), (c) implies (a). Again trivially, (a) implies (b). To prove that (b) implies (d), note that (b) implies part (c) of Theorem 5 and (b) implies, by the Corollary to Theorem 4, that each nonzero eigenvalue of D has magnitude equal to unity. Thus, (b) implies (d). Q.E.D.

The next theorem characterizes value stability.

THEOREM 6. *Suppose D is a weighted digraph. Then the following are equivalent:*

(a) *D is value stable under all autonomous pulse processes.*

(b) *D is value stable under all simple pulse processes.*

(c) *D is pulse stable under all simple pulse processes and unity is not an eigenvalue of D.*

Proof. The proof uses Lemmas analogous to Lemmas 1 and 2 for

$$\left\{\left\| \sum_{t=0}^{N} A^t \right\| : N = 0,1,2,\cdots\right\} \tag{8}$$

and

$$\left\{\left\| \sum_{t=0}^{N} J^t \right\| : N = 0,1,2,\cdots\right\}. \tag{9}$$

Clearly, (a) implies (b). We shall prove (b) implies (c) and (c) implies (a).

To prove the latter, it is sufficient to show that under assumption (c), $\{\| \sum_{t=0}^N J^t \|\}$ is bounded. To show that $\{\| \sum_{t=0}^N J^t \|\}$ is bounded, it is sufficient to show that for each j, $\{\| \sum_{t=0}^N B_j^t \|\}$ is bounded. Let λ be the eigenvalue appearing on the diagonal of B_j. If $\delta_j = 0$, then $\sum_{t=0}^N B_j^t$ is $\sum_{t=0}^N \lambda^t$. Since $|\lambda| \leqq 1$ by pulse stability and $\lambda \neq 1$ by hypothesis, $\sum_{t=0}^N \lambda^t$ converges. We conclude that $\{| \sum_{t=0}^N \lambda^t |\} = \{\| \sum_{t=0}^N B_j^t \|\}$ is bounded. Suppose next that $\delta_j = 1$. Then by pulse stability, $|\lambda| < 1$. If $k > l$, then $\sum_{t=0}^N b_{k,l}^{j,t} = 0$. If $k \leqq l$, then by the proof of Theorem 5, if $t > 2e$, we have

$$|b_{k,l}^{j,t}| \leqq \frac{t^e}{e!} |\lambda|^{t-e},$$

where e is one less than the dimension of B_j. Thus, for $t > 2e$,

$$\left| \sum_{t=0}^{N} b_{k,l}^{j,t} \right| \leqq \frac{1}{e!} \sum_{t=0}^{N} t^e |\lambda|^{t-e}. \tag{10}$$

Applying the ratio test to the sum on the right hand side of (10) and using the fact that $|\lambda| < 1$, we find that the series converges. Thus, $\{| \sum_{t=0}^{N} b_{k,l}^{j,t}|\}$ is bounded, and hence so is $\{\| \sum_{t=0}^{N} B_j^t \|\}$.

To complete the proof, we assume (b) and prove (c). Now value stability implies pulse stability. Finally, we shall assume that unity is an eigenvalue and reach a contradiction. It is sufficient to prove that $\{\| \sum_{t=0}^{N} J^t \|\}$ is unbounded. In particular, suppose $\lambda = 1$ and B_j is a diagonal block in which λ is the diagonal entry. By pulse stability λ is unlinked, and so $\sum_{t=0}^{N} B_j^t$ is $\sum_{t=0}^{N} \lambda^t = N+1$. Thus, $\{\| \sum_{t=0}^{N} B_j^t \|\}$ is unbounded, and hence so is $\{\| \sum_{t=0}^{N} J^t \|\}$. This contradicts value stability. Q.E.D.

6. Rosettes. Although the results of Section 5 can be used to determine stability properties of a signed or weighted digraph, they do not relate stability to any structural properties of the digraph. If we could understand what it is about the structure which causes instabilities, we could exploit this knowledge to find stabilizing strategies, or to evaluate proposed strategies. Unfortunately, not much is known about the relation of structure to stability. In this section, we shall give a few sample theorems, which hold for a special class of digraphs called advanced rosettes. Although this class may appear very special, a surprisingly large number of systems which have been encountered by the authors belong to this category.

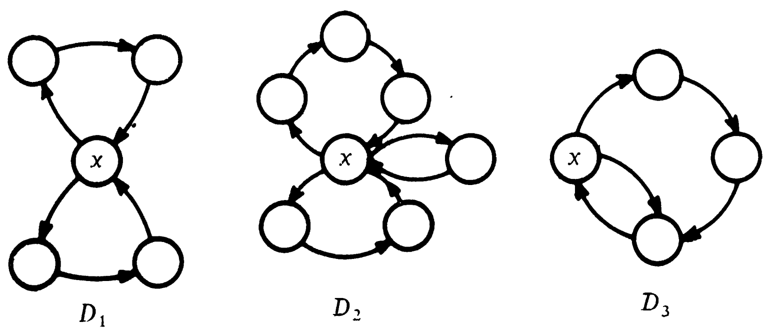

FIG. 5. Advanced rosettes with central vertex x. Digraphs D_1 and D_2 are rosettes, but D_3 is not.

A digraph D is a **rosette** if it consists of a central vertex x and nonintersecting cycles leading out of x. Digraphs D_1 and D_2 of Fig. 5 are rosettes. More generally, a digraph D is an **advanced rosette** if for every pair of vertices x and y of D, there is a sequence in D from x to y (we say D is **strongly connected**) and there is a **central vertex** x which is on all cycles of D. All the digraphs of Fig. 5 are advanced rosettes.

If D is a signed advanced rosette, let a_i denote the sum of the signs* of the cycles of length i and let s be the largest integer such that $a_s \neq 0$. If $a_i = 0$ for all i, we take $s = 0$. If $s = 0$, it is easy to show that D is pulse and value stable under all

* A sign is thought of as a number $+1$ or -1.

simple pulse processes. If $s > 0$ then the stability properties of D are mirrored in the properties of the **rosette sequence** $(a_1, a_2, \cdots, a_s)$. The next two theorems were first discovered by Joel Spencer.

THEOREM 7. *Suppose D is a signed advanced rosette with $s > 0$ and rosette sequence $(a_1, a_2, \cdots, a_s)$. If D is pulse stable under all simple pulse processes, then*

(a) $a_s = \pm 1$, *and*

(b) $a_i = (-a_s)a_{s-i}$, $1 \leqq i \leqq s-1$.

Proof. Let

$$R(\lambda) = \lambda^s - \sum_{i=1}^{s} a_i \lambda^{s-i}.$$

It is not hard to prove that the characteristic polynomial $C(\lambda)$ of D is given by

$$C(\lambda) = \lambda^{n-s} R(\lambda).$$

Note that $R(0) = a_s \neq 0$. Thus, 0 is not a root of $R(\lambda)$ and so $R(\lambda)$ has as roots exactly the nonzero eigenvalues of D. By the corollary to Theorem 4, if D is pulse stable under all simple pulse processes, then every root of $R(\lambda)$ has magnitude 1. The product of the roots is $\pm a_s$. Since a_s is an integer, it follows that a_s is ± 1, which proves (a).

An old lemma of Kronecker's states that if $f(x)$ is a monic polynomial with integer coefficients each of whose roots has magnitude unity, then each root of $f(x)$ is a root of unity. Thus, each root of $R(\lambda)$ is a root of unity. We shall prove that $R(\alpha) = 0$ if and only if $R(\alpha^{-1}) = 0$. This result will give us (b). For, applying it, one proves that $R(\lambda) = R(0)\lambda^s R(\lambda^{-1})$. By comparing the coefficients of the terms λ^{s-i}, one derives (b).

It is left to show that $R(\alpha) = 0$ if and only if $R(\alpha^{-1}) = 0$. Factor $R(\lambda)$ into irreducible factors $P_j(\lambda)$. Let β be any root of P_j. Then P_j is the unique irreducible polynomial with β as a root. If β is a primitive n_j-th root of unity, then P_j is the n_j-th cyclotomic polynomial, the unique monic polynomial whose roots are the primitive n_j-th roots of unity. Thus, if α is a root of R, then α is a root of some P_j, whence α^{-1} is a root of P_j, whence α^{-1} is a root of R. And conversely. Q.E.D.

THEOREM 8. *Suppose D is a signed advanced rosette with $s > 0$ and rosette sequence $(a_1, a_2, \cdots, a_s)$, and suppose D is pulse stable under all simple pulse processes. Then D is value stable under all simple pulse processes if and only if $\sum_{i=1}^{s} a_i \neq 1$.*

Proof: By Theorem 6, D is value stable under all simple pulse processes if and only if 1 is not an eigenvalue, i.e., if and only if $R(1) \neq 0$, i.e., if and only if $1 - \sum_{i=1}^{s} a_i \neq 0$, i.e., if and only if $\sum_{i=1}^{s} a_i \neq 1$. Q.E.D.

7. Applications. In this section, we shall apply the theorems about stability

to the digraphs of Fig.'s 1 and 2. Let us consider first the signed digraph D of Fig. 1, which represents energy demand in the electrical power area. A calculation shows that the characteristic polynomial $C(\lambda)$ is $\lambda^2(\lambda^5-\lambda^3-\lambda^2-1)$. Now $f(\lambda)=\lambda^5-\lambda^3-\lambda^2-1$ has a real root strictly between 1 and 2, since $f(1)=-2$ and $f(2)=19$. Thus, Theorem 4 implies that the signed digraph is pulse unstable under some simple pulse process. If one believes the model, one can interpret the instability result as follows: the system is sensitive enough to external influences that certain external influences lead to arbitrarily large values at some of the variables, and indeed to increasingly large changes in value. These results are not too surprising after our analysis in Section 2 of the cycles of D, or in view of observation of exponential increases in such variables as energy use. However, the results are based on our oversimplified pulse process model and should be tested using other methods. The same will be true of the other conclusions in this section. In practice, one usually assumes that no variable in a real-life system can reach arbitrarily large levels. In particular, no variable can change by larger and larger amounts in successive time periods. Long before changes (pulses) or values reach very large levels, the structure of the system itself will become modified. It would be wise to try to minimize the impact of that modification by foreseeing it or consciously choosing one of a possible set of modifications.

In any case, it has been suggested that one promising strategy is to invert the rate structure, that is, charge large users of electricity more rather than less. This corresponds to changing the sign of arc (U,R) from $-$ to $+$. Dealing with the new signed digraph obtained from that of Fig. 1 by making this change of sign, one calculates that $C(\lambda)$ is given by $\lambda^2(\lambda^5+\lambda^3-\lambda^2-1)=\lambda^2(\lambda-1)(\lambda^2+1)(\lambda^2+\lambda+1)$. Hence, the eigenvalues are $0,0,1,\pm i,-1/2\pm(\sqrt{3}/2)i$. Each nonzero eigenvalue has magnitude 1. Since all the nonzero eigenvalues are distinct, none of them can be linked. We conclude, by Theorem 5, that the new signed digraph is pulse stable under all autonomous pulse processes. However, 1 is an eigenvalue, so Theorem 6 implies that we still have value instability. The strategy of inverting the rate structure does not prevent the system from being sufficiently sensitive to external changes that some of these changes can lead to arbitrarily large values. What inverting the rate structure has accomplished is to make sure that changes at any given time cannot be too large.

The strategy of inverting the rate structure is value-stabilizing as well as pulse-stabilizing if we also make a second change, changing the sign of arc (C,F) from $+$ to $-$. This corresponds to forcing factories to move out of an area every time a new power plant is constructed! (Whether or not this strategy corresponds to a feasible real-world strategy is highly doubtful.) To show that it is value-stabilizing, let us consider the signed digraph of Fig. 1 with both (U,R) changed from $-$ to $+$ and (C,F) changed from $+$ to $-$. Now the characteristic polynomial is $C(\lambda)=\lambda^2(\lambda^5+\lambda^3+\lambda^2+1)=\lambda^2(\lambda+1)(\lambda^2+1)(\lambda^2-\lambda+1)$, and we have as roots $0,0,-1,\pm i,1/2\pm(\sqrt{3}/2)i$. By Theorems 5 and 6, the new signed digraph is pulse and value stable under all autonomous pulse processes. Once again, the reader is

reminded that this conclusion depends on our oversimplified model, and should be subjected to test using other techniques.

The theorems of Sec. 6 apply to the signed digraph D of Fig. 1, since it is an advanced rosette with central vertex U. A simple calculation shows that the rosette sequence is $(0,1,1,0,1)$. Now $a_2 \neq -a_5a_3$, so Theorem 7 implies that D is pulse unstable under some simple pulse process. If we change (U,R) from $-$ to $+$, the rosette sequence turns out to be $(0,-1,1,0,1)$. The necessary conditions of Theorem 7 are satisfied, though this does not verify pulse stability, which must be checked by calculating eigenvalues. Value instability follows from Theorem 8, since $a_1 + a_2 + a_3 + a_4 + a_5 = 1$. Finally, if (U,R) is changed from $-$ to $+$ and (C,F) from $+$ to $-$, then the rosette sequence is $(0,-1,-1,0,-1)$. The conditions of Theorem 7 are again satisfied, and here $a_1 + a_2 + a_3 + a_4 + a_5 = -3 \neq 1$. Thus, one discovers that changing (C,F) from $+$ to $-$ could be value-stabilizing by modifying the rosette sequence so that the necessary conditions of Theorem 7 and the condition of Theorem 8 are satisfied. It is left to check this conclusion by calculating eigenvalues. Other value-stabilizing strategies can be discovered by using the conditions of Theorems 7 and 8 as necessary conditions. Any change of sign in the signed digraph of Fig. 1 will leave $a_1 = 0$ and $a_4 = 0$. Moreover, $a_2 = \pm 1$, $a_3 = \pm 1$ or ± 3, and $a_5 = \pm 1$. The necessary conditions $a_2 = -a_5a_3$ and $a_1 + a_2 + a_3 + a_4 + a_5 \neq 1$ then imply that $a_2 = a_3 = a_5 = -1$. Potential value-stabilizing strategies are those corresponding to change of signs which set $a_2 = a_3 = a_5 = -1$. Using this observation, one discovers that there is no single sign change which will value-stabilize the signed digraph. The only other value-stabilizing strategies which involve change of two signs are changing signs of arcs (R,U) and (J,P) or changing signs of arcs (R,U) and (F,J).

Turning next to the weighted digraph of Fig. 2, we find that the characteristic polynomial is $C(\lambda) = \lambda^8(\lambda^4 + .19)$. The roots are 0 with multiplicity 8, $-.47 \pm (.47)i$, $.47 \pm (.47)i$. By Theorems 5 and 6, the digraph is pulse and value stable under all autonomous pulse processes. The reader will recall that Fig. 2 describes energy use and air pollution resulting in the short-term from San Diego's transportation system. As we remarked earlier, in recent years San Diego has been exhibiting rising rates of fuel consumption by automobiles and rising levels of air pollution. The pulse process model suggests that any spiralling of fuel consumption or air pollution, at least in the short-term, would have to come from repeated external impulses, rather than from the operation of feedback within the system. Indeed, it is fairly easy to see from the weighted digraph why this is the case. There is only one cycle, that from Number of People-trips T to Vehicle Miles V to Accidents A to Average Delay D to People-trips T. This is an inhibiting (negative feedback) cycle. Such cycles, as a general rule, produce oscillations. Here, the product of the weights of the arcs on this cycle is small enough so that the oscillations eventually die out. In any case, the results suggest that to stabilize the rising levels of variables

in this system, we shall have to search for external influences and perhaps counter these influences by introducing countervailing pulses into the system.

Although these conclusions about transportation once again depend on our specific model, they suggest something interesting about the system of energy use in transportation. Namely, they suggest that the source of rising levels of energy use in transportation is different from the source of rising levels of usage of electrical power, where the operation of feedback within the system tends to lead to spiralling levels. Perhaps this conclusion is true only of San Diego, but even then it is interesting.

Signed and weighted digraphs like those of Fig.'s 1 and 2 have been constructed for a wide variety of problems. Other signed digraphs for energy use in transportation are studied in Roberts [7]. Kruzic [6] builds a weighted digraph for the energy and environmental impact of deep water ports; Coady, *et al.* [1] build a weighted digraph for assessing the use of the coastal zone for urban recreation; Kane, *et al.* [4] build a weighted digraph for analyzing the allocation of scarce resources to health care delivery; and the Organization for Economic Cooperation and Development is building signed digraphs for analyzing the impact of various governmental funding decisions on the scientific community. (In references [1], [4] and, [6], the analysis was done using the KSIM model of Julius Kane, which differs in a number of respects from the specific change of value rule we have adopted. However, the general spirit of the approach is the same.) These examples and others are summarized in Chapter 4 of the forthcoming book Roberts [8].

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References

1. S. K. Coady, G. P. Johnson and J. M. Johnson, Effectively conveying results: a key to the usefulness of technology assessment (Submitted to *Journal of Technology Assessment*), Mimeographed, Institute for Water Resources, Corps of Engineers, paper delivered at the First International Congress on Technology Assessment, The Hague, May 31, 1973.

2. F. Harary, R. Z. Norman and D. Cartwright, Structural Models: An Introduction to the Theory of Directed Graphs, Wiley, New York, 1965.

3. J. Kane, Intuition, policy and mathematical simulation, Mimeographed, Resource Science Center, University of British Columbia, Vancouver, 1973, prepared for the International Symposium on Uncertainties in Hydrological and Water Resource Systems.

4. J. Kane, W. Thompson and I. Vertinsky, Health care delivery: A policy simulation, Socio-Econ. Plan. Sci., 6 (1972) 283–293.

5. J. Kane, I. Vertinsky and W. Thompson, KSIM: a methodology for interactive resource policy simulation, Water Resources Research 9 (1973) 65–79.

6. P. G. Kruzic, A Suggested Paradigm for Policy Planning, Stanford Research Institute Technical Note TN–OED–016, Menlo Park, Calif., June 1973.

7. F. S. Roberts, Building and analyzing an energy demand signed digraph, Environment and Planning, 5 (1973) 199–221.

8. ———, Discrete Mathematical Models, with Applications to Social, Biological and Environmental Problems, Prentice-Hall, Englewood Cliffs, N.J., 1976.

9. ———, Signed digraphs and the growing demand for energy. Environment and Planning, 3 (1971) 395–410.

10. ———, Weighted digraph models for energy use and air pollution in transportation systems, Environment and Planning, submitted. (Available as Report R–1578–NSF from the RAND Corp., Santa Monica, Calif.)

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TOPICS IN ORTHOGONAL FUNCTIONS

J. J. PRICE

1. Introduction. The study of general orthogonal functions has been a field of active research since about 1920. It arose from the classical theory of Fourier series, Bessel functions, orthogonal polynomials, etc., that branch of analysis concerned with detailed investigation of orthonormal sets derived from physics. By 1900, the literature on these sets was already extensive and well on the way to becoming vast.

As the new ideas of functional analysis circulated in the early part of this century, mathematicians began looking at orthonormal sets more abstractly, as bases of certain spaces. Working from this point of view, they were led to investigate general properties of orthonormal systems as well as special properties of particular systems.

In this article, we discuss two types of questions in the general theory. The first deals with completeness of orthonormal sets, the second with rearrangements of orthogonal series. To convey the flavor of these topics as quickly as possible, let us state several typical theorems at once, leaving the necessary definitions for later.

THEOREM (Talalyan). *Suppose you delete one function from a complete orthonormal set of functions in* $L^2[0,1]$. *The remaining functions, although no longer complete, have the following property: For each* $\varepsilon > 0$, *there is a set* $S_\varepsilon \subset [0,1]$ *of measure exceeding* $1-\varepsilon$ *on which they are complete.*

The same conclusion holds if any finite number of functions are deleted. Roughly speaking, the remaining orthonormal set is nearly complete, or is complete in some weaker sense. We intend to discuss several types of near completeness and relations among them. In particular, we shall consider possible extensions of Talalyan's theorem: can you delete *infinitely* many functions from a complete orthonormal set and still retain some kind of near completeness? If so, then "how many"?

To illustrate our interest in rearrangements, we recall the celebrated theorem of Carleson:

THEOREM (Carleson). *The trigonometric Fourier series of a function in $L^2[0,2\pi]$ converges almost everywhere.*

Thus the Fourier series of each function in L^2 behaves well. Yet there exist functions in L^2 whose Fourier series behave very badly when rearranged. In fact, Uljanov has proved the following remarkable result:

THEOREM (Uljanov). *Given any complete orthonormal system of functions Φ, there exists a function in L^2 whose Fourier series with respect to Φ can be rearranged so as to diverge almost everywhere.*

Thus no complete orthonormal set can be a "system of unconditional convergence." It is natural to ask whether there do indeed exist orthonormal systems of unconditional convergence. If so, since they must be incomplete, are there any that are at least nearly complete? We shall discuss these questions and some of the interesting ideas in the proof of Uljanov's Theorem.

2. Some preliminaries. First of all, let us agree that all functions will be real-valued, Lebesgue measurable on $[0,1]$, and finite almost everywhere. We assume the reader is familiar with the basic properties of the spaces $L^2[a,b]$ and with orthonormal sets in these spaces. Excellent sources for the general theory of orthogonal functions are the books of Alexits and of Kacmarz and Steinhaus. A highly readable account of material relevant to Sections 2–5 is contained in the book of Goffman and Pedrick.

For other sources, see the bibliography. We shall not give numbered references since it will be clear from the context which article in the bibliography applies.

DEFINITION 1. A set of functions $\Phi = \{\phi_n(x)\}_1^\infty$ is **complete** in $L^2[a,b]$ if the finite linear combinations of elements of Φ are dense in $L^2[a,b]$. In other words, given any $f(x)$ in $L^2[a,b]$ and any $\varepsilon > 0$, there is a linear combination

$$g(x) = a_1\phi_{n_1}(x) + a_2\phi_{n_2}(x) + \cdots + a_k\phi_{n_k}(x)$$

such that $\| f(x) - g(x) \| < \varepsilon$.

If Φ is an *orthonormal* set, then Definition 1 is equivalent to the following one:

DEFINITION 2. If

$$\int_0^1 f(x)\phi_n(x)dx = 0, \qquad n = 1,2,3\cdots,$$

then $f(x) = 0$ almost everywhere.

It is a consequence of Definition 2 that deleting a single element destroys completeness of an orthonormal set. This is not true, however, for non-orthogonal sets. For example, take the non-orthogonal set $\Phi = \{1, x, x^2, x^3, \cdots\}$. Note that Φ is complete because the linear combinations of elements of Φ are dense in the continuous functions by the Weierstrass Approximation Theorem, and the continuous

functions are dense in $L^2[0,1]$. If you discard all the odd powers from Φ, then the remaining set, $\{1, x^2, x^4, x^6, \cdots\}$, will still be complete by the Weierstrass Theorem and the change of variable $y = x^2$.

To account for the apparent redundancy of the system Φ, we offer an intuitive geometric argument. Interpret x^{2n} and x^{2n+1} as vectors in Hilbert space. The cosine of the angle θ between these vectors is computed in terms of scalar products:

$$\cos\theta = \frac{(x^{2n}, x^{2n+1})}{(x^{2n}, x^{2n})^{\frac{1}{2}}(x^{2n+1}, x^{2n+1})^{\frac{1}{2}}}$$

$$= \frac{\int_0^1 x^{4n+1}dx}{\left(\int_0^1 x^{4n}dx\right)^{\frac{1}{2}}\left(\int_0^1 x^{4n+2}dx\right)^{\frac{1}{2}}} = \frac{(4n+1)^{\frac{1}{2}}(4n+3)^{\frac{1}{2}}}{4n+2}$$

$$= \left(1 - \frac{1}{(4n+2)^2}\right)^{\frac{1}{2}}.$$

When n is large, $\cos\theta \approx 1$. Hence the vectors x^{2n} and x^{2n+1} point in nearly the same direction, so it is not surprising that either one does the work of both. Orthogonal vectors, on the other hand, are at right angles to each other; they are as unalike as possible.

There is a moral in all this: you generally give up a lot more when you discard elements from an orthogonal set than from a non-orthogonal set. So even though the theorems we shall state concern deletions from general systems, they are most dramatic when applied to orthogonal systems.

We remark that we can strengthen the above example by invoking Müntz's Theorem, a generalization of the Weierstrass Approximation Theorem. According to Müntz, the system $\{x^{n_1}, x^{n_2}, x^{n_3}, \cdots\}$ is complete if and only if

$$\Sigma \frac{1}{n_i} = \infty.$$

For instance, the choices $n_i = 100i$ or $n_i =$ the ith prime number, will yield complete systems.

3. Near completeness. The first result on deletion of functions is due to the Armenian mathematician Talalyan. In order to state his theorem, we define a kind of near completeness.

DEFINITION 3. A set of functions in $L^2[0,1]$ has **property (T)** if for each $\varepsilon > 0$, there is a set $S_\varepsilon \subset [0,1]$ of measure exceeding $1 - \varepsilon$ such that the restrictions of the given functions to S_ε are complete in $L^2(S_\varepsilon)$.

Obviously a complete set has property (T). Now we can state Talalyan's Theorem as follows.

THEOREM 1. *If a finite number of elements are deleted from a complete set in* $L^2[0,1]$, *the remaining set has property* (T).

Talalyan showed that property (T) is equivalent to another property which is interesting in its own right.

DEFINITION 4. A set $\Phi = \{\phi_n(x)\}_1^\infty$ of finite measurable functions is **complete in measure** on $[0,1]$ if it satisfies the following condition. Given any finite measurable function $f(x)$ and any $\varepsilon > 0$, there is a linear combination

$$g(x) = a_1\phi_{n_1}(x) + \cdots + a_k\phi_{n_k}(x)$$

such that $|f(x) - g(x)| < \varepsilon$, except perhaps on a set of measure less than ε.

In the language of probability theory, $g(x)$ is close to $f(x)$ with high probability. Note the similarity between Definitions 1 and 4. For brevity, we shall refer to completeness in measure as **property (M)**.

THEOREM 2. *Properties* (T) *and* (M) *are equivalent.*

Now a question that arises naturally: can you delete *infinitely many* functions from a complete system so that the remaining system has properties (T) and (M)? If so, then obviously there would have to be infinitely many survivors. But that alone is no guarantee. For example, suppose you remove the sines from the trigonometric system $\{1, \sin 2\pi nx, \cos 2\pi nx\}$. The surviving cosines are complete in $L^2[0,1/2]$. But they are not complete on any set of measure more than 1/2. Reason: the cosines are even functions with respect to the point 1/2. Hence with linear combinations of cosines, you can approximate only even functions on $[0,1]$. But not every function on a set of measure greater than 1/2 is the restriction of an even function.

More extreme examples exist. In Section 4, we exhibit some deleted systems that are not complete on any set of positive measure. The question, therefore, is whether you can delete *some* infinite subset of a complete system so that the remaining set has properties (T) and (M). A positive answer was given by Goffman and Waterman in an elegant paper on the space of measurable functions:

THEOREM 3. *Given any set with property* (M), *it is possible to make infinitely many deletions without destroying property* (M).

Of course the deletions must be made with care. We urge the reader to look up the proof of Theorem 3. It is surprisingly short and simple.

Let us mention another kind of near completeness. Boas and Pollard found that if a finite number of functions are deleted from a complete orthonormal set, then completeness can be recovered by multiplying the remaining functions by a suitable non-negative function.

DEFINITION 5. A system of functions $\{\phi_n(x)\}_1^\infty$ has **property (BP)** if there

exists a non-negative bounded measurable function $m(x)$ such that the system $\{\phi_n(x)m(x)\}_1^\infty$ is complete in $L^2[0,1]$.

We can state the Boas-Pollard theorem as follows.

THEOREM 4. *If a finite number of elements are deleted from a complete orthonormal set in* $L^2[0,1]$, *then the remaining set has property* (BP).

Boas and Pollard gave an example of a system that retains property (BP) even after a certain infinite set of deletions, and they asked whether this is always possible. Motivated by their question, Robert E. Zink and the author obtained the following result.

THEOREM 5. *Properties* (T), (M) *and* (BP) *are equivalent.*

It follows from Theorems 3 and 5 that every complete orthonormal set permits certain infinite deletions without loss of property (BP).

We shall refer to the three equivalent properties as **near completeness.**

4. The Haar functions. It is time to look at some explicit examples of orthonormal sets and see what we can say about deletions and near completeness. We shall consider three: the Haar, the Rademacher and the Walsh orthonormal functions.

We begin with the Haar functions, in many ways the simplest complete orthonormal set. We let $h_0(x) \equiv 1$. Then we define functions in blocks: $h_1(x)$, then $h_2(x)$, $h_3(x)$, then $h_4(x)$, $h_5(x)$, $h_6(x)$, $h_7(x)$, etc. For each integer $m \geqq 0$, we partition $[0,1]$ into 2^m equal intervals. If $n = 2^m + k$, where $0 \leqq k \leqq 2^m - 1$, we let $h_n(x) \equiv 0$ outside of the kth interval, and we let $h_n(x)$ take two values on the interval:

$$h_{2^m+k}(x) = \begin{cases} 2^{m/2} & x \in \left[\dfrac{k}{2^m}, \dfrac{k+\frac{1}{2}}{2^m}\right) \\ -2^{m/2} & x \in \left[\dfrac{k+\frac{1}{2}}{2^m}, \dfrac{k+1}{2^m}\right) \\ 0 & \text{otherwise.} \end{cases}$$

The Haar functions fall naturally into blocks of 2^m functions. Let H_m denote the mth block,

$$H_m = \{h_n(x)\colon 2^m \leqq n \leqq 2^{m+1} - 1\}.$$

If $h_n \in H_m$, then the set where $h_n(x)$ is non-zero is a small interval of length 2^{-m}. We denote this interval by σ_n, the support of h_n. As m increases the supports shrink, hence the graphs of the Haar functions become taller and thinner.

The Haar system is simple enough that questions concerning deletions are easily settled by a criterion of Robert E. Zink and the author.

THEOREM 6. *A family of Haar functions* $\{h_{n_i}(x)\}$ *is nearly complete if and only if almost every point of* $[0,1]$ *is contained in infinitely many of the supports* σ_{n_i}.

This condition is the absolute minimum you could hope for: infinitely many non-zero functions almost everywhere. What counts then, is not how many functions you throw away, but how many you keep.

As a quick application of Theorem 6, we get a sufficient condition for near completeness. We simply note that the supports of the functions in each block H_m cover $[0,1]$ exactly once.

COROLLARY 1. *A family of Haar functions is nearly complete if it contains infinitely many blocks* H_m.

Next we derive a necessary condition. It is an exercise to show that a family of sets $\{S_i\}$ cannot cover a set of positive measure infinitely often unless $\Sigma\,|S_i|$ diverges. Here $|S_i|$ denotes the measure of S_i. By Theorem 6 therefore, a necessary condition for near completeness of a family of Haar functions $\{h_{n_i}(x)\}$ is

$$\Sigma\,|\sigma_{n_i}| = \infty.$$

Now we observe that $\sigma_n \approx 1/n$. More precisely,

$$\frac{1}{n} \leqq |\sigma_n| < \frac{2}{n},$$

because $|\sigma_n| = 2^{-m}$ if $2^m \leqq n < 2^{m+1}$. Therefore the condition is equivalent to

$$\Sigma\frac{1}{n_i} = \infty.$$

COROLLARY 2. *A necessary condition for a family of Haar functions* $\{h_{n_i}(x)\}$ *to be nearly complete is* $\Sigma 1/n_i = \infty$.

This condition is not sufficient, however. For instance, it is satisfied by the set of all Haar functions whose supports lie in a fixed interval $[a,b]$ no matter how small.

If $\Sigma 1/n_i$ converges, say $n_i = 2^i$, then $\{h_{n_i}(x)\}$ cannot be nearly complete. But even more is true in this case: the supports σ_{n_i} do not cover any set of positive measure infinitely often. Therefore, at almost every point of $[0,1]$ only a finite number of the functions are non-zero. Hence the family cannot be complete on any set of positive measure.

5. The Rademacher functions. This is a simple but interesting system of step functions. For each integer $n \geqq 0$, partition $[0,1]$ into 2^{n+1} equal subintervals. Define $r_n(x) = +1$ or -1 alternately on these subintervals starting with $+1$ at the left. Equivalently, take the binary expansion $x = 0.d_1d_2d_3\cdots$, where the binary

digit $d_n(x)$ is either 0 or 1, and set

$$r_n(x) = \begin{cases} 1 & \text{if } d_{n+1}(x) = 0, \\ -1 & \text{if } d_{n+1}(x) = 1, \end{cases}$$

that is,

$$r_n(x) = 1 - 2d_{n+1}(x), \qquad n \geqq 0.$$

(Actually a binary rational number $k \cdot 2^{-n}$ has two expansions, one terminating and the other repeating. Use either one; in this and following discussions we can afford to neglect countable sets.)

The Rademacher functions are orthonormal but they are not complete; for example, the product $r_0(x)r_1(x)$ is orthogonal to each of them. In fact every finite product of distinct Rademacher functions is orthogonal to each Rademacher function, as is easily checked. Thus the system appears far from being complete. On the basis of this circumstantial evidence, therefore, the next assertion should not be too surprising.

THEOREM 7. *The system of Rademacher functions is not complete on any set of positive measure.*

Let us sketch the proof. It depends on a basic property of Rademacher functions: given a dyadic rational point $k \cdot 2^{-n}$, the functions $r_j(x)$ for $j \geqq n-1$ are all odd with respect to that point.

Suppose the Rademacher system were complete on some set E of positive measure. Then it would have property (M) relative to E and so would each deleted system $R_n = \{r_j(x): j \geqq n-1\}$.

Take a subset F of E that has positive measure and is symmetric about some point $k \cdot 2^{-n}$. Then the characteristic function of F is an even function relative to $k \cdot 2^{-n}$, hence cannot be approximated in measure by odd functions. It follows that R_n does not have property (M) on E, a contradiction. Therefore the Rademacher functions are not complete on E. We remark that the existence of F requires an argument involving points of density of measurable sets and the density of dyadic rationals on the line.

Precisely the same argument applies to the trigonometric systems $\{\sin 2^n \pi x\}$ and $\{\cos 2^n \pi x\}$.

THEOREM 8. *The orthonormal sets* $\{\sin 2^n \pi x\}_1^\infty$ *and* $\{\cos 2^n \pi x\}_0^\infty$ *are not complete on any set of positive measure.*

We may say that these sets are nowhere complete.

DEFINITION 6. A system of functions is *nowhere complete* if there exists no set E of positive measure for which the system is complete in $L^2(E)$.

We have seen that the systems composed of Haar functions or trigonometric functions with subscripts 2^n are nowhere complete. These are examples where extravagant deletions from a complete set can destroy all semblance of completeness.

It may not appear so, but the nowhere completeness of the Rademacher functions follows the same pattern. In the next section, we describe a complete orthonormal system $\{w_n(x)\}$ which contains the Rademacher functions in such a way that $r_n(x) = w_{2^n}(x)$.

6. The Walsh functions. How to complete the Rademacher functions? We know that any finite product of them is orthogonal to each individual one. Hence we need at least all such products. In fact, they are exactly what we need; the set of finite products of Rademacher functions, together with the constant function $w_0(x) \equiv 1$, is a complete orthonormal set in $L^2[0,1]$ called the **Walsh functions.**

The accepted enumeration of the Walsh functions is due to Paley. If

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}, \qquad 0 \leqq k_1 < k_2 < \cdots < k_s,$$

then

$$w_n(x) = r_{k_1}(x) \cdot r_{k_2}(x) \cdots r_{k_s}(x).$$

For example, $23 = 2^0 + 2^1 + 2^2 + 2^4$ so $w_{23}(x) = r_0(x)r_1(x)r_2(x)r_4(x)$. In particular,

$$w_{2^n}(x) = r_n(x).$$

Enumerated in this way, the Walsh functions fall naturally into blocks of length 2^n. Let W_n denote the nth block,

$$W_n = \{w_j(x)\colon 2^n \leqq j \leqq 2^{n+1} - 1\}.$$

The blocks follow a simple rule of formation: each block W_n is generated by multiplying all preceding Walsh functions by $r_n(x)$.

There is a close connection between W_n and the corresponding block of Haar functions, H_n. The elements of W_n are linear combinations of the elements of H_n and conversely. Therefore we get, free of charge, an immediate translation of Corollary 1 into a statement about Walsh functions:

THEOREM 9. *A family of Walsh functions is nearly complete if it contains infinitely many blocks* W_n.

For a while I thought the condition of Theorem 9 was very close to a necessary condition for near completeness. But that turned out to be false. I recently found an example of a much thinner nearly complete family of Walsh functions $\{w_{n_i}(x)\}$. The indices $\{n_i\}$ form a sequence of finite arithmetic progressions of increasing length and increasing differences. Therefore they grow very sparse among the integers, although not as sparse as a geometric sequence.

For lacunary sequences of indices, that is, sequences $\{n_i\}$ satisfying

$$\frac{n_{i+1}}{n_i} \geqq q > 1,$$

the corresponding family of Walsh functions is nowhere complete, as is the case for the Rademacher functions. I do not know a necessary condition for near completeness of a family of Walsh functions although a natural conjecture is

$$\Sigma \frac{1}{n_i} = \infty.$$

7. Rearrangements. Let us recall a few basic facts. Given an orthonormal set $\Phi = \{\phi_n(x)\}_1^\infty$ and a function $f(x)$ in $L^2[a, b]$, one assigns to $f(x)$ the Φ-Fourier series or expansion

$$\sum_{n=1}^{\infty} a_n \phi(x), \qquad a_n = \int_a^b f(x)\phi_n(x)dx.$$

The Bessel Inequality asserts that

$$\sum_{n=1}^{\infty} a_n^2 \leqq \left(\int_a^b [f(x)]^2 dx \right)^{\frac{1}{2}} < \infty.$$

Conversely, given a sequence $\{a_n\}$ with $\Sigma_1^\infty a_n^2 < \infty$, the Riesz-Fischer Theorem asserts that

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

is the series of a function in L^2. Thus the preceding series represents an L^2 function if and only if $\Sigma_1^\infty a_n^2$ converges.

In 1915, Lusin posed a question that became one of the leading problems in analysis for a half century. He asked whether the trigonometric Fourier series of each function in L^2 converges almost everywhere. The problem can be stated in slightly different terms via the following definition:

DEFINITION 8. An orthonormal set $\{\phi_n(x)\}_1^\infty$ is a **system of convergence** if the convergence of $\Sigma_1^\infty a_n^2$ implies the convergence of $\Sigma_1^\infty a_n \phi_n(x)$ almost everywhere.

Thus Lusin's question asks whether the trigonometric functions form a system of convergence. In 1966, Carleson gave an affirmative answer:

THEOREM 10. *The trigonometric system is a system of convergence.*

In contrast to Carleson's positive result is an important negative result of Uljanov. In 1960, Zahorski asserted the existence of an L^2-function whose trigonometric Fourier series could be rearranged to diverge almost everywhere. His construction was extremely intricate and sketchy, however. Then in 1961, Uljanov gave a difficult

but comprehensible proof of a more general theorem: *For every complete orthonormal system Φ, there is an L^2 function whose Φ-Fourier series can be rearranged to diverge almost everywhere.*

Let us look at this remarkable theorem from a different angle. Instead of rearranging the Φ-Fourier series of a function f, let us think of expanding f with respect to a rearrangement of Φ. Seen in this way, the trouble with the series may not be the fault of the function, but rather of a clumsy ordering of the system. Hence, Uljanov's theorem can be expressed as follows.

THEOREM 11. *For every complete orthonormal set, there is an ordering under which the set is not a system of convergence.*

There is one other version of the theorem worth stating.

DEFINITON 7. An orthonormal set $\{\phi_n(x)\}_1^\infty$ is a **system of unconditional convergence** if the convergence of $\Sigma_1^\infty a_n^2$ implies the almost everywhere convergence of $\Sigma_1^\infty a_n\phi_n(x)$ and all rearrangements.

COROLLARY. *No complete orthonormal set is a system of unconditional convergence.*

8. Examples of systems of unconditional convergence. Before going on, let us see that such systems actually exist. We shall give two examples.

The first is the family of Haar functions $\{h_{2^i}(x)\}$. This is a system of unconditional convergence by default, for according to the argument at the end of Section 3, the series

$$\sum_{i=1}^{\infty} c_i h_{2^i}(x)$$

has only a finite number of non-zero terms for almost every x. Therefore it trivially converges unconditionally almost everywhere.

The second example, and a non-trivial one, is the Rademacher system. The example depends on the fact that the Rademacher system is first of all a system of convergence.

THEOREM 12. *If Σa_n^2 converges, then the Rademacher series $\Sigma a_n r_n(x)$ converges almost everywhere.*

We omit the proof, although it is not hard. However, we cannot resist digressing for a moment to mention an application of the theorem.

For each x, the sequence $\{r_n(x)\}$ is a sequence of $+1$'s and -1's, so $\Sigma\, a_n r_n(x)$ is a collection of series $\Sigma \pm a_n$ for various sequences of signs. Now a sequence of signs corresponds to a sequence of 0's and 1's which, in turn, correspond to the binary digits of some x in $[0, 1]$. Conversely, the binary digits of each x determine a sequence of signs. Therefore, there is an essentially one-to-one correspondence

between the points of $[0,1]$ and the set of all sequences of signs. (We continue to ignore a countable number of exceptions.)

In addition, since $\{r_n(x)\} = \{1 - 2d_{n+1}(x)\}$, there is also a one-to-one correspondence between the points of $[0,1]$ and the set of sequences $\{r_n(x)\}$. We conclude that as x runs over $[0,1]$, the sequences $\{r_n(x)\}$ run once through all possible sequences of signs. Now Theorem 12 can be interpreted as an assertion about series with arbitrary choices of signs.

COROLLARY. *If Σa_n^2 converges, then $\Sigma \pm a_n$ converges for almost every choice of signs (or, with probability 1).*

Thus the divergence of the harmonic series $\Sigma 1/n$ is practically impossible! To be serious, $\Sigma \pm 1/n$ converges for almost every choice of signs due to large scale cancellations of terms. This cancellation occurs because almost every sequence of signs contains "as many" plusses as minuses, or equivalently, almost every sequence of 0's and 1's contains "as many" 0's as 1's. We are paraphrasing a standard result: almost every x in $[0,1]$ is **normal** relative to the base 2. That means that if p_n and q_n are the numbers of 0's and 1's among the first n binary digits of x, then

$$\lim_{n\to\infty} \frac{p_n}{n} = \lim_{n\to\infty} \frac{q_n}{n} = \frac{1}{2}.$$

(See Halmos, page 206 or Hardy and Wright, page 123.)

This ends the digression. Let us return to unconditional convergence.

THEOREM 13. *The Rademacher system is a system of unconditional convergence.*

The idea of the proof is that an observer at x looking at a rearranged Rademacher series sees the same thing as an observer at y looking at the given series, and there is a one-to-one correspondence between x and y. Therefore, permuting the series amounts simply to changing the variable.

It is known that a rearrangement of dyadic digits induces a rearrangement of the interval $[0,1]$. To be precise, if $x = 0.d_1(x)d_2(x)d_3(x)\cdots$ and $\{n_i\}$ is a permutation of the positive integers, define

$$y = Tx = 0.d_{n_1}(x)d_{n_2}(x)d_{n_3}(x)\cdots.$$

Then T is a one-to-one measure preserving transformation of $[0,1]$ onto itself.

Clearly $d_i(y) = d_{n_i}(x)$. This relation translates into a corresponding one between Rademacher functions. For

$$r_i(y) = 1 - 2d_{i+1}(y);$$

hence

$$r_i(y) = 1 - 2d_{n_{i+1}}(x) = r_{n_{i+1}-1}(x).$$

Now suppose $\Sigma a_m r_m(x)$ is a given Rademacher series with $\Sigma a_m^2 < \infty$. Consider any rearrangement

$$\Sigma a_{m_i} r_{m_i}(x),$$

where $\{m_i\}_{i=0}^{\infty}$ is a permutation of the non-negative integers. Write $m_i = n_{i+1} - 1$. Then $\{n_i\}_{i=1}^{\infty}$ is a permutation of the positive integers. If T is the transformation of $[0,1]$ associated with $\{n_i\}$, then

$$\Sigma a_{m_i} r_{m_i}(x) = \Sigma a_{m_i} r_{n_{i+1}-1}(x) = \Sigma a_{m_i} r_i(y).$$

By Theorem 12, the series on the right converges for almost every y because

$$\Sigma a_{m_i}^2 < \infty.$$

Therefore the series on the left converges for almost every x. This completes the proof of Theorem 13.

We remark that both systems of unconditional convergence given in this section are not only incomplete as required by Theorem 11, but are nowhere complete.

9. Sketch of the proof of Uljanov's theorem. The proof of Theorem 11 is not easy, but the ideas involved are interesting and accessible. We shall outline Uljanov's argument, indicating at one point a beautiful simplification due to Olevskii.

The principal step is establishing the theorem for the Haar functions. Once that is done, a clever argument shows that the result carries over to any complete orthonormal set.

Here is the strategy of the proof. Try to construct a sequence $\{Q_j(x)\}_1^{\infty}$ with the following properties:

(1) $Q_j(x)$ is a finite linear combination of Haar functions with terms permuted from their natural order.

(2) The Haar functions involved in $Q_j(x)$ all have higher subscripts than those in $Q_1(x), \cdots, Q_{j-1}(x)$.

(3) $\Sigma_{j=1}^{\infty} (\int_0^1 Q_j^2(x)dx) < \infty$.

(4) There is a set E_j of positive measure such that for each $x \in E_j$, some partial sum of the finite series $Q_j(x)$ exceeds $\frac{1}{4}N_j^{\frac{1}{4}}$ in absolute value, where $\{N_j\}$ is a rapidly increasing sequence.

(5) Almost every point in $[0,1]$ belongs to infinitely many of the sets E_j.

If such a sequence can be found, then $\Sigma_1^{\infty} Q_j(x)$ is the desired series. Because of (1) and (2), it is a rearranged Haar series; because of (3), the series represents a function in $L^2[0,1]$. Because of (4) and (5), its partial sums are unbounded for almost every x. Hence the series diverges almost everywhere.

So much for strategy, now let us look at tactics. Here we come to the heart of the problem: how to find a nasty permutation of a set of Haar functions. Let us start with the set $S_1 = \{h_n(x): 1 \leqq n \leqq 2^{N_1} - 1\}$. We seek a rearrangement of S_1,

that is, a permutation $\{n_i\}$ of the integers $\{1,2,\cdots,2^{N_1}-1\}$, and a linear combination

$$Q_1(x) = \sum_{i=1}^{2^{N_1}-1} a_i h_{n_i}(x)$$

with small L^2 norm, but with large partial sums on a set of substantial measure.

Uljanov constructed such a permutation, but it is extremely complicated and difficult to describe. Olevskii found a much simpler permutation which is so neat we can describe it in a few words. Each Haar function in S_1 is uniquely identified by the midpoint of its support, a dyadic rational of the form $k \cdot 2^{-N_1}$. We simply reorder S_1 according to increasing order of the corresponding midpoints. That's all there is to it! For example, if $N_1 = 4$, the ordering goes as follows:

$$\begin{array}{ccccccccccccccc} \frac{1}{16} & \frac{1}{8} & \frac{3}{16} & \frac{1}{4} & \frac{5}{16} & \frac{3}{8} & \frac{7}{16} & \frac{1}{2} & \frac{9}{16} & \frac{5}{8} & \frac{11}{16} & \frac{3}{4} & \frac{13}{16} & \frac{7}{8} & \frac{15}{16} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ h_8 & h_4 & h_9 & h_2 & h_{10} & h_5 & h_{11} & h_1 & h_{12} & h_6 & h_{13} & h_3 & h_{14} & h_7 & h_{15} \end{array}$$

Let $\{n_i\}$ denote this permutation of S_1. Uljanov defines

$$P_1(x) = \sum_{i=1}^{2N_1-1} a_{n_i} h_{n_i}(x),$$

where the coefficients are chosen so that each summand takes the values 1, -1, and 0. He proves that

$$\int_0^1 P_1^2(x)dx = N_1, \tag{6}$$

(7) there is a set E_1 of measure at least $1/4$ on which some partial sum of $P_1(x)$ exceeds $N_1/4$.

Assertion (6) follows directly from orthogonality. Assertion (7) is harder to prove; we shall just sketch the idea. Let

$$P_1^*(x) = \max_{k \leqq 2^{N_1}-1} \left| a_{n_1} h_{n_1}(x) + \cdots + a_{n_k} h_{n_k}(x) \right|,$$

the maximum partial sum of $P_1(x)$. This function is difficult to handle, but Uljanov obtains some important information about its *average* value. He shows that

$$\int_0^1 P_1^*(x)dx > \tfrac{1}{2} N_1 .$$

The argument is ingenious, yet amazingly simple. It uses a nice property of the permutation $\{n_i\}$: If I_i is the interval $[i \cdot 2^{-N_1}, 1]$ for $1 \leqq i \leqq 2^{N_1}-1$, then

$$I_1 \supset I_2 \supset I_3 \supset \cdots,$$

and I_i contains exactly half of the support of $h_{n_i}(x)$, the half where $h_{n_i}(x) < 0$. (Probably Uljanov discovered the argument first, then found a suitable permutation.)

Next observe that $P_1^*(x) \leqq N_1$ since the summands are either 0 or of absolute value 1 and their supports cover $[0,1]$ exactly N_1 times. But if the average value of $P_1^*(x)$ exceeds $\frac{1}{2}N_1$ while $P_1^*(x) \leqq N_1$, then $P_1^*(x)$ must be fairly large on some set E_1. Assertion (7) follows easily.

Now define

$$Q_1(x) = \frac{1}{N_1^{3/4}} P_1(x).$$

Then by (6) and (7),

$$\int_0^1 Q_1^2(x)dx = \frac{1}{N_1^{3/2}} \int_0^1 P_1^2(x)dx = \frac{1}{N_1^{3/2}} N_1 = 1/\sqrt{N_1},$$

and on E_1, some partial sum of $Q_1(x)$ exceeds $N_1^{-3/4} \cdot \frac{1}{4}N_1 = \frac{1}{4}N_1^{1/4}$.

The rest of the proof consists of technical details; we shall wave our hands. For each $j > 1$, we construct $Q_j(x)$ in a similar manner, using new Haar functions each time. We obtain

$$\int_0^1 Q_j^2(x)dx = 1/\sqrt{N_j},$$

and a set E_j of measure at least 1/4 on which some partial sum exceeds $\frac{1}{4}N_j^{1/4}$. In the sequence $\{N_j\}$ increases so fast that $\Sigma(1/\sqrt{N_j}) < \infty$, then properties (1)–(4) are clearly satisfied.

That property (5) holds still requires an argument. Let us say only that the set E_j is a finite union of small intervals and E_{j+1} covers at least 1/4 of each interval comprising E_j. Stated technically, the E_j's are independent sets so that (5) follows by the Borel-Cantelli lemma. (See Halmos, page 201.)

This then, is the proof of Uljanov's theorem for the Haar functions. Note that the construction does not require the complete Haar system, but only a nearly complete family of Haar functions whose supports cover $[0,1]$ infinitely often.

To establish the theorem for an arbitrary complete orthonormal set $\Phi = \{\phi_j(x)\}$, Uljanov uses an idea of Marcinkewicz. He proves that the convergence of a Haar series

$$\Sigma c_i h_i(x), \qquad \Sigma\, c_i^2 < \infty,$$

is equivalent to that of a certain series

$$\Sigma\, a_n \phi_n(x), \qquad \Sigma a_n^2 < \infty,$$

constructed by using skillfully chosen finite segments of the Φ-Fourier series of $h_i(x)$ each involving different elements of Φ. Since there exists a Haar series $\Sigma c_i h_i(x)$ which diverges after suitable rearrangement, the corresponding series $\Sigma a_i \phi_i(x)$ has the same property.

Final remark. Uljanov's construction is even more remarkable in view of a beautiful theorem of Garsia on rearrangements. For simplicity, we state only a weak form of Garsia's result as it applies to Haar series.

THEOREM 14. *Let* $\Sigma\ c_n h_n(x)$ *be a Haar series with* $\Sigma c_n^2 < \infty$. *Let* $\{N_j\}$ *be an increasing sequence of positive integers. If within each block* $2^{N_j} \leqq n < 2^{N_j+1}$, *the terms of the series are randomly and independently rearranged, then with probability* 1 *the resulting rearranged Haar series converges almost everywhere.*

Thus, Uljanov's construction was a nearly impossible feat.

10. Some questions. What can we say about a system of unconditional convergence? By Theorem 11, it cannot be complete. But probably it cannot even be nearly complete. For the construction in the proof of Theorem 11 requires only a nearly complete subset of the Haar functions, not the full system. Hence an unconditional system of Haar functions is not nearly complete, and this most likely carries over to arbitrary orthonormal sets.

The examples given in Section 8 suggest that even more may be true: a system of unconditional convergence may be nowhere complete. This is almost certainly the case. For if a family of Haar functions is complete on a set E of positive measure, the supports cover E infinitely often. The proof of Theorem 11 can then probably be imitated yielding a rearranged Haar series that diverges on E. It seems likely, although not obvious, that the same holds for general orthonormal systems.

Another question is the converse: Is a nowhere complete orthonormal set a system of convergence? In view of the preceding discussion, a natural problem is the following:

QUESTION. *Is nowhere completeness necessary and sufficient for an orthonormal set to be a system of unconditional convergence?*

The properties of orthonormal sets discussed in this paper such as completeness, nowhere completeness, and unconditionality, have an interesting common feature: they do not depend on the enumeration of the set. Generally, theorems concerning convergence, summability, Fourier coefficients, etc. are definitely tied to the enumeration. They are really results about *ordered sets* of orthogonal functions, not just *sets* of orthogonal functions. It seems worthwhile to keep these distinctions in mind.

In this mental set, we interpret Carleson's theorem as saying that the natural ordering of the trigonometric system is a good one for convergence. However, Uljanov's theorem then says that every complete orthonormal set has a bad ordering. So we are led to ask whether every complete orthonormal set has a good ordering, that is, whether an analogue of Carleson's theorem holds.

QUESTION. *Can every orthonormal set be ordered so that it is a system of convergence?*

A theorem of Menshov suggests that the answer may be affirmative:

THEOREM 15. *Every orthonormal set Φ can be ordered so that the Φ-Fourier series of each L^2 function is almost everywhere (C,α) summable for $\alpha > 0$.*

Recall that $(C,1)$ summable means that the averages of the partial sums converge. In a way, (C,α) summability for all $\alpha > 0$ is the next best thing to convergence.

For a much more extensive account of problems in orthogonal series, we refer the reader to the survey article of Uljanov.

References

1. G. Alexits, Konvergenzprobleme der Orthogonalreihen, Budapest, 1960.

2. R. P. Boas and Harry Pollard, The multiplicative completion of sets of functions, Bull. Amer. Math. Soc., 54 (1948) 518–522.

3. L. Carleson, On convergence and growth of partial sums of Fourier Series, Acta Math., 116 (1966) 135–157.

4. A. M. Garsia, Existence of almost everywhere convergent rearrangements for Fourier series, of L_2 functions, Ann. Math., 79 (1964) 623–629.

5. C. Goffman and G. Pedrick, A First Course in Functional Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1965.

6. C. Goffman and D. Waterman, Basic sequences in the space of measurable functions, Proc. Amer. Math. Soc., 11 (1960) 211–213.

7. P. R. Halmos, Measure Theory, Van Nostrand, Princeton, N.J., 1950.

8. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 2nd edition, Oxford University Press, New York, 1945.

9. S. Kacmarz and H. Steinhaus, Theorie der Orthogonalreihen, 2nd edition, Stechert-Hafner, New York, 1951.

10. N. Lusin, Integral and Trigonometric Series, M. L. Gostekhizdat, Moscow, 1951.

11. J. Marcinkiewicz, Sur la convergence des séries orthogonales, Studia Math., 6 (1936) 39–45.

12. D. Menchoff, Sur la sommation des séries orthogonales par des méthodes de Cesàro, Recueil Math. Moscou (Math. Sbornik) 8 (1940) 121–134.

13. C. Müntz, Über den Approximationsatz von Weierstrass, Schwarz Festschrift, (1914) 303–314.

14. A. M. Olevskii, Divergent L^2 series with respect to a complete system, Akad. Nauk SSSR, 138 (1961) 545–548.

15. J. J. Price, Sparse subsets of orthonormal systems, Proc. Amer. Math. Soc., 35 (1972) 161–164.

16. J. J. Price and Robert E. Zink, On sets of functions that can be multiplicatively completed, Ann. Math., 82 (1965) 139–145.

17. ———, On sets of completeness for families of Haar functions, Trans. Amer. Math. Soc., 119 (1965) 262–269.

18. A. A. Talalyan, The representation of measurable functions by series, Russian Math. Surveys, 15 (5), (1960) 77–136.

19. P. L. Uljanov, Divergent Fourier series, Russian Math. Surveys, 16 (3), (1961) 1–74.

20. ———, Solved and unsolved problems in the theory of trigonometric and orthogonal series, Russian Math Surveys, 19 (1), (1964) 1–63.

21. Z. Zahorski, Une série de Fourier permutée d'une fonction de classe L_2 divergente presque partout, C.R. Acad. Sci. Paris, 251 (1960) 501–503.

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MARTIN'S AXIOM

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Introduction. During the past decade, many new axioms of set theory have appeared. The principal object of these axioms is to settle important problems which cannot be settled without new axioms. Thus Gödel and Cohen have shown that the Continuum Hypothesis cannot be settled on the basis of the presently accepted axioms; so it is natural to look for reasonable new axioms which will settle it.

A second purpose of new axioms is more mundane: to assist in the proof of independence results. To see how this happens, we need a couple of definitions.

Let ZFC be the usual set of axioms for set theory (including the Axiom of Choice). The reader need not be familiar with these axioms if he will accept the fact that such a set of axioms exists and leads to the usual results of set theory. We shall assume that ZFC is consistent, i.e., that no contradictions can be derived from the axioms of ZFC.

The independence results in which we are interested have the form 'A is not provable from ZFC' (where A is some statement of set theory). We show that such a result can be reformulated as a result on consistency.

LEMMA 1. *The statement A is not provable from ZFC iff the axiom system ZFC + **not**-A is consistent.*

Proof. If A is provable from ZFC, then A and not-A are provable from ZFC + not-A; so ZFC + not-A is inconsistent. Assume ZFC + not-A is inconsistent; we must prove A from the axioms of ZFC. One way to do this is by indirect proof; we assume not-A and derive a contradiction. We can in fact derive such a contradiction from not-A in ZFC because ZFC + not-A is inconsistent. Q.E.D.

It follows that the results of interest to us can be put in the form '$ZFC + A$ is consistent'. Now suppose that we wish to prove several such results; say the statements '$ZFC + A_i$ is consistent' for $i = 1, \cdots, k$. We can do this by devising a new axiom A; proving that $ZFC + A$ is consistent; and proving $A \to A_i$ in ZFC for $i = 1, \cdots, k$. We shall then have proved more than the consistency of each $ZFC + A_i$; we shall have proved the consistency of $ZFC + A_1 + \cdots + A_k$.

This suggests we look for axioms A such that: (a) we can prove that $ZFC + A$ is consistent; (b) we can prove $A \to B$ in ZFC for many interesting statements B. (It does not matter if A itself is not very interesting.) A good example is Gödel's axiom $V = L$. The purpose of this article is to study another such example.

1. Formulation of the Axiom. We shall formulate our axiom in terms of partially ordered sets (henceforth called **posets**). A poset P is a **net** if for every p and q in P, there is an r in P such that $p \leqq r$ and $q \leqq r$. A **subnet** of a poset P is a subset of P which is a net.

A subset D of a poset P is **dense** in P if for every $p \in P$, there is a $q \in D$ such that

$p \leqq q$. (This is actually density in the topological sense for a suitable topology on P, viz., that having as a base all sets of the form $\{p: p \geqq q\}$.) We are interested in subnets of P which intersect many dense subsets of P.

THEOREM 1. *Let P be a non-empty poset, and let $\{D_n\}$ be a sequence of dense subsets of P. Then there is a subnet of P which intersects each D_n.*

Proof. Choose p_n inductively so that $p_{n+1} \geqq p_n$ and $p_{n+1} \in D_n$. This is possible because D_n is dense. Since $n \leqq m$ implies $p_n \leqq p_m$, the p_n constitute a subnet. Q.E.D.

In what follows, κ and λ are infinite cardinals and α and β are ordinals. We identify κ with the first ordinal having κ predecessors. Thus $\aleph_0$ is identified with ω and $\aleph_1$ with the first uncountable ordinal.

Theorem 1 becomes false if $\{D_n\}$ is replaced by a collection $\{D_\alpha: \alpha < \aleph_1\}$ of $\aleph_1$ dense sets. To see this, let A have cardinal $\aleph_0$ and let B have cardinal $\aleph_1$. Let P be the set of mappings from a finite subset of A to B. For $p, q \in P$, let $p \leqq q$ mean that q is an extension of p. If Q is a subnet of P, there is a mapping f from A to B which extends every member of Q. Since f is not onto, there is a $b \in B$ not in the range of f; so Q does not intersect $D_b = \{p: b \in \mathrm{range}(p)\}$. On the other hand, each D_b is dense in P.

In view of this example, we put a restriction on P. We say that the elements p and q of P are **incompatible** if there is no r in P such that $p \leqq r$ and $q \leqq r$. We say that P satisfies the **countable chain condition** (abbreviated *CCC*) if every pairwise incompatible subset of P is countable.

For each κ, we introduce an axiom.

($A\kappa$) *If P is a non-empty poset satisfying the CCC, and $\{D_i: i \in I\}$ is a family of dense subsets of P having cardinal $\leqq \kappa$, then there is a subnet Q of P which intersects each D_i.*

Theorem 1 shows that $(A\aleph_0)$ is true. We will see in the next section that $(A\kappa)$ is false for $\kappa \geqq 2^{\aleph_0}$. Thus the interesting case is $\aleph_1 \leqq \kappa < 2^{\aleph_0}$.

Martin's Axiom is the statement that $(A\kappa)$ is true for $\aleph_1 \leqq \kappa < 2^{\aleph_0}$. Clearly the Continuum Hypothesis implies Martin's Axiom. The remarkable fact is that many consequences of the Continuum Hypothesis, if properly stated, are also consequences of Martin's Axiom. This is the viewpoint of [4]. From our viewpoint, it is best to consider $(A\kappa)$ for various κ, especially $\kappa = \aleph_1$. According to this viewpoint, we should establish the consistency of $ZFC + (A\aleph_1)$ and then derive interesting consequences of $(A\aleph_1)$.

THEOREM 2 [6]. *The axiom system $ZFC + (A\aleph_1) + 2^{\aleph_0} = \aleph_2$ is consistent.*

Unfortunately, the proof of Theorem 2 is too complicated to even sketch here. It requires a considerable extension of Cohen's forcing technique. We shall devote the rest of the paper to consequences of $(A\kappa)$.

2. Applications to Measure Theory. We know that $(A\aleph_0)$ is provable in ZFC. This suggests the consequences of $(A\kappa)$ will be statements about sets of cardinal $\leqq \kappa$ which are known to be true about countable sets. This is frequently (but not always) the case.

THEOREM 3 [4]. *If* $(A\kappa)$, *then the union of* $\leqq \kappa$ *sets of reals of (Lebesgue) measure zero is of measure zero.*

Proof. Let $\{E_i : i \in I\}$ be the $\leqq \kappa$ sets of measure 0, and let $E = \cup E_i$. Let $\varepsilon > 0$. We must find an open set including E which has measure $\leqq \varepsilon$.

Let P be the collection of all open sets of measure $< \varepsilon$; and for $p, q \in P$, let $p \leqq q$ mean $p \subseteq q$. We first show that the poset P satisfies the CCC. Let Q be a pairwise incompatible subset of P. Let $Q_n = \{p \in Q : m(p) \leqq (1 - 2^{-n})\varepsilon\}$ (where $m(p)$ is the measure of p). It is enough to show that Q_n is countable.

For each $p \in Q_n$, choose $\bar{p} \subseteq p$ so that $\bar{p}$ is a finite union of open intervals with rational endpoints and $m(p - \bar{p}) < 2^{-n}\varepsilon$. Since there are only countably many such finite unions, it is enough to show that if p and q are distinct members of Q_n, then $\bar{p} \neq \bar{q}$. Suppose $\bar{p} = \bar{q}$. Then $p \cup q \subseteq (p - \bar{p}) \cup q$; so $m(p \cup q) < 2^{-n}\varepsilon + (1 - 2^{-n})\varepsilon = \varepsilon$. This implies that p and q are compatible, contradicting $p, q \in Q$.

For $i \in I$, let $D_i = \{p \in P : E_i \subseteq p\}$. Using $m(E_i) = 0$, we easily conclude that D_i is dense in P. It follows by $(A\kappa)$ that there is a subnet Q of P which meets every D_i.

Let G be the union of the members of Q. Then G is open. From $D_i \cap Q \neq \emptyset$ we find that $E_i \subseteq G$; so $E \subseteq G$.

It remains to show that $m(G) > \varepsilon$ leads to a contradiction. By Lindelöf's Theorem, G is a countable union of sets in Q. It follows that there is a finite union G_1 of sets in Q such that $m(G_1) \geqq \varepsilon$. But since Q is a net, some member of Q includes G_1 and hence has measure $\geqq \varepsilon$. This is a contradiction. Q.E.D.

Let us see what this means about independence. By Theorem 3 and Theorem 2, the following axiom system is consistent: ZFC + 'the union of $\aleph_1$ sets of measure zero is of measure 0.' Thus by Lemma 1, we cannot prove the following in ZFC: 'there are $\aleph_1$ sets of measure 0 whose union is not of measure 0.' We leave to the reader the reformulation of our remaining results as independence results.

COROLLARY. *If* $(A\kappa)$, *then the union of* $\leqq \kappa$ *measurable sets is measurable.*

Proof. Let $\{E_i : i \in I\}$ be the sets, $E = \cup E_i$. Then E has a measurable kernel, i.e., a measurable subset F such that every measurable subset of $E - F$ has measure zero. In particular, $m(E_i - F) = 0$; so by Theorem 3, $E - F = \cup (E_i - F)$ has measure 0. It follows that $E = F \cup (E - F)$ is measurable. Q.E.D.

Since $[0,1]$ is the union of $2^{\aleph_0}$ sets which contain only one point and hence have measure 0, we see from Theorem 1 that (as previously mentioned) $(A\kappa)$ is false for $\kappa \geqq 2^{\aleph_0}$.

It follows that $(A\kappa)$ is only interesting for $\aleph_0 < \kappa < 2^{\aleph_0}$. Since such κ do not

frequently occur in mathematics, one might expect the consequences of $(A\kappa)$ to be only of interest to set theorists. We show this is not so by an example.

If $\boldsymbol{X}$ is a collection of sets of reals, let $C\boldsymbol{X}$ be the collection of complements of sets in $\boldsymbol{X}$ and let $P\boldsymbol{X}$ be the collection of continuous images of sets in $\boldsymbol{X}$. Let A be the collection of continuous images of Borel sets. The sets in A, CA, PCA, $CPCA$, $PCPCA, \cdots$ are called **projective sets,** and have been studied by topologists (see [3]). Lusin proved that every set in A or CA is measurable. Gödel showed that it is consistent with ZFC to assume that there is a non-measurable set in PCA. Now a rather deep result says that every set in PCA is the union of $\aleph_1$ Borel sets. Hence $(A\aleph_1)$ implies that every set in PCA is measurable (by the Corollary to Theorem 3).

This can be carried further. Martin has shown that the existence of a measurable cardinal implies that every set in $PCPCA$ is the union of $\aleph_2$ Borel sets. Thus if there is a measurable cardinal and $(A\aleph_2)$ holds, then every set in $PCPCA$ is measurable.

3. **Combinatorial Consequences.** Let ω be the set of natural numbers, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be collections of subsets of ω. An **$(\boldsymbol{A},\boldsymbol{B})$-set** is a subset C of ω such that $C \cap A$ is finite for every $A \in \boldsymbol{A}$ and $C \cap B$ and $C^c \cap B$ are infinite for every $B \in \boldsymbol{B}$. (We use C^c for the complement of C.)

A set D is **$\boldsymbol{A}$-small** if there are sets $A_1, \cdots, A_n$ in $\boldsymbol{A}$ such that $D-(A_i \cup \cdots \cup A_n)$ is finite. If an $(\boldsymbol{A},\boldsymbol{B})$-set exists, then clearly no set in $\boldsymbol{B}$ is $\boldsymbol{A}$-small.

THEOREM 4 [4]. *Let $(A\kappa)$ hold. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be collections of subsets of ω having cardinal $\leqq \kappa$. If no set in $\boldsymbol{B}$ is $\boldsymbol{A}$-small, then there is an $(\boldsymbol{A},\boldsymbol{B})$-set.*

Proof. Let P be the set of all mappings f from an $\boldsymbol{A}$-small subset of ω to $\{0,1\}$ such that $\{i \mid f(i) = 1\}$ is finite. We make P into a poset by letting $f \leqq g$ hold if g is an extension of f.

We first show P satisfies the CCC. If f and g are members of P which have a common extension, then the smallest common extension of f and g is in P. Thus if f and g are incompatible, there is an i such that one of $f(i)$ and $g(i)$ is 0 and the other is 1. It follows that $\{f\colon f(i) = 1\} \neq \{g : g(i) = 1\}$. Since there are only countably many finite subsets of ω, a pairwise incompatible subset of P must be countable.

For $A \in \boldsymbol{A}$, let D_A be the set of $f \in P$ whose domain includes A. It is clear that D_A is dense in X.

For $B \in \boldsymbol{B}$, let $D_{B,n}$ be the set of $f \in P$ such that $B \cap \{i : f(i) = 0\}$ and $B \cap \{i : f(i) = 1\}$ both have cardinal $\geqq n$. We show that $D_{B,n}$ is dense in P. Let $f \in P$. Since B is not $\boldsymbol{A}$-small, we can find $2n$ numbers $i_1, \cdots, i_n, j_1, \cdots, j_n$ in B but not in the domain of f. Extend f by setting $f(i_\kappa) = 0, f(j_\kappa) = 1$. The extended f is in $D_{B,n}$.

By $(A\kappa)$, there is a subnet Q of P which intersects all the D_A and $D_{B,n}$. Since Q is a net, there is a mapping g from ω to $\{0,1\}$ which extends every member of Q. Let $C = \{x \mid g(x) = 1\}$. If $A \in \boldsymbol{A}$, g extends some $f \in D_A$. Then $C \cap A = \{x \in A \mid f(x) = 1\}$; so $C \cap A$ is finite. If $B \in \boldsymbol{B}$ and $n \in \omega$, then g extends

some member of $D_{B,n}$; so $C \cap B$ and $C^c \cap B$ have at least n members. Since this holds for all n, $C \cap B$ and $C^c \cap B$ are infinite. Q.E.D.

We say $\boldsymbol{A}$ is **proper** if ω is not $\boldsymbol{A}$-small and if for each $A \in \boldsymbol{A}$, A is not $(\boldsymbol{A}-\{A\})$-small.

LEMMA 2. *Assume* $(A\kappa)$. *Let* $\boldsymbol{A}$ *be proper and have cardinal* $\leqq \kappa$, *and let* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *be disjoint subsets of* $\boldsymbol{A}$. *Then there is an* $(\boldsymbol{A}_1, \boldsymbol{A}_2)$-*set* C *such that* $\boldsymbol{A} \cup \{C\}$ *is proper.*

Proof. Let $\boldsymbol{B}$ consist of all sets $D-(A_1 \cup \cdots \cup A_n)$, where $A_i, \cdots, A_n \in \boldsymbol{A}$ and D is either ω or a member of $\boldsymbol{A}-\boldsymbol{A}_1$ distinct from $A_1, \cdots, A_n$. Since $\boldsymbol{A}$ is proper, no set in $\boldsymbol{A}_2 \cup \boldsymbol{B}$ is $\boldsymbol{A}_1$-small. Since $\boldsymbol{B}$ clearly has cardinal $\leqq \kappa$, it follows from Theorem 4 that there is a $(\boldsymbol{A}_1, \boldsymbol{A}_2 \cup \boldsymbol{B})$-set C. We must show that $\boldsymbol{A} \cup \{C\}$ is proper.

Let $A_1, \cdots, A_n \in \boldsymbol{A}$. Since $\omega-(A_1 \cup \cdots \cup A_n)$ is in $\boldsymbol{B}$, $C-(A_1 \cup \cdots \cup A_n)$ and $\omega-(A_1 \cup \cdots \cup A_n \cup C)$ are infinite. This shows that C is not $\boldsymbol{A}$-small and that ω is not $(\boldsymbol{A} \cup \{C\})$-small. We must now show that if $A \in \boldsymbol{A}$, then A is not $(\boldsymbol{A} \cup \{C\}-\{A\})$-small. Let $A_1, \cdots, A_n \in \boldsymbol{A}-\{A\}$. If $A \notin \boldsymbol{A}_1$, then $A-(A_1 \cup \cdots \cup A_n)$ is in $\boldsymbol{B}$ and hence $A-(A_1 \cup \cdots \cup A_n \cup C)$ is infinite. If $A \in \boldsymbol{A}_1$, $A \cap C$ is finite and $A-(A_1 \cup \cdots \cup A_n)$ is infinite; so again, $A-(A_1 \cup \cdots \cup A_n \cup C)$ is infinite. Q.E.D.

THEOREM 5 [4]. *If* $(A\kappa)$, *then* $2^{\kappa}=2^{\aleph_0}$.

Proof. Using Lemma 2 and transfinite induction, we can choose A_α for $\alpha<\kappa$ so that $\{A_\alpha: \alpha<\kappa\}$ is proper. For each $I \subseteq \{\alpha: \alpha<\kappa\}$, we may choose a $(\{A_\alpha: \alpha \in I\}, \{A_\alpha: \alpha \notin I\})$-set C_I by Theorem 4. Since $\alpha \in I$ iff $C_I \cap A_\alpha$ is finite, the mapping from I to C_I is one-one. Since there are 2^κ possible I's and $2^{\aleph_0}$ subsets of ω, $2^\kappa \leqq 2^{\aleph_0}$; so $2^\kappa=2^{\aleph_0}$. Q.E.D.

We now consider a problem which appeared in this MONTHLY (Advanced Problem 5845). Let $O_\kappa=\{\alpha: \alpha<\kappa\}$, and let $\boldsymbol{S}_\kappa$ be the σ-ring in $O_\kappa \times O_\kappa$ generated by the set of all rectangles (i.e., sets of the form $A \times B$). Let $(R\kappa)$ be the statement that every subset of $O_\kappa \times O_\kappa$ is in $\boldsymbol{S}_\kappa$. For what uncountable κ is $(R\kappa)$ true? The published answer gives references to show that $(R\aleph_1)$ is true and that $(R\kappa)$ is false if $\kappa>2^{\aleph_0}$. This does not completely solve the problem unless we assume the Continuum Hypothesis.

THEOREM 6 [2]. *If* $(A\lambda)$ *holds for all* $\lambda<\kappa$, *then* $(R\kappa)$ *holds.*

Proof. Let $W \subseteq O_\kappa \times O_\kappa$. Using transfinite induction, we select subsets A_α and B_α of ω for $\alpha<\kappa$ so that $A_\alpha \cap B_\beta$ is infinite iff $(\alpha, \beta) \in W$. Suppose A_α and B_α have been selected for $\alpha<\gamma$, and suppose (as an induction hypothesis) that $\{A_\alpha: \alpha<\gamma\} \cup \{B_\alpha: \alpha<\gamma\}$ is proper. Using Lemma 2, we first select A_γ so that $A_\gamma \cap B_\alpha$ is infinite iff $(\gamma, \alpha) \in W$ and $\{A_\alpha: \alpha \leqq \gamma\} \cup\{B_\alpha: \alpha<\gamma\}$ is proper; we then select B_γ so that $A_\alpha \cap B_\gamma$ (for $\alpha \leqq \gamma$) is infinite iff $(\alpha, \gamma) \in W$ and $\{A_\alpha: \alpha \leqq \gamma\} \cup \{B_\alpha: \alpha \leqq \gamma\}$ is proper.

Set $X_n=\{\alpha \mid n \in A_\alpha\}$, $Y_n=\{\alpha \mid n \in B_\alpha\}$. Then $(\alpha, \beta) \in X_n \times Y_n$ iff $n \in A_\alpha \cap B_\beta$; so

$(\alpha,\beta) \in W$ iff $(\alpha,\beta) \in X_n \times Y_n$ for infinitely many n. Hence

$$W = \bigcap_m \bigcup_{n \geqq m} (X_n \times Y_n);$$

so $W \in \boldsymbol{S}_\kappa$. Q.E.D.

Note that Theorem 6 gives a new proof of $(R\aleph_1)$. Kunen [2] has also shown that $(R\aleph_2)$ is not provable in ZFC from $2^{\aleph_0} \geqq \aleph_2$.

4. Topological consequences. Let X be a topological space. Recall that a set A in X is **nowhere dense** if every non-empty open set includes a non-empty open set disjoint from A. A set is **meager** if it is the union of countably many nowhere dense sets. A set A has the **property of Baire** if there is an open set B such that $A-B$ and $B-A$ are meager.

There is an analogue between meager sets and sets of measure zero; likewise between sets having the property of Baire and measurable sets. We shall prove the analogue of Theorem 3.

THEOREM 7 [**4**]. *Assume* $(A\kappa)$. *If X is a topological space having a countable base, then the union of* $\leqq \kappa$ *meager sets in X is meager.*

Proof. It is clearly enough to show that if $\{E_i : i \in I\}$ is a collection of $\leqq \kappa$ nowhere dense sets, then their union E is meager. Let $\{G_n : n \in \omega\}$ be a sequence in which every non-empty set in some countable base of X appears infinitely often (and no other set appears). For $i \in I$, let $A_i = \{n : E_i \cap G_n \neq \varnothing\}$. For $n \in \omega$, let $B_n = \{m : G_m \subseteq G_n\}$. Let $\boldsymbol{A} = \{A_i : i \in I\}$, $\boldsymbol{B} = \{B_n : n \in \omega\}$.

We show no B_n is $\boldsymbol{A}$-small. If I_0 is a finite subset of I, $E_{I_0} = \bigcup_{i \in I_0} E_i$ is nowhere dense. Hence there are infinitely many m such that $G_m \subseteq G_n$ and $G_m \cap E_{I_0} = \varnothing$. This means there are infinitely many m in B_n which are not in A_i for any $i \in I_0$.

By Theorem 4, there is an $(\boldsymbol{A},\boldsymbol{B})$-set C. Let H_k be the union of the G_m for $m \geqq k$ and $m \in C$. Then H_k is open. We show that H_k^c is nowhere dense. It suffices to show that for each n, there is a G_m included in $H_k \cap G_n$, i.e., an $m \geqq k$ such that $m \in C$ and $m \in B_n$. This holds because $C \cap B_n$ is infinite.

We must show now that $E \subseteq \cup_k H_k^c$. It will suffice to show that for each i, there is a k such that $E_i \cap H_k = \varnothing$. Since $A_i \cap C$ is finite, we can choose k greater than every member of $A_i \cap C$. If $m \geqq k$ and $m \in C$, then $m \notin A_i$; so $E_i \cap G_m = \varnothing$. It follows that $E_i \cap H_k = \varnothing$. Q.E.D.

COROLLARY. *Assume* $(A\kappa)$. *If X is a topological space having a countable base, then the union of* $\leqq \kappa$ *sets in X having the property of Baire has the property of Baire.*

The Corollary to Theorem 7 has consequences similar to the Corollary to Theorem 3 (for X the space of reals).

Suppose that in the topological space X, x is a limit point of the sequence $\{x_n\}$. Then x need not be the limit of a subsequence of $\{x_n\}$. This will be the case, however, if there is a countable base at x.

THEOREM 8. *Assume* $(A\kappa)$. *In the topological space* X, *let* x *be a limit point of the sequence* $\{x_n\}$. *Suppose that there is a base at* x *having cardinal* $\leqq \kappa$. *Then there is a subsequence of* $\{x_n\}$ *converging to* x.

Proof. Let $\{G_i: i \in I\}$ be the base at x. Let $A_i = \{n: x_n \notin G_i\}$, $A = \{A_i: i \in I\}$. We show that ω is not A-small. Let I_0 be a finite subset of I, $G = \bigcap_{i \in I_0} G_i$. Then G is a neighborhood of x; so $x_n \in G$ for infinitely many n. Each such n is in A_i^c for $i \in I_0$.

By Theorem 4, there is an infinite set C such that $C \cap A_i$ is finite for all i. The subsequence $\{x_n: n \in C\}$ then converges to x. Q.E.D.

A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence. It is well known that a countable product of sequentially compact spaces is sequentially compact. On the other hand, it is easy to show that the product of $2^{\aleph_0}$ two-point discrete spaces is not sequentially compact.

COROLLARY. *If* $(A\kappa)$, *then the product of* $\leqq \kappa$ *compact metric spaces is sequentially compact.*

Proof. Let $\{X_i: i \in I\}$ be the collection of spaces, X their product. Since X is compact, every sequence in X has a limit point. In view of Theorem 8, it is enough to show that X has a base of cardinal $\leqq \kappa$. This is a simple exercise in cardinal arithmetic (noting that each X_i has a countable base). Q.E.D.

This corollary generalizes a theorem of Booth. Booth [**1**] has also applied Martin's axiom to questions about dimension.

We state without proof an unpublished result of Silver bearing on a famous topological problem.

THEOREM 9 (Silver). *If* $(A\aleph_1)$, *then there is a separable normal non-metrizable Moore space.*

5. Souslin's Problem. We now discuss briefly the problem which inspired Theorem 2. Souslin's problem was originally stated in terms of axioms for the ordered set of reals. We use instead a problem about posets, which was proved equivalent to the original problem by Miller [**5**].

A poset P is a **chain** {**antichain**} if every two distinct elements in P are comparable {incomparable}. A **tree** is a poset P such that for each $x \in P$, $\{y: y \leqq x\}$ is a chain. A **Souslin tree** is an uncountable tree P such that every chain or antichain in P is countable. We let (SH) be the statement that there is no Souslin tree.

Soon after the invention of forcing, Tennenbaum and Jech (independently)

showed that SH is not provable in ZFC. The next theorem, in conjunction with Theorem 2, shows that not-(SH) is not provable in ZFC.

LEMMA 3. *If there is a Souslin tree, there is a Souslin tree Q having cardinal $\aleph_1$ such that for every $x \in Q$, $\{y \in Q: y \geqq x\}$ is uncountable.*

Proof. Since every uncountable subset of a Souslin tree is a Souslin tree, we may assume that we have a Souslin tree P of cardinal $\aleph_1$. Let Q be the set of $x \in P$ such that $\{y \in P: y \geqq x\}$ is uncountable.

Using Zorn's lemma, choose a maximal antichain Z in $P-Q$. Since P is a Souslin tree, Z is countable. If $z \in Z$, $\{y \in P: y \geqq z\}$ is countable and $\{y \in P: y \leqq z\}$ is a chain and hence is countable. Thus only countably many y are comparable with some $z \in Z$. By choice of Z, these include all elements of $P-Q$; so $P-Q$ is countable. Thus Q has cardinal $\aleph_1$ and hence is a Souslin tree. If $x \in Q$, $\{y \in Q: y \geqq x\}$ includes all members of $\{y \in P: y \geqq x\}$ not in $P-Q$; so $\{y \in Q: y \geqq x\}$ is uncountable. Q.E.D.

THEOREM 10 [6]. *If ($A\aleph_1$), then (SH).*

Proof. Suppose otherwise. Let Q be as in Lemma 3. We can write $Q = \{x_\alpha : \alpha < \aleph_1\}$. Let $Q_\beta = \{x_\alpha : \beta \leqq \alpha < \aleph_1\}$. Our choice of Q implies that each Q_β is dense in Q. Every pairwise incompatible set is an antichain; so Q satisfies the CCC. Hence by ($A\aleph_1$), there is a subnet N of Q such that N intersects each Q_β. Since Q is a tree, the subnet N is a chain. But clearly N is uncountable; so we have a contradiction. Q.E.D.

Since ($A\aleph_1$) implies that the Continuum Hypothesis is false, Theorem 10 leaves open the question of whether not-(SH) is provable in ZFC from the Continuum Hypothesis. Recently Jensen has shown that it is not; his proof requires a considerable extension of the methods used to prove Theorem 2.

References

1. D. Booth, Generic covers and dimension, Duke Math. J., 38 (1971) 667–670.

2. K. Kunen, Inaccessibility properties of cardinals, Dissertation, Stanford University, 1968.

3. C. Kuratowski, Topologie, Warsaw, (1948).

4. D. A. Martin and R. M. Solovay, Internal Cohen extensions, Ann. Math. Logic, 2 (1970) 143–178.

5. E. W. Miller, A note on Souslin's problem, Amer. J. Math., 65 (1943) 673–678.

6. R. M. Solovay and S. Tennenbaum, Cohen extensions and Souslin's problem, Ann. Math., 94 (1971) 201–245.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

24. Lindsay Childs. I believe that the question of deciding whether a given integer a is a primitive root for infinitely many primes p, is difficult and that it relates to both Fermat's Last Theorem as well as the Riemann hypothesis. Is this really true? Where can I read about the present state of knowledge of the problem and its connection with the two famous unsolved problems?

25. B. R. Ust. I would be very grateful for a list of references, accessible to college students, that would illustrate the uses of mathematics in making maps.

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

INVERTIBILITY OF FUNCTION SPACES

JAMES CHEW

Let us first review the pertinent definitions. A space is *invertible* at $a \in X$ if given an open set U containing a, there exists a homeomorphism h of X onto X such that $h(X - U) \subset U$. If a space is invertible at each of its points, it is said to be invertible. Let $\mathscr{S}$ be a sub-base for the topology of X. We say X is *sub-invertible* at a point $a \in X$ if given $S \in \mathscr{S}$ with $a \in S$, there exists a homeomorphism h of X onto X such that $h(X - S) \subset S$. A space is sub-invertible if it is sub-invertible at each of its points. A space X is *near-homogeneous* at a point $a \in X$ if given an open set U containing a and given $b \in X$, there exists a homeomorphism h of X onto X such

that $h(b) \in U$. A space is near-homogeneous if it is near-homogeneous at each of its points. The qualifier 'sub' in front of near-homogeneity has the same effect as in the corresponding situation for invertibility.

The following theorem from [1] gives the basic characterization of invertibility.

THEOREM 1. *A space X is invertible iff given a non-empty open set U and a proper closed set C, there exists a homeomorphism h of X onto X such that $h(C) \subset U$.* ▮

By variously specializing one or more of the three constituents h, C and U appearing in Theorem 1, previous authors have isolated such properties as weak-invertibility, sub-near-homogeneity, near-homogeneity and localizations of these properties. In order to make transparent the connections among the properties which are of concern to us at this time, it is convenient to make the following:

DEFINITION 1. Let X be a topological space (with $\operatorname{card}(X) > 1$!) and let P be a property which can be assigned to subsets of X. Let S_P be a proper subset of X having property P. We say X *satisfies* H_P if given a non-empty open set U, there exists a homeomorphism h of X onto X such that $h(S_P) \subset U$.

We then have the following hierarchy of properties with invertibility at one extreme and near-homogeneity at the other. X satisfies H_P where:

(a) (invertible) *P is the property of being closed,*
(b) (weakly invertible) *P is the property of being compact,*
(c) (?) *P is the property of being finite,*
(d) (near-homogeneous) *P is the property of being a singleton set.*

The phrase 'weakly invertible' was coined in [3]. Clearly (a) $\overset{T_2}{\Rightarrow}$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) while $\mathbb{R}^n$ is an example showing that the first implication cannot be reversed. We have not studied property (c) at all.

We acknowledge our gratitude to the referee for pointing out that the full force of invertibility is not required in the hypotheses of some of the theorems that follow. The referee also points out that the two theorems from [2] whose proofs are incomplete may be salvaged by adding the hypothesis that the spaces under consideration are not compact. The referee's comments and suggestions have also played a part in increasing the clarity of the present article.

Let F denote the set of all continuous functions from a space X into itself. The compact-open topology on F will be denoted by $\mathscr{C}$, the point-open topology by $\mathscr{P}$ and the subset of F consisting of constant maps by M.

The following lemma is stated in [2] for the compact-open topology only. The proof for the point-open topology goes through with minor modifications only.

LEMMA 1. *Let h be a homeomorphism of X onto X. Then h induces a homeomorphism h_* of F onto F where h_* is defined by $h_*(f) = h \circ f$. Here F may be considered as a space with either topology $\mathscr{C}$ or $\mathscr{P}$.* ▮

The next two statements are theorems from [**2**].

Let X be an invertible Hausdorff space. Then:

(i) $(F,\mathscr{C})$ is sub-near-homogeneous,

(ii) $(F,\mathscr{C})$ is near-homogeneous at each point of M.

We have been unable to prove (i) or (ii). The arguments given in [**2**] for these statements are not valid. However, we are able to prove the next two theorems which are alternatives of statements (i) and (ii).

THEOREM 2. *Let X be a non-compact weakly invertible space. Then $(F,\mathscr{C})$ is sub-near-homogeneous.*

Proof. Let f and g be given elements of F and let $W = \{\phi \in F: \phi(C) \subset U\}$ be a given sub-basic open set containing f. Since $W \neq \varnothing$, we see that $U \neq \varnothing$. Now $K = g(C)$ is a compact subset of X and hence $K \neq X$. Let h be a homeomorphism of X onto X such that $h(K) \subset U$. Then h_* is a homeomorphism of F onto F such that $h^*(g) \in W$. ▮

The proof of our Theorem 2 above is precisely that of statement (i) as found in [**2**]. To see where this proof breaks down when applied to the case where X is compact, we need only consider $g \in F$ such that $g(C) = X$. Clearly no homeomorphism of X onto X can carry $g(C) = X$ into a proper open subset U of X. The proof given in [**2**] for statement (ii) has a similar deficiency. The appropriately revised version of statement (ii) is as follows:

THEOREM 3. *Let X be a non-compact weakly invertible space. Then $(F,\mathscr{C})$ is near-homogeneous at each point of M.* ▮

When the compact-open topology $\mathscr{C}$ is replaced by the point-open topology $\mathscr{P}$, we can get by with even less than weak-invertibility. Notice also that the requirement that X be non-compact has been dropped in the next two theorems.

THEOREM 4. *Let X be a near-homogeneous space. Then $(F,\mathscr{P})$ is sub-near-homogeneous.*

Proof. Let $W = \{\phi \in F: \phi(p) \in U\}$ be a given sub-basic open set containing $f \in F$, and let $g \in F$ be given. Then there exists a homeomorphism h of X onto X such that $h(g(p)) \in U$. Clearly $h_*(g) \in W$. ▮

THEOREM 5. *Let X be a space of infinite cardinality satisfying H_P where P = the property of being finite. Then $(F,\mathscr{P})$ is near-homogeneous at each point of M.*

Proof. Let $f \in M$ be given. Then there exists $p \in X$ such that $f(X) = \{p\}$. Let $B = \{\phi \in F\colon \phi(p_i) \in U_i,\ i = 1,2,\cdots,n\}$ be a given basic open set in F containing f. Note that $f \in B$ implies $p \in \bigcap_{i=1}^{n} U_i$. Let $g \in F$ be given. Then $C = \{g(p_i)\colon i = 1,2,\cdots,n\}$ is a proper finite subset of X. Hence there exists a homeomorphism h of X onto X such that $h(C) \subset U$ where $U = \bigcap_{i=1}^{n} U_i$. Thus $h^*(g) \in B$. ▮

A question in [2] asks: Is $(F,\mathscr{C})$ sub-invertible if X is an invertible Hausdorff space? The answer for $(F,\mathscr{P})$ is given below. No separation axiom on X is required.

THEOREM 6. *If X is an invertible space, then $(F,\mathscr{P})$ is sub-invertible.*

Proof. Let $W = \{f \in F: f(p) \in U\}$ be a non-empty sub-basic open set in F. Let h be a homeomorphism of X onto X such that $h(X - U) \subset U$. Then $g \notin W$ implies $g(p) \in X - U$ and so $h(g(p)) \in U$. Hence $h_*(F - W) \subset W$. ▌

Let X be an arbitrary space and let F be equipped with the compact-open topology. For each $x \in X$, let m_x denote that element of M mapping all of X to x. The proof of the following lemma is straightforward.

LEMMA 2. *Let X be an arbitrary space and let F be given the compact-open topology. Then the canonical mapping $x \to m_x$ is a homeomorphism from X to F.* ▌

The next theorem is reminiscent of Theorem 2.4 of [2].

THEOREM 7. *Let X be an invertible space. As a closed subspace of $(F,\mathscr{C})$, the space M of constant maps is invertible.* ▌

A remark in [1] states that invertibility, by itself, seems to have very little to do with the separation axioms. Our inability to prove that an invertible homogeneous Hausdorff space is regular supports this belief. We close this note by suggesting a way in which to connect invertibility with the separation axioms.

DEFINITION 2. We say a space is *homogeneously invertible* if given a non-empty open set U and given $a \in U$ and $b \in X - U$, there exists a homeomorphism of X onto X such that $h(b) = a$ and $h(X - U) \subset U$.

Obviously a homogeneously invertible T_1-space is both homogeneous and invertible. Is a Hausdorff space homogeneously invertible if it is both homogeneous and invertible?

THEOREM 8. *A homogeneously invertible Hausdorff space is regular.*

Proof. Let U be an open set and let $a \in U$. Let $b \in X - U$. Since X is Hausdorff, there exist disjoint open sets V and W containing a and b respectively. We may assume $V \subset U$. Let h be a homeomorphism of X onto X such that $h(X - V) \subset V$ and $h(b) = a$. Since V and $\bar{W}$ are disjoint, it follows that $h(\bar{W}) \subset V \subset U$. But $h(\bar{W}) = \overline{h(W)}$ and so $h(W)$ is an open set containing a such that $\overline{h(W)} \subset U$. This shows that X is regular. ▌

Is a homogeneously invertible Hausdorff space normal?

References

1. P. H. Doyle and J. G. Hocking, Invertible spaces, this MONTHLY, 68 (1961) 959–965.
2. S. A. Naimpally, Function spaces of invertible spaces, this MONTHLY, 73 (1966) 513–515.
3. J. G. Hocking, Schwache lokale Invertierbarkeit, Archiv der Math., XV (1964) 46–49.

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CANTOR'S SERIES FOR VECTORS

M. S. WATERMAN

Decimal expansions have been generalized in several directions. First, there are the well-known expansions with arbitrary integer bases. For a sequence of integers $q_1, q_2, \cdots$ $(q_i \geqq 2)$ Cantor [1] obtained the expansion

$$x = \sum_{m=1}^{\infty} \frac{a_m}{q_1 q_2 \cdots q_m}, \tag{1}$$

where $x \in [0,1)$ and $a_m \in \{0, 1, \cdots, q_m - 1\}$. See [5] for a survey on expansions with references to Cantor's series. There are also expansions with non-integer bases [4, 5, 8, 9], negative bases [7], and similar expansions exist for complex numbers [2]. In n-dimensional Euclidean space, matrix expansions and their associated transformations have been studied extensively [3, 8]. The purpose of this note is to generalize Cantor series (1) to matrix expansions and to give a list of examples of such expansions.

Let n be a fixed positive integer and $Q_1, Q_2, \cdots$ be a sequence of nonsingular $n \times n$ matrices. We take $|\ |$ to be a norm on n-dimensional Euclidean space and take $\|\ \|$ to be a compatible matrix norm (i.e., $|Qx| \leqq \|Q\| \cdot |x|$). (See Lancaster [6] for material on matrix norms.) Let $I = \times_{i=1}^{n} [0,1)$ and assume $|x| \leqq B$ for $x \in I$. Next we make some fundamental definitions.

$$\begin{aligned} T^{(i)}(x) &= Q_i x - [Q_i x] && (i \geqq 1), \\ T_m &= T^{(m)} \circ T^{(m-1)} \circ \cdots \circ T^{(1)} && (m \geqq 1), \\ a_1(x) &= [Q_1 x], \\ a_m(x) &= [Q_m(T_{m-1}(x))] && (m \geqq 2), \end{aligned}$$

where $[(y_1, y_2, \cdots, y_n)^T] = ([y_1], [y_2], \cdots, [y_n])^T$ and $[\ \ ]$ is the usual greatest integer function in this last expression.

THEOREM. *Assume* $\sup_i \|Q_i^{-1}\| < 1$. *Then*

$$x = \sum_{i=1}^{\infty} (Q_i Q_{i-1} \cdots Q_1)^{-1} a_i(x). \tag{2}$$

Proof. $x = Q_1^{-1} Q_1 x = Q_1^{-1}(a_1(x) + T_1(x)) = Q_1^{-1} a_1(x) - Q_1^{-1} T_1(x)$. For our induction, assume for a positive integer m that

$$x = \sum_{i=1}^{m} Q_1^{-1} Q_2^{-1} \cdots Q_i^{-1} a_i(x) + Q_1^{-1} Q_2^{-1} \cdots Q_m^{-1} T_m(x). \tag{3}$$

Then $Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}T_m(x) = Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}Q_{m+1}^{-1}(a_{m+1}(x) + T_{m+1}(x))$ and (3) holds for all m. Therefore, since $T_m(x) \leqq B$,

$$\lim_{m\to\infty} \left| x - \sum_{i=1}^{m} Q_1^{-1}Q_2^{-1}\cdots Q_i^{1}a_i(x) \right| = \lim_{m\to\infty} \left| Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}T_m(x) \right|$$

$$\leqq \lim_{m\to\infty} B \| Q_1^{-1} \| \| Q_2^{-1} \| \cdots \| Q_m^{-1} \| \leqq B \lim_{m\to\infty} (\sup_i \| Q_i^{-1} \|^m) = 0.$$

Thus (2) holds and the theorem is proved.

Clearly the theorem holds when $\|(Q_mQ_{m-1}\cdots Q_1)^{-1}\| \to 0$ as $m \to \infty$. Under the assumptions of the theorem, the convergence is geometric with rate $B(\sup_i \| Q_i^{-1} \|)^m$.

Next we give some examples, both general and numerical, of the theorem. As noted below, examples 1, 2, and 3 are known. In the case $n > 1$ and the Q_i are not all identical, the result was not known. Example 4 gives a numerical example of this situation.

Example 1. If $n = 1$ and $|\ \ |$, $\|\ \ \|$ are both the usual absolute value on the real line, we can obtain the results mentioned above for any $x \in [0,1)$. If $Q_i \equiv q$ is a positive integer, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{q^m}, \quad a_m(x) \in \{0, 1, \cdots, q-1\}. \tag{4}$$

This is the usual q-adic expansion. If $Q_i \equiv \beta$ where $\beta > 1$ and β is not an integer, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\beta^m}, \quad a_m(x) \in \{0, 1, \cdots, [\beta]\}. \tag{5}$$

These β-expansions have been extensively studied [**3**, **4**, **5**, **9**, **10**]. If we let $Q_i = \gamma_i$, $\gamma_i \in R$, $\inf_i |\gamma_i| > 1$, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\gamma_1\gamma_2\cdots\gamma_m}. \tag{6}$$

This last formulation allows expansions with negative bases (e.g., -10) as well as mixtures of integral and non-integral positive and negative numbers. For some material on expansions with negative radices, see [**7**].

Example 2. Let $n = 2$. If $Q_i \equiv \begin{bmatrix} r & -q \\ q & r \end{bmatrix} = Q$, then $Qx = (rx_1 - qx_2, qx_1 + rx_2)^T$ which is equivalent to $(x_1 + ix_2)(r + iq) = (rx_1 - qx_2) + i(qx_1 + rx_2)$. Therefore if we take

$$|x| = \sqrt{x_1^2 + x_2^2} \text{ and } \|Q\| = \sqrt{r^2 + q^2},$$

we have $|Qx| \leqq \|Q\| \, |x|$ by the theory of complex numbers. Of course, $\|Q^{-1}\| = (r^2 + q^2)^{-1/2}$. Thus

$$x = \sum_{m=1}^{\infty} Q^{-m} a_m(x), \tag{7}$$

and the expansion is valid for complex numbers $x = x_1 + x_2 i$ ($|x_j| < 1$) with complex base satisfying $r^2 + q^2 > 1$. This transformation and the expansion have been studied by Fischer [2, 3].

Example 3. If $|x|^2 = \Sigma_{i=1}^{n} x_i^2$, then $\|Q\|^2 = \Sigma_{i,j} q_{ij}^2$ is a compatible matrix norm. As a specific numerical example with $n = 2$, we take $x^T = (1/2, 3/4)$ and $Q = \begin{bmatrix} 2 & 0 \\ 2/3 & 4/3 \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/4 & 3/4 \end{bmatrix}$ and $\|Q^{-1}\|^2 = 7/8 < 1$. We compute

$$\begin{aligned} a_1(x) &= (1,1)^T \\ a_2(x) &= a_3(x) = a_4(x) = (0,0)^T \\ a_5(x) &= (0,1)^T. \end{aligned}$$

The fifth order approximation to x is $Q^{-1}a_1(x) - Q^{-5}a_5(x) = (1/2,\ 755/1024)^T$. Finally we check the rate of convergence.

$$|x - Q^{-1}a_1(x) - Q^{-5}a_5(x)| = |(0, 13/1024)^T| = (13/1024) < 2(7/8)^{5/2}\ .$$

The last inequality is by the guaranteed rate of convergence given in the proof of the theorem where $B = \sup_{x \in I} |x| = 2$.

Example 4. If $|x| = \max\{|x_i| : 1 \leqq i \leqq n\}$, then $\|Q\| = n \max\{|q_{ij}| : 1 \leqq i, j \leqq n\}$ is a compatible matrix norm. For the matrices given in Table 1 below, we have

TABLE 1. Expansion of $(1/2, 1/2)^T$

m	1	2	3	4
Q_m	$\begin{bmatrix} 3/2 & 3/2 \\ -3/2 & 3/2 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 \\ 4/3 & 8/3 \end{bmatrix}$	$\begin{bmatrix} 8 & -4 \\ -8 & 8 \end{bmatrix}$	$\begin{bmatrix} 3/2 & -3/2 \\ 3/2 & 3/2 \end{bmatrix}$
Q_m^{-1}	$\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$	$\begin{bmatrix} 1/4 & 0 \\ -1/8 & 3/8 \end{bmatrix}$	$\begin{bmatrix} 1/4 & 1/8 \\ 1/4 & 1/4 \end{bmatrix}$	$\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$
$\|Q_m^{-1}\|$	2/3	3/4	1/2	2/3
$a_m^T(x)$	(0, 1)	(2,0)	(−3, 5)	(0, 1)
$A_m^T(x)$	(1/3, 1/3)	(7/12, 5/12)	(97/192, 91/192)	(1/2, 1/2)

$$(1/2, 1/2)^T = Q_1^{-1}(1,0)^T + (Q_2Q_1)^{-1}(2,0)^T + (Q_3Q_2Q_1)^{-1}(-3,5)^T + (Q_4Q_3Q_2Q_1)^{-1}(0,1)^T.$$

Let $A_m(x) = \Sigma_{m=1}(Q_i \cdots Q_1)^{-1} a_i(x)$ in Table 1.

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References

1. G. Cantor, Über die einfachen Zahlensysteme, Z. Math. und Physik, 14 (1869) 121–128.

2. R. Fischer, Ergodische Eigenschaften komplexer Ziffernentwicklungen, S.-B. Osterr. Akad. Wiss. Math.-Natur. Kl. Abt. II, 180 (1972) 49–68.

3. ———, Mischungsgeschwindigkeit für Ziffernentwicklungen nach reellen Matrizen, Acta Arith., 23 (1973) 5–12.

4. J. Galambos, A generalization of a theorem of Borel concerning the distribution of digits in dyadic expansions, this MONTHLY, 78 (1971) 774–779.

5. ———, Probabilistic theorems concerning expansions of real numbers, Periodica Math. Hungar., 3 (1973) 101–113.

6. P. Lancaster, Theory of Matrices, Academic Press, New York, 1969.

7. A. Oppenheim, Criteria for irrationality of certain classes of numbers, this MONTHLY, 61 (1954) 235–241.

8. W. Philipp, Mischungseigenschaften gewisser auf dem Torus definierter Endomorphismen, Math. Zeitschr., 101 (1967) 369–374.

9. W. Parry, On the β-expansions of a real number, Acta Math. Acad. Sci. Hungar., 15 (1964) 107–113.

10. A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar., 8 (1957) 477–493.

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THE GAUSS-GREEN AND CAUCHY INTEGRAL THEOREMS

W. F. EBERLEIN

1. Introduction. The early proofs of Cauchy's integral theorem $\int_\gamma f(z)dz = 0$ separated real and imaginary parts and appealed to the *classical* Gauss-Green theorem

$$\int_{\partial R} Pdx + Qdy = \iint_R (\partial Q/\partial x - \partial P/\partial y)dxdy$$

to express the resulting line integrals as double integrals, which then vanish because of the Cauchy-Riemann equations. This procedure is valid only under the assumptions that f' is continuous and that the contour γ satisfies severe regularity conditions. As a legacy of new ideas introduced by E. Goursat and simplified by such mathematicians as E. H. Moore, A. Pringsheim, and K. Knopp, we now possess a clean elementary proof of Cauchy's theorem when f is analytic in a disk containing (any

rectifiable) γ under *no* assumptions about the continuity of f'. (Bochner [1] has carried over the Goursat decomposition of a rectangle or triangle to general regions with rectilinear boundaries to establish a Gauss-Green theorem in which the classical condition that P, Q have continuous partial derivatives is weakened to total differentiability of P, Q plus continuity of the combination $\partial Q/\partial x - \partial P/\partial y$ [$= 0$ in the Cauchy-Goursat case].) But beyond the disk lies a topological morass. There is an illusion among analysts that homology is the natural and only staff, but any topologist will point out that in an open subset of the plane homology is the Abelianization of homotopy performed automatically when one integrates over a closed curve: integration over a circuit homologous to 0 equals integration over a circuit homotopic to 0 plus integrals of the form $\int_\gamma + \int_{-\gamma} = 0$.

In this paper we shall complete the historical cycle by deriving simultaneously the two homotopy versions of Cauchy's theorem (*Theorem* 3) from the disk theorem via a *discrete* Gauss-Green theorem (*Theorem* 1). In the process the topological troubles evaporate and leave only the definition of homotopy behind. Our starting point was a conversation with J. L. Ullman in which he drew our attention to the proof of the first half of *Theorem* 2 due to F. M. Stewart sketched in [2].

2. A Discrete Gauss-Green Theorem. Consider any additive Abelian group G — say the complex numbers C — and the (old-fashioned) lattice $L_{mn} = [(j,k)\colon j = 0,1,\cdots,m;\ k = 0,1,\cdots,n]$ $(m,n \geqq 1)$. If $F\colon L_{mn} \to G$ set $(\Delta_x F)(j,k) = F(j+1,k) - F(j,k)$ when $(j,k) \in L_{m-1,n}$ and $(\Delta_y F)(j,k) = F(j,k+1) - F(j,k)$ when $(j,k) \in L_{m,n-1}$.
When $F\colon L_{m-1\ n-1} \to G$ set

$$\iint_{L_{mn}} F\Delta x\Delta y = \sum_{\substack{j=0,\ldots,m-1\\ k=0,\ldots,n-1}} F(j,k).$$

Now consider any pair of functions P, Q with $P\colon L_{m-1,n} \to G$ and $Q\colon L_{m,n-1} \to G$ and set

$$\int_{\partial L_{mn}} P\Delta x + Q\Delta y = \sum_{j=0}^{m-1} P(j,0) + \sum_{k=0}^{n-1} Q(m,k) - \sum_{j=0}^{m-1} P(j,n) - \sum_{k=0}^{n-1} Q(0,k).$$

THEOREM 1. $$\int_{\partial L_{mn}} P\Delta x + Q\Delta y = \iint_{Lmn} (\Delta_x Q - \Delta_y P)\Delta x\Delta y.$$

Proof.

$$\iint_{L_{mn}} (\Delta_x Q - \Delta_y P)\Delta x\Delta y = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} [Q(j+1,k) - Q(j,k)]$$

$$- \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} [P(j,k+1) - P(j,k)] = \sum_{k=0}^{n-1} [Q(m,k) - Q(0,k)]$$

$$- \sum_{j=0}^{m-1} [P(j,n) - P(j,0)] = \sum_{j=0}^{m-1} P(j,0) + \sum_{k=0}^{n-1} Q(m,k)$$

$$- \sum_{j=0}^{m-1} P(j,n) - \sum_{k=0}^{n-1} Q(0,k) \equiv \int_{\partial L_{mn}} P\Delta x + Q\Delta y.$$

COROLLARY. *If* $\Delta_x Q - \Delta_y P = 0$ *on* $L_{m-1,n-1}$ *then* $\int_{\partial L_{mn}} P\Delta x + Q\Delta y = 0$.

3. The homotopy versions of Cauchy's theorem. In what follows we consider functions f analytic in an open subset X of the complex numbers C. By an *admissible* path in X we shall mean a continuous mapping γ: $[0,1] \to X$ that is either piecewise smooth or, more generally, of bounded variation (rectifiable). γ is closed if $\gamma(0) = \gamma(1)$. Let $D(c;r)$ denote the open disk with center c and radius r. Our starting point is then

THEOREM 0. (Cauchy theorem for the disk). *Let f be analytic in $D(c;r)$ and let γ be any closed admissible path in $D(c;r)$. Then $\int_\gamma f(z)dz = 0$.*

The first step is to consider continuous mappings of the unit square $S = [0,1] \times [0,1]$ into our open set X. We split the boundary ∂S of S into the four obvious pieces — B(ottom), R(ight), T(op), L(eft) — and parametrize in standard counterclockwise fashion.

THEOREM 2. (Cauchy theorem for the singular cell). *Let H: $S \to X$ be continuous and let f: $X \to C$ be analytic. Assume $H(\partial S_B)$ and $H(\partial S_T)$ are admissible. If* (1) *$H(\partial S_R)$ and $H(\partial S_L)$ are admissible, then $\int_{H(\partial S)} f(z)dz = 0$. If* (2) $H(0,t) = H(1,t)$ $(0 \leqq t \leqq 1)$, *then* $\int_{H(\partial S_B)} f(z)dz + \int_{H(\partial S_T)} f(z)dz = 0$.

Proof. The case $X = C$ is trivially subsumed in Theorem 0 for sufficiently large r. Assume then that $X \neq C$ and set $r =$ distance $(C - X, H(S))$. Since $C - X$ and $H(S)$ are disjoint closed sets and $H(S)$ is compact it follows that $r > 0$. Because H is uniformly continuous on S there exists a positive integer N such that $|H(p) - H(q)| < r$ whenever $|p - q| \leqq 2^{\frac{1}{2}}/N$. Set

$$z_{jk} = H(j/N, k/N) \qquad j,k = 0,1,\cdots,N$$

$$S_{jk} = [j/N,(j+1)/N] \times [k/N,(k+1)/N] \qquad j,k = 0,1,\cdots,N-1.$$

Then $H(S_{jk}) \subset D(z_{jk}; r) \subset X$.

Now make contact with Theorem 1 by setting

$$P(j,k) = \int_{z_{jk}}^{z_{j+1,k}} f(z)dz \qquad (j,k) \in L_{N-1,N}$$

$$Q(j,k) = \int_{z_{jk}}^{z_{j,k+1}} f(z)dz \qquad (j,k) \in L_{N,N-1}.$$

Both integrals are of the form $\int_c^d f(z)dz$, where c, d lie in some disk $D(z_{jk}; r)$ and the path of integration is the line segment $\overline{cd}$. It follows from the Cauchy theorem for this disk that the line segment $\overline{cd}$ can be replaced by any admissible path in the disk from c to d.

Now compute the boundary integral in Theorem 1:

$$\begin{aligned}
\int_{\partial L_{NN}} P\Delta x + Q\Delta y &= \sum_{j=0}^{N-1} P(j,0) + \sum_{k=0}^{N-1} Q(N,k) - \sum_{j=0}^{N-1} P(j,N) - \sum_{k=0}^{N-1} Q(0,k) \\
&= \sum_{j=0}^{N-1} \int_{H(j/N,0)}^{H((j+1)/N,0)} f(z)dz + \sum_{k=0}^{N-1} \int_{H(1,k/N)}^{H(1,(k+1)/N)} f(z)dz \\
&\quad - \sum_{j=0}^{N-1} \int_{H(j/N,1)}^{H((j+1)/N,1)} f(z)dz - \sum_{k=0}^{N-1} \int_{H(0,k/N)}^{H(0,(k+1)/N)} f(z)dz \\
&= \int_{H(\partial S)} f(z)dz
\end{aligned}$$

in Case 1. In Case 2 the second and fourth sums cancel, under *no* assumptions about the admissibility of $H(\partial S_R)$ and $H(\partial S_L)$, to yield

$$\int_{\partial L_{NN}} P\Delta x + Q\Delta y = \int_{H(\partial S_B)} f(z)dz + \int_{H(\partial S_T)} f(z)dz.$$

Thus *Theorem* 2 becomes an immediate consequence of the *Corollary to Theorem 1* once we show that $\Delta_x Q - \Delta_y P = 0$ on $L_{N-1,N-1}$. But for any (j,k) in $L_{N-1,N-1}$ we have

$$\begin{aligned}
(\Delta_x Q - \Delta_y P)(j,k) &= [Q(j+1,k) - Q(j,k)] - [P(j,k+1) - P(j,k)] \\
&= \int_{z_{j+1,k}}^{z_{j+1,k+1}} f(z)dz - \int_{z_{jk}}^{z_{j,k+1}} f(z)dz - \int_{z_{j\,k+1}}^{z_{j+1,k+1}} f(z)dz + \int_{z_{jk}}^{z_{j+1,k}} f(z)dz \\
&= \left[\int_{z_{jk}}^{z_{j+1,k}} + \int_{z_{j+1,k}}^{z_{j+1,k+1}} + \int_{z_{j+1,k+1}}^{z_{j,k+1}} + \int_{z_{j,k+1}}^{z_{jk}}\right] f(z)dz = 0
\end{aligned}$$

by the Cauchy theorem for the disk $D(z_{jk}\,; r)$ and the quadrilateral contour $z_{jk} \cdots z_{jk}$ contained in it.

The final steps on the road to the homotopy versions of Cauchy's theorem are formal definitions. Given a topological space X and continuous mappings $\gamma\colon [0,1] \to X$ (paths), one says that two paths γ_0 and γ_1 are *fixed endpoint* (FEP) homotopic in X if there exists a continuous mapping $H\colon S = [0,1] \times [0,1] \to X$ such that $H(s,0) = \gamma_0(s)$ and $H(1,s) = \gamma_1(s)$ $(0 \leqq s \leqq 1)$ while $\gamma_0(0) = H(0,t) = \gamma_1(0)$ and $\gamma_0(1) = H(1,t) = \gamma_1(1)$ $(0 \leqq t \leqq 1)$. (Continuous deformation of the path γ_0 into the path γ_1 keeping the endpoints fixed.) If one drops the FEP subsidiary condition but requires instead that the curves be closed, one says that the *closed* paths γ_0, γ_1 are *freely* homotopic in X if now $H(0,t) = H(1,t)$ $(0 \leqq t \leqq 1)$. When X is an open subset of C one is back in the situation of Theorem 2: The FEP condition is the special case of Part (1) in which the left and right sides of S map into points; the hypothesis of Part (2) is simply that the closed paths $\gamma_0 = H(\partial S_B)$ and $\gamma_1 = -H(\partial S_T)$ be freely homotopic.

THEOREM 3. (Joint homotopy version of Cauchy's theorem). *Let f be analytic in an open set X and let γ_0, γ_1 be two admissible paths in X. Then* $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$ *if either* (1) *γ_0 and γ_1 are FEP homotopic in X or* (2) *γ_0 and γ_1 are closed and freely homotopic in X.*

Added in proof. Theorems 2 and 3 generalize automatically to all dimensions if one replaces open subsets X of $C = R^2$ by open subsets X of R^n, integration of analytic functions along curves by integration of closed 1-forms along curves, and Cauchy's theorem for the disk as Theorem 0 by Poincaré's lemma for the ball: *Every closed differential form defined in an open ball of R^n is exact.*

References

1. S. Bochner, Functions in one complex variable as viewed from the theory of functions in several variables, pp. 315–333 of Lectures on Functions of a Complex Variable (W. Kaplan, M. O. Reade, G. S. Young, editors), University of Michigan Press, Ann Arbor, 1955.
2. M. Heins, Complex Function Theory, Academic Press, New York, 1968.
3. E. Hille, Analytic Function Theory, vol. 1, Ginn and Company, Boston, 1959.

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THE QUADRILATERAL INEQUALITY IN TWO DIMENSIONS

A. SUDBERY

A normed linear space V is called a *quadrilateral space* if the inequality

$$|a+b| + |b+c| + |c+a| \leqq |a| + |b| + |c| + |a+b+c| \tag{1}$$

holds for all $a, b, c \in V$. Such spaces were defined and studied by Smiley and Smiley [**1**] who conjectured that every two-dimensional real normed linear space is a quadrilateral space. It is the purpose of this note to prove their conjecture.

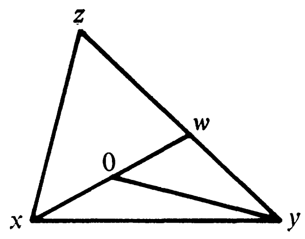
We need a preliminary definition and two lemmas. The proofs of these are intuitively obvious once one has drawn a diagram, but we shall spell out the details to make sure that they depend on no more than the idea of a norm.

DEFINITION. Given $x, y, z \in V$, we say that *the triangle xyz contains* 0 if there exist non-negative real numbers α, β, γ, not all zero, such that

$$\alpha x + \beta y + \gamma z = 0.$$

LEMMA 1. *If the triangle xyz contains* 0, *then*

$$|x| + |y| \leqq |x-z| + |y-z|.$$



Proof. Suppose $\alpha x + \beta y + \gamma z = 0$, where $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma \geqq 0$. We may suppose that at least two of α, β, γ are non-zero, for otherwise one of x, y, z would be zero and the stated inequality would either reduce to the triangle inequality or become trivial. Thus $\beta + \gamma$ may be taken to be non-zero, so we can define

$$w = -\frac{\alpha}{\beta+\gamma}x = \frac{\beta}{\beta+\gamma}y + \frac{\gamma}{\beta+\gamma}z.$$

Then $y - w = (\gamma/(\beta+\gamma))(y-z)$ and $w - z = (\beta/(\beta+\gamma))(y-z)$, so

$$|y-z| = \frac{\gamma}{\beta+\gamma}|y-z| + \frac{\beta}{\beta+\gamma}|y-z| = |y-w| + |w-z|$$

since β and γ are both non-negative. Also

$$|x-w| = \left| x + \frac{\alpha}{\beta+\gamma}x \right| = |x| + \frac{\alpha}{\beta+\gamma}|x| = |x| + |w|.$$

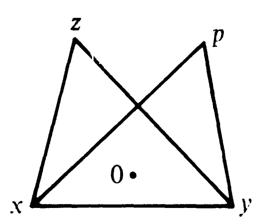
Hence

$$\begin{aligned} |x-z| + |y-z| &= |x-z| + |w-z| + |y-w| \\ &\geqq |x-w| + |y-w| \\ &= |x| + |w| + |y-w| \\ &\geqq |x| + |y|. \end{aligned}$$

LEMMA 2. *Let V be a two-dimensional real normed linear space, and let $x, y, z \in V$. If the triangle xyz contains 0, then*

$$|x| + |y| + |z| \leqq |p| + |x-p| + |y-p| + |z-p| \text{ for all } p \in V.$$

(Note: For the Euclidean plane, this result is stated in [2].)



Proof. First we show that at least one of the triangles pxy, pyz, pzx contains 0. If any two of x, y, z are linearly dependent, then, since xyz contains 0, there must be some pair, say (x, y)—not necessarily the same as the first pair—between which there is a linear dependence of the form

$$\alpha x + \beta y = 0$$

with α and β non-negative. Then the triangle pxy contains 0.

If no two of x, y and z are linearly dependent, then in the relation

$$\alpha x + \beta y + \gamma z = 0$$

α, β and γ must all be strictly positive, and any other linear dependence between x, y and z must be a multiple of this one. In particular, if $\alpha' x + \beta' y + \gamma' z = 0$ then α', β' and γ' must all have the same sign.

Since V has dimension 2, there exists a linear dependence between p and each pair of x, y and z. Moreover, if no two of x, y, and z are linearly dependent, the coefficient of p in each of these dependences must be non-zero and so may be taken to be 1. Thus we can write

$$\begin{aligned} p &= \kappa x + \lambda y \\ &= \mu y + \nu z \\ &= \rho z + \sigma x. \end{aligned}$$

By subtracting the first two of these equations and using the result of the previous paragraph, it follows that κ and ν cannot have the same sign unless they are both zero. Thus one of them, say κ, must be non-positive. If λ is also non-positive, then pxy contains 0; if λ is positive, we can substitute $y = -(1/\beta)(\alpha x + \gamma z)$ to find that pxz contains 0.

Thus in every case we have one of the triangles, say pxy, containing 0. Applying Lemma 1 to this triangle,

$$|x| + |y| \leqq |x-p| + |y-p|.$$

Also, from the triangle inequality,

$$|z| \leqq |p| + |z-p|;$$

and the sum of these two is the stated inequality.

We are now in a position to prove our main result.

THEOREM. *If V is a two-dimensional real normed linear space, the inequality* (1) *holds for all $a, b, c \in V$.*

Proof. Let $x = a + b$, $y = b + c$, $z = c + a$. If the triangle xyz contains 0, the result follows from Lemma 2 by taking $p = a + b + c$. If xyz does not contain 0, we can obtain a triangle which does by reversing one of the vectors x, y, z; for since V has dimension 2, there is some linear dependence

$$\alpha x + \beta y + \gamma z = 0,$$

and if xyz does not contain 0, one of the coefficients, say α, has the opposite sign to the other two. Writing the equation as $(-\alpha)(-x) + \beta y + \gamma z = 0$ then shows that the triangle $(-x)yz$ contains 0. The result now follows from Lemma 2 by taking $p = c$.

References

1. D. M. Smiley & M. F. Smiley, this MONTHLY, **71** (1964) 755.
2. O. Bottema, R. Ž. Djordjevič, R. R. Janič, D. S. Mitrinovič and P. M. Vasič, Geometric Inequalities, Wolters-Noordhoff, Groningen 1969, p. 132.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

HOW MANY LATIN SQUARES ARE THERE?

RONALD ALTER

A latin square of order n is an arrangement of the first n integers in an $n \times n$ array so that every integer appears exactly once in every row and exactly once in every column. Let L_n be the number of latin squares of order n and let R_n be the number of reduced latin squares of order n. (A reduced latin square has the first row and first column in natural, i.e., lexicographic, order.) It is easy to see that

$$L_n = n!(n-1)!R_n. \tag{1}$$

To answer the title question it suffices to determine R_n. Clearly $R_1 = 1$ for $n = 1, 2$, and 3. Also it is not difficult to compute that $R_4 = 4$ and $R_5 = 56$. In fact Euler [**3**], who first raised the question of the enumeration of latin squares, showed this although he failed to enumerate R_6.

By applying the properties of systems of distinct representatives, M. Hall [**5**], established the following lower bound, b_n on R_n.

$$R_n \geqq (n-2)!(n-3)!\cdots(2!)(1!) = b_n. \tag{2}$$

As a crude upper bound on R_n, the following can be established without much difficulty:

$$R_n \leqq (n-1)!\,[(n-2)\cdot(n-2)!]\,[(n-3)^2\cdot(n-3)!]\cdots[(n-k)^{k-1}\cdot(n-k)!]\cdots (2^{n-3}\cdot 2!)(1^{n-2}\cdot 1!).$$

Thus,

$$R_n \leqq \prod_{i=1}^{n-1} (n-i)^i b_n. \tag{3}$$

Many articles have appeared on latin squares. Among the authors who have studied the problem of enumerating finite latin squares are: A. Cayley, M. Frolov, P. A. MacMahon, G. Tarry, H. W. Norton, A. Sade, R. A. Fisher, F. Yates, S. M. Jacob, E. Schönhardt, J. W. Brown, and M. B. Wells, to name a few. Reference to their work and the related works of others can be found in the encyclopedic treatise on latin squares and their applications by J. Dénes and A. D. Keedwell [2], which has recently appeared.

In 1890 M. Frolov [4] enumerated latin squares of order 6 and showed that $R_6 = 9408$. This result may have been obtained earlier by Clausen (1842), but the evidence for this is a little sketchy. For reference to this work see [2, p. 140]. In 1948 A. Sade [6] calculated R_7 by use of a clever computational technique. The method involves the construction of a special set of latin rectangles (a rectangular array which can be completed to a latin square) which are inequivalent under any combination of row and column permutations. While the set of latin rectangles is being constructed, the number of different rectangles in each equivalence class is counted. Using a computer adaptation of Sade's method, Wells [7] determined R_8 by using over eight hours computing time. This method was recently improved by Bammel and Rothstein [1] whose method enabled them to compute R_8 using only 4 minutes of computer time. They further claim to have computed R_9 in less than five hours. Table 1 summarizes these results and also gives the prime decomposition of the R_n.

TABLE 1

n	R_n	Prime Factorization
1	1	1
2	1	1
3	1	1
4	4	2^2
5	56	$2^3 . 7$
6	9 408	$2^6 . 3 . 7^2$
7	16 942 080	$2^{10} . 3 . 5 . 1103$
8	535 281 401 856	$2^{17} . 3 . 1\ 361\ 291$
9	377 597 570 964 258 816	$2^{21} . 3^2 . 5\ 231 . 3\ 824\ 477$

Looking at the prime factors of R_n one observes that

(i) For R_4 to R_9 increasing powers of 2 appear, and

(ii) $3 \mid R_n$ for $n = 6, 7, 8$ and 9.

This raises some questions.

(i) Do increasing powers of 2 divide R_n?

(ii) If the answer to (i) is yes, what is the highest power of 2 that will divide R_n?

(iii) Does $3 \mid R_n$ for all $n \geqq 6$?

With current computing facilities, Table 1 will undoubtedly be extended. This in turn may shed some light on the above divisibility questions. However, the determination of R_n (and thus also L_n) still appears to be extremely difficult.

The author would like to thank Professor Richard K. Guy for his encouragement and helpful suggestions during the preparation of this article.

References

1. S. E. Bammel and J. Rothstein, The number of 9×9 latin squares, J. Discrete Math., 11 (1975) 93–95.

2. J. Dénes and A. D. Keedwell, Latin Squares and Their Applications, English Universities Press, London; Akadémiai Kiadó, Budapest, and Academic Press, New York, 1974.

3. L. Euler, Recherches sur une nouvelle espèce de quarrés magiques, Verh. Zeeuwsch. Genootsch. Wetensch. Vlissingen, 9 (1782) 85–239.

4. M. Frolov, Recherches sur les permutations carrées, J. Math. Spéciales, (3) 4 (1890) 8–11, 25–30.

5. M. Hall, Distinct representatives of subsets, Bull. Amer. Math. Soc., 54 (1948) 922–926.

6. A. Sade, Énumération des carrés latins, application au 7^e ordre, conjecture pour les ordres supérieurs, Published by the author, Marseille, 1948, 8 pp.

7. M. B. Wells, The number of latin squares of order 8, J. Combinatorial Theory, 3 (1967) 98–99.

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CLASSROOM NOTES

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UPPER TRIANGULAR RINGS, IDEALS, AND CATALAN NUMBERS

L. W. Shapiro

Let T_n denote the ring of $n \times n$ upper triangular matrices over a field F. We want to determine the number of ideals in T_n. The answer is

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1},$$

where C_n is the nth Catalan number. Catalan numbers have a long and varied history as shown by two extensive surveys; Alter [1] and Brown [2]. Ideal means two

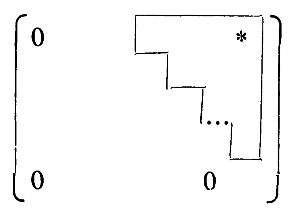
sided ideal throughout. Let E_{ij} denote the matrix consisting of all zero entries, except for a one in the (i,j) position and let S_{ij} be the subring of all scalar multiples of E_{ij}.

LEMMA 1. *Let I be an ideal of T_n and $X = (x_{ij}) \in I$ with, say, $x_{kl} \neq 0$. Then $S_{kl} \subseteq I$.*

Proof. We have that $fE_{kk}XE_{ll}$ is in I for all f in F.

LEMMA 2. *Using the same notation as in Lemma 1 we have that $x_{kl} \neq 0$ implies $S_{k-i,l+j} \subseteq I$ for $i,j \geqq 0$.*

Proof. Let $E_{k-i,k}XE_{l,l+j} = Y$ and the $(k-i, l+j)$ entry of Y will be nonzero. Since $Y \in I$ we can apply Lemma 1. The effect of Lemmas 1 and 2 is that any ideal of T_n has the form



If any entry of any matrix in I is non-zero then the entries to the right or above (or both) that entry may vary freely over F as does the given entry.

LEMMA 3 (D. Andre). *The number of paths from $(0,0)$ to $(2n,0)$ such that each edge goes from (n,m) to $(n+1, m \pm 1)$ and where each y coordinate is non-negative is*

$$\frac{1}{n+1}\binom{2n}{n} = C_n.$$

This pretty result has a clever, but well-known and accessible proof and will not be proved here. See, for instance, Feller [4].

Tilting our matrices 45° counter-clockwise shows that Lemma 3 is what we need to count the ideals. Let the (i,j)th position of the matrix correspond to $(j+i, j-i+2)$ on the path. Since each "non-negative" path must start at $(0,0)$, $(1,1)$ and end at $(2n-1,1)$, $(2n,0)$ we see that each ideal pattern corresponds to a unique "non-negative" path.

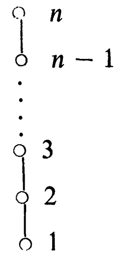
Putting these lemmas together we have shown the following.

PROPOSITION 1. *The ring T_n has*

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} \qquad \textit{ideals.}$$

We have essentially just determined the ideals of the incidence algebra of the

partially ordered set



Except that equivalents to Lemma 3 may be lacking, a procedure exists to determine the ideals of the incidence algebra of any locally finite, partially ordered set. See Doubilet, Rota, and Stanley [3], section 3 for this generalization.

We now list several similar results.

PROPOSITION 2. *The number of nilpotent ideals of* T_n *is* $C_n = 1/(n+1)\binom{2n}{n}$.

PROPOSITION 3. *If R is a ring of all matrices of the form*

$$\begin{bmatrix} \boxed{M_1} & & & * \\ & \boxed{M_2} & & \\ & & \ddots & \\ & & & \boxed{M_k} \end{bmatrix}$$

where each M_i *is all* $k_i \times k_i$ *matrices over F then the number of ideals of R is* C_{k+1}.

PROPOSITION 4. T_n *has* 2^{n-1} *commutative ideals for* $n \geqq 2$.

Proof. An ideal will be commutative if and only if its nonzero entries occur in a rectangle in the upper left hand corner and this rectangle includes no entries on the main diagonal. We index these rectangles by the position of their lower left corner entry and count each ideal in the smallest rectangle containing it. Then each ideal corresponds to a shortest path across the rectangle. There are

$$\binom{n-l+k-1}{k-1}$$

suitable paths across the rectangle indexed by (k,l). Thus we have

$$1 + \sum_{l=2}^{n} \sum_{k=1}^{l-1} \binom{n-l+k-1}{k-1} = 1 + \sum_{l=2}^{n} \binom{n-1}{l-1} = 2^{n-1}$$

ideals where the 1 counts the zero ideal.

A more direct proof of this proposition would be of interest.

References

1. R. Alter, Some remarks and results on the Catalan numbers, Proc. Louisiana Conf. on Combinatorics, Graph Theory, and Computing, 1971, pp. 109–132.

2. W. G. Brown, Historical note on a recurrent combinatorial problem, this MONTHLY, 72 (1965) 973–977.

3. P. Doubilet, G. C. Rota, and R. Stanley, On the foundations of combinatorial theory (VI): The idea of a generating function, Berkeley Symp. on Stat. and Probability VI (1972) 267–318.

4. W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, Wiley, New York, 1957, pp. 66–71.

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BINARY OPERATIONS ON FAMILIES OF CONTINUOUS FUNCTIONS

Dedicated to Paul Cauffman

KENNETH D. MAGILL, JR.

Let R denote the space of real numbers and $C(R)$ the family of all continuous functions mapping R into R. Let m be any continuous binary operation on R and define a mapping M on $C(R) \times C(R)$ by

$$(M(f,g))(x) = m(f(x), g(x)).$$

It is well known that for $f, g \in C(X)$, the function $M(f,g)$ again belongs to $C(X)$, i.e., M is a binary operation on $C(X)$ and a routine computation shows that

$$M(f,g) \circ h = M(f \circ h, g \circ h)$$

for all $f, g, h \in C(X)$. In other words, composition is right distributive over M. Although it is quite easy to prove, it seems to be less well known that every binary operation on $C(R)$ over which composition is right distributive, is obtained in exactly this manner. We prove such a result for any first countable space which allows continuous extensions of certain types of functions.

DEFINITION. A space X has the *weak extension property* if for any two compact, countable subsets A and B of X, each having exactly one limit point, any homeomorphism from A onto B can be extended to a continuous function mapping X into X.

Remarks: The spaces described above include all locally Euclidean spaces, all 0-dimensional metric spaces and all normal absolute retracts. The proofs of Theorems (3.4) and (3.5) of [1] serve to verify this for the first two types of spaces. These proofs are straightforward and the case for normal absolute retracts is even more straightforward.

For any space X, $C(X)$ denotes the family of all continuous functions which map X into X.

PROPOSITION. *Let X be any first countable space with the weak extension property. Let m be any continuous binary operation on X and define a mapping M on $C(X) \times C(X)$ by*

$$(M(f,g))(x) = m(f(x),g(x)). \tag{1}$$

Then M is a binary operation on $C(X)$ over which composition is right distributive. Conversely, every binary operation on $C(X)$ over which composition is right distributive, is obtained in exactly this manner.

Proof. It is well known that if m is a continuous binary operation on X, then M defined as in (1) is a binary operation on $C(X)$ and, as in the case for the space of real numbers, a straightforward computation shows that

$$M(f,g) \circ h = M(f \circ h, g \circ h)$$

for all $f, g, h \in C(X)$. We prove only that each such M is obtained in this way. For any such M, we fix any point $a \in X$ and define a binary operation m on X by

$$m(x,y) = (M(\langle x \rangle, \langle y \rangle))(a), \tag{2}$$

where $\langle x \rangle$ and $\langle y \rangle$ denote the constant functions on X which send everything into the points x and y, respectively. We next show that for any $f, g \in C(X)$ and $x \in X$, we have

$$(M(f,g))(x) = m(f(x),g(x)). \tag{3}$$

We use (2) and the fact that composition is right distributive over M and we get

$$\begin{aligned} m(f(x),g(x)) &= (M(\langle f(x) \rangle, \langle g(x) \rangle))(a) \\ &= (M(f \circ \langle x \rangle, g \circ \langle x \rangle))(a) \\ &= (M(f,g)) \circ \langle x \rangle(a) = (M(f,g))(x). \end{aligned}$$

In order to complete the proof, we need only show that m is continuous. Suppose $\lim(x_n, y_n) = (v,w)$. Then $\lim x_n = v$ and $\lim y_n = w$. There is no loss in generality in assuming that all points x_n are distinct from each other and v, and that all the points y_n are distinct from each other and w. Define a function f by $f(x_n) = y_n$ for each positive integer n and $f(v) = w$. Then since X has the weak extension property, the function f can be extended to a continuous selfmap g of X. Now let i denote the identity map and use (3) and the fact that $M(i,g)$ is continuous to get

$$\begin{aligned} \lim m(x_n, y_n) &= \lim (M(i,g))(x_n) \\ &= (M(i,g))(v) = m(i(v),g(v)) = m(v,w). \end{aligned}$$

This verifies the continuity of m.

Reference

1. K. D. Magill, Jr., I-subsemigroups and α-monomorphisms, J. Australian Math. Soc., 16 (1973) 146–166.

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AN ELEMENTARY PROOF OF VORONOVSKAYA'S THEOREM

JOHN P. COLEMAN

In a course which includes Bernstein's proof of the Weierstrass polynomial approximation theorem it is desirable to demonstrate the slow convergence of the sequence of Bernstein polynomials. This is easily done for some simple functions, such as $\exp(x)$, by examining the explicit form of the Bernstein polynomial of degree n, but it is more satisfactory to use the general result embodied in

VORONOVSKAYA'S THEOREM. *If f is twice differentiable on $[0,1]$, then, for all $x \in [0,1]$,*

$$\lim_{n\to\infty} n[f(x) - B_n(f,x)] = -\tfrac{1}{2}x(1-x)f''(x). \tag{1}$$

(The theorem may also be stated in a local form which only requires the existence of f'' at a particular point, rather than throughout the interval, see Todd [**1**, p. 21].)

The proof of this theorem is not difficult but neither is it particularly instructive, and one may be reluctant to spend lecture time on it. The purpose of this note is to suggest a proof of a restricted version of the theorem, which takes very little time to describe.

The Bernstein polynomial of degree n for a function f defined on $[0,1]$ is

$$B_n(f,x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k}{n}\right)$$

with

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Bernstein's proof of the Weierstrass theorem (Todd [**1**, p. 19] or Ralston [**2**, p. 28]) requires the identities

$$\sum_{k=0}^{n} p_{nk}(x) = 1, \quad \sum_{k=0}^{n} p_{nk}(x)(k-nx) = 0, \tag{2}$$

$$\sum_{k=0}^{n} p_{nk}(x)(k-nx)^2 = nx(1-x), \tag{3}$$

which may be deduced from the binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and its derivatives with respect to x, by replacing y by $1-x$ after differentiating. Similarly it can be shown that

$$\sum_{k=0}^{n} p_{nk}(x)(k-nx)^3 = nx(1-x)(1-2x) \tag{4}$$

and

$$\sum_{k=0}^{n} p_{nk}(x)(k-nx)^4 = nx(1-x)[3(n-2)x(1-x)+1]. \tag{5}$$

(Alternatively, (4) and (5) may be obtained from a recursion formula [**1**, p. 15].)

THEOREM. *If f has a bounded fourth derivative on $[0,1]$ then eqn.* (1) *is true for all $x \in [0,1]$.*

Proof. From Taylor's theorem we have, for $x \in [0,1]$ and $k \leqq n$,

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right)f'(x) + \frac{1}{2}\left(\frac{k}{n} - x\right)^2 f''(x)$$
$$+ \frac{1}{6}\left(\frac{k}{n} - x\right)^3 f'''(x) + \frac{1}{24}\left(\frac{k}{n} - x\right)^4 f^{\mathrm{IV}}(\zeta),$$

where ζ (which depends on k and x) lies between k/n and x (and therefore between 0 and 1). Then use of the identities (2)–(4) gives

$$B_n(f,x) - f(x) = \sum_{k=0}^{n} p_{nk}(x)\left[f\left(\frac{k}{n}\right) - f(x)\right]$$
$$= \frac{1}{2n}x(1-x)f''(x) + \frac{1}{6n^2}x(1-x)(1-2x)f'''(x) + R_n(x),$$

where

$$R_n(x) = \frac{1}{24n^4}\sum_{k=0}^{n} p_{nk}(x)(k-nx)^4 f^{\mathrm{IV}}(\zeta).$$

Since $f^{\mathrm{IV}}(x)$ is bounded on $[0,1]$, there is some finite number M such that

$$|f^{\mathrm{IV}}(\zeta)| < M.$$

Furthermore, $p_{nk}(x) \geqq 0$ for all $x \in [0,1]$ so

$$|R_n(x)| \leqq \frac{M}{24n^4}\sum_{k=0}^{n} p_{nk}(x)(k-nx)^4$$

$$= \frac{M}{24n^3}x(1-x)[3(n-2)x(1-x)+1]$$

from eqn. (5), and the required result follows.

The class of functions to which the proof applies is sufficiently general to show that for most practical purposes the Bernstein polynomials do not provide useful approximations.

Acknowledgment: I am grateful to the referee for pointing out an error in an earlier version of this paper.

References

1. J. Todd, Introduction to the Constructive Theory of Functions, Birkhauser Verlag, 1963.
2. A. Ralston, A First Course in Numerical Analysis, McGraw Hill, New York, 1965.

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DIRECTIONAL DIFFERENTIATION IN THE PLANE AND TANGENT VECTORS ON C^r MANIFOLDS

H. G. ELLIS

If at a certain point each of the functions f and g is differentiable, then so is their product fg. The product can, however, be differentiable at that point without both, or even either of f and g being so, as the example $f(x) = x^{1/3}$, $g(x) = x^{2/3}$ shows. This simple observation suggests the possibility of refining Chevalley's widely adopted "algebraic" or "intrinsic" definition of a tangent vector at a point P of a C^ω(that is to say, an analytic) manifold [1, p. 76] to make it applicable also to every C^r manifold with $r \geqq 1$. That definition, abstracted from the notion of directional differentiation, says that a tangent vector at P is a real-valued linear operator on the real-valued functions analytic at P which obeys the algebraic rule for differentiation at P of products of functions differentiable at P. The analyticity assumption ensures that the tangent vectors so defined are in fact directional differentiations at P, and that the dimensionality of the tangent space made up of these tangent vectors is the same as that of the manifold. Here I shall show how to arrive at the same goals, relaxing the requirement from analyticity at P to that of mere differentiability at P of the functions operated upon, but compensating for this loss by incorporating into the definition an additional algebraic condition. The revised definition will work for all C^r manifolds with $r \geqq 1$; it will also be "intrinsic" in that it will make no direct reference to a coordinate system.

The basic ideas can be explained more readily for the vector space $\mathbb{R}^2$ than for an abstract manifold, so I shall begin there, saving the generalities for later. It will be convenient to have at our disposal the coordinate functions x and y, such that

THE ODDS IN SOME ODD-EVEN GAMES

E. F. SCHUSTER AND A. N. PHILIPPOU

1. Introduction. Early in a first course in probability we feel that one should give some problems with nonintuitive answers in which the student can employ the basic axioms and properties in discrete probability theory, especially set theoretic ideas including union and intersection in connection with the countable additivity axiom. In this direction we present several "odd-even games" which, we feel, are of independent interest. Our present emphasis is on the nonintuitive nature of the games.

2. Some odd-even games. In the sequel we shall consider discrete probability distributions on the nonnegative integers and we shall consider 0 to be even. An "odd-even game" consists of a single trial of a random experiment which produces a nonnegative integer. The outcome of the game is then "odd" or "even," depending on whether the outcome of the random experiment is an odd or an even integer, respectively. The problem will be to determine the "best bet," hence, the odds or probabilities in a number of odd-even games. Our first example is:

Example 1. Consider the experiment: Record the waiting time (in tosses) till the first appearance of a "six" in repeated tosses of an unbiased six-sided die. Is the best bet "odd" or "even"? What if the die is biased?

The answer to this problem is contained in the following

THEOREM 1. (Bernoulli Odd-Even Game). *In an odd-even game based on the experiment, "record the waiting time in trials till the first success in repeated independent Bernoulli trials with constant positive probability p of success," the best bet is always "odd." In particular,* $P(\text{"even"}) = q/(1+q)$ *and* $P(\text{"even"}) - P(\text{"odd"}) = (q-1)/(1+q)$, *where* $q = 1-p$.

Note: The only fair game occurs when $p = 0$ since (with probability one) the experiment never terminates and hence nobody wins.

Example 2. Consider the experiment: Record the number of dead jackrabbits on or along the road in a 10 mile stretch of highway 54 between Orogrande, New Mexico and Alamogordo, New Mexico. Is the best bet "odd" or "even" in a 7.5 mile stretch?

Assuming the incidence of dead rabbits in the above example follows a Poisson distribution (which is frequently the case) we have our answer in:

THEOREM 2. (Poisson Odd-Even Game). *Let an odd-even game be based on the experiment: Record the number of occurrences of an event in an interval of specified length (time). If the number of occurrences of the event in an interval of positive length t follows the Poisson distribution with mean rate λt then for every λ and t the best bet is "even." In particular,* $P(\text{"even"}) = (1 + \exp(-2\lambda t))/2$ *and* $P(\text{"even"}) - P(\text{"odd"}) = \exp(-2\lambda t)$.

Lest one think that life is always so simple, consider a sucker bet based on the following:

Example 3. Consider the game based on the following generalization of Example 1: Record the waiting time in tosses till the rth "six" in repeated tosses of an unbiased die. Which is the sucker bet? What if the die is biased?

Our answer is contained in the rather remarkable

THEOREM 3 (Negative Binomial Odd-Even Game). *Let an odd-even game be based on the experiment: Record the waiting time in trials till the r-th success in repeated independent Bernoulli trials with constant probability p of success. Then the best bet is "odd" or "even" depending on whether r is odd or even, respectively. In particular,*

$$P(\text{"even"}; r) = [1 + ((q-1)/(q+1))^r]/2 \tag{2.1}$$

and $P(\text{"even"}; r) - P(\text{"odd"}; r) = ((q-1)/(1+q))^r$.

Additional sucker bets can be based on our

THEOREM 4 (Binomial Odd-Even Game). *In an odd-even game based on the number of successes in n independent Bernoulli trials with constant positive probability p of success, the best bet is "even," except when n is odd and $p \geqq 1/2$. In particular,*

$$P(\text{"even"}) = (1 + (q-p)^n)/2$$

and

$$P(\text{"even"}) - P(\text{"odd"}) = (q-p)^n. \tag{2.2}$$

3. Proofs of theorems. Let E and 0 represent the events "even" and "odd," respectively. One can easily establish Theorems 1 and 2 by computing $P(E)$ by summing the probabilities massed at the even integers. However, a computationally simpler proof of Theorem 1 follows by observing that if E occurs, then the first trial must be a failure and there must be an odd number of trials thereafter till the first success. This leads to $P(E) = q(1-p(E))$, i.e., $P(E) = q/(1+q)$.

Theorem 4 follows directly from (2.2) which can be established by inducting on n as follows: If $n = 1$, then $P(E; n = 1) = q$ and $P(0; n = 1) = p$, so that (2.2) holds for $n = 1$. Let S and F be the events the $(k+1)$th Bernoulli trial is "success" and "failure," respectively. Then $P(E; n = k+1) = P(E; n = k)P(F) + P(0; n = k)P(S)$ and $P(0; n = k+1) = P(0; n = k)P(F) + P(E; n = k)P(S)$. Hence

$$P(E; n = k+1) - P(0; n = k+1) = \{P(E; n = k) - P(0; n = k)\}\{P(F) - P(S)\},$$

so that (2.2) follows from the induction hypothesis.

We establish Theorem 3 by finding finite difference equations for the probabilities in question and then establish (2.1) by inducting on r. Let E_r and 0_r be the outcome

"even" and "odd", respectively, in the game of Theorem 3 and let S and F be as above with $k = 0$. Then $P(E_r) = P(E_r \mid S)P(S) + P(E_r \mid F)P(F)$

$$= P(0_{r-1})P(S) + P(0_r)P(F) = (1 - P(E_{r-1}))p + (1 - P(E_r))q.$$

Thus, $P(E_r) = (1 - pP(E_{r-1}))/(1 + q)$ for integers $r \geqq 1$ with $P(E_0) = 1$, so that (2.1) follows easily by inducting on r.

The work of A. N. Philippou was supported by the University of Texas at El Paso Research Institute.

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POLYGONS HAVE EARS

G. H. MEISTERS

We refer to a simple closed polygonal plane curve with a finite number of sides as a *Jordan polygon*. We assume the truth of the famous Jordan Curve Theorem only for Jordan polygons. (For elementary proofs see Appendix 2 of Chapter V of [4] or Appendix B1 of [5].) Three consecutive vertices V_1, V_2, V_3 of a Jordan polygon $P = V_1V_2V_3V_4 \cdots V_nV_1$ ($n \geqq 4$) are said to form an *ear* (regarded as the region enclosed by the triangle $V_1V_2V_3$) at the vertex V_2 if the (open) chord joining V_1 and V_3 lies entirely inside the polygon P. We say that two ears are non-overlapping if their interior regions are disjoint; otherwise they are *overlapping*. If we *remove* or cut off an ear $V_1V_2V_3$ (by drawing the chord V_1V_3) from the Jordan polygon P, then there remains the Jordan polygon $P' = V_1V_3V_4 \cdots V_nV_1$ which has one less vertex than P.

The property of Jordan polygons expressed by the following theorem seems to provide a particularly simple and conceptual bridge from the Jordan Curve Theorem for Polygons to the Triangulation Theorem for Jordan Polygons; at least simpler perhaps than that given in Appendix B2 of [5].

TWO EARS THEOREM. *Except for triangles, every Jordan polygon has at least two non-overlapping ears.*

Proof. Our proof is by induction on the number n of vertices of the Jordan polygon P. Since the proof for quadrilaterals as well as the proof for the general case ($n > 4$) can be divided into the same two cases, for the sake of brevity, we deal with quadrilaterals at the beginning of each of these two cases.

Let P denote a Jordan polygon with at least four vertices, select a vertex V of P at which the interior angle is less than 180°, and let V_- and V_+ denote the vertices of P which are adjacent to V. (Any V on any minimal triangle enclosing P will do.)

Case 1. The polygon P has an ear at V. If we remove this ear, then the remain-

ing polygon P' is either a triangle (and hence forms another ear for P which is non-overlapping with the ear at V) or else is a Jordan polygon with more than three vertices, but with one less vertex than P, so that the induction hypothesis yields two non-overlapping ears E_1 and E_2 for P'. Since they are non-overlapping, at least one of these two ears, say E_1, is not at either of the vertices V_- and V_+. Since all ears of P' (except for any which might occur at V_- or V_+) are also ears of P, the two ears E_1 and V_-VV_+ are non-overlapping ears for P.

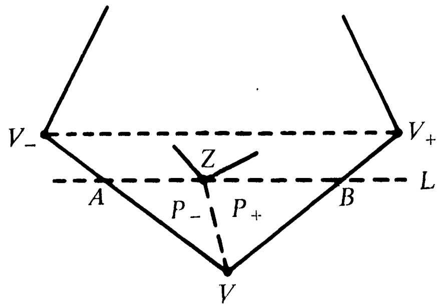

FIG. 1.

Case 2. Polygon $P = VV_+ \ldots Z \ldots V_- V$ does not have an ear at V. Chord VZ divides P into two Jordan polygons $P_- = VZ \ldots V_- V$ and $P_+ = VV_+ \ldots ZV$.

Case 2. The polygon P does not have an ear at V. (See Fig. 1.) Then the triangle V_-VV_+ must contain in its interior or on the open chord V_-V_+ at least one vertex of P. Let Z denote one such vertex with the additional property that the line L through it and parallel to V_-V_+ is as close to V as possible. Clearly such a Z must exist. If A and B denote the points of intersection of this line L with the chords VV_- and VV_+, respectively, then the open triangular region VBA cannot contain any vertex (or edge points) of P. Hence the (open) chord VZ lies entirely inside the Jordan polygon P and so divides it into two Jordan polygons $P_- = VZ \cdots V_-V$ and $P_+ = VV_+ \cdots ZV$, each with fewer vertices than P. (The polygon P_- does not contain V_+, and the polygon P_+ does not contain V_-.) If P is a quadrilateral, then VV_-Z and VV_+Z are two non-overlapping ears for P. On the other hand, if P is not a quadrilateral, then at least one of the two polygons, say P_+ is not a triangle.

Case 2a. The polygon P_- is a triangle. Then VV_-Z is an ear for P and the polygon P_+ must possess (by the induction hypothesis) two non-overlapping ears E_1 and E_2, at least one of which, say E_1, is not at either of the vertices V or Z. This ear E_1 for P_+ is then also an ear for P and obviously does not overlap with the ear VV_-Z.

Case 2b. The polygon P_- is not a triangle. The induction hypothesis now yields four mutually non-overlapping ears, two (E_1^+, E_2^+) for P_+ and two (E_1^-, E_2^-) for P_-.

At least one of the pair (E_1^-, E_2^-), say E_1^-, is not at either of the vertices V and Z; and, similarly, at least one of the pair (E_1^+, E_2^+), say E_1^+, is not at either of these vertices either. (The vertices V and Z of P are the only vertices common to P_- and P_+.) Consequently, E_1^- and E_1^+ will be (non-overlapping) ears for the original polygon P. This completes the proof of the theorem.

APPLICATION. A procedure for triangulating a Jordan polygon without introducing any new vertices is now immediately obvious: Namely, locate an ear $V_1V_2V_3$ and cut it off, then locate an ear of the remaining polygon (of one less vertex!) and cut it off, and continue this process until the remaining polygon is itself a triangle.

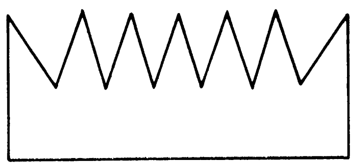

FIG. 2 — A polygon with many ears.

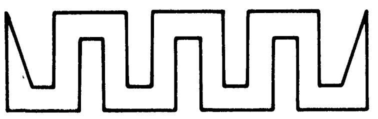

FIG. 3 — A polygon with many vertices, but only two ears.

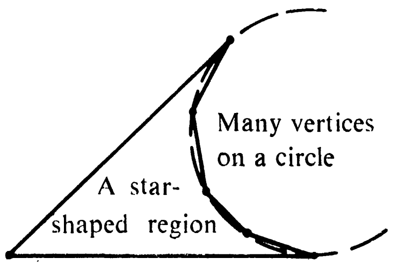

FIG. 4 — A star-shaped polygon with many vertices, but only two ears.

Note that while some Jordan polygons may have many ears (as in Fig. 2), others have only two ears no matter how many vertices they may have (as in Figs. 3 and 4). Figure 4 is also an example of a Jordan polygon with many vertices but only three interior angles less than π, and this is the minimum that is possible. Of course a Jordan polygon is convex if and only if it has an ear at every vertex.

Now consider the following two statements for Jordan polygons P with $n > 3$ vertices.

(I) *The polygon P has at least two non-overlapping ears.*

(II) *The polygon P admits a triangulation $\mathscr{T}$ with no new vertices.*

We have seen above that on the one hand (I) can be proved without assuming (II) and on the other hand (II) is an immediate consequence of (I). We now show that, conversely, (II) implies (I). Let t denote the number of triangles in $\mathscr{T}$. Then $t = t_0 + t_1 + t_2$ where t_j denotes the number of triangles in $\mathscr{T}$ which have exactly

j sides in common with P. Note that each triangle of $\mathscr{T}$ which has exactly two sides in common with P is an ear for P and no two such ears overlap. Now it can be shown that the sum Σ of the interior angles of a Jordan polygon is equal to $(n-2)\pi$. But also $\Sigma = t\pi$. Consequently, $t_0 + t_1 + t_2 = n-2$. Furthermore, it is clear that $t_1 + 2t_2 = n$. Eliminating t_1 we obtain $t_2 = t_0 + 2 \geqq 2$.

The "Shelling Theorem" on page 32 of [**1**] is closely related to our Two Ears Theorem, but our proof is entirely different and was carried out before we discovered this reference.

We close with two applications of our Two Ears Theorem.

(1) From Euler's formula $V - E + F = 1$ and the fact (just proved) that $t = n-2$, it follows that every triangulation with no new vertices contains $n-3$ non-crossing diagonals.

(2) From the existence of an ear for P and by Mathematical Induction it follows immediately that the sum Σ of the interior angles of P is equal to $[(n-1)-2]\pi + \pi = (n-2)\pi$.

For some interesting applications of triangulation of polygons see [**2**] and [**3**].

References

1. R. H. Bing, Elementary point set topology, this MONTHLY, 67 (1960) No. 7 Part II.

2. V. G. Boltyanskii, Equivalent and Equidecomposable Figures, Topics in Mathematics, Translated from the Russian, Heath, Boston, 1963.

3. J. H. Conway and H.S. M. Coxeter, Triangulated polygons and frieze patterns, Mathematical Gazette, 57 (1973) 88–94.

4. Richard Courant and Herbert Robbins, What is Mathematics?, Oxford University Press, N.Y., 1941.

5. Einar Hille, Analytic Function Theory, Volume I, Blaisdell, New York, 1965.

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

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INDUSTRIAL MATHEMATICS: A COURSE IN REALISM

C. A. HALL

Introduction. There is an anxiety experienced by most mathematics majors as they approach graduation and contemplate a career in industry. The basic cause for this anxiety is the desire to foresee this *environment* in which they hope to exist as an industrial mathematician. During academic year 1972–73 at the University of

Pittsburgh, a course, Industrial Mathematics (M111), was designed to relieve this anxiety to some extent by exposing the student to a *simulated industrial environment*. I take this opportunity to report on its seeming success as well as some views on preparation of industrial mathematicians.

What is "industrial" mathematics and what is the primary function of an industrial mathematician? Contrary to popular belief, the primary function of an industrial mathematician is not just problem solving! I once heard a colleague, when asked by a student what a mathematician does, make the statement that pure mathematicians prove theorems and applied mathematicians solve problems. I do not accept either as a close approximation to reality if one considers them literally. An industrial mathematician does much more than solve problems.

E. H. Bareiss [1] says "mathematics is an *art*, a *language*, a *tool* and a *means of accounting*." A metaphor might be that mathematics is an *existence-uniqueness theorem*, a *differential equation*, a *computer algorithm* and an *analysis of heat transfer*. Barreis continues ... "From the point of view of the professional mathematician, the art of mathematics ranks most prestigious, followed by language and tool. Accounting (or accountability) is often not considered mathematics at all. From the standpoint of industry, the order of importance is reversed, for the art has often no necessary connection with the physical world and is therefore of little immediate value, whereas the language and the tool clearly have value, and, without accounting our techno-society could not exist."

I assume that each member of the class in industrial mathematics has a preconception of what mathematics *is*, and submit that the above quotes serve to distinguish "industrial mathematics" from mathematics as most seniors are conditioned to envision it. Perhaps modern industrial mathematics should more aptly be called *scientific computing*, which Garrett Birkhoff [2] defines as "the art of utilizing physical intuition, mathematical theorems and algorithms, and modern computer technology to construct and explore realistic models of problems arising in the natural sciences and engineering."

A question related to: *What is industrial mathematics*? and one which is somewhat easier to answer is: *What is the primary function of an industrial mathematician*? M. S. Klamkin [3], delineates five aspects or stages of the "evolution and dispatch" of an industrial problem:

I. Recognition
II. Formulation
III. Solution
IV. Computation
V. Explanation

I recommend that the reader refer to [3] for an elaboration of these stages of "problem solving." It is precisely these five aspects of an industrial environment which are simulated in M111.

The uses of mathematics in industry are as varied as the number of industrial

mathematicians themselves, so no single course could hope to survey more than but a limited few applications (undoubtedly biased by the interests of the instructor). Only one real world problem is analyzed by an M111 class during the fifteen week term, however, great care is taken to introduce the student to all the ramifications of being asked to "solve" such a problem.

The primary goal of M111 is the exposure of each student to a simulated industrial environment and the specific mathematical problem or use of mathematics discussed is but a means to this end.

Let me conclude this section with some truisms that are worthy of the readers' consideration and which are subtly revealed during the M111 course:

T_1: An industrial mathematician cannot afford to be just a mathematician, he must also be a "scientist."

T_2: An industrial mathematician normally works in a group with two or three other mathematicians or scientists. (Bareiss calls this the *team approach.*)

T_3: An industrial mathematician spends considerable time communicating his ideas and findings in writing and orally.

T_4: An industrial mathematician is often viewed and justified as a parasite on more product-oriented scientists and engineers.

T_5: An industrial mathematician may be hired primarily to fulfill the function of an economist, physicist or engineer, if he is so qualified.

T_6: An industrial mathematician, just like any other human being, experiences moments of ecstasy as well as moments of frustration and boredom.

$T_7 \equiv T_6$: Not all aspects of "problem evolution and dispatch" are glamorous. Different aspects appeal to different people.

T_8: An industrial mathematician is a buffer between the physical scientist or engineer and the computer.

Industrial mathematics course ingredients. The following ingredients are suggested as a guide to a realistic course in industrial mathematics:

(1) A class of problems is chosen which contains some *new mathematical material* for the students, is of sufficient depth to consume fifteen weeks of concentrated effort and is amenable to illustrating the five aspects of problem evolution and dispatch. A realistic model also allows for varying of problem parameters.

(2) *The class works as a team* under the supervision of the instructor to recognize, formulate, solve, compute and explain the problem.

(3) *Explicit assignments* are made to each student according to his capabilities. Some students undoubtedly contribute more to the team than others—such is reality.

(4) *Computer implementation* of an algorithm on a *realistic level* is essential. The problem is not rigged to give pathological or "textbook solutions." It need not require sophisticated mathematical techniques for solution, but must be sufficiently complex, so as to require careful and skillful analysis and development to avoid catastrophic results.

(5) Problems which are of current industrial (research) interest are highly desirable in that they tend not to yield easily to "clean" textbook solutions and also tend to generate more enthusiasm and a sense of real accomplishment on the student's part when he realizes he is treading over "virgin territory" or at best "roughly mapped terrain."

(6) *Progress reports*, one page in length, are written by each student at appropriate stages of development and some class time is allocated to a critique of them.

(7) *Guest speakers* from industry, with varying expertise, greatly enhance the course.

(8) One-third to one-half of the class periods are spent in the *discussion* of specific problems the students encounter and approaches to resolve them.

(9) The class is split into groups of 4–5 students to handle specific tasks such as programming a particular subroutine. Each such group then *reports orally* to the entire class on their progress and approach to their task.

(10) *Documentation*, to include the physical problem, mathematical model, algorithms, computer program and sample problems is the primary concern during the last few days of the term.

Sample course modules. M111 was taught twice during academic year 1972–73 at the University of Pittsburgh, in the Fall Term to 9 students and in the Winter Term to 20 students. The students were primarily seniors, mathematics majors and computer science minors. In general terms the two problems considered were:

1. FALL TERM: REDUCTION OF CINETHEODOLITE DATA. Cinetheodolites are angle-measuring instruments used to determine the trajectories of rapidly moving aerial targets or missiles. Two or more cinetheodolites are placed at known positions on PITT FIELD and record, on 35 mm film, *azimuth* and *elevation* angles to the missile for various values of time. They are synchronized with the Student Union clock so that for each time value, each cinetheodolite determines a "line of sight" to the missile. Due to errors, these lines of sight do not intersect, and the most likely missile position is determined by the method of least squares. That is, given a set of observed azimuths $\{A_i\}_{i=1}^N$ and observed elevations $\{E_i\}_{i=1}^N$, one seeks perturbations δA_i and δE_i such that the lines of sight determined by $\{(A_i + \delta A_i, E_i + \delta E_i)\}_{i=1}^N$ intersect in a unique point (x^*, y^*, z^*) and such that the weighted sum of the squares of the residuals

$$S(x, y, z) \equiv \sum_{i=1}^{N} \{(\mathrm{Cos}^2 E_i)(\delta A_i)^2 + (\delta E_i)^2\} \tag{1}$$

is a minimum. Replacing the perturbations of angles by the appropriate Arctangent functions, the gradient of S is easily computed. S has a minimum as a function of x, y, z only if this gradient vector is zero at (x^*, y^*, z^*), hence one seeks a solution of the nonlinear system of equations

$$\begin{aligned} \partial S/\partial x &= 0 \\ \partial S/\partial y &= 0 \qquad (2) \\ \partial S/\partial z &= 0. \end{aligned}$$

After some discussion of the cinetheodolite system and data collections and film reading procedures, two mathematical models were formulated. The first yielded approximate solutions of the nonlinear system above, and the second model was a linearized version in which the Arctangent functions were approximated by their Taylor expansions of degree one.

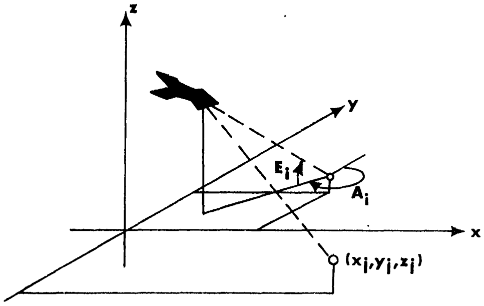

FIG. 1: Cinetheodolite data are azimuth A_i and elevation E_i to the target for each time value.

Using trigonometry, linear algebra and calculus the mathematical solutions to these models were determined and the necessary computer programs were written and debugged.

Several examples were designed to illustrate the programs and models. A 50 page report was written to document the project.

SAMPLE PROBLEM: The following camera co-ordinates and observed azimuth and elevation data are given for a fixed value of time.

Camera number	Camera coordinates x	y	z	Observed Azimuth	Elevation
1	4000	5000	1000	237.0364	0.0
2	1300	1200	0	255.9639	38.9539
3	1300	−1200	0	107.0361	38.9538
4	2800	1700	5	219.3633	21.8942
5	2800	−1700	5	136.6367	21.8942
6	200	2500	0	287.7447	20.8553
7	1600	100	10	190.4622	58.4325
8	−2000	−800	15	15.9314	17.6014
9	− 600	700	0	336.3706	29.7953
10	1050	90	0	245.9453	84.1217

The 'best' approximation to the position of the target or missile for this time frame was computed to be

$$x^* = 985.4284, \quad y^* = -0.2429, \quad z^* = 1004.4879 \tag{3}$$

and the associated weighted sum of the squares of the residuals is

$$S(x^*, y^*, z^*) = .006 \text{ deg}^2 \approx 10^{-6} \text{ rad}^2. \tag{4}$$

2. WINTER TERM: FINITE DIFFERENCE SOLUTION OF STEADY STATE HEAT FLOW. Heat flow through a thin sheet of metal gives rise to a temperature distribution which is position dependent. The governing laws of physics for steady state reduce to a second order elliptic partial differental equation (PDE) which can rarely be solved in closed form. Finite difference methods are implemented to approximate the PDE at a finite number of points in the problem domain.

Depending on the shape of the sheet of metal, one coordinate system may be more natural than another. Quite often the given PDE can be transformed via a change of coordinate systems to an equivalent PDE with domain the unit square. The class formulated such transformed heat flow problems on the unit square as solutions of

$$-\begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} + cu = f, \quad 0 \leqq x,\ y \leqq 1, \tag{5}$$

subject to Dirichlet boundary conditions. The functions a_{ij}, c and f were of course dependent upon the transformation.

A finite difference model was developed and programmed. The resulting program HEAT utilized block Gaussian elimination with up to 625 mesh points. The output included an approximation to the solution of the PDE at each mesh point, along with a CALCOMP plot of a piecewise bilinear surface which passed through this set of data. This plot then represented graphically a "good" approximation to the temperature distribution of the transformed problem. Finally a plot of the approximate temperature distribution of the original problem was generated.

Several examples were designed to illustrate the flexibility of the program HEAT and a 45 page report was written to document the project.

SAMPLE PROBLEM: The partial differential equation

$$-u_{xx} - u_{yy} + (x^2 + y^2)u = 40(2-x-y+4xy)\exp(xy), \quad 0 \leqq x, y \leqq 1, \tag{6}$$

with u fixed on the boundary of the unit square so that the true solution is

$$u(x, y) = \{10 - 20[(x - \tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2]\exp(xy)\}, \tag{7}$$

was solved numerically by HEAT with maximum mesh point errors as indicated below. See Figure 2.

Number of meshpoints	*Maximum error*
16	.0506
81	.0139
361	.0035

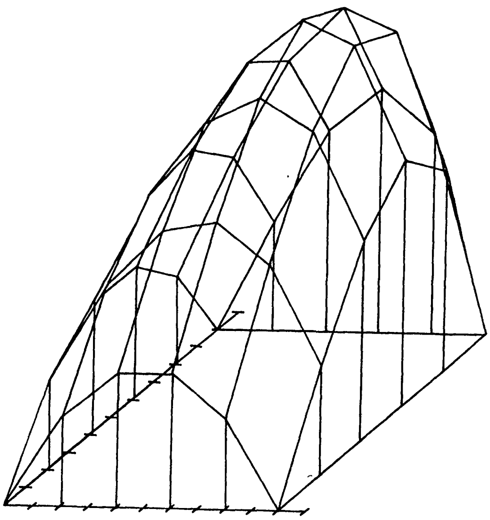

FIG. 2: The finite difference solution to (6) over a uniform 4 by 4 grid.

SAMPLE PROBLEM: The partial differential equation

$$-u_{xx} - u_{yy} + 12u = -4(x^3y + xy^3)\exp(x^2 + y^2), \quad (x, y) \in \mathscr{R}, \tag{8}$$

where $\mathscr{R}$ is the quarter annulus $\{(x, y): x \geqq 0, y \geqq 0, \ 1/2 \leqq (x^2 + y^2)^{\frac{1}{2}} \leqq 3/2\}$ with u fixed on the boundary of $\mathscr{R}$, so that the true solution is

$$u(x, y) = xy\exp(x^2 + y^2) \tag{9}$$

was transformed via the change of variables

$$\begin{aligned} x &= (r + 1/2)\mathrm{Cos}(\theta\pi/2) \\ y &= (r + 1/2)\mathrm{Sin}(\theta\pi/2) \end{aligned} \tag{10}$$

into the partial differential equation

$$-\begin{bmatrix} \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} \end{bmatrix} \begin{bmatrix} \dfrac{\pi(r + 1/2)}{2} & 0 \\ 0 & \dfrac{2}{\pi(r + 1/2)} \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} + 6(r + 1/2)u \tag{11}$$

$= -\pi(r + 1/2)^5\mathrm{Sin}(\theta\pi)\exp(r + 1/2)^2$, where $0 \leqq r, \ \theta \leqq 1$. This transformed

boundary value problem was then solved numerically over the square, using HEAT with a 24 by 24 finite difference grid with a maximum mesh point relative error of 0.61%. See Figures 3 and 4.

During academic years 1973–74 and 1974–75 other material used dealt with plate bending, network analysis of a steam generator and diffusion of electrons in fluorescent lamps.

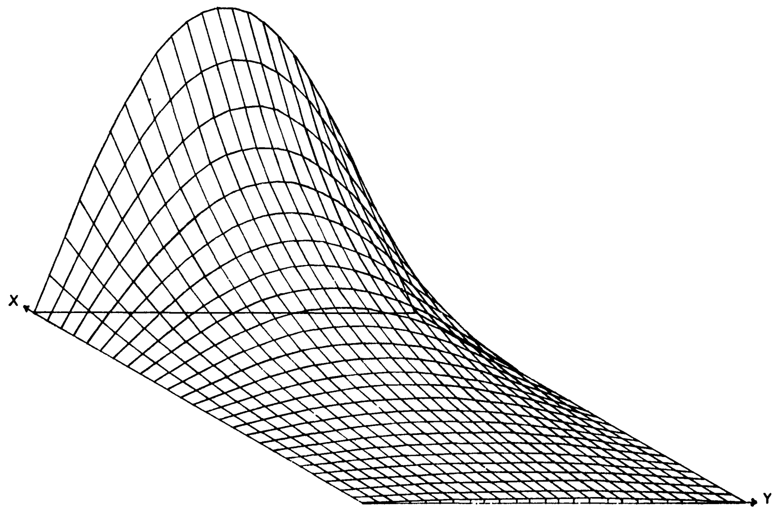

FIG. 3: The finite difference solution to the transformed problem (11) with a uniform 24 by 24 grid.

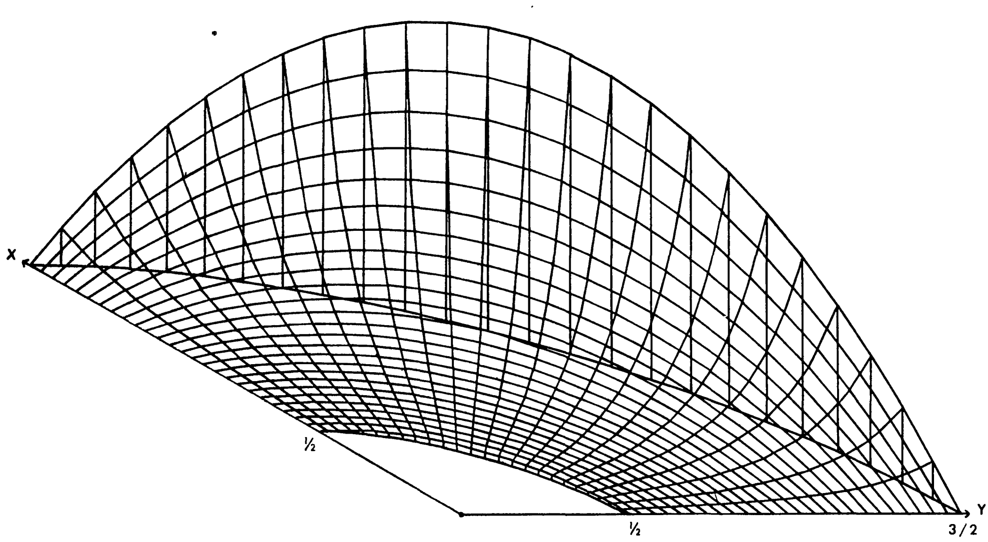

FIG. 4: Approximate temperature distribution generated from the finite difference solution in Fig. 3 by inverting the coordinate transformation in (10). Note that the surfaces in Figures 3 and 4 represent the same function relative to different coordinate systems.

References

1. E. H. Bareiss, The college preparation for a mathematician in industry, this MONTHLY, 79 (1972) 972–984.

2. Garrett Birkhoff, The numerical solution of elliptic equations, Regional Conference Series in Applied Mathematics, SIAM, 1971.

3. M. S. Klamkin, On the ideal role of an industrial mathematician and its educational implications, this MONTHLY, 78 (1971) 53–76.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before September 30, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E2540. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Let F be a finite field of order q, let n be a divisor of $q-1$, and let α be a nonzero element of F. Evaluate

$$S(n,q;\alpha) = \Sigma\,(t^n-\alpha)^{-1},$$

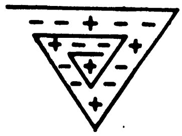
the sum being over all $t \in F$ with $t^n \neq \alpha$.

E2541. *Proposed by E. T. H. Wang, Wilfrid Laurier University*

A *Steinhaus triangle* is formed as follows: Start with a row of n plus and minus

signs. Under each pair of like signs, a plus sign is written and under each pair of unlike signs, a minus sign is written. Continuing on, one finally obtains a triangle of $\frac{1}{2}n(n+1)$ plus and minus signs.

Prove that if the first row pattern of a Steinhaus triangle is $- - + - - + \cdots$ (i.e., two minuses followed by a plus), then the same pattern repeats itself when one traverses all of the entries in a clockwise spiral fashion as shown in the figure:



E2542. *Proposed by Ron Evans, University of Wisconsin*

Let G be the group generated by the transformations T and S on the extended complex plane, where $zT = -1/z$ and $zS = z + 2i$. Suppose that z_0 is fixed by some non-identity transformation in G. Prove that z_0 must lie on the extended imaginary axis.

E2543. *Proposed by the late C. W. Anderson, University of California, Berkeley*

For the natural number n, let $\tau(n)$ be the number of divisors and $\sigma(n)$ the sum of these divisors. Show that there exists a constant $k > 0$ such that if $x = \sigma(n)/n$ is sufficiently large, then

$$\tau(n) > 2^{\exp kx}.$$

Show also that there exist n for which $\tau(n)$ is arbitrarily large, but for which $\sigma(n)/n$ is arbitrarily close to unity.

E2544. *Proposed by Harvey Cohn, City College of C.U.N.Y.*

Consider the sequence of words formed by "Fibonaccian juxtaposition": $w_1 = 0$, $w_2 = 1$, $w_{n+2} = w_n w_{n+1}$ for $n \geqq 1$. (Thus $w_3 = 01$, $w_4 = 101$, $w_5 = 01101$, etc.) Form the sequence S by

$$S = w_1 w_2 w_3 w_4 \cdots = 010110101101 \cdots.$$

Now let $\gamma = \frac{1}{2}(5^{\frac{1}{2}} - 1)$ be the reciprocal of the "golden mean" and define $t_n = [n\gamma] - [(n-1)\gamma]$ for $n = 1, 2, \cdots$ where the square brackets denote the greatest integer function. Form the sequence $T = t_1 t_2 t_3 \cdots$. Show that the sequences S and T are identical.

Generalize.

E2545. *Proposed by Ron Evans, University of Wisconsin*

Let V be an invertible $n \times n$ matrix with rational entries and let G denote the group of all $n \times n$ matrices with integral entries and determinant 1. Prove that if H and VHV^{-1} are subgroups of G of finite index p, q respectively, then $p = q$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Broken-Line Brachistochrone Paths

E 1255*. [1957, 109]. *Proposed by J. P. Ballentine*

A constant gravitational field operates in the direction of the negative y-axis. A particle starts from point (a, b) with a given initial velocity, travels along a straight line to the point (x, y), thence without loss of velocity along another straight line to the point (c, d). The point (x, y) is so chosen that the total time is a minimum. Prove or disprove that the particle reaches the point (x, y) when the time is precisely half gone.

Comment by the Editors. The statement of the problem should really read "without loss of *speed*" rather than "without loss of *velocity*" since the (vector) velocity must necessarily change direction at the "break-point." The following simplifying assumptions can be made: The initial point can certainly be chosen to be the origin $(0,0)$, and the y-axis can be oriented so that the gravitational force operates in the *positive* y-direction. Further, the terminal point can be chosen to be $(1,1)$ without any real loss of generality. Let the break-point be at (x_0, y_0) and suppose that the particle reaches this break-point at t_0 and reaches the terminal point at t_1. By brute force computation, t_0 and t_1 can be computed for a mesh of break-points covering the square, and the ratio t_0/t_1 can be calculated for that break-point making t_1 a minimum. If this ratio is not close to $\frac{1}{2}$ (and this Editor can see no reason why it should be), continuity can be invoked to answer the original statement of the problem in the negative. (Note that one need only consider break-points for which $y_0 > x_0$.)

A more significant problem would be to locate precisely the break-point making t_1 a minimum and to calculate the ratio t_0/t_1 for this point. This minimum-time path should then in some sense approximate the brachistochrone, but in what sense: least squares, minimax? More break-points could be added and the terminal point could be varied. The general problem would seem to be very complicated.

Largest Cross-Section of a Tetrahedron

E 1298* [1958, 43]. *Proposed by H. D. Grossman*

It is not difficult to show that the longest linear section of a triangle is the longest side of the triangle. Is the greatest planar section of a tetrahedron the largest face of the tetrahedron?

Solution by Murray S. Klamkin, University of Waterloo. The answer is yes, as was shown in the solution to the identical Advanced Problem 5006 [1962, 63; 1963, 338; 1963, 1108]. Curiously enough, the corresponding result for a 5-simplex is false: See D. W. Walkup, *A simplex with a large cross-section*, this MONTHLY 75 (1968), 34–36.

Tritangent Circles Kissing Precisely

E2475 [1974, 516]. *Proposed by W. J. Berger, District of Columbia Teachers College*

Under what conditions can the four tritangent circles of a triangle be rearranged so as to be mutually tangent?

Solution by A. W. Walker, Toronto, Canada. If the mutual tangency has precisely six distinct contacts, the triangle must be right-angled. Let the triangle be ABC with angle A the largest, and let the radii of the incircle and the three excircles be r, r_a, r_b, r_c respectively. Then

$$r^{-1} = r_a^{-1} + r_b^{-1} + r_c^{-1} \tag{1}$$

$$2(r^{-2} + r_a^{-2} + r_b^{-2} + r_c^{-2}) = (r^{-1} \pm r_a^{-1} + r_b^{-1} + r_c^{-1})^2, \tag{2}$$

where (2) is the Descartes-Soddy tangency condition. (See Daniel Pedoe, *On a theorem in geometry*, this MONTHLY 74 (1967), 627–640.) The $+$ sign is taken if the contacts are all external, whereas the $-$ sign is taken if the circle with radius r_a surrounds the other three. Taking the $+$ sign in (2) and combining (1) and (2) gives $r_a + r_b + r_c = 0$, an impossibility, so that mutual external tangency is not possible. Using the $-$ sign in (2), substitute (1) in (2) and eliminate r to get

$$r_a^{-2} + r_a^{-1}r_b^{-1} + r_a^{-1}r_c^{-1} = r_b^{-1}r_c^{-1},$$

which reduces to $rr_a = r_b r_c$ by (1). Now if K is the area of ABC and if s is its semi-perimeter, we know that

$$K = rs = r_a(s-a) = r_b(s-b) = r_c(s-c),$$

$$\tan^2 \tfrac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)},$$

from which follows

$$\tan^2 \tfrac{1}{2}A = \frac{r\,r_a}{r_b r_c}.$$

Thus $\tan^2 \frac{1}{2}A = 1$, implying that $A = 90°$.

For any triangle, the four tritangent circles can be moved into mutual tangency with a single contact point; also, any isosceles triangle will have mutually tangent circles if we say that two coincident circles are mutually tangent.

Also solved by the proposer. Partial solution by Michael Goldberg.

Editor's comment. The proposer observes that if mutual tangency is possible, then it can be realized by placing the four circles with centers at the vertices of a rectangle whose sides are the same as the legs of the original (right) triangle. He also notes that if the original triangle has integral sides, then the four circles will have integral radii. The interested reader is urged to attempt the construction for himself, using the 3–4–5 triangle. (The four circles have radii 1, 2, 3, 6.)

Tetratangent Spheres Kissing Precisely

E 2476* [1974, 516]. *Proposed by W. J. Berger, District of Columbia Teachers College*

Under what conditions can the five tetratangent spheres of a tetrahedron be rearranged so as to be mutually tangent?

Partial solution by A. W. Walker, Toronto, Canada. The general tetrahedron has eight tetratangent spheres, three of which may not exist. Consider then the other five and denote their radii by r, r_a, r_b, r_c, r_d. (The notation is analogous to that in the preceding problem.—Ed.) Let the volume of the tetrahedron be V and let its facial areas be A, B, C, D. Then

$$2rS = 3V = 2r_a(S-A) = \cdots = 2r_d(S-D),$$

where $S = \frac{1}{2}(A+B+C+D)$. (See N. A. Court, *Modern Pure Solid Geometry*, Macmillan, 1935, p. 82.)

We assume that the proposer desires nontrivial tangency, i.e., precisely ten distinct contacts. If we assume that all contacts are external, then

$$\begin{aligned}(1)\qquad & 3r^{-2}S^{-2}(S^2+A^2+B^2+C^2+D^2)\\ &= 3(r^{-2}+r_a^{-2}+r_b^{-2}+r_c^{-2}+r_d^{-2})\\ &= (r^{-1}+r_a^{-1}+r_b^{-1}+r_c^{-1}+r_d^{-1})^2 = 9r^{-2},\end{aligned}$$

where the outer equalities are readily verified (Court, *op. cit.*, pp. 83–85) and the middle equality is the Soddy condition for mutual tangency of the five spheres (Pedoe, *op. cit.*). Then

$$A(S-A)+B(S-B)+C(S-C)+D(S-D) \equiv 2S^2 - A^2 - B^2 - C^2 - D^2 = 0,$$

an impossibility. But if four spheres are in mutual external contact and the fifth one with radius r_a surrounds and touches them, then r_a^{-1} is replaced by $-r_a^{-1}$ in (1) above, giving

$$3(S^2+A^2+B^2+C^2+D^2) = r^2S^2(3r^{-1}-2r_a^{-1})^2 = (S+2A)^2,$$

the possibility and significance of which require further investigation.

Partial solutions also by Michael Goldberg, and by the proposer.

Editor's comment. The proposer observes that the expected generalization of the result of Problem E 2475 from two to three dimensions is not valid; if the tetrahedron contains a solid right angle, then mutual tangency need not hold. Consider the tetrahedron with vertices $A=(0,0,0)$, $B=(1,0,0)$, $C=(0,1,0)$, and $D=(0,0,1)$. Then $r=(3+\sqrt{3})^{-1}$, $r_a=(3-\sqrt{3})^{-1}$, and $r_b=r_c=r_d=(1+\sqrt{3})^{-1}$ and the Soddy condition is not met for either mutual external contact or for the largest sphere (with radius r_a) containing the other four.

Goldberg reminds us that Soddy's original formulation of the two- and three-dimensional tangency conditions was in the form of a poem entitled "The Kiss Precise" published in *Nature* 137

(1936), p. 1021 and that the tangency conditions were generalized to n-space by T. Gossett in another poem with the same title (*Nature* 139 (1937), p. 62).

A New Perspective

E2477 [1974, 516]. *Proposed by A. W. Walker, Toronto, Canada*

A straight line L meets the sides BC, CA, AB of a triangle ABC with orthocenter H, at X, Y, Z; DE is a diameter of the circle ABC. Through X, Y, Z lines $B'C'$, $C'A'$, $A'B'$ are drawn parallel to AE, BE, CE to form a triangle $A'B'C'$ oppositely similar to ABC. If D', E', H' are the images of D, E, H for this similarity, prove that in general

(a) The lines AA', BB', CC', DD', HH' concur, so that $ABCD$ and $A'B'C'D'$ ($ABCH$ and $A'B'C'H'$) are oppositely similar perspective cyclic (orthocentric) quadrangles;

(b) The lines DH' and HD' meet at the invariant point of the similarity, and $DHD'H'$ is a cyclic quadrangle;

(c) The axis L is perpendicular to DD' and bisects EE'.

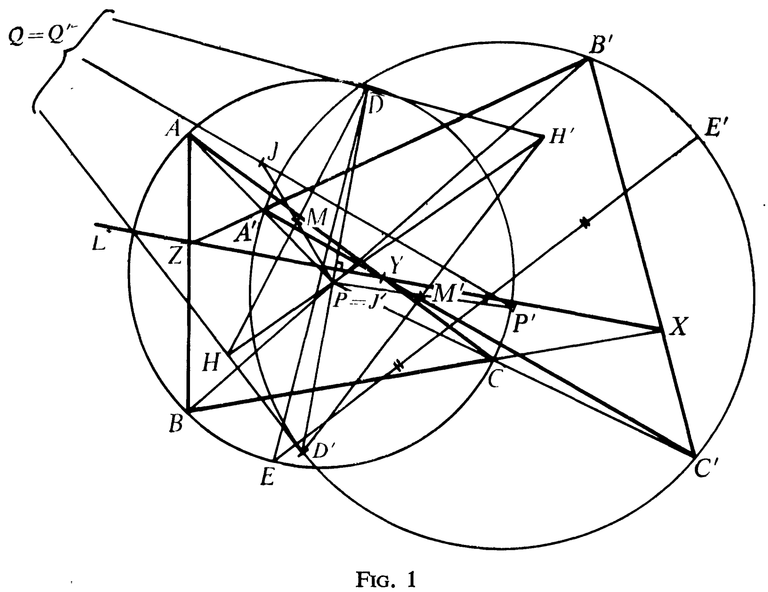


FIG. 1

ABC, *A'B'C'* are oppositely similar perspective triangles with orthocenters H, H' and center of perspective P. The invariant point is $Q \equiv Q'$.

Solution by the proposer. (a) The oppositely similar triangles ABC and $A'B'C'$ have D and D' as orthologic centers [1], i.e., the lines through A, B, C (A', B', C') parallel to $A'H'$, $B'H'$, $C'H'$ (AH, BH, CH) concur at D (D'). (See Fig. 1.) Since these triangles are perspective with axis L and therefore (in general) with center P, the points P, D, D' lie on a line perpendicular to L [2, 3]. It is also known [4] for any

pair of triangles ABC, $A'B'C'$ that if the lines AU, BU, CU are parallel to the lines $A'V$, $B'V$, $C'V$, the loci of U and V are the homothetic circumconics Σ and Σ'; if the triangles are perspective from a center P, then Σ and Σ' meet at P and at another real point Q, and all the UV lines pass through Q. (This is a generalization of the projectively related case for which Σ and Σ' are circles and the inscribed perspective triangles are directly similar, with $Q \equiv Q'$ as the invariant point of the similarity. See Fig. 2.)

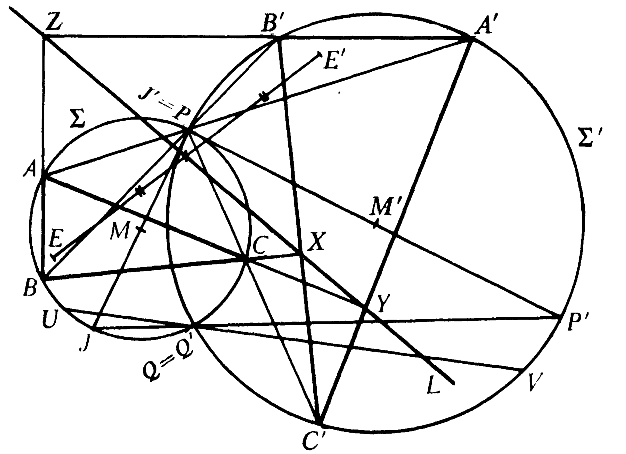


FIG. 2

Circles Σ, Σ' cut orthogonally. ABC, $A'B'C'$ are paralogic triangles with orthocenters E, E' and center of perspective P. The invariant point of the direct similarity is $Q \equiv Q'$. There is an affine projective relationship between Fig. 1 and Figs. 2, 3, (but it is not real).

In the present case (Fig. 1) D and H are two positions of U, with H' and D' as corresponding positions of V, so Σ and Σ' are the rectangular hyperbolas $ABCDHP$ and $A'B'C'D'H'P'$ [**2**, **3**] with centers M and M' at the midpoints of DH and $D'H'$ [**5**]. But it is evident in the circular case (Fig. 3) that if a line through P meets the circle Σ (Σ') again at D (D'), and H (H') is the reflection of D (D') in the center of Σ (Σ'), then P, H, H' are collinear; this affine projective result is therefore also valid for the hyperbolic case, completing the proof. (Note that A, A' are the orthologic centers of triangles BCD and $B'C'D'$ (Fig. 1); the reflections of A in M and of A' in M' are the orthocenters of these triangles and also the orthologic centers of BCH and $B'C'H'$ collinear with P.)

(b) It can be seen that for the circular case DH' and HD' are UV lines passing through Q (as required) if and only if the circles Σ and Σ' cut orthogonally, $DHD'H'$ then being an orthocentric quadrangle (Fig. 3). In the projectively equivalent hyperbolic case (Fig. 1) each pair of opposite sides of $DHD'H'$ is therefore a pair of conjugate directions for Σ and Σ'; using a readily verified property of the rectangular hyperbola, the bisectors of the angles between these sides are then parallel to the

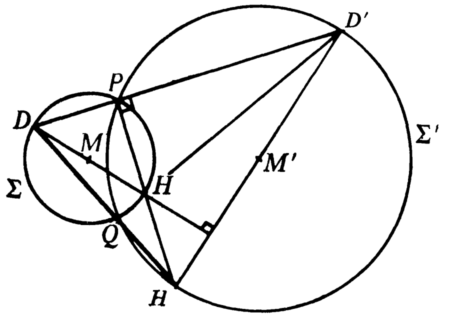


Fig. 3

asymptotes. The parallelism of the three pairs of angle bisectors is a necessary [6] and sufficient condition that the quadrangle $DHD'H'$ is cyclic.

Note that the line through the reflections of P in M and M', the line through A' and the reflection of A in M, etc., concur at Q. (Figs. 1 and 2.) Also the bisectors of the angles between any pair of mutual image segments for the opposite similarity (e.g., AB and $A'B'$, or DH and $D'H'$) are parallel to the axes of similitude and to the asymptotes of Σ and Σ', and [6] the bisectors of the angles between any pair of opposite sides (e.g., AB and CD) of the cyclic quadrangle $ABCD$ inscribed in the rectangular hyperbola Σ are parallel to the axes of Σ. Since $A'B'$ is perpendicular to CD, these two sets of angle bisectors are at 45° as required.

The above argument does not, of course, assume that the projection or the homothety is real. It may be confirmed (e.g., by analytic geometry) that the line of centers MM' meets only one of the hyperbolas Σ, Σ' at real points, and their homothetic centers are not real.

Note that AQA', BQB', CQC', DQD', EQE', HQH', MQM' are all 90° in the circular case and therefore all have the same angle bisectors (the axes of similitude) in the hyperbolic case, so that $Q \equiv Q'$ is the invariant point of either similarity.

Also, any line through P meets Σ and Σ' again in mutual image points for the direct or opposite similarity.

(c) In the circular case with circles Σ and Σ' cutting orthogonally (Fig. 2), each side of either of the inscribed perspective triangles is parallel to the corresponding altitude of the other, so that their orthocenters are the projective equivalents of E and E'; also, it is known that the axis of perspective of these paralogic triangles [7, 8] bisects the line segment joining the orthocenters.

References

1. J. A. Third, Proc. Edinburgh Math. Soc. (1) 31 (1913), p. 23.
2. P. Sondat, Interméd. des Mathématiciens (1) 1 (1894), pp. 10, 44, 45.
3. V. Thébault, this MONTHLY 59 (1952), pp. 25, 26.
4. J. Neuberg, Mathesis 38 (1924), (supp.) p. 180.
5. H. Schaal, Elem. der Math. 19 (1964), p. 54.

6. H. G. Forder, Higher Course Geometry, 1949, p. 148.
7. R. A. Johnson, Modern Geometry, 1929, pp. 258, 259.
8. N. M. Gibbins, Math. Gazette 12 (1925), p. 440; also 30 (1946), p. 293.

Also solved by M. G. Greening (Australia).

Editor's comment. The proposer remarks that the above results can be "summarized" by the following: The orthocenters and center of perspective of two *oppositely* similar perspective triangles are collinear. From Figure 2, it can be seen that these points are not necessarily collinear for *directly* similar perspective triangles. Justification of this collinearity and associated results depends on the projective relationship between this figure and a pair of (directly similar) perspective triangles with corresponding sides perpendicular. The projection is apparently affine, as midpoints are preserved and two homothetic conics project into two circles; however, the conics are hyperbolas with imaginary homothetic centers, any orthocentric quadrangle is the projection of a cyclic quadrangle, and the projection is obviously unreal.

He goes on to say that the only investigation of oppositely similar perspective triangles known to him is a paper by H. A. W. Speckman, *Nieuw Archief voor Wiskunde* (2) 6 (1903), 179–188 (in Dutch); the review of this paper in *Jahrbuch über die Fortschritte der Math.* 34 (1905), p. 560 gives no details, so that the above results may there be included, but are hardly well known nor readily accessible.

Similar Functions

E 2478 [1974, 516]. *Proposed by Charles Wells, Case Western Reserve University*

All functions map the reals to the reals. Call f and g *similar* if $f = h^{-1}gh$ for some bijective h.

(a) Are sin and cos similar?

(b) Characterize those a, b such that $f(x) = x^2$ and $g(x) = x^2 + ax + b$ are similar.

I. *Solution by Karl Heuer, Student, Moorhead (Minnesota) Senior High School.* (a) Suppose for some h, $\sin x = h(\cos(h^{-1}(x)))$ for all x. Set $x_1 = h(1)$ and $x_2 = h(-1)$. Since $x_1 = h(\cos 0) = \sin h(0)$ and $x_2 = h(\cos\pi) = \sin(h(\pi))$, both x_1 and x_2 are in $[-1,1]$. But

$$\begin{aligned}\sin x_1 &= h(\cos(h^{-1}(x_1))) = h(\cos 1) = h(\cos(-1))\\ &= h(\cos(h^{-1}(x_2))) = \sin x_2,\end{aligned}$$

which is impossible since $x_1 \neq x_2$ and sin is increasing on $[-1,1]$. Thus sin and cos are not similar.

(b) Suppose for some a and b there is a bijective h such that

$$h(x^2) = [h(x)]^2 + ah(x) + b \tag{*}$$

for all x. Then

$$[h(x)]^2 - [h(-x)]^2 + a[h(x) - h(-x)] = 0$$

or

$$[h(x) - h(-x)][h(x) + h(-x) + a] = 0.$$

Since h is one-to-one, $h(x) \neq h(-x)$ for all $x \neq 0$. Therefore $h(x)+h(-x)+a=0$ for all $x \neq 0$. Since h is onto, there is some x_0 such that $h(x_0) = -a/2$. If $x_0 \neq 0$, then $h(x_0)+h(-x_0)+a=0$ which implies that $h(-x_0) = -a/2$ contradicting the fact that h is one-to-one. Hence $h(0) = -a/2$. Setting $x=0$ in (*) gives us the necessary condition,

$$4b = a^2 - 2a,$$

for f and g to be similar. This condition is also sufficient as can be seen by considering $h(x) = x - a/2$.

II. *Solution by Victor Norton, Bowling Green State University.*

(a) The functions sin and cos are not similar. If $h(\cos x) = \sin h(x)$ for some one-to-one mapping h of the reals onto the reals, then clearly $h([-1,1]) = [-1,1]$. But

$$\cos x = h^{-1}(\sin h(x))$$

and the right side is one-to-one on $[-1,1]$ whereas the left side is not.

(b) The one value of f with a unique preimage (viz. 0) is also a fixed point of f. If g is similar to f, then g must also have a single value with a unique preimage, and this value will likewise be a fixed point of g. Since $g(x) = x^2+ax+b$, this implies $-a/2 = g(-a/2) = b - a^2/4$. This condition is also sufficient for similarity: take $h(x) = x - a/2$.

Also solved by James Bookey, Benjamin Burrell, J. P. Comiskey, P. G. de Buda, T. E. Elsner, G. T. Frey, Dennis Jespersen, Ralph Jones, Michael Josephy, B. G. Klein, O. P. Lossers (Netherlands), J. P. Matelski, L. E. Mattics, J. G. Mauldon, T. J. Miles, Ram Murty & Kumar Murty, E. M. Norris, D. J. Samuelson, Nan-Shan Shou, T. A. Slobko, Southern Colorado State College Problem Solving Group, Jan Verster, Jeanne Wright, and the proposer. One solution was unsigned.

Editor's comment. Samuelson observes that the condition that $f(x) = x^2$ and $g(x) = x^2+ax+b$ be similar is geometrically that the vertex of g lie on the line $y=x$. More generally, it seems that (distinct) $f(x) = x^2 + a_1x + b_1$ and $g(x) = x^2 + a_2x + b_2$ are similar if and only if the line which connects their vertices has slope 1. A more interesting question might be: When are $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_2$ similar? Another question that suggests itself: Can an even function ever be similar to an odd function?

A Functional Equation with only Obvious Solutions

E 2479 [1974, 517]. *Proposed by J. H. Blau, Antioch College*

Let n be a fixed natural number. It is well known that the functional equation

$$f(x+y^n) = f(x) + (f(y))^n$$

has many discontinuous solutions if $n=1$. Discuss the situation for other values of n. (The function f is a real-valued function of a real variable.)

I. *Solution by P. R. Chernoff, University of California, Berkeley.* By taking

$x = y = 0$, we see that $f(0) = 0$, and therefore $f(y^n) = (f(y))^n$ for all y. Now any nonnegative z can be written as y^n. Hence

$$f(x+z) = f(x+y^n) = f(x) + f(y)^n = f(x) + f(y^n) = f(x) + f(z).$$

In particular, take $x = -z$; then $f(-z) = -f(z)$. Hence, if $z \geqq 0$,

$$f(x-z) = -f(-x+z) = -[f(-x) + f(z)] = f(x) - f(z).$$

It follows that $f(x+w) = f(x) + f(w)$ for all x, w; that is, f is additive.

Since f is additive, we have $f(tw) = tf(w)$ if t is rational. Now, for rational t, consider the identity

$$f((t+x)^n) = [f(t+x)]^n = (f(t) + f(x))^n.$$

Expanding both sides, and exploiting the rationality of t, we have

$$\sum_{k=0}^{n} \binom{n}{k} t^k f(x^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} t^k f(1)^k f(x)^{n-k}.$$

Equating the coefficients of like powers of t, we get

$$f(x^k) = f(1)^{n-k} f(x)^k, \qquad k = 0, 1, \cdots, n.$$

In particular, for $k = 2$, we have

$$(*) \qquad f(x^2) = f(1)^{n-2} f(x)^2.$$

(Note that we have used the fact that $n > 1$.) Depending on the sign of $f(1)^{n-2}$, this says that f is either order-preserving or order-reversing. In any case, it follows that the additive function f is continuous. Hence $f(x) = xf(1)$.

Setting $x = 1$ in (*), we see that if n is even, we must have $f(1) = 0$ or 1, and if n is odd, we have $f(1) = 0$, 1, or -1. Hence the given equation has only continuous solutions for $n > 1$.

II. Comment by Ignace Kolodner, Carnegie-Mellon University. The question can be handled with any $n \in \mathbb{R}$ provided that y and $f(y)$ are restricted so that the equation makes sense. Aside from the well-known cases $n = 0$ and $n = 1$, the only solutions are the obvious ones. If $n > 0$ but is not an integer, we get $f(0) = 0$, $f(1) = 0$ or $f(1) = 1$, and $f(x) \geqq 0$ if $x > 0$. If $n < 0$, proving that $f(0) = 0$ presents a slight problem. If n is not an odd integer, one proves first that $f(x) = cx + f(0)$ on $\mathbb{R}$ and then it follows that $f(0) = 0$, $c = 1$; if n is a negative odd integer one shows in succession that $f(x) = cx + f(0)$ on Q (the rationals) and $f(0) = 0$, and then the matter for $n \leqq -3$ is reduced to that for $n \geqq 9$. The case $n = -1$ requires an additional twist to show that f is bounded on $[0, \frac{1}{2}]$, and thus continuous.

Also solved by Anders Bager (Denmark), Ram Murty and Kumar Murty, Ignace Kolodner, Gy. Maksa (Hungary), Nan-Shan Shou, and the proposer. Partial solution by Charles Rees.

A Consequence of Jensen's Inequality

E 2480 [1974, 659]. *Proposed by M. S. Klamkin, University of Waterloo, Canada*

If $a_i \geqq 0$, $\Sigma a_i = 1$, and $0 \leqq x_i \leqq 1$ for $i = 1, 2, \cdots, n$, prove that

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} + \cdots + \frac{a_n}{1+x_n} \leqq \frac{1}{1+x_1{}^{a_1}x_2{}^{a_2}\cdots x_n{}^{a_n}}.$$

When does equality hold?

Solution by Miriam Beesing, Junior, Hamline University. Assume without loss of generality that $a_i > 0$ for all i. The proposed inequality follows from a straightforward application of Jensen's Inequality for concave functions: If f is strictly concave (downwards) on an interval I, then for all $y_i \in I$,

$$\Sigma a_i f(y_i) \leqq f(\ \Sigma a_i y_i)$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$. (See Roberts and Varberg, *Convex Functions*, Academic Press, 1973, pp. 189, 192.) To apply this to the proposed inequality, let $y_i = \log x_i$ and $f(y) = (1+e^y)^{-1}$, assuming for the moment that $x_i > 0$ for all i. Since $0 < x_i \leqq 1$, it follows that $-\infty < y_i \leqq 0$ and since $f''(y) = e^y(e^y - 1)(1+e^y)^{-3} < 0$ if $y < 0$, we see that f is strictly concave downwards on $(-\infty, 0]$. Then

$$\sum_{i=1}^{n} \frac{a_i}{1+x_i} = \sum_{i=1}^{n} a_i(1+\exp y_i)^{-1} \leqq \left\{1 + \exp\left(\sum_{i=1}^{n} a_i y_i\right)\right\}^{-1}$$

$$= \left\{1 + \prod_{i=1}^{n} \exp(a_i y_i)\right\}^{-1} = \left\{1 + \prod_{i=1}^{n} x_i^{a_i}\right\}^{-1},$$

with equality if and only if $y_1 = \cdots = y_n$, i.e., if and only if $x_1 = \cdots = x_n$.

If some $x_i = 0$, the above proof breaks down, but this case is easily handled on its own merits. Again, equality holds if and only if $x_1 = \cdots = x_n$, which in this case means they are all 0.

If we allow $a_i = 0$ (and assume $0^0 = 1$), then the condition for equality becomes $x_i =$ constant for all i for which $a_i > 0$.

Finally we remark that the inequality is reversed if $x_i \geqq 1$ for all i. This is because $f''(y) > 0$ on $(0, \infty)$ and thus f is concave upwards on $[0, \infty)$.

Also solved by F. F. Abi-Khuzam, Paul Chauveheid (Belgium), J. P. Crawford, R. J. Evans, T. H. Foregger, J. R. Gard, Ellen Hertz, John Horváth, Hans Kappus (Switzerland), P.G. Kirmser, P. W. Lindstrom, O. P. Lossers (Netherlands), L. E. Mattics, M. R. Modak (India), William Nuesslein, E. A. Power, E. B. Rockower, St. Olaf College Students, Kenneth Schilling, Allen Stenger, Phil Tracy, John Tung, John Williams (Australia), and the proposer.

Editor's comment. The proposer notes that the special case $n=2$ of our problem is a lemma of D. Borwein, used in his solution to Problem 5333 [1965, 1030; 1966, 1022]. Borwein's lemma

was used to prove the special case $a_1 = \ldots = a_n = n^{-1}$ of our problem; once this was shown, 5333 followed from a trivial application of the arithmetic mean-geometric mean inequality. Kappus uses Borwein's lemma as a starting point and obtains our result by an easy induction. Kappus also notes that this problem generalizes Problem 254 in Elem. Math. 11 (1956), p. 112.

Evans shows that the problem generalizes to infinite sequences $\{a_i\}$ and $\{x_i\}$ with $a_i \geqq 0, \Sigma\ a_i = 1$, and $0 \leqq x_i \leqq 1$. Rockower shows, using Riemann sums, that the following continuous analog holds: Suppose that $a(t)$ and $x(t)$ are continuous functions on the interval [0, 1] (the actual interval is not important) and that $a(t) \geqq 0$, $\int a(t)dt = 1$, and $0 \leqq x(t) \leqq 1$. Then

$$\int_0^1 a(t)[1 + x(t)]^{-1}dt \leqq \left\{1 + \exp\int_0^1 a(t)\log x(t)dt\right\}^{-1},$$

with equality if and only if $x(t)$ is constant on the set where $a(t) > 0$. The above results can be subsumed by the following generalization, which can be shown by first considering simple functions and then passing to the limit: Let (X, A, μ) be a measure space with $\mu(X) = 1$ and let f be a measurable real-valued function on X which satisfies $0 \leqq f(x) \leqq 1$ almost everywhere. Then

$$\int(1 + f)^{-1}d\mu \leqq \left\{1 + \exp\int \log f d\mu\right\}^{-1}$$

with equality if and only if $f(x)$ is almost everywhere constant.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before September 30, 1975.

6036. *Proposed by Carl Pomerance, University of Georgia*

If n is a natural number, we let $\sigma(n)$ denote the sum of the divisors of n, $S(n)$ the set of prime divisors of n, and $\omega(n)$ the cardinality of $S(n)$. Clearly if n is an even perfect number, then $S(n) = S(\sigma(n))$ and $\omega(n) = 2$. Prove the converse.

6037. *Proposed by Jim Lawrence, University of Washington*

Show that any graph H is isomorphic to an induced subgraph of some finite graph H' which has a group of automorphisms that acts transitively on its vertices.

6038. *Proposed by Oto Strauch, Bratislava, Czechoslovakia*

Let $f(x) = \Sigma a_i x^i$ and $s_n(x) = \Sigma_{i \leqq n} a_i x^i$. Let $r \neq 0$ be an interior point of the interval of convergence of the power series $\Sigma a_i x^i$. Prove that if $s_n(r) < f(r)$ for every $n = 0, 1, 2, \cdots$, then the derivative $f'(r) \neq 0$.

6039. *Proposed by Robert Gilmer, Florida State University*

Let R be an associative ring and let $\{X_i\}_1^n$ be a finite set of commuting indeterminates over R. Prove that each central idempotent of the power series ring $R[[X_1, \cdots, X_n]]$ is in R.

6040. *Proposed by Jan Mycielski, University of Colorado*

Let f be a continuously differentiable map of the unit cube I^n into the Euclidean space R^n which maps the boundary of I^n into one point. Let $J(f,x)$ be the Jacobian determinant of f at x. Prove

$$\int_{I^n} J(f,x)dx = 0.$$

6041. *Proposed by S. W. Golomb, University of Southern California*

There are n horses in a "random" horse race, in which all $n!$ orders of finish are equally probable *a priori.* A gambler is allowed to select k horses, to finish first, second, $\cdots$, k-th.

1. What is the probability $Q_n(k,i)$ that exactly i of his k selections will finish among the first k?

2. What is the probability $P_n(k,i)$ that exactly i of his k selections will finish in the precise positions predicted for them?

SOLUTIONS OF ADVANCED PROBLEMS

Editor's Note. Some time ago Professor Joseph B. Keller and a number of colleagues at the Courant Institute of Mathematical Sciences of New York University set for themselves the project of finding solutions to some of the previously unsolved problems from this Department. It is hoped that this significant activity may stir up some friendly competition with other institutions. The editors take great pleasure in reporting success with a good many of these problems and in including the solutions in this and succeeding issues of the MONTHLY.

A Multiple Integral of $\sin x/x$

5314 [1965, 795]. *Proposed by D. R. Brillinger, University of California at Berkeley*

Show that the following integral is absolutely convergent:

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\sin x_1}{x_1}\,\frac{\sin x_2}{x_2}\cdots\frac{\sin x_k}{x_k}\,\frac{\sin(x_1+x_2+\cdots+x_k)}{(x_1+x_2+\cdots+x_k)}\,dx_1\cdots dx_k.$$

Solution by Peter Ungar, Courant Institute, New York University. This problem is a simple application of inequality 282 in Hardy, Littlewood and Pólya, *Inequalities*, p. 201, which says: If

$$\phi(x) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1)\cdots f(x_n)f(x-x_1-\cdots-x_n)dx_1\cdots dx_n,$$

then

$$\int_{-\infty}^{\infty}\phi^2(x)dx \leqq \left(\int_{-\infty}^{\infty} f^{(2n+2)/(2n+1)}(x)dx\right)^{2n+1}$$

Apply this to

$$f(x)=\min\left(1,\frac{1}{|x|}\right) \geqq \left|\frac{\sin x}{x}\right|, \qquad x \text{ real.}$$

Then the right side of the inequality is finite. Now for $0 \leqq x \leqq 1$ and any real $x_1, x_2, \cdots, x_n$,

$$f(x - x_1 - \cdots - x_n) \geqq \tfrac{1}{2}f(-x_1 - x_2 - \cdots - x_n) = \tfrac{1}{2}f(x_1 + x_2 + \cdots + x_n).$$

Hence $\phi(x) \geqq \frac{1}{2}\phi(0)$ for $0 \leqq x \leqq 1$. It follows that $\phi(0)$ is finite since $\int_0^1 \phi^2(x)dx \leqq \int_{-\infty}^{\infty} \phi^2(x)dx < \infty$. This yields the desired result.

Three Balls and an Intersecting Line

5427 [1966, 897]. *Proposed by J. W. Wyman*

Let X_1, X_2, X_3 be points in E^n, Euclidean n-space, with

$$d(X_1,X_3) = d(X_1,X_2) + d(X_2,X_3). \tag{1}$$

Let $S_{\varepsilon_i}(X_i)$ denote the open ball of radius ε_i and center X_i in E^n. If

$$\varepsilon_3 < d(X_2,X_3), \tag{2}$$

determine how small ε_1 and ε_2 must be so that

$$x_1 \in S_{\varepsilon_1}(X_1),\ x_2 \in S_{\varepsilon_2}(X_2),\ x_1 \neq x_2 \Rightarrow x_1x_2 \cap S_{\varepsilon_3}(X_3) \neq \varnothing. \tag{*}$$

Find the least upper bound (l.u.b.) of all ε' such that

$$\varepsilon' = \varepsilon_1 + \varepsilon_2 \tag{3}$$

and (*) is met.

Solution by Ralph Jones, University of Massachusetts, Amherst. For a sensible answer, let $n \geqq 2$. Let L be the line X_1X_3, and coordinatize L so that $X_1 = 0$, $X_2 = a > 0$, and $X_3 = b$. Let C be the $(n-1)$-cone with vertex coordinate v on L and with axis L such that both spheres $S_{\varepsilon_1}(X_1)$ and $S_{\varepsilon_3}(X_3)$ are tangent to C. Then choose ε_2 so that $S_{\varepsilon_2}(X_2)$ is also tangent to C. The condition (*) translates into this situation: $\varepsilon_1 < v < a$. The tangency requirements are

$$\frac{\varepsilon_1}{v} = \frac{\varepsilon_3}{b-v} = \frac{\varepsilon_2}{a-v}.$$

Thus $v = b\varepsilon_1/(\varepsilon_1 + \varepsilon_3)$, and the condition (*) is that $\varepsilon_1 < a\varepsilon_3/(b-a)$. Now $\varepsilon_2 < (a\varepsilon_3 - (b-a)\varepsilon_1)/b$ actually (but does not exceed this value according to the conditions of the problem) and $\varepsilon' = \varepsilon_1 + \varepsilon_2 < a\varepsilon_3/(b-a)$; so the number $a\varepsilon_3/(b-a)$ is clearly the least upper bound of all such ε'.

Also solved by R. Burridge, Courant Institute, New York University.

Some Bernstein-type Operators

5575 [1968, 790]). *Proposed by Alexandru Lupas, Cluj, Rumania*

Let $C[0,\infty)$ be the class of bounded continuous functions on $[0,\infty)$ and define a family of linear operators mapping $C[0,\infty)$ into $C[0,\infty)$ as follows

$$(*) \qquad (L_{n,t}f)(x) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{t^k}{(1+t)^{n+k}} f\left(x+\frac{k}{n}\right).$$

(a) Show that for fixed $x \in [0,\infty)$ and all $T \geqq 0$,

$$\sup_{0 \leqq t \leqq T} \left|(L_{n,t}f)(x) - f(x+t)\right| \to 0, \qquad \text{as } n \to \infty.$$

(b) A function is P-convex of order s on the reals if the divided differences of order $s+2$, $f[x_1, x_2, \cdots, x_{s+2}]$ are positive for all real x_i, $i = 1, 2, \cdots, s+2$. Show that if $f(t)$ is convex of order s then $(L_{n,t})(0)$ is also a convex function of t of order s.

(c) The sequence of operators $\{L_{n,t}\}$ defined by (*) is decreasing on the class of convex functions of order 1 and satisfies $L_{n+1,t} - L_{n,t} = 0$, for sufficiently large n, on the class of linear functions.

Solution by G. C. Papanicolau, Courant Institute, New York University. Instead of the proposer's original L_n, we shall use the form $(L_{n,t}f)(x)$ as given in (*) above.

(a) First note that

$$\text{(i)} \qquad \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n = 1,$$

$$\text{(ii)} \qquad \sum_{k=0}^{\infty} k \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n = nt,$$

$$\text{(iii)} \qquad \sum_{k=0}^{\infty} (k-nt)^2 \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n = n(t^2+t),$$

obtained by successive differentiation of

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k.$$

We have thus the estimates:

$$\begin{aligned} &\left|(L_{n,t}f)(x) - f(x+t)\right| \\ &\quad = \left| \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n \left(f\left(x+\frac{k}{n}\right) - f(x+t)\right)\right| \\ &\quad = \sum_{|k/n-t|<\delta} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n \left| f\left(x+\frac{k}{n}\right) - f(x+t) \right| \end{aligned}$$

$$+ \sum_{|k/n-t|\geq\delta} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n \left| f\left(x+\frac{k}{n}\right) - f(x+t) \right|.$$

The first term on the right side of the inequality is arbitrarily small if δ is chosen sufficiently small. With $\delta > 0$ so chosen and fixed the second term on the right of the inequality is estimated as follows:

$$\sum_{|k/n-t|\geq\delta} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n \left| f\left(x+\frac{k}{n}\right) - f(x+t) \right|$$

$$\leqq 2M \sum_{|k/n-t|\geq\delta} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^n \frac{(k-nt)^2}{n^2\delta^2}$$

$$\leqq 2M\frac{n(t^2+t)}{n^2\delta} = \frac{2M(t^2+t)}{n\delta}, \qquad M = \sup_{x\geqq 0} |f(x)|.$$

Letting n go to infinity we obtain the desired result (a).

To prove (b) we proceed as follows: First we note that for each $n \geqq 1$ and $x \in [0,\infty)$ fixed $(L_{n,t}f)(x)$ is an infinitely differentiable function of t. By repeated use of the mean value theorem divided differences of any order are equal to a constant multiple times derivatives of corresponding order evaluated at intermediate points. Thus it suffices to show that

$$\left(\frac{d}{dt}\right)^p (L_{n,t}f)(x) \geqq 0, \qquad p = 1,2,\cdots,$$

whenever $f(t)$ is convex of order p. By direct computation we find that

$$\left(\frac{d}{dt}\right)^p (L_{n,t}f)(x) = \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right)^{n+p} C_n(\Delta_{1/n}^p f)\left(x+\frac{k}{n}\right),$$

where Δ_h denotes the difference operator $(\Delta_h f)(x) = (1/h)[f(x+h) - f(x)]$, and Δ_h^p is the pth iterate of this operator. Assertion (b) follows now from this explicit formula for the p derivatives of $(L_{n,t}f)(x)$.

(c) We wish to show that for all $t \geqq 0$ and $x \geqq 0$

(iv) $\quad (L_{n,t}f)(x) - (L_{n+1,t}f)(x) \geqq 0, \; n = 1,2,\cdots$

provided the second divided differences of f

$$f[x_1,x_2,x_3], \quad 0 \leqq x_1 \leqq x_2 \leqq x_3,$$

are nonnegative. In order to prove (iv) we verify by direct computation the identity

$$(L_{n,t}f)(x) - (L_{n+1,t}f)(x)$$

$$= \frac{1}{n(n+1)} \sum_{k=0}^{\infty} \binom{n+k+1}{k} \left(\frac{t}{1+t}\right)^{k+1} \left(\frac{1}{1+t}\right)^n f\left[x+\frac{k}{n+1}, x+\frac{k+1}{n+1}, x+\frac{k+1}{n}\right],$$

from which (iv) follows immediately. When $f(x)$ is linear then, from the above identity we have that $L_{n,t} - L_{n+1,t} = 0$ for all n (and not for just sufficiently large n).

The preceding results can be interpreted in the context of finite difference equations.

Let $u(x,t)$, $x \geqq 0$, $t \geqq 0$ be the solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad u(x,0) = f(x). \tag{1}$$

Clearly $u(x,t) = f(x+t)$. Let us introduce a finite difference approximation for (1). Let $t_n = \delta t n$, $x_j = \delta x j$, $n = 0,1,2,\cdots$, $j = 0,1,2,\cdots$, and let U_j^n denote an approximation to $u(x_j, t_n)$. Because of the form of (1) one may consider an explicit difference scheme

$$U_j^{n+1} - U_j^n = \frac{\delta t}{\delta x}(U_{j+1}^n - U_j^n), \ \ j \geqq 0,\ n \geqq 0,\ U_j^0 = f(x_j), \tag{2}$$

or on implicit difference scheme

$$U_j^n - U_j^{n-1} = \frac{\delta t}{\delta x}(U_{j+1}^n - U_j^n),\ j \geqq 0,\ n \geqq 1,\ U_j^0 = f(x_j). \tag{3}$$

Then (2) can be solved to yield

$$U_j^n = \left(\left[1 - \frac{\delta t}{\delta x} + \frac{\delta t}{\delta x} T_{\delta x}\right]^n f\right)(x_j) = \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{\delta t}{\delta x}\right)^{n-k} \left(\frac{\delta t}{\delta x}\right)^k f(x_j + k\delta x)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\delta t)^k (\Delta_{\delta x}^k f)(x_j),$$

where $\Delta_h = (1/h)(f(x+h) - f(x))$. U_j^n converges to $u(x,t) = f(x+t)$ for general continuous f as $n,j \to \infty$, $x = j\delta x$, $t = n\delta t$ fixed provided $0 \leqq \delta t/\delta x \leqq 1$. On setting $\delta x = 1/n$ we have $\delta t/\delta x = t$, and hence

$$U_j^n = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k f\left(x + \frac{k}{n}\right) \to f(x+t), \quad n \to \infty, \quad 0 \leqq t \leqq 1. \tag{4}$$

This is the classical theorem of Bernstein concerning polynomial approximations. Note that the restriction $0 \leqq t \leqq 1$ is simply the stability condition associated with the explicit difference scheme (2). Note also that (4) is the analog of (*) of part (a). The analogs of part (b) and (c) hold for the approximation (4) also.

Similar results obtain for the implicit difference scheme (3) where the expansions start with the solution

$$U_j^n = \left(\left[1 + \frac{\delta t}{\delta x} - \frac{\delta t}{\delta x} T_{\delta x}\right]^{-n} f\right)(x_j).$$

For probabilistic interpretations of the problem see W. Feller, *An Introduction to Probability Theory*, vol. II, p. 239.

Also solved by Peter Lax, and by the proposer.

Lax and the proposer solve (a) by proceeding directly from the convergence of (a) for quadratic functions, provable from (i) (ii), (iii), above and use of a known theorem of Korovkin for positive operators. (G. G. Lorentz, *Approximation of Functions*, p. 71; P. Korovkin, *Linear Operators and Approximation Theory*, Mathematica 8 (31), 1966.)

Representing the Square Root of a Fourier Transform

5643 [1968, 1125]. *Proposed by S. Zaidman, University of Montreal*

Is there a positive function $a(x)$, $x \in R^n$, $a(x) > \alpha > 0$, such that

$$a(x) = (2\pi)^{-n/2} \int e^{ix\lambda} A(\lambda) d\lambda,$$

with $A(\lambda) \in L^1$, however $\sqrt{a(x)}$ does not admit any such representation?

Solution by Peter D. Lax, Courant Institute, New York University. According to the Riemann-Lebesgue lemma there is no function $a(x)$ which is the Fourier transform of an $L_1(R^n)$ function and which is bounded from below by a positive constant. However, if $a(x)$ is periodic, represented by an absolutely convergent Fourier series, and $a(x) > \alpha > 0$ for all x, then $\sqrt{a(x)}$ is also represented by an absolutely convergent Fourier series, for $\sqrt{z}$ is analytic in an open set containing all $y \geqq \alpha$, and by Paul Levy's extension of Wiener's theorem according to which $f(a(x))$ has an absolutely convergent Fourier series if $a(x)$ does and if f is analytic in an open set of the complex plane containing the closure of the range of $a(x)$.

Additionally if $a(x)$ is the Fourier transform of an $L_1(R)$ function, and $a(x) > 0$ for all x, then $\sqrt{a(x)}$ is not necessarily the Fourier transform of an L_1 function. This is seen by considering the converse theorem to Levy's result, given by Katznelson, Kahane, Helson and Rudin. These authors found that the *only* functions f that have the property that they map *all* functions $a(x)$ of the algebra $F(L_1)$ whose range is contained in a closed set K, into elements of $F(L_1)$ are those analytic in an open set containing K. A special discussion of the square root function in a Banach algebra setting is contained in an article by Katznelson in the Annales de l'Ecole Normale, 1963; it is known therefore that there exists a function $a(x) \geqq 0$ which is the Fourier transform of an $L_1(R)$ function while $\sqrt{a(x)}$ is not.

NOTE: The proposer's actual problem was meant to require only that $a(x)$ be non-negative, and this is resolved in the above.

Generating Subsets of the Plane

5670 [1969, 423]. *Proposed by Stephan Silverman*

Let $M = \{A \times B \mid A \text{ and } B \text{ are subsets of the reals}\}$. Then, is the σ-algebra generated by M all subsets of the plane?

Solution by Peter Ungar, Courant Institute, New York University, and Maurice Machover, St. John's University. The answer is yes, assuming the continuum

hypothesis. Let us call a set of the form $A \times B$ a *rectangle*. We shall show in fact that every set in the plane is a $\sigma\delta\sigma$-set (a countable union of countable intersections of countable unions of rectangles). If $y = f(x)$ is any real-valued function on a subset D of the reals, its graph is the set $\{(x, y): y = f(x), x \in D\}$. Similarly, each function $x = g(y)$ has a graph. It is a known consequence of the continuum hypothesis [W. Sierpinski, *Hypothèse du Continu*, 2nd edition, p. 11] that the plane, and hence any of its subsets, is a countable union of graphs. Thus it suffices to show that the graph G of an arbitrary function $y = f(x)$, $x \in D$, is a $\delta\sigma$-set. Let n be any positive integer. Let

$$C_{nj} = \left\{x \in D: \frac{j}{n} \leqq f(x) < \frac{j+1}{n}\right\} \times \left[\frac{j}{n}, \frac{j+1}{n}\right)$$

$j = \cdots, -2, -1, 0, 1, 2, \cdots$. Then $C_n = \bigcup_j C_{nj}$ is the set obtained by putting a vertical line segment of length $1/n$ through every point of the graph. G itself is the intersection $G = \bigcap_{n=1}^{\infty} C_n$. A similar construction shows that the graph of any function $x = g(y)$ is a $\delta\sigma$-set.

The question of the problem may be generalized to ask whether M can generate $F \times G$ when F, G are any two sets. If F (or G) is countable, then every subset of $F \times G$ is a σ-set. If F, G are two sets of cardinality $\aleph_1$, then M generates $F \times G$ (provable without the continuum hypothesis as above). But if card $F > c$, card $G \geqq c$, then it is possible to show that there are subsets of $F \times G$ not in the σ-algebra generated by $\{A \times B\}$.

$\Gamma^{(n)}(1)$

5878 [1972, 1043; 1974, 176]. *Proposed by Václav Konečný, Jarvis Christian College, Hawkins, Texas*

Show that

$$\int_0^{\infty} e^{-x}\ln^2 x\, dx = \gamma^2 + \frac{\pi^2}{6} \text{ and } K - \frac{5}{2} < \int_0^{\infty} e^{-x}\ln^3 x\, dx < K - \frac{9}{4},$$

where $K = -\gamma(\gamma^2 + \pi^2/2)$ and γ is Euler's constant.

II. *Editor's Notes*. B. C. Berndt, University of Illinois, calls attention to the fact that the problem (generalized) appears in the Journal of the Indian Mathematical Society, V. 15 (1923), p. 178.

See also Problem 3766, this MONTHLY, v. 45 (1938), p. 57.

Some Trigonometric Integrals

5951 [1974, 90]. *Proposed by J. Gilles, University of Charleroi, Belgium*

Show that

$$(1) \qquad \int_0^{\infty} \frac{x^3 \sin \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}}-1} dx = \frac{17}{16} - \frac{8}{3\pi} - \frac{7}{\pi^2} + \frac{35}{2\pi^3} - \frac{105}{16\pi^4},$$

$$\text{(2)} \qquad \int_0^\infty \frac{x^3 \cos \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}}-1}dx = \frac{-8}{3\pi} + \frac{7}{\pi^2} + \frac{35}{2\pi^3} + \frac{873}{16\pi^4}.$$

Comment by Torleiv Kløve, Bergen, Norway. A method for calculating integrals of the form

$$\int_0^\infty \frac{x^m}{e^{2\pi\sqrt{x}}-1}\cos \pi n x\, dx, \qquad \int_0^\infty \frac{x^m}{e^{2\pi\sqrt{x}}-1}\sin \pi n x\, dx$$

for rational values of n and positive integral values of m is given by Ramanujan in the paper *Some definite integrals connected with Gauss's sums,* Messenger of Mathematics, 44 (1915), pp. 75–85 (= Collected Papers, pp. 59–67).

Solution submitted by M. L. Glasser. The above reference also cited by B. C. Berndt.

$\sum \frac{1}{n^3 \sin(n\pi\theta)}$ and $\Gamma\left(\frac{8}{9}\right)$

5952 [1974; 175, 1033]. *Proposed by J. Gilles, University of Charleroi, Belgium*
Prove the following results:

$$\text{(3)} \qquad \sum_{n=1}^\infty \frac{1}{n^3 \sin n\pi\sqrt{2}} = -\frac{13\pi^3}{360\sqrt{2}}.$$

$$\text{(4)} \qquad \cdot\ \Gamma\left(\frac{8}{9}\right) = \frac{9-\sqrt{14}+\sqrt{75-32\sqrt{3}}}{33}\cdot \sqrt[4]{182}.$$

I. *Solution by Bruce C. Berndt, Institute for Advanced Study.* We prove formula (3). First, if θ is irrational, it can be shown by a method of H. J. S. Smith, *On some discontinuous series considered by Riemann,* Messenger of Math. 11 (1882), 1–11, that

$$\sum_{n=1}^\infty \frac{1}{n^3 \sin(n\pi\theta)}$$

converges.

Suppose a/b is irrational. Let $f(z) = z^{-3}\csc(\pi a z)\csc(\pi b z)$, and let R_α denote the residue of f at a pole α. Let $\{C_N\}$, $1 \leqq N < \infty$, be a sequence of circles of radii ρ_N centered at the origin chosen so that ρ_N tends to ∞ as N tends to ∞ and so that the circles are positioned at a distance greater than some fixed positive number from all of the points $(n/a, 0)$ and $(n/b, 0)$, $-\infty < n < \infty$. By the residue theorem,

$$\text{(1)} \qquad \frac{1}{2\pi i}\int_{C_N} f(z)dz = \sum_{0<|n/a|<\rho_N} R_{n/a} + \sum_{0<|n/b|<\rho_N} R_{n/b} + R_0.$$

For $n \neq 0$, we have

$$R_{n/a} = \frac{(-1)^n a^2}{n^3 \pi \sin(n\pi b/a)}, \quad R_{n/b} = \frac{(-1)^n b^2}{n^3 \pi \sin(n\pi a/b)}.$$

Since

$$f(z) = z^{-3}\left\{\frac{1}{\pi a z} + \frac{\pi a z}{6} + \frac{7(\pi a z)^3}{360} + \cdots\right\}\left\{\frac{1}{\pi b z} + \frac{\pi b z}{6} + \frac{7(\pi b z)^3}{360} + \cdots\right\},$$

we see that

$$R_0 = \frac{7\pi^2 a^3}{360 b} + \frac{7\pi^2 b^3}{360 a} + \frac{\pi^2 ab}{36}.$$

By our choice of the circles C_N, it is easy to see that the integral on the left side of (1) tends to 0 as N tends to ∞. Hence, letting N tend to ∞ in (1), we find that

$$\frac{a^2}{\pi} \sum_{|n|>0} \frac{(-1)^n}{n^3 \sin(n\pi b/a)} + \frac{b^2}{\pi} \sum_{|n|>0} \frac{(-1)^n}{n^3 \sin(n\pi a/b)} + \frac{7\pi^2 a^3}{360 b} + \frac{7\pi^2 b^3}{360 a} + \frac{\pi^2 ab}{36} = 0,$$

or, putting $a/b = \theta$, we conclude that

(2) $$\theta^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \sin(n\pi/\theta)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \sin(n\pi\theta)} = -\frac{\pi^3}{72}\left(\frac{7}{10}\theta^3 + \frac{7}{10}\theta^{-1} + \theta\right).$$

Now suppose that $\theta = r + s\sqrt{d}$, where r, s and d are integers with $d > 1$ and square-free, and where $r^2 - ds^2 = \delta$ with $\delta = \pm 1$. Then $1/\theta = \delta(r - s\sqrt{d})$. For such θ, we deduce from (2) that

(*) $$(1 - \delta\theta^2) \sum_{n=1}^{\infty} \frac{(-1)^{n(r+1)}}{n^3 \sin(n\pi s\sqrt{d})} = \frac{-\pi^3}{72}\left(\frac{7}{10}\theta^3 + \frac{7}{10}\theta^{-1} + \theta\right).$$

In particular, let $r = s = 1$ and $d = 2$. Then (*) yields the desired result (3).

Also solved (Formula (3)) by Günter Bach (Germany), Finbarr Holland (England), O. P. Lossers (Netherlands), H. L. Montgomery & P. J. Weinberger, Jonathan Rosenberg, and Robert Spira.

II. *Solution by H. L. Montgomery and P. J. Weinberger, University of Michigan.* The proposed formula (4) is incorrect as the following calculations show: $\Gamma(8/9) = 1.077758833133\cdots$, whereas the proposed expression is equal to $1.077699109236\cdots$. Had the formula been correct, it would have given the first known nonintegral rational x for which $\Gamma(x)$ was not transcendental.

Also solved (Formula 4)) by Finite Differences Class of F. Alberti, R. S. Lehman and O. P. Lossers (Netherlands).

Editor's notes. (1) Lossers points out that (3) above may be found in E. C. Titchmarsh, *Theory of Functions*, 2nd Ed., p. 135.

(2) Rosenberg notes that similar methods of contour integration may be used to obtain similar formulas, e.g.,

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \cot n\pi \left(\frac{1+\sqrt{5}}{2}\right) = \frac{-\pi^3}{45\sqrt{5}}.$$

(3) Bach proves the following generalization:

$$\sum_{\upsilon=1}^{\infty} \frac{1}{\upsilon^{2m+1} \sin\sqrt{2}\pi\upsilon}$$

$$= \frac{(-1)^m \pi^{2m+1}}{2\alpha(1+\alpha^{2m})} \sum_{\mu=0}^{m+1} \frac{(2^{2\mu}-2)(2^{2m+2-2\mu}-2)}{(2\mu)!(2m+2-2\mu)!} \alpha^{2\mu} B_{2\mu} B_{2m+2-2\mu},$$

where $m \geqq 1$, $\alpha = \sqrt{2} - 1$, and $B_{2\mu}$ are the Bernoulli numbers (i.e., $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$).

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Introduction to Finite Mathematics, Third Edition. By John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson. Prentice-Hall, Englewood Cliffs, New Jersey, 1974. xi + 484 pp. $12.50. (Telegraphic Review, August-September 1974.)

Mathematics: With Applications in the Management, Natural, and Social Sciences. By Margaret L. Lial, Charles D. Miller. Scott, Foresman, Glenview, Illinois, 1974. ii + 564 pp. $11.95. (Telegraphic Review, November 1974.)

Graphs, Models, and Finite Mathematics. By Joseph Malkevitch, Walter Meyer. Prentice-Hall, Englewood Cliffs, New Jersey, 1974. x + 515 pp. $11.95. (Telegraphic Review, August-September 1974.)

Although many mathematics books intended for students in liberal arts and/or

business have claimed to discuss significant applications of elementary mathematics, few have progressed beyond the most superficial and artificial examples. The winds of curriculum change have unfurled many publishers' banners emblazoned with the words "models" and "applications to practical problems." Of the most recent crop of such books, the three texts under review are distinguished by the breadth and depth of their applications, as well as by their writing styles.

The intersection of the tables of contents of these three books consists of the following elements: matrix theory, probability (including Markov chains and statistics), linear programming, and game theory. Each text discusses other topics and treats common topics in an individualistic way.

The book by Kemeny, Snell, and Thompson is a classic. In its first appearance (1957), it stood out as a radical and happy departure from other texts at this level and, indeed, seemed to have introduced the term "finite mathematics." It remains a paragon: "Age cannot wither her, nor custom stale/Her infinite variety . . ." One reason for this perpetual youth is the continuing careful revision of the text and exercises.

I have always had the impression that many students found the previous editions rather formidable. Written clearly and carefully, with a wealth of examples and exercises, the book nevertheless required a certain maturity and sophistication of its readers. Its format and style were Spartan and monochromatic (both literally and figuratively). Well, according to a publisher's blurb, this new edition "has been specifically revised in light of the average student at the freshman/sophomore level." Indeed, on a superficial level, everything from the new format (larger, less cramped pages) to the use of more colorful illustrations makes this a very attractive book. On a deeper level; there are a number of substantive changes: a condensation of the (traditional and often boring) discussion of sets and logic, an introduction to finite statistics (including the Central Limit Theorem applied to independent trials, tests of hypotheses, confidence intervals, and a discussion of statistical pitfalls), and an introduction to the computer language BASIC (with applications). This introduction to programming is the best I've seen in such an elementary text and should be useful even if a computer is not available.

However, despite all these changes, the prospective user of this text should be warned that this is not a watered down version of the previous editions. The book still requires a certain ability on the part of the student and careful preparation on the part of the instructor to bring out its excellent features.

The comments made thus far are directed to the reader of this review as teacher; but there are also professional reasons for considering this well-tempered text. As mathematicians we should read with delight of the many applications of finite mathematics to sociometrics, genetics, learning theory, anthropology (marriage rules in a primitive society), and economics. (One criticism: in exercises 6–8 on p. 461, it is not clear why there should be only eight — instead of sixteen — second-cousin relationships between a man and a woman.) Computer simulations (with sample

BASIC programs) of craps, poker, and baseball are discussed, as well as a linear programming model for a decision problem involving the deer hunting season.

As was the practice in previous editions, suggested readings are given at the end of each chapter; and, whenever possible, two of each kind of exercise are given, one with and one without an answer printed in the text. There is also a solutions manual, which will be discussed below.

The book by Lial and Miller is in some ways the most ambitious and least lively of the three texts under review. In addition to the core topics mentioned in my second paragraph, this text discusses functions (including the exponential and logarithm functions), sequences and limits, and calculus (both differential and integral). The book is divided into four essentially independent parts — a review and reinforcement portion, a part focusing on matrix theory and linear programming, a treatment of probability and statistics, and the calculus portion. The text seems to be very flexible and could be used for several types of courses, ranging from one-quarter to full-year introductions to mathematics. It contains a generous supply of exercises at all levels.

Realistic examples from the general areas mentioned in the book's title are used to motivate and illustrate new mathematical ideas. For example, matrices and their operations are nicely motivated by inventory problems, while decision making is illustrated by a problem taken from the field of life insurance. The calculus is developed intuitively, the basic ideas of derivative and integral motivated by economics and business considerations. Of special interest are the "cases" (seventeen in all) which appear at the end of each major division of the book. These are more elaborate applications of the mathematics in the text to real-world situations, applications gleaned from responses to "over 200 letters to leading people in business and biology." These self-contained illustrations of mathematical modeling from such sources as Anheuser-Busch, Montgomery Ward, Levi Strauss, Quaker Oats, American Airlines, and Signal Oil enliven the text and make it a contender for adoption by teachers desiring to include some non-finite mathematics and whose students might not be prepared for the maturity of the book by Kemeny, Snell, and Thompson. (One additional comment: the book has a rather funereal cover; and the text material is presented in purple and sepia tones. In my opinion this detracts from the attractiveness and readability of the text.)

Graphs, Models, and Finite Mathematics is a delightful addition to the literature of finite mathematics. Written for one- or two-semester courses and avoiding formality and excessive rigor, it presents mathematics that is "interesting, useful, and accessible." (Set theory — two pages worth — is relegated to an appendix.)

The major emphasis (the first four chapters and part of the fifth) is on graph (network) theory. Motivating important concepts by problems such as mail distribution, the scheduling of speakers at a symposium, and molecular structure in chemistry (essentially self-contained), the authors introduce the student to graphs, digraphs, Euler circuits, Hamilton circuits, and coloring problems, among other

topics. The modelling process is emphasized in these discussions as well as throughout the rest of the book. Whether or not an instructor is particularly interested in graph theory, he should realize that a strong case can be made for its inclusion in a modern finite mathematics course. In the text, Chapters 6–11 are essentially independent of graph theory and of each other.

In addition to treatments of the standard topics, the text has a nice chapter on the theory of elections, including a discussion of Arrow's theorem and the influence of his work on the theory of social choice. There is also a beautiful chapter illustrating the use of difference equations in modeling growth phenomena. In this way examples, such as predator-prey interaction, can be presented to those with no knowledge of calculus. Included in this chapter is a nice exercise on chain letters.

The book is well-written and shows that the authors have a lively sense of humor. Among the names (real and fictitious) sprinkled throughout the book can be found the Presto Pick Puncher (for making eight varieties of guitar pick), the prophet Isaiah, zingers, zonkers, and zappers, the Little Varmint Undergarment Co., Shakespeare, Robert Frost, the publisher's mathematics editor, widgets, and the Ferro-Hippus Auto Co. There are many tables and illustrations, reproductions of newspaper clippings and cartoons, and pictures of mathematical practitioners such as George Dantzig, John von Neumann, and Kenneth Arrow. At the end of each chapter, there is a list of books for further reading. This book's general level of sophistication is not as high as that of the previous two books; but there is considerable mathematical content, presented in a way that should be enjoyed by student and teacher alike.

Each of the three books under consideration contains answers to some of the exercises and has a companion solutions manual and/or instructor's guide. Kemeny, Snell, and Thompson provide 260 pages of answers to exercises, while Lial and Miller give sample chapter tests (with answers) as well as answers to even-numbered exercises. The *Instructor's Manual* by Malkevitch and Meyer contains solutions to problems and teaching suggestions for each section of the text.

In summary, then, I recommend that anyone faced with teaching a mathematics course to liberal arts students or to students with interests in business, the natural sciences, or the social sciences consider these texts carefully.

HENRY J. RICARDO, Manhattan College

Complex Variables. By George Polya and Gordon Latta. Wiley, New York, 1974. xiv + 327 pp. $13.95. (Telegraphic Review, November 1974.)

Introductory Complex Analysis and Applications. By William R. Derrick. Academic Press, New York, 1972. xi + 218 pp. $9.00 (Telegraphic Review, April, 1972.)

About five years ago the instructor of an undergraduate course in complex variables had a choice of two or three texts. Recently, however, a surprisingly large number of new books has appeared. Not long ago I considered for adoption five

books, all published in 1972 or later. The two books reviewed here are those I considered most suitable for our course — a one-semester course intended for majors in engineering, physics, and mathematics. Though a number of students in the class had taken a course in advanced calculus, this was *not* a prerequisite. I was therefore particularly careful to see how the authors handled the topics which are often presented using theorems or techniques not covered in elementary calculus, such as uniform convergence and Green's Theorem. The sections on physical applications also present a difficulty. The applications should be given in enough detail to interest the physics and engineering student, but should also be comprehensible to the mathematics student (and perhaps the teacher!) who may have little background in physics.

I adopted the Polya-Latta text and am quite enthusiastic about it. Perhaps the best feature of the book is the treatment of the physical applications. When a complex-valued function of a complex variable is first considered (in Chapter 2), a vector-field is given as one possible geometrical (or physical) interpretation. Soon after analytic functions are defined, the authors show that if $w = f(z)$ is analytic in a domain then the vector-field $\bar{w}$ is sourceless and irrotational. The initial introduction to Cauchy's Theorem is an intuitive discussion which could almost be described as a "proof by physics." I found this close interweaving of the mathematics and the physical interpretation to be very successful. There should, however, be more exercises in Chapter 5 on applications.

A course in elementary calculus is quite adequate as a prerequisite to reading the book. The authors have made this possible by substituting in several places heuristic discussions for rigorous proofs. They also soft-pedal epsilonics and topological niceties. These departures from strict mathematical rigor are always done in good taste, and the student is usually warned of the deficiencies in the argument. Nevertheless, mathematical purists may object to some features of the book; for example, the terms "curve" and "domain" are used in the text a number of times before they are formally defined.

As might be expected, the problems are an outstanding feature of the book. After each section there is a generous selection of exercises, and there is a further long list at the end of each chapter. Some of the latter are taken directly from Polya-Szegö, *Aufgaben*, and are quite difficult. I found that in order to cover the material at a reasonable pace, I assigned less than half the problems at the end of the sections and very few indeed of the problems at the end of the chapters.

I discovered only one really troublesome spot in the book. On about page 150, I feel the authors should have said more about the expression of a complex line integral over a differentiable arc C, as a Riemann integral. Though I had hoped to cover the first six chapters, I reluctantly omitted Chapter 4 for lack of time. The last three chapters are on conformal mapping and analytic continuation, hydrodynamics, and asymptotic expansions. I will probably adopt the book again the next time I teach the course.

The text by Derrick is a more traditional book and seemed to me to be one of the best of its kind. Though one semester of advanced calculus would be a desirable prerequisite for a course based on this book, I feel that it could be adapted to a class without advanced calculus by omitting a few of the more difficult proofs. Nevertheless, Derrick does assume more mathematical maturity and more interest in pure mathematics for its own sake than does Polya-Latta.

The author brings us quickly to the heart of the subject by giving very brief discussions of topics, such as limits and continuity, for which the theorems are analogous to those for real variables. Thus, we reach the Cauchy-Riemann equations on p. 13. The complex line integral is defined on p. 29, for piecewise differentiable arcs only, as a Riemann integral. A relatively simple, concise proof of the Cauchy-Goursat Theorem comes next, followed by standard theorems, which are stated under more restrictive conditions than would be suitable for a graduate course. Thus, their proofs are simpler and more concise than is usual. Uniform convergence and the limit superior are defined and used in these proofs.

The excellent sections on physical applications come rather late in the book, beginning on p. 134. The author discusses fluid flow, heat flow, and electrostatic problems. The treatment is concise, but clearly written. It is emphasized that these are really the same mathematical problem, and the analogies of the three applications are clearly summarized in a table on p. 142. The last three chapters of the book are on boundary-value problems, Fourier and Laplace transformations, and asymptotic expansions.

To summarize: Derrick is more concise and covers more material, both mathematically and in the applications. Its proofs are more rigorously presented, and more mathematical background is required of the student. Polya-Latta, on the other hand, gives the student a better intuitive grasp of the subject and provides better motivation to the engineering and physics student by integrating the applications with the mathematics.

BURNETT MEYER, University of Colorado

Number Systems and the Foundations of Analysis. By Elliott Mendelson. Academic Press, New York and London, 1973. $12.95 xii + 358 pp. (Telegraphic Review, June–July 1973.)

This book presents the construction of the basic number systems of mathematics, starting from scratch. I used this book as a textbook for an undergraduate course. The students liked it, especially because no steps are omitted from proofs. There are five chapters, on set theory, the natural numbers, the integers, the rational numbers, and the real numbers. In addition, there are seven appendices, dealing with cardinal numbers, axiomatic set theory, and complex numbers, among other topics. The development is along standard lines, starting with the Peano Postulates, and extending by equivalence relations. The real numbers are constructed by means of

Cauchy sequences of rational numbers, and an appendix gives the method of Dedekind cuts. All steps are given, sometimes with excruciating detail. The material is presented so as to show points of contact with other areas of mathematics. For example, the simplest concepts of number theory are introduced in the chapter on the integers, and the first concepts of ring theory are also presented in this chapter. Fields are introduced in the chapter on real numbers, and this same chapter has the basic topology of the real line, as well as properties of continuous functions. Rings and fields are used in the statement of characteristic properties of the basic number systems. However, characterization of the complex numbers is given. One characterization is the following: Any field possessing a non-trivial involutorial automorphism such that the "self-conjugate" elements form a complete ordered field, is isomorphic to the complex number system. This omission is the stranger because the bibliography lists a paper of Bosch and Krajkiewicz which presents this result in detail in a completely elementary setting.

The exercises are well-chosen and usually illustrate some point; they range from routine to hard. There are few errors, and most of the ones which do occur are in the exercises. The one drawback of the book is the overly elaborate notation, used to distinguish analogous operations (e.g., addition) in different systems. As the students pointed out, this is annoying to the point of distraction. Surely following standard mathematical practice in this regard would cause no harm.

This book is well-written and is a good introduction to abstract mathematics as well as being a textbook on number systems. It is certainly one of the best books on the subject.

JOHN A. SYNOWIEC, Indiana University Northwest

FILMS

(For general information about this series of films, see the introduction on page 416 of this MONTHLY, Vol. 82, Number 4.)

Equidecomposable Polygons: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: J. D. E. Konhauser. 16mm sound and color; 10 1/2 minutes. Available for rent or purchase from International Film Bureau, Inc. — sale \$145; rental \$10. Also available for rent from numerous University Film Libraries.

This is a good mathematical film. It considers one interesting accessible theorem and related matters which serve to enhance the understanding of that result. Namely, the film indicates the proof of the theorem: any two polygons having the same area are equidecomposable, with the congruent pieces being related by a translation or a 180° rotation.

Equidecomposable means that the two polygons can be cut into pieces such that the pieces are matched into congruent pairs, in a one-to-one fashion. Along the way it is mentioned that any polygon which is T-equidecomposable with the parallelogram has its additive invariant $J_l = 0$ for every direction l. The additive invariant J_l in a direction l is defined as the algebraic sum of the oriented lengths of all pieces of the boundary which lie parallel to the direction l. T-equidecomposable means that the congruence between corresponding parts of the polygon can be accomplished by use of translation only. It is somewhat disappointing that the film does not mention that the converse of this theorem is also true, and that the theorem and converse generalize to arbitrary polygons.

The sound of the film is very well planned. The voice is clear and distinct and the choice of words is good, to the point, and gives a full explanation without being wordy (although Dehn's name is mispronounced). There are many quiet times with no voice to distract one's thoughts from the meaning of the pictures that one is watching. During these quiet times the sound level of the background music is sufficiently low so that the music is quite unobtrusive to those who are concentrating while it is clearly audible for those who care to listen.

There is very heavy use of colors for the different pieces of the polygons and dissections. However, it seems to this viewer that a more systematic use of the color scheme could be helpful. For example, if the parts of one polygon had been red with different shadings, stripes, dots, etc., and if the other polygon had its parts blue, then the overlay of congruent pieces could have been shown by the color combination — namely, purple.

A final comment, which applies to this and to most mathematical films which this author has seen, concerns the lack of a bibliography on the film. It would take so little effort on the part of the film maker to show a few feet at the end listing three or four references in which the details of the principles and results discussed in the film could be read. This is especially important since film distributors are loath to send enclosed notes along with the film when they rent them. For this film one should surely include as references the two books: *Equivalent and Equidecomposable Figures*, V. G. Boltianskii, D. C. Heath & Company, Boston, 1963, and "Zerlegungsgleichheit ebener Polygone," H. Hadwiger and P. Glur, *Elemente der Mathematik*, Vol. VI, No. 5, pages 97–106.

Notwithstanding all the comments above, this is one of the best mathematical films I have had the pleasure of seeing and it is to be highly recommended to geometry classes or classes of prospective teachers.

M. N. BLEICHER, University of Wisconsin, Madison

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively, Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, *The Yale Mathematics Building Competition: Architecture for a Time of Questioning*. Ed: Charles W. Moore, Nicholas Pyle. Yale U Pr, 1974, viii + 117 pp, $15. An extensively illustrated record of the 1969-70 architectural competition for Yale's new mathematics building. Includes sketches of some of the 467 unsuccessful entries, together with extensive coverage of the winning Venturi and Ranch design "which has the character of an ingenious, elegant proof of a difficult theorem." Contains a wealth of imaginative ideas for harmonizing the mathematician's environment with his working habits. LAS

GENERAL, S**, P*, L**, *Women in Mathematics*. Lynn M. Osen. MIT Pr, 1974, xii + 185 pp, $4.95 (P). Traces the impact women have had on the development of mathematical thought through biographical sketches (Hypatia, Agnesi, du Châtelet, Herschel, Germain, Somerville, Kovalevsky, Noether) interwoven with historical context and mention of specific mathematical contributions. Includes an essay "The Feminine Mathtique," concerning the complex of attitudes that have grown up both in and about women in mathematics. The obstacles and prejudices that were a part of the lives of the women depicted here are presented as a reminder and a challenge to resist the bias against women still strongly evident among some mathematicians. PJC

GENERAL, L**, *The Arithmetic Teacher: 1954-1973: Cumulative Index, V. 1-20*. NCTM, 1974, 124 pp, $5.40 (P). Author, title and subject indices. LAS

BASIC, T(13: 1), S, *College Mathematics for Business, Second Edition*. Flora M. Locke. Wiley, 1974, xi + 365 pp, $9.95. Application of arithmetic to elementary personal and business problems--interest, taxes, depreciation. Good style for level: one idea, then examples, then many problems. LH

PRECALCULUS, T*(13: 1), *Precalculus: A Short Course*. Saturnino L. and Charles G. Salas. Xerox, 1975, xii + 289 pp, $9.95. Includes those topics most useful for calculus. The topics include numbers and algebraic expressions, analytic geometry, functions, trigonometry, induction, logarithms, and complex numbers. The text consists of brief explanations with lots of drawings, worked examples and problems. CEC

EDUCATION, S, L, *Finite Differences: A Problem Solving Technique*. Dale Seymour, Margaret Shedd. Creat Pub, 1973, 116 pp, $3.95 (P). An attractive empirical Polya-like approach to elementary number relationships intended for school children: triangular numbers, mosaics, networks, regions in a circle, etc., with solutions in the back. LAS

HISTORY, T*(12-13: 1), S*, L*. *History of Mathematics.* Arthur Gittleman. Merrill, 1975, x + 291 pp, $14.95. A streamlined chronological history of mathematics, emphasizing mathematics as an aspect of human culture. Topics are visited tantalizingly briefly, so that an enthusiastic teacher can enjoy supplementing the treatment. Well-chosen exercises integrate easily with the spirit of the book. Altogether, only high-school mathematics is presumed; consequently, the book is eminently suitable for prospective elementary teachers, a liberal arts course in mathematics, or possibly an alternative senior high math course--but it is far too "thin" for prospective high-school teachers or math majors. Incidentally, it's priced too high. PJC

HISTORY, P, L. *The Emergence of Probability.* Ian Hacking. Cambridge U Pr, 1975, 209 pp, $15.95. A philosophical investigation of the evolution of the concept of probability from an inexplicable pre-Renaissance void to the seventeenth century work of Bernoulli and Hume. Focuses on the emergence of equipossible cases as a key concept in understanding probabilistic issues. LAS

FOUNDATIONS, P. *Axiomatic Set Theory.* Ed: Thomas J. Jech. Proc. of Symp. in Pure Math., V. XIII, Part II. AMS, 1974, viii + 222 pp, $22.50. Belated completion of the proceedings of the 1967 U.C.L.A. symposium. Part I was published in 1971 (TR, February 1972). Many of the results and ideas in this volume are now nearly ten years old! LAS

FOUNDATIONS, T(15-16: 1), L. *Completeness, Compactness, and Undecidability: An Introduction to Mathematical Logic.* Alfred B. Manaster. P-H, 1975, vi + 154 pp, $8.50. Spiral approach to basic metalogical results, using Gentzen formulation and Turing machines. Sophisticated bare-bones treatment, e.g., includes undecidability of predicate calculus and of arithmetic but not the full Gödel incompleteness result. Small number of problems (of which #17b, p. 53, requires an emendation to yield a true result). PJC

FOUNDATIONS, P, L. *Deviant Logic: Some Philosophical Issues.* Susan Haack. Cambridge U Pr, 1974, xiv + 191 pp, $11.95. An "overdue" examination of the philosophical rather than the purely formal consequences of non-classical logics. Following a detailed study of the possibility of rival systems and an articulation of standards by which such systems may be judged, author Haack investigates certain "deviant" systems (e.g., intuitionism, quantum logic) in detail. An appendix provides formal properties of most well-known deviant systems. LAS

FOUNDATIONS, T(18), P*, L. *Set Theory: An Introduction to Large Cardinals.* Frank R. Drake. Stud. in Logic and Found. of Math., V. 76. North-Holland, 1974, xii + 351 pp, $23.10. Collects together the most significant results of the past ten years on large cardinals, drawn from papers and lecture courses of Scott, Jensen, Shoenfield, Silver, and others. Motivation furnished for questions and discoveries. By intention, no independence results, forcing, or Boolean-valued models. Some exercises, with hints. PJC

COMBINATORICS, P. *Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes.* Thomas Zaslavsky. Memoirs No. 154. AMS, 1974, vii + 102 pp, $3.30 (P). Develops formulas for the number of k-faces of the partition of Euclidean or projective d-space induced by a finite set of hyperplanes. SG

NUMBER THEORY, S(18), P. *Einführung in Siebmethoden der analytischen Zahlentheorie.* Dr. Wolfgang Schwarz. Bibliographisches Inst, 1974, 215 pp, (P).

NUMBER THEORY, P, *Automorphic Forms on Adele Groups.* Stephen S. Gelbart. Annals of Math. Stud., No. 83. Princeton U Pr, 1975, x + 267 pp, $9 (P). An exposition of recent results in the theory of automorphic forms and group representations, stressing the Jacquet-Langlands theory and the decomposition of $L^2(Gl(2,Q)\backslash Gl(2,A(Q)))$. SG

LINEAR ALGEBRA, T?(14: 1), S?, *Vector Algebra.* L. Marder. Prob. Solvers, No. 3. Allen & Unwin (U.S. Distr: Crane, Russak), 1971, 88 pp, $2.75 (P). Minimal textual material, numerous exercises. Topics: vector operations, systems of forces, differentiation and integration, vectors, curves, surfaces, trajectories. Of possible use for physics student review. TAV

ALGEBRA, T*(14-17: 1), *Groupes, Observation, Théorie, Pratique.* Alain Bouvier, Denis Richard. Hermann, 1974, xi + 307 pp, 48F (P). An introduction to group theory in French. Many examples, applications, and exercises motivate the theory. Goes up through the Sylow theorems and then concludes with a chapter on applications of groups to geometry. An excellent text. CEC

ALGEBRA, T(18: 1), S, P, *Homologische Algebra.* Götz Brunner. Bibliographisches Inst, 1973, 213 pp, sfr. 24.60 (P). A view of basic category theory including representable functors as seen through the eyes of an R-module. This background provides the context for derived functors including a special study of the cases Ext and Tor. Offers a clean lecture style, a basic bibliography, and a reasonable index of terminology. JAS

ALGEBRA, T(17-18: 1), P, L, *The Mathematical Theory of Coding.* Ian F. Blake, Ronald C. Mullin. Acad Pr, 1975, xi + 356 pp, $28. Unified treatment of diverse algebraic and combinatorial methods used in the construction of codes. Subject is discrete and continuous channel coding, but approach remains at a theoretical level, embracing discussions of finite fields, combinatorial constructions, matroids, structure of semisimple rings, and group representations. Presumes knowledge of abstract algebra, including some familiarity with Galois theory. PJC

FINITE MATHEMATICS, T(13-14: 2), *Fundamental Mathematics for the Social and Management Sciences.* Lloyd S. Emerson, Laurence R. Paquette. Allyn, 1975, xii + 420 pp, $11.95. One-year course for business students. Vectors, matrices, linear programming, probability and calculus. Problems stress business applications. LH

FINITE MATHEMATICS, T(13-14: 2), S, *Finite Mathematics, A Discrete Approach.* Karl J. Smith. Scott F, 1975, 378 pp, $10.95. Logic, probability, vectors, systems of equations, Markov chains, game theory. Elementary. Student discovers patterns via many examples and exercises. Instructors manual. LH

FINITE MATHEMATICS, T(13: 1), *Mathematics for Decision Making: An Introduction.* Frank Fleming, Roy Luke. Merrill, 1974, x + 321 pp, $11.95. Includes material on mathematics of finance, probability and statistics, and linear programming, together with the necessary prerequisite topics. RSK

CALCULUS, T*(13: 2), *Introduction to Calculus.* Lynn Loomis. A-W, 1975, xiv + 782 pp, $12.95. A two semester subset of *Calculus* (TR, January 1975). In this version, derivatives are followed immediately by antiderivatives. Omitted: differential equations, multiple integrals, concluding chapter on theory. LAS

CALCULUS, T(13-14). *Calculus and Analytic Geometry, Third Edition.* John A. Tierney. Allyn, 1975, xiii + 720 pp, $16.50; *Student Solutions Manual,* 231 pp, (P); *Instructor's Supplement,* 215 pp, (P). Some changes from earlier editions: more problems, expanded exposition of limits, rearrangement of certain chapters; greater stress on numerical methods. For reviews of earlier editions, see TR, December 1968, and June 1972. SG

CALCULUS, T(13-14). *Calculus and Analytic Geometry, Third Edition.* Robert C. Fisher, Allen D. Ziebur. P-H, 1975, xiii + 817 pp, $14.95. Revised edition with none of the hard proofs and "more routine exercises." PJM

CALCULUS, S*(13). *Prof. E. McSquared's Original, Fantastic & Highly Edifying Calculus Primer.* H. Swann, J. Johnson. William Kaufmann, Inc., One First St., Los Altos, CA 94022. 1975, 111 pp, $2.95 (P). An "energized fantasy world" of calculus cartoons to aid in the understanding of limits, not just for those who have been educated on a diet of cartoons, but for any beginning student who needs help dramatizing mathematical arguments. More medium than message, it will delight both young and old. Calculus will just never be the same again. LAS

CALCULUS, T(13: 1). *Calculus for Business and Social Science.* William J. Adams. Xerox, 1975, xiii + 226 pp, $10.75. A very short calculus. Six chapters (many rather brief) on functions, derivative, optimization, curve sketching, the integral, and partial derivatives. LLK

REAL ANALYSIS, T*(14-16: 2), S, L. *Analysis in Euclidean Space.* Kenneth Hoffman. P-H, 1975, xiv + 432 pp, $15.95. A lucid, contemporary "advanced calculus" focused on the general principles of mathematical analysis rather than on applied or "hard" classical analysis. Discussion integrates theory and examples involving real and complex numbers and matrices; the notation slips unobtrusively from R^n to normed linear spaces. Concludes with an especially well motivated introduction to the Lebesgue integral and with a chapter on differentiability of functions from R^n to R^m. LAS

REAL ANALYSIS, T(17: 1). *General Integration and Measure.* Alan J. Weir. Cambridge U Pr, 1974, xi + 298 pp, $17.50. Continuation (Ch. 9-14) of the author's *Lebesgue Integration and Measure* (TR, January 1974) in which the integral is based upon Daniell's construction. General integration; Lebesgue-Stieltjes integrals and measures; Riesz representation theorem; general measures; classical approach; uniqueness and approximation theorems; product measures; Borel measures; real and complex measures; Radon-Nikodym theorem. Many exercises with solutions included! References. Index. RSK

COMPLEX ANALYSIS, T***(17: 2), S, P**, L. *Applied and Computational Complex Analysis, Volume I: Power Series, Integration, Conformal Mapping, Location of Zeros.* Peter Henrici. Wiley, 1974, xv + 682 pp, $24.95. Written in the algorithmic spirit: no problem is solved unless an algorithm for its solution is found. The criterion for inclusion of topics is utility, e.g., Goursat's proof of the Cauchy theorem is omitted, but extensive treatment is given to the location of zeros of polynomials. "Seminar assignments" at the end of each chapter can be used for independent student research using experimental computation. This radical, exciting departure from traditional complex variable texts could signal a revitalization of the whole area. TAV

DIFFERENTIAL EQUATIONS, P. *Variational Methods for Eigenvalue Approximation.* Hans F. Weinberger. CBMS Reg. Conf. in Appl. Math., No. 15. SIAM, 1974, v + 160 pp, $10.65 (P). Lecture notes of an NSF-CBMS Regional

Conference held at Vanderbilt in 1972. After introductory chapters, the author covers such topics as the existence and characterization of eigenvalues, improvable bounds, approximation, finite difference equations. A well-written, complete set of notes. SG

DIFFERENTIAL EQUATIONS, T(17-18), S, P. *Similarity Methods for Differential Equations.* G.W. Bluman, J.D. Cole. Appl. Math. Sci., V. 13. Springer-Verlag, 1974, ix + 332 pp, $9.50. With notable clarity and remarkable comprehension, the first part of the book reproduces the pioneering work of Sophus Lie (1842-99) in the study of continuous groups of transformations of ODE's, based on the infinitesimal property of the group. The second part extends the method to PDE's. Invariance under a group reduces the number of independent variables and connects the resulting solutions with the usual "similarity solutions" of PDE's. I-CH

NUMERICAL ANALYSIS, T(15-16: 1), S. *Principles of Numerical Analysis.* Alston S. Householder. Dover, 1974, x + 274 pp, $4 (P). Slightly corrected, unabridged republication of the original (1953) edition. Special attention given to the Graeffe process, Bernoulli's method and the Monte Carlo method, among the standard material: solutions of linear and nonlinear equations, interpolation polynomials, matrices, numerical integration, and differentiation. An excellent reference and text, despite lack of computer programs. I-CH

NUMERICAL ANALYSIS, T(16-18: 1), S*, P*, L. *Solving Least Squares Problems.* Charles L. Lawson, Richard J. Hanson. P-H, 1974, xii + 340 pp, $14.50. Many algorithms. Includes Householder's, Given's and singular value decomposition. Considers linear inequality and equality constraints and practical computational problems. Fortran code. LH

ANALYSIS, T?, S?. *Laplace Transforms.* J. Williams. Prob. Solvers, No. 10. Allen & Unwin (U.S. Distr: Crane, Russak), 1973, 93 pp, $4.95 (P). A potpourri of rules, examples and problems in Laplace transforms with little reason for the selection. Underorganized and overpriced. TAV

GEOMETRY, T**(16-18: 2, 3), S, P*, L**. *A Comprehensive Introduction to Differential Geometry, Volume 3.* Michael Spivak. Publish or Perish, 1975, ix + 474 pp, $16.25 (P). Volume Three of this endeavor to bring differential geometry to the masses (of mathematicians at least) now promises volumes four and five. This volume concerns itself with classical surface theory but takes place in the rather modern setting of the preceeding volumes. There is also an extensive errata section for the earlier volumes. Volumes Four and Five are to contain generalizations of surface theory, a study of isometric imbeddings and the ultimate (?) generalization entitled "The Generalized Gauss-Bonnet Theorem and What It Means for Mankind." The entire work seems to continue well its intent of bringing twentieth century style together with a lot of nineteenth century geometry. Few areas of mathematics need it more. JAS

GEOMETRY, T*(17-18: 1), P, L*. *Regular Complex Polytopes.* H.S.M. Coxeter. Cambridge U Pr, 1974, x + 185 pp, $28.50. "I have made an attempt to construct it like a Bruckner symphony, with crescendos and climaxes, little foretastes of pleasure to come... . Its relationship to my earlier *Regular Polytopes*...resembles that of *Through the Looking-Glass* to *Alice's Adventures in Wonderland.* The sequel is more profound; it is essentially self-contained, but some of the same characters reappear with recognizable but slightly changed names, and there are many new characters of the same sort, but even more fantastic." (Preface, p. ix.) A continuation of the author's life-long romance with regular solids, conjuring up unexpected relationships with various branches of mathematics. PJC

TOPOLOGY, P. *Continuous Flows in the Plane.* Anatole Beck. Grund. math. Wissenschaften, B. 201. Springer-Verlag, 1974, xi + 462 pp, $46.80. A continuous flow in the plane is a continuous map $\phi: R \times R^2 \to R^2$ with $\phi(s + t, x) = \phi(s, \phi(t,x))$. This monograph is a study of properties of such maps. PJM

TOPOLOGY, T(15-16: 1), L. *Introduction to Topology, Third Edition.* Bert Mendelson. Allyn, 1975, ix + 206 pp, $11.95. A modest first course in topology. Chapter titles: sets, metric spaces, topological spaces, connectedness, compactness. No separation axioms. One section on categories and functors. Not many exercises. PJM

TOPOLOGY, T(15-17). *Elements of Modern Topology.* Ronald Brown. McGraw, 1968, xvi + 351 pp, $13.50. More geometric than most books. The majority of the book is dedicated to cell complexes; homotopy, including a detailed study of homotopy extension; and covering spaces. A nicely done and unusual text with good exercises. JAS

TOPOLOGY, P, L. *Topology and its Applications.* Ed: S. Thomeier. Lect. Notes in Pure and Appl. Math., V. 12. Dekker, 1975, vi + 203 pp, $15.75 (P). Proceedings of a May, 1973 conference in Newfoundland, consisting primarily of four invited papers (including a 50 page exposition of catastrophe theory by R. Thom); also includes contributed shorter papers and/or abstracts. LAS

TOPOLOGY, P*, L*. *Lectures on Set Theoretic Topology.* Mary Ellen Rudin. CBMS Reg. Conf. in Math., No. 23. AMS, 1975, 76 pp, $4 (P). A brisk and lively portrait of the frontier of general topology between normality and metrizability. A stylistic *tour de force* in which the necessary gruesome detail is embedded painlessly in a rich, straightforward tale--often humorous, always personal--of the encounter between some of this century's most creative mathematical imaginations and some of mathematics' most subtle and unimaginable concepts. LAS

TOPOLOGY, P. *Links of Codimension Two, Second Edition.* Mauricio A. Gutiérrez. Universidad Nacional de Colombia, 1970, ii + 58 pp, (P). "To the admired memory of Colombia's worst soldier: Gen. Francisco de Paulo Diago y Diago. May all generals imitate his example and be, like him, overwhelmed in battle." Higher-dimensional links (embeddings of m copies of S^n in S^{n+2}) in codimension 2, with a homotopy-theoretic criterion for a link to be trivial. Lots of fundamental groups, covering spaces, and surgery. PJC

PROBABILITY, T*(15-17: 1, 2), S, L. *Probability with Applications.* Michael Woodroofe. McGraw, 1975, x + 372 pp, $16.50. A careful mathematical treatment that also attempts to develop the intuition about objective and subjective probability. Appendices on set theory and integration. Final chapters on random walks and martingales. FLW

PROBABILITY, T(16-17: 1), P. *Complex Stochastic Processes: An Introduction to Theory and Application.* Kenneth S. Miller. A-W, 1974, xi + 238 pp, $9.50 (P); $16.50. Introduces complex valued random variables as a tool to study the properties of familiar stochastic processes. Emphasizes the nature of the complex Gaussian distribution. Concludes with complex variable arguments in the theory of linear least squares analysis. Bibliography. TAV

STATISTICS, S*, P, L*. *Statistical Distributions: A Handbook for Students and Practitioners.* N.A.J. Hastings, J.B. Peacock. Halsted Pr, 1975, xiii + 130 pp, $5.95. The basic facts about 25 common families of distributions that arise in practice. FLW

COMPUTER SCIENCE, T(13-16), S, L. *Computers in Modern Society.* Ralph J. Kochenburger, Carolyn J. Turcio. Hamilton, 1974, xv + 266 pp, $9.95. Slightly more substantial than other books of this type. It is "based on the idea that one can attain a real understanding of the social implications of computers only by directly encountering and interacting with computers." Basic concepts of computers, programming, problem solving and the interaction with society. Needs to be supplemented by a manual. Glossary. Index. RWN

APPLICATIONS (BIOLOGY), P, L. *Some Mathematical Questions in Biology, V.* Ed: Jack D. Cowan. Lect. on Math. in Life Sci., V. 6. AMS, 1974, vi + 141 pp, $13 (P). Six lectures, including two by Fields medalists René Thom and Steven Smale, from a July 1973 conference in Mexico City. LAS

APPLICATIONS (BIOLOGY), S(15-18), P, L*. *Mathematical Theories of Populations: Demographics, Genetics and Epidemics.* Frank Hoppensteadt. CBMS Reg. Conf. in Appl. Math., No. 20. SIAM, 1975, vii + 72 pp, (P). Highly readable introduction to basic deterministic models of single-species population dynamics, genetics, and epidemics in three essentially independent chapters, each concluding with references to more advanced literature. Although lacking any data for confirmation of the theories, it is an excellent source for an undergraduate seminar in the theory of populations. LAS

APPLICATIONS (PHYSICS), P. *Cooperative Phenomena.* Ed: H. Haken. M. Wagner. Springer-Verlag, 1973, xv + 458 pp, $54.20. A collection of 42 papers dedicated to Herbert Fröhlich concerning phenomena in which the whole is more than and different from a simple addition of its parts. LAS

APPLICATIONS (PHYSICS), T(18), S, P, L. *Physical Cosmology.* P.J.E. Peebles. Princeton U Pr, 1971, xvi + 282 pp, $12.50; $6.50 (P). Notes from a physics graduate course. Following a historical survey of "golden moments" between 1912 and 1950, the monograph examines in detail selected topics, e.g., Hubble's constant, mean mass density, microwave background. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-31: Transport Phenomena.* Ed: G. Kirczenow, J. Marro. Springer-Verlag, 1974, xiv + 517 pp, $16 (P). Lectures from the June 1974 international school of statistical mechanics in Sitges, Spain. LAS

APPLICATIONS (SOCIAL SCIENCE), S**(13-16), L**. *Initiation aux Mathématiques des Processus de Diffusion, Contagion et Propagation.* J.P. Monin, R. Benayoun, B. Sert. Gauthier-Villars, 1973, 97 pp, (P). By way of motivation, considerable description of examples of diffusion, contagion, and propagation. Next, the mathematical treatment of deterministic (differential equations) and stochastic (Markov chains) models. Then, three specific applications, comparing the models to real data: the evolution of demand for automobiles in Spain, the diffusion of an innovation among physicians, and the diffusion and evolution of opinions. Last but not least, an extensive bibliography. (Lacking: an index.) A good model for a book on modelling! PJC

Reviewers Whose Initials Appear Above

Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Fontbonne College: Mr. Denis Beverage has been appointed Instructor; Sister Adele Rothan has been promoted from Instructor to Assistant Professor.

University of Iowa: Dr. D. D. Anderson, University of Chicago, has been appointed Visiting Assistant Professor; Dr. H. L. Schoen, Virginia Polytechnic Institute and State University, has been appointed Associate Professor; Instructor Matilde Macagno has been promoted to Assistant Professor; Professor Edwin N. Oberg retired in June 1974.

Texas A & M University: Associate Professors C. K. Chui and C. J. Maxson have been promoted to Professors; Assistant Professors J. R. Boone and L. F. Guseman, Jr., have been promoted to Associate Professors; Associate Professors B. C. Moore and W. S. McCulley retired on August 31, 1974.

Towson State College: Dr. Ohoe Kim, University of Rochester, has been appointed Assistant Professor; Associate Professors M. G. Horak and C. L. Zimmerman have been promoted to Professors; Instructor Joyce C. Neubert has been promoted to Assistant Professor.

Wayne State University: Assistant Professor David Jonah has been promoted to Associate Professor; Professor A. G. Laurent was elected Fellow of the American Statistical Association; Associate Professor (and Associate Dean) William M. Borgman retired in September 1974 with the title of Associate Professor (and Associate Dean) Emeritus.

University of Western Ontario: Dr. S. Z. Ditor, Louisiana State University, has been appointed Assistant Professor; Professor R. H. Cole retired on June 30, 1974, with the title of Professor Emeritus.

Assistant Professor P. C. Deliyannis, Illinois Institute of Technology, has been promoted to Associate Professor.

Dr. James F. Gray, S. M., St. Mary's University, has been appointed Vice President for University Planning.

Professor B. F. Jones, Rice University, who won the George R. Brown Award for Superior Teaching three times, has been named Noah Harding Professor of Mathematics.

Professor L. D. Kovach, Naval Postgraduate School, has been appointed Chairman of the Department of Mathematics.

Dr. K. O. Leland, who completed a two-year Postdoctoral appointment at the Air Force Aerospace Research Laboratories in Dayton, Ohio, has been appointed a mathematician at the Navy Personnel Research and Development Center, San Diego, California.

Dr. Perry Scheinock has left his position as Executive Director of the Delaware Health Services Authority, Inc., Dover, Delaware, and has joined the staff of the University City Science Center in Philadelphia as Project Director of the Data Management Center on the National Breast Cancer Screening Program.

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Utilizing a wide range of resources, this free telephone-activated referral service maintains an information bank of programs which are innovative and people who are knowledgeable about various aspects of postsecondary education. Project Director Jane Lichtman stated that her staff doesn't presume to be the "experts"; rather, they seek to connect inquirers with other people who are actively engaged in similar pursuits. If the information is not on hand, staff members will research the question, check the resource and referral availability and supply you with the information quickly, all for the price of your phone call.

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Project NEXUS was established by the American Association for Higher Education through a grant from the Fund for the Improvement of Postsecondary Education. Operations were begun during January, 1974.

SECOND NATIONAL CONVENTION OF TWO-YEAR COLLEGE MATHEMATICS EDUCATORS

The Second National Convention of Two-Year College Mathematics Educators will be held in Chicago on October 29–November 1, 1975, at the Sheraton-Chicago.

Plans are being made to have speakers, discussion leaders, and authors from the TYC Mathematics Community. For further information, please write to Professor William L. Drezdzon, Convention Committee Chairman, Oakton Community College, 7900 North Nagle, Morton Grove, Illinois 60053.

SYMPOSIUM ON CALCULUS OF VARIATIONS AND CONTROL THEORY

The Mathematics Research Center of the University of Wisconsin-Madison will hold a Symposium on Calculus of Variations and Control Theory during the period September 22–24, 1975. The Symposium will consist of about 15 invited lectures dealing with classical calculus of variations, optimal control theory and control theory in general and will honor Professor L. C. Young on the occasion of his official retirement.

Professor D. L. Russell of the University of Wisconsin-Madison is acting as Chairman of the Program Committee for the Symposium. Other individuals who have been asked to act as members of the Program Committee are Professors T. J. Higgins, D. J. Patil, V. Rideout and D. Rudd. A detailed program and information on registration and accommodations will be available about July 15, 1975. Requests for the program should be directed to Professor D. L. Russell, Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, Wisconsin 53706.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE NORTH CENTRAL SECTION

The fall meeting of the North Central Section of the MAA was held at Moorhead State College, Moorhead, Minnesota, October 25–26, 1974.

On Friday evening Gerald Heuer, Concordia College, Moorhead, spoke on *What is the "Solution" of a Bimatrix Game?*.

The invited speaker for the Saturday morning session was Dr. H. O. Pollak, MAA President-Elect, whose topic was *On the Nature of Mathematical Research in Industry.* Other papers presented at the meeting were:

1. *Additive partitions of the positive reals* (MONTHLY Problem 5971), by M. B. Gregory, University of North Dakota.
2. *The Smirnov compactification of the real numbers*, by Don Mattson, Moorhead State College.
3. *Greatest common divisors of ax-b and cx-d*, by L. R. Tanner, Jamestown College.
4. *On choosing the correct function*, by Hubert Walczak, College of Saint Thomas.
5. *The wall problem: a movie*, by Pierre Malraison, Carleton College.
6. *Numerator polynomial coefficient array for the convolved Fibonacci sequence*, by G. E. Bergum, South Dakota State University.
7. *Some student projects in group theory*, by J. A. Gallian, University of Minnesota.

The chairmen of the contributed papers sessions were: Derald Rothmann, Moorhead State College, and Sylvan Burgstahler, University of Minnesota at Duluth, Chairman of the NCS/MAA.

LOUIS GUILLOU, *Secretary*

NOVEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Fall meeting of the Maryland—District of Columbia—Virginia Section of the MAA was held November 23, 1974, at Montgomery College, Rockville, Maryland. One hundred twenty-five persons attended. Professor Geraldine Coon, chairman of the Section, presided.

During the morning there was a session of the contributed papers, an invited speaker, and a short business meeting. The invited speaker, Dr. James Yorke of the University of Maryland, presented a talk titled "Period Three Implies Chaos." Following lunch, there was another session of contributed papers.

The contributed papers presented were:

1. *Non-parametric statistics*, by E. G. Swafford, U.S. Naval Academy.
2. *Non-constant variance regression analysis*, by M. S. Hellman, Potomac, Maryland.
3. *Planes, cubes, and center representable polyhedra*, by L. S. Joel, D. R. Shier, and Marjorie L. Stein, National Bureau of Standards.
4. *Matrix representation of linear codes*, by H. M. Beck, Oxon Hill, Maryland.

5. *The personalized system of instruction and the learning vector*, by Norman Locksley, University of Maryland.
6. *Mini-courses in mathematics*, by David Russell, Prince George's Community College.
7. *Pre-service mathematics for teachers*, by J. M. Smith, George Mason University.
8. *Small-group instruction in mathematics courses*, by R. E. Hildebrand, University of Maryland.
9. *Use of the computer in a traditional calculus course*, by Linda R. May, Salisbury State College.
10. *Individualization of mathematics for college certificate programs*, by H. V. Ellis, Jr., Paul D. Camp Community College.
11. *Position versus momentum in quantum mechanics*, by Andrew Vogt, Georgetown University.
12. *Flexing the hexaflexagon*, by Bruce McLean, Madison College.
13. *The much-maligned divergent series*, by D. L. Parker, Salisbury State College.
14. *Smoothness of norms and Banach spaces*, by J. F. Kent, University of Richmond.
15. *Spiral-like injections of* $[0, \infty)$ *in the plane*, by W. L. Young, U.S. Naval Academy.
16. *Hamilton orbit generator (HOG)*, by H. Siddalingaiah, Computer Sciences Corporation.
17. *The "Canonization" of conicoids on Univac* 1106, by Lew Kowarski, Morgan State College.
18. *Prohibition against time*, by Dean Spencer, Q. E. D. Systems, Inc.

J. M. Smith, *Secretary*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twentieth annual meeting of the Northeastern Section of the MAA was held at Lowell Technological Institute, Lowell, Massachusetts, on November 30, 1974; there were seventy-six people in attendance. The section chairperson, L. Aileen Hostinsky, presided at the morning session at which the following talks were given:

Grinberg's condition for Hamilton circuits in planar graphs, by Norton Starr, Amherst College.
Almost congruent triangles, by B. B. Peterson, Middlebury College.

At the afternoon business meeting the following officers were elected for the coming year: Chairperson, Anne F. O'Neill, Wheaton College; Vice-Chairperson, G. P. Murphy, University of Maine; Secretary-Treasurer, G. W. Best, Phillips Academy. In addition to various reports, the Section passed a resolution expressing appreciation to Leslie Korper, Llewellyn Jensen, J. R. Stauffer, and William Crawford for their continued service as Regional Chairman of the MAA High School Examination. The business meeting concluded with brief reports by Professor P. J. Davis, Governor of the Section, and by Dr. A. B. Willcox, the guest speaker for the Association.

The following talks completed the program:

Some bridges to and from mathematics, by A. B. Willcox, The Mathematical Association of America.
Generators for finite groups, linear and abstract, by Harriet Pollatsek, Mount Holyoke College.

G. W. Best, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 wks. bef. mtg.

FLORIDA, Florida A & M University, Tallahassee, March 5–6, 1976.

ILLINOIS, second Friday/Saturday in May.

INDIANA

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI

NEBRASKA, April.

NEW JERSEY

NORTH CENTRAL, Southwest Minnesota State College, Marshall, October 25, 1975.

NORTHEASTERN, Simmons College, Boston, November 29, 1975.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA–ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 wks bef. mtg.

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 22, 1975.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, May 1–2, 1976.

SEAWAY, State University College, Cortland, New York, October 31–November 1, 1975.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, North Carolina, Spring 1976.

SOUTHERN CALIFORNIA

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 wks. bef. mtg.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, Western Michigan University, Kalamazoo, August 19–22, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Western Michigan University, Kalamazoo, August 19, 1975.

FIBONACCI ASSOCIATION, California State University, San Francisco, October 18, 1975.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, Seattle University, Washington, August 11–14, 1975.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, MGM Grand Hotel, Las Vegas, Nevada, November 17–19, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975

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by Ivan Niven.
A clear interesting exposition of number systems, beginning with the natural numbers and extending to the rational and real numbers.

WHAT IS CALCULUS ABOUT? (NML-02)
by W. W. Sawyer
Develops the basic concepts of calculus intuitively, exploiting such familiar concepts as speed, acceleration, and volume.

AN INTRODUCTION TO INEQUALITIES (NML-03)
by E. F. Beckenbach and R. Bellman
Includes an axiomatic treatment of inequalities, proofs of the classical inequalities and powerful applications to optimum problems.

GEOMETRIC INEQUALITIES (NML-04)
by N. D. Kazarinoff
An informal presentation of the arithmetic mean-geometric mean inequality and discussions of the famous isoperimetric theorems, the reflection principle, and Steiner's symmetrization—with problem solving emphasis.

THE CONTEST PROBLEM BOOK (NML-05)
Problems from the Annual High School Mathematics Contests sponsored by the MAA and four other organizations. Covers the period 1950-1960. Compiled and with solutions by Charles T. Salkind.
A complete collection of examination questions and solutions from the first decade of this national high school mathematics competition. No mathematics beyond intermediate algebra required. Elementary procedures are consistently used in solutions, but more sophisticated alternatives are included where appropriate.

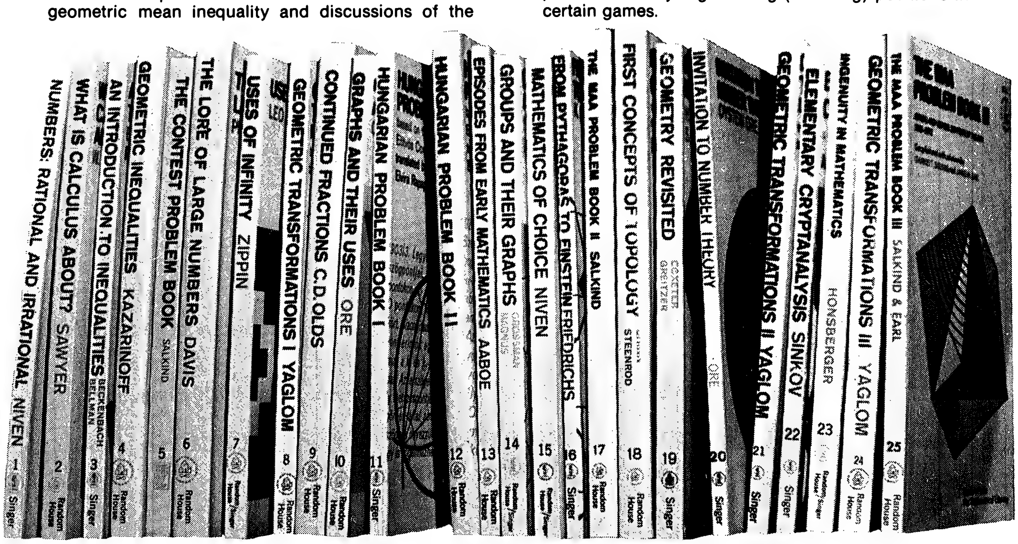
THE LORE OF LARGE NUMBERS (NML-06)
by P. J. Davis
How to work with numbers, big and small, and understand some of their less obvious properties.

USES OF INFINITY (NML-07)
by Leo Zippin
How mathematics have transformed the almost mystic concept of infinity into a precise tool essential in all branches of mathematics.

GEOMETRIC TRANSFORMATIONS (NML-08)
by I. M. Yaglom translated by Allen Shields
Concerned with isometries (distance-preserving transformations), a completely different way of looking at familiar geometrical facts that supplies the students with methods by which powerful problem solving techniques may be developed.

CONTINUED FRACTIONS (NML-09)
by Carl D. Olds
Shows how rational fractions can be expanded into continued fractions and introduces such diverse topics as the solutions of linear Diophantine equations, expansion of irrational numbers into infinite continued fractions and rational approximation to irrational numbers.

GRAPHS AND THEIR USES (NML-10)
by Oystein Ore
Develops enough graph theory to approach such problems as scheduling the games of a baseball league or a chess tournament, solving some ancient puzzles or analyzing winning (or losing) positions in certain games.



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THE AMERICAN MATHEMATICAL MONTHLY

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NOTICE TO THE READERSHIP

Due to the present serious financial situation of the MAA, all possible steps to reduce expenditures have been considered. As a result, the format of a MONTHLY page is being changed, beginning with this issue. By using somewhat less space between lines and decreasing the size of margins, it has proved possible to print the same amount of material on 96 pages as on 128. The resulting savings in paper and printing costs are substantial. It is hoped that our readers will show understanding for this step.

ALEX ROSENBERG, *Editor*

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THE QUOTA METHOD OF APPORTIONMENT

M. L. BALINSKI AND H. P. YOUNG

Abstract: The problem of apportionment is explained together with an account of the methods used by the United States Congress beginning with the first decennial apportionment of 1792. Fairness and historical precedent dictate that several properties must be satisfied by any method which may be deemed acceptable. It is shown that the method presently used violates one of these and that a new procedure, the quota method, is the unique method satisfying the essential properties.

1. Introduction. Article I, Section 2 of the Constitution of the United States provides, "Representatives and direct taxes shall be apportioned among the several states which may be included within this Union, according to their respective Numbers...," a phrase which was later supplanted in 1868 by the Fourteenth Amendment, Section 2 with, "Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each State, excluding Indians not taxed," and (again Article I, Section 2) "The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at Least one Representative."

The precise interpretation of the unchanging Constitutional provision "according to their respective numbers" has been the subject of both political and theoretical debate since the founding of the Republic. The first Presidential veto was exercised by George Washington to quash an "act for an apportionment of Representatives... according to the first enumeration." In so doing he followed the advice (and used the words) of his Secretary of State, Thomas Jefferson, while disregarding that of his Secretary of the Treasury, Alexander Hamilton. Debates, reports, methods and bills have succeeded themselves decennially ever since, following each census.

The clear intent of the Constitution is well captured by Daniel Webster's definition: "To *apportion* is to distribute by right measure, to set off in just parts, to assign in due and proper proportion." ([19], p. 107). If fractional numbers of representatives were allowed to be allocated to the various states, then the problem would have a completely natural solution—namely, the number of representatives accorded to a state would be strictly proportional to its population. But since "a

fraction is the broken part of some integral number" ([19], p. 113) and such representation is not allowed, "that which cannot be done perfectly must be done in a manner as near perfection as can be." ([19], p. 108.) And the question is: How?

Consider, for example, the five-state country with populations as given in Table 1 (from [13], p. 103). The "exact proportional solutions" (or *exact quotas*) are given for a house of 25, 26, and 27 seats.

For a house of 25 seats the as-near-perfection-as-can-be integer solution is evidently 9,7,5,3,1 seats for A,B,C,D,E respectively. But, in a house of 26 or 27 seats which state or two states should receive the extra seat or seats?

State	*Population*	*25 Seats* *Exact quota*	*26 Seats* *Exact quota*	*27 Seats* *Exact quota*
A	9061	8.713	9.061	9.410
B	7179	6.903	7.179	7.455
C	5259	5.057	5.259	5.461
D	3319	3.191	3.319	3.447
E	1182	1.137	1.182	1.227
	26000	25	26	27

TABLE 1.

2. Formulation. Let $\boldsymbol{p} = (p_1, \cdots, p_s)$ be the populations of s states, where each $p_i > 0$, and $h \geqq 0$ the number of seats in the house. The problem is to find, for each house size $h \geqq 0$, an **apportionment for** h: an s-tuple of non-negative integers $(a_1, \cdots, a_s)$ whose sum is h. A **solution** of the apportionment problem is therefore a function f which to every $\boldsymbol{p}$ and h associates a unique apportionment for h, $a_i = f_i(\boldsymbol{p}, h) \geqq 0$ where $\Sigma_i a_i = h$. If f is a solution and h a house size then f^h will represent the function f restricted to the domain $(\boldsymbol{p}, h')$ where $0 \leqq h' \leqq h$. f^h is called a **solution up to** h, and f is called an **extension** of f^h.

A specific apportionment method may give several different solutions, for "ties" may occur when using it, for example when two or more states have identical populations. For this reason it is useful to define an **apportionment method** M as a non-empty set of solutions. Notice that, in particular, a solution up to a given house size h may have several different extensions in M.

The "ideal" or "strictly proportional" number of representatives "due" state j, called the **exact quota** of state j, is $q_j(\boldsymbol{p}, h) = p_j h / \Sigma_i p_i$. Given $\boldsymbol{p}$ and h, if $q_i = q_i(\boldsymbol{p}, h)$ is integer for all i, then $a_i = q_i$ is the perfect solution. Otherwise, each state i should receive at least as many seats as its **lower quota** $\lfloor q_i \rfloor$ (the largest integer less than or equal to q_i) and certainly no more than its **upper quota** $\lceil q_i \rceil$ (the smallest integer greater than or equal to q_i), since these result from "rounding" the exact quota q_i down or up. In general an apportionment method is said to **satisfy lower quota** if, for each of its solutions f, $f_i(\boldsymbol{p}, h) \geqq \lfloor q_i(\boldsymbol{p}, h) \rfloor$, to **satisfy upper quota** if $f_i(\boldsymbol{p}, h) \leqq \lceil q_i(\boldsymbol{p}, h) \rceil$, and to **satisfy quota** if it satisfies both lower and upper quota.

3. United States Apportionment History 1792–1901: Satisfying Quota. The first apportionment of seats in Congress among the thirteen original states was declared in the Constitution itself. Following the census of 1790, Congress passed the first act of apportionment in 1792 alloting a total of 120 seats to the 15 states then in the Union. George Washington questioned the fairness of the proposed apportionment, and turned to his Secretary of State, Thomas Jefferson, for advice. Jefferson also found it wanting and pointed out that, "No invasions of the Constitution are fundamentally so dangerous as the tricks played on their own numbers, apportionment," ([15], p. 470). Washington vetoed the bill, after having "maturely" considered it, saying: "First ... there is no one proportion or division which, applied to the respective numbers of the States, will yield the

number and allotment proposed by the bill. Second ... the bill has allotted to eight of the states more than one [Representative] for thirty thousand." ([**18**], pp. 16–17.) Jefferson's reasoning about the problem was as follows: "it will be said that, though, for taxes, there may always be found a divisor which will apportion them among the States according to numbers exactly, ..., yet, for *representatives*, there can be no such common ratio, or divisor which ... will divide them exactly without a remainder or fraction. I answer, then, that taxes must be divided *exactly*, and representatives as *nearly* as the *nearest ratio* will admit; and the fractions must be neglected ..." ([**15**], p. 463). More precisely, Jefferson was proposing the following method. Given a "ratio" or "divisor" λ, each state i should receive $a_i = \lfloor p_i/\lambda \rfloor$ seats. If h seats are to be apportioned, then ideally $\lambda = \Sigma_i p_i/h$, but as we must also have $\Sigma_i a_i = h$, it is necessary to adjust λ, and choose the λ "nearest" to the ideal that will achieve this result. Allowing for the possibility of ties, the **Jefferson method** may therefore be stated as follows: given $\boldsymbol{p}$ and h, choose the largest λ (at most $\Sigma_i p_i/h$) such that $h' = \Sigma_i \lfloor p_i/\lambda \rfloor \geqq h$. Let $E' = \{i : p_i/\lambda \text{ is integer}\}$, (clearly $|E'| \geqq 1$), and let E be an $(h'-h)$-cardinality subset of E'. Then $f_i(\boldsymbol{p},h) = \lfloor p_i/\lambda \rfloor$ for $i \notin E$ and $f_i(\boldsymbol{p},h) = p_i/\lambda - 1$ for $i \in E$. Thus if $h' - h > 0$ there exists more than one apportionment for h, hence more than one solution. The unique Jefferson apportionment for house size 26 in the example of Table 1 is found to be $f(\boldsymbol{p},26) = (10,7,5,3,1)$, (obtained with $\lambda = 906.1$). This method is known in the United States literature as the **method of greatest divisors,** and in the European literature as the **method of d'Hondt** (a nineteenth-century Belgian mathematician), but has not heretofore been ascribed to Jefferson.

Hamilton, also consulted by Washington, argued that the bill should be signed: "It is inferred from the provisions of the Act—that the following process has been pursued. (I) The aggregate numbers of the United States are divided by 30,000, which gives the total number of representatives, or 120. (II) This number is apportioned ... by the following rule: as the *aggregate* numbers of the *United States* are to the *total number* of representatives found as above, so are the *particular numbers of each state* to the number of representatives of such state. But (III) as this process leaves a residue of eight out of the 120 members unapportioned, these are distributed among those states which upon that second process have the largest fractions or remainders ... And hence results a strong argument for its constitutionality." ([**12**], pp. 228–229.)

The **Hamilton method** is, therefore: First, give to each state i its lower quota $\lfloor q_i \rfloor$; then order the states by their fractional remainders $d_j = q_j - \lfloor q_j \rfloor \geqq 0$ in a priority list $d_{j_1} \geqq d_{j_2} \geqq \cdots \geqq d_{j_s}$. Second, give one additional seat to each of the first $h - \Sigma \lfloor q_j \rfloor = \Sigma d_j$ states on the list. If there are ties, say if $d_{j_t} = d_{j_{t+1}}$, then there exist distinct arrangements of the priority list each of which leads to a solution of the given problem. It should immediately be stated that this method is generally known as the **Vinton method of 1850,** although first proposed, it appears, by Hamilton. The (unique) Hamilton apportionment for the example of Table 1 at house 26 is therefore $f(\boldsymbol{p},26) = (9,7,5,4,1)$. It is clear that the Hamilton method satisfies quota. In fact it is easy to see that any Hamilton solution $\{a_i\}$ solves: $\min_{a_i} \Sigma |a_i - q_i|$, $\min_{a_i} \Sigma (a_i - q_i)^2$, and $\min_{a_i} \max_i |a_i - q_i|$, where $\Sigma_i a_i = h$ and the a_i are nonnegative integers.

While it is true that the bill vetoed by Washington gave an apportionment that agreed with Hamilton's method for that particular situation, the bill did not specify what (if any) method was used to arrive at this apportionment. Jefferson considered this a serious weakness of the bill: "The bill does not say that it has given the residuary representatives *to the greatest fractions*; though in fact it has done so. It seems to have avoided establishing that into a rule, lest it might not suit on another occasion. Perhaps it may be found the next time more convenient to distribute them *among the smaller states*; at another time *among the larger states*; at other times according to any other crochet which ingenuity may invent and the combinations of the day give strength to carry ... whereas the other construction [Jefferson's] reduces the apportionment always to an arithmetical operation, about which no two men can ever possibly differ." ([**15**], p. 469.)

The apportionment scheme actually used for the censuses of 1790 through 1830 was a diluted

form of Jefferson's proposal: a λ was chosen (without first specifying a house size) and the a_i's determined by $a_i = \lfloor p_i/\lambda \rfloor$. The house was then given by $h = \Sigma_i a_i$. Since the choice of λ was decided by political maneuvering, this was not, however, an apportionment *method* in the sense used here.

On 5 April 1832 Daniel Webster entered the lists of apportionment on the floor of the Senate. "Representation founded on numbers must have some limit, and being, from its nature, a thing not capable of indefinite subdivision, it cannot be made precisely equal ... the Constitution, therefore, must be understood ... as requiring of Congress to make the apportionment of Representatives among the several states according to their respective numbers, *as near as may be* ... the nearest approach to relative equality of representation among the states ... the number nearest to the exact proportion of that state." ([**19**], pp. 107–109.) Webster then proposed: "... let the rule be, that the population of each state shall be divided by a common divisor, and, in addition to the number of members resulting from such division, a member shall be allowed to each state whose fraction exceeds a moiety of the divisor." ([**19**], p. 120.) Webster's construction was not used until 1842, and was then applied to obtain the apportionment based upon the 1840 census. As in the case of the previous scheme used (based on Jefferson) the size of the House was not first determined, rather it came as part of the over-all calculation. However, Webster's construction can be turned into a method giving solutions for house sizes determined in advance.

The **Webster method** is: Choose the largest $\lambda > 0$ such that $h' = \Sigma_i \lfloor p_i/\lambda + \frac{1}{2} \rfloor \geqq h$. Let $E' = \{i; p_i/\lambda + \frac{1}{2} = \text{integer}\}$, (clearly $|E'| \geqq 1$), and let E be any $(h'-h)$-cardinality subset of E'. Then $f_i(\mathbf{p}, h) = \lfloor p_i/\lambda + \frac{1}{2} \rfloor$ for $i \not\in E$ and $f_i(\mathbf{p}, h) = p_i/\lambda + \frac{1}{2} - 1 = p_i/\lambda - \frac{1}{2}$ for $i \in E$. If $h' - h > 0$, then there exists more than one apportionment for h, hence more than one solution. The (unique) Webster apportionment in the example of Table 1 is therefore $f(\mathbf{p}, 26) = (9,8,5,3,1)$ (obtained with $\lambda = 957.2$). This method is known as the **method of major fractions** but, again, has not heretofore been credited to Webster even though Webster used the term "major fractions."

The apportionment act of 23 May 1850 (9 Stat., L. 428), sponsored by Samuel F. Vinton of Ohio, fixed upon the Hamilton method and directed the Secretary of the Interior to thereafter determine the apportionment following each census, once given by Congress the number of seats to be allocated. This law, although in force through the census of 1900, did not still discussion in the House. On 25 October 1881 C.W. Seaton, Chief Clerk of the Census Office, Department of the Interior, wrote to the Chairman of the Committee on the Census that he had completed various apportionments according to the populations ascertained by the census of 1880 and "... made upon assumptions as to the total number of Representatives ranging from 275 to 350 ... While making these calculations I met with the so-called 'Alabama paradox' where Alabama was allotted 8 Representatives out of a total of 299, receiving but 7 where the total became 300." ([**1**], p. 18.) Note that the Hamilton method applied to the example of Table 1 gives $f(\mathbf{p}, 26) = (9,7,5,4,1)$ while $f(\mathbf{p}, 27) = (9,8,6,3,1)$, that is, state D *loses* a seat as the House *gains* a seat.

"This atrocity which [mathematicians] have elected to call a 'paradox' ... this freak [which] presents a mathematical impossibility" (Representative John C. Bell of Colorado, 8 January 1901; [**10**], pp. 724–725) proved to be particularly upsetting in 1901. The majority, victoriously led by Albert J. Hopkins of Illinois, Chairman of the Census Committee, opted for a House of 357 members: but every apportionment for 350 through 400 gave to Colorado 3 seats save for one, namely 357, which gave her 2. Representative Charles E. Littlefield of Main was also considerably upset: "Not only is Maine subjected to the assaults of the chairman [Hopkins] of this committee, but it does seem as though mathematics and science has combined to make a shuttlecock and battle door of the State of Maine in connection with the scientific basis upon which this bill is presented ... God help the State of Maine when mathematics reach for her ..." ([**10**], pp. 592–593). By the apportionment act of 1891 Maine received 4 seats, whereas for the populations of 1900 she would receive only 3 in a house of 357. Moreover, "Maine loses on 382. She does not lose when the House is increased to 383, 384, or 385. She loses again with 386, and does not lose with 387 or 388. Then she loses again on 389 and 390, and then ceases to lose." ([**10**, p. 592.) Perhaps the gentleman from Maine should be

excused his exasperation with mathematicians for, several days later, in the continuing debate concerning apportionment, Hopkins explained: "It is true that under the majority bill Maine is entitled to only three Representatives, and, if Dame Rumor is to be credited, the seat of the gentleman who addressed the House on Saturday last is the one in danger ... [He] takes a modest way to tell the House and the country how dependent the State of Maine is upon him ... Maine crippled! Maine, the State of Hannibal Hamlin, of William Pitt Fessenden, of James G. Blaine ... That great State crippled by the loss of LITTLEFIELD! Why, Mr. Speaker, if the gentleman's statement be true ... I can see much force in the prayer he uttered here when he said, 'God help the State of Maine' [laughter]," ([**10**], pp. 729–730).

Although several voices spoke out for other methods in the period 1800–1901, the primary discussion centered on the size of the House. "Mr. Speaker, in the reapportionment of members of Congress the first question that arises should be as to the seating capacity of the hall in which they are to meet and do business." (Representative Galusha A. Grow of Pennsylvania, January 1901, [**10**], p. 664.) In a more realistic vein the force of most arguments were as stated in the "Views of the Minority" in 1901: "We also believe that in the new apportionment no State should lose a Representative. We therefore recommend a House of 386 members" ([**1**], p. 116). In fact, the apportionment act of 16 January 1901 used the Hamilton method and fixed the House at 386. Despite this, the obvious malaise was due to the so-frequent occurrence of the Alabama paradox.

An apportionment solution is said to be **house monotone** if $f(p, h+1) \geqq f(p, h)$ for all h, that is, if it does not admit the Alabama paradox. An apportionment method is house monotone if all its solutions are. Clearly, house monotonicity is an essential property of any acceptable apportionment method. As stated by Seaton, in his letter of 1881, after discovering the paradox, "Such a result as this is to me conclusive proof that the process employed in obtaining it is defective ... [The] result of my study of this question is the strong conviction that an entirely different process should be employed." ([**1**], p. 18.)

Several attempts were made to alter the prevailing Hamilton (called Vinton) method to produce a house monotone method satisfying quota. For example, the **modified Vinton method** is: first, give to each state i its lower quota $\lfloor q_i \rfloor$; then order the states by

$$e_j = (q_j - \lfloor q_j \rfloor)/p_j \geqq 0$$

in a priority list $e_{j_1} \geqq e_{j_2} \geqq \cdots \geqq e_{j_s}$. Second, give one additional seat to each of the first $h - \Sigma \lfloor q_j \rfloor$ states on the list. The unique solution given by this method for the example of Table 1 at house size 26 is $f(p, 26) = (9, 7, 5, 3, 2)$. However, it is not difficult to construct an example for which this method produces the Alabama paradox.

4. United States Apportionment History 1910–1973: Avoiding the Alabama Paradox. The modern era of apportionment dawns with the act of 8 August 1911. The House settled on a membership of 433, and chose this number for the usual reason: "It is proper to say in this connection that a membership of 433 in the House is the lowest number that will prevent any State from losing a Representative." ([**2**], p. 1.) The bill provided that if either Arizona or New Mexico were admitted as states before the next apportionment, each would be given 1 representative, thus bringing the total to 435. The method used, and presented as essentially original by Professor W. F. Willcox of Cornell University in his letter of 21 December 1910 to Representative E. D. Crumpacker, Chairman of the Committee on the Census, was the Webster method but was dubbed by Willcox "the method of major fractions." Two arguments were put forward for its acceptance. First, that the Alabama paradox property of the Hamilton (or Vinton) method "... is so eminently unfair that in several instances Congress has modified it to prevent palpable injustice." ([**2**], p. 3.) Second, "The history of reports, debates, and votes upon apportionment seems to show a settled conviction in Congress that every major fraction gives a valid claim to an additional Representative" ([**2**], p. 9, from Willcox's

letter), where a "major fraction" is any fraction above 1/2. Willcox (see also [20]) must be credited with having turned the Webster construction into a method. This is shown by the fact that he supplied Congress with tables giving the apportionments based on the census of 1910 for memberships of the House ranging from 390 through 440 inclusive.

No apportionment was accepted on the basis of the census of 1920. Many members of the House contended that the 1920 census figures were not accurate, that due to a bad winter certain rural areas were undercounted and, also, that temporary migrations caused distortions in the totals reported. But much discussion was generated.

In 1921 E. V. Huntington, Professor of Mathematics at Harvard, initiated an investigation [14] of a class of house monotone methods. His general point of view is summarized as follows: "... between any two states there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored state will ordinarily reverse the *sign* of this inequality, so that the more favored state now becomes the less favored, and *vice versa.* Whether such a transfer should be made or not depends on whether the *amount* of inequality between the two ... is less or greater than it was before; if ... reduced ... it is obvious that the transfer should be made. The fundamental question therefore at once presents itself, as to how the '*amount of inequality*' between two states is to be measured" ([13], p. 85). He therefore asks for an apportionment which is stable in the sense that no inequality, computed according to the chosen measure, T, can be reduced by transferring one seat from one state delegation to another.

Given population $\boldsymbol{p} = (p_1, \cdots, p_s)$, and an apportionment $\boldsymbol{a} = (a_1, \cdots, a_s)$ for h, consider the numbers p_i/a_i and a_i/p_i. These represent the "average district size" and "average share of representatives" in state i. If $p_i/a_i > p_j/a_j$, or $a_i/p_i < a_j/p_j$ or $a_j > a_i(p_j/p_i)$ or $a_j(p_i/p_j) > a_i$, then state j is **better off** than state i. Define the **relative difference** between two numbers x and y to be $|x-y|/\min(x,y)$. Huntington puts forth as the proper measure of inequality T the relative difference between any of these pairs since the relative difference, $(p_i a_j/p_j a_i) - 1$, is always the same. If a transfer of one representative from state j to state i lessens the inequality then it should be made. The apportionment is **stable** if no transfer is justified, i.e., if any such transfer from j to i makes i advantaged, j disadvantaged and the inequality at least as great (as bad) as before. Thus, the condition for Huntington stability is that

$$\frac{p_j(a_i+1)}{p_i(a_j-1)} - 1 \geqq \frac{p_i a_j}{p_j a_i} - 1$$

or

$$\text{(1)} \qquad \frac{p_j^2}{(a_j-1)a_j} \geqq \frac{p_i^2}{a_i(a_i+1)} \quad \text{or} \quad \frac{p_j}{\sqrt{(a_j-1)a_j}} \geqq \frac{p_i}{\sqrt{a_i(a_i+1)}}$$

for *all* pairs of states i and j (clearly, if j is less well off than i or if $i = j$ the inequality must hold).

An apportionment satisfying (1) is easily constructed. One way is as follows. At $h = 0$ every state has $f_i(\boldsymbol{p}, 0) = 0$. If $f(\boldsymbol{p}, h) = \boldsymbol{a} = (a_1, \cdots, a_s)$ is an apportionment for $h \geqq 0$, an apportionment for $h+1$ is obtained by assigning the additional seat to any one state j which maximizes the **rank-index** $p_j/\sqrt{a_j(a_j+1)}$. Not only does such an apportionment satisfy (1), but any apportionment satisfying (1) can be obtained in this manner. The solution is therefore, except for ties, unique ([5], although this key point seems to have been missed by Huntington).

Another way to obtain solutions is to observe that (1) implies the existence of a **divisor** λ satisfying

$$\text{(2)} \qquad \min_j \frac{p_j}{\sqrt{(a_j-1)a_j}} \geqq \lambda \geqq \max_i \frac{p_i}{\sqrt{a_i(a_i+1)}}.$$

Conversely, given λ, if $a_j(\lambda)$ is chosen for each j to be the smallest integer satisfying the **divisor criterion** $\sqrt{(a_j+1)a_j} \geqq p_j/\lambda$, then $\boldsymbol{a} = (a_1, \cdots, a_s)$ is an apportionment for $\Sigma_j a_j(\lambda) = h$ satisfying (1).

This gives a "local" condition for verifying that a given apportionment satisfies Huntington's criterion (1). As λ is decreased, apportionments satisfying (1) may be obtained for all h, although because of "ties" several apportionments for different house sizes may correspond to the same λ (for the precise procedure see below). Huntington cleverly baptized his candidate **the method of equal proportions**, or EP. The unique EP apportionment for 26 in the example of Table 1, obtained for example with $\lambda = 960$, is $f(\boldsymbol{p}, 26) = (9,7,6,3,1)$.

Huntington's method presents, however, several serious difficulties. These must be aired despite the risk of persuading readers of the force of Representative Gillett's 8 January 1901 statement: "It has been abundantly proved that mathematics cannot determine any apportionment which shall be universally fair and equal." ([**10**], p. 742.) Most seriously, EP does not satisfy quota. In some examples it accords more than rounding the exact quota q_i up, in others less than rounding the exact quota down. While explicitly recognized by the many proponents of EP, this flaw was conveniently painted over. It is for them fortunate, indeed, that no census figures since 1930 have provided an example exhibiting the non-quota phenomenon. It is also fortunate for EP that no careful investigation has heretofore been made of how badly non-quota EP solutions can become, but more on this point anon. Furthermore, there are other natural definitions of a measure T of the inequality between two states besides the relative inequality. There is nothing sacred about Huntington's notion. For example, why not consider $p_i/a_i - p_j/a_j$ or $a_j/p_j - a_i/p_i$ or $a_j - a_i(p_j/p_i)$ or $a_j(p_i/p_j) - a_i$, where j is in each case the advantaged state? Each of these leads to a different priority list method and to a different divisor test method. And still other tests may yield still different methods.

In general, let $r(p, a)$ be any real valued function of two real variables called a **rank-index** (possibly including $\pm\infty$ for certain values of p and a). Given a rank-index, a **Huntington method** $\boldsymbol{M}$ of apportionment is the set of all solutions obtained recursively as follows:

(i) $f_i(\boldsymbol{p}, 0) = 0, \quad 1 \leqq i \leqq s$

(ii) If $a_i = f_i(\boldsymbol{p}, h)$ is an apportionment for h of $\boldsymbol{M}$, and k is some one state for which $r(p_k, a_k) \geqq r(p_i, a_i)$ for $1 \leqq i \leqq s$, then,

$$f_k(\boldsymbol{p}, h+1) = a_k + 1, \quad f_i(p_i, h+1) = a_i \quad \text{for} \quad i \neq k.$$

By definition, any Huntington method is house-monotone (avoids the Alabama paradox). Moreover, each of the measures of inequality listed above yields a Huntington method. In fact, in trying various difference measures [**13**] it was found that either a measure does not guarantee the existence of a stable solution or one of five distinct methods result, one of which is equal proportions, one of which is Jefferson's, and one of which is Webster's. These five are commonly referred to as the "modern workable methods," because they avoid the Alabama paradox (see Table 3).

Instead of focusing on a rank-index, one can take the divisor test idea and generalize it to produce a class of methods instead. Let $d(a)$, called a **divisor criterion**, be any real-valued monotone-increasing function of the one variable a with $d(0) \geqq 0$, and $\lim_{a\to\infty} d(a) = \infty$. Given a divisor criterion, a **divisor method** $\boldsymbol{M}$ of apportionment is the set of solutions obtained as follows: Given h, for each λ, $0 < \lambda \leqq \infty$, let $a_i(\lambda)$ be the smallest integer satisfying $d(a_i(\lambda)) \geqq p_i/\lambda$. Choose λ so that $\Sigma_1^s a_i(\lambda) = h' \leqq h$ and, for all sufficiently small $\varepsilon > 0$, $\Sigma_1^s a_i(\lambda - \varepsilon) = h'' > h$. Let

$$E(\lambda) = \{i;\, d(a_i(\lambda)) = p_i/\lambda\}, \quad |E(\lambda)| = h'' - h' \geqq 1.$$

If $h'' - h' = \delta > 1$ then order (arbitrarily) the elements of $E(\lambda)$, and let $E_\alpha(\lambda)$ be the first α of the elements of $E(\lambda)$ $(= E_\delta(\lambda))$. Then

$$f_i(\boldsymbol{p}, h') = a_i(\lambda), \quad 1 \leqq i \leqq s;$$

and, for $h' + \alpha < h''$,

$$f_i(\boldsymbol{p}, h'+\alpha) = a_i(\lambda) + 1 \quad \text{for} \quad i \in E_\alpha(\lambda),$$
$$= a_i(\lambda) \quad \text{otherwise}.$$

Clearly, any divisor method is house-monotone. In fact, any divisor method is a Huntington method, as is easily seen. The rationale for a divisor criterion is this: the numbers p_i/λ "should" be proportional to the numbers of seats received by the states, but because of the integer problem, the specific sense of this proportionality is interpreted through the particular divisor criterion chosen. If we take $d(a)=\sqrt{a(a+1)}$ then we obtain the method of equal proportions. Jefferson's method is obtained with $d(a)=a+1$, and Webster's with $d(a)=a+\frac{1}{2}$. In fact, these are three of the five so-called "modern workable methods," all of which are divisor methods. Table 2 lists the five methods, their various names, the measures of difference or stability criteria, rank-index and divisor criteria associated with each. Table 3 gives, for the example of Table 1, the unique apportionment for $h=26$ obtained by each of the five methods in question.

Method	Stable for test T (where $p_i/a_i \geqq p_j/a_j$)	Rank-index $r(p,a)$	Divisor criterion $d(a)$
Smallest Divisors (SD)	T_1: $a_j - a_i(p_j/p_i)$	p/a	a
Harmonic Mean (HM)	T_2: $p_i/a_i - p_j/a_j$	$p/\{2a(a+1)/(2a+1)\}$	$2a(a+1)/(2a+1)$
Equal Proportions (EP)	T_3: $(p_ia_j/p_ja_i)-1$	$p/\{a(a+1)\}^{\frac{1}{2}}$	$(a(a+1))^{\frac{1}{2}}$
Webster (W) (also known as Major Fractions)	T_4: $a_j/p_j - a_i/p_i$	$p/(a+\frac{1}{2})$	$a+\frac{1}{2}$
Jefferson (J) (also known as Greatest Divisors or d'Hondt)	T_5: $a_j(p_i/p_j)-a_i$	$p/(a+1)$	$a+1$

TABLE 2.

State	p	$q(p,26)$	SD	HM	EP	W	J
A	9061	9.061	9	9	9	9	10
B	7179	7.179	7	7	7	8	7
C	5259	5.259	5	5	6	5	5
D	3319	3.319	3	4	3	3	3
E	1182	1.182	2	1	1	1	1
	26000	26	26	26	26	26	26

TABLE 3.

Let two states in some apportionment problem have populations p^* and $\bar{p}$ with $p^*>\bar{p}$. Suppose that $f'\in \boldsymbol{M}'$ accords a^* seats to the star-state and $\bar{a}$ seats to the bar-state at some house size h', and that $f''\in \boldsymbol{M}''$ accords a total of $a^*+\bar{a}$ seats to this pair of states at some house h''. Then f'' **favors the large state over** f' if it accords at least a^* seats to the star-state at h'', for any such choice of p^*, $\bar{p}$, h' and h''. A method $\boldsymbol{M}''$ **favors large states over** $\boldsymbol{M}'$ if any solution $f''\in \boldsymbol{M}''$ favors the large state over any $f'\in \boldsymbol{M}'$. Table 4 suggests that the "modern workable methods" are listed, in Table 3, in the order of increasing favoritism to large states, SD tending to most favor small states, J to most favor large states. This is, in fact, the case and can be verified by using the following theorem.

THEOREM 1. *Let $\boldsymbol{M}'$ and $\boldsymbol{M}''$ be methods determined by divisor criteria d' and d'' respectively where, for all integers $a>b$ $(\geqq 0)$, $d''(a)/d''(b)<d'(a)/d'(b)$. Then $\boldsymbol{M}''$ favors large states over $\boldsymbol{M}'$.*

Proof: Suppose $p^* > \bar{p}$ and some $f' \in M'$ accords the star-state a^* seats and the bar-state $\bar{a}$ seats. Then, $d'(\bar{a}) \geqq \bar{p}/\lambda' \geqq d'(\bar{a}-1)$ and $d'(a^*) \geqq p^*/\lambda' \geqq d'(a^*-1)$ implying $d'(a^*-1)/d'(\bar{a}) \leqq p^*/\bar{p}$. Suppose, contrary to what is to be shown, that for some $f'' \in M''$ the bar-state is accorded $\bar{a}+k$ seats and the star-state a^*-k seats with $k \geqq 1$. Then

$$d''(\bar{a}+k) \geqq \bar{p}/\lambda'' \geqq d''(\bar{a}+k-1) \quad \text{and} \quad d''(a^*-k) \geqq p^*/\lambda'' \geqq d''(a^*-k-1)$$

implying $d''(a^*-k)/d''(\bar{a}+k-1) \geqq p^*/\bar{p}$. Since d'' is monotone the derived inequalities imply

$$d''(a^*-1)/d''(\bar{a}) \geqq d''(a^*-k)/d''(\bar{a}+k-1) \geqq d'(a^*-1)/d'(\bar{a}),$$

contradicting the condition of the theorem.

For example, compare EP and J. If $a > b$ then $(a+1)/(b+1) < \{a(a+1)\}^{\frac{1}{2}}/\{b(b+1)\}^{\frac{1}{2}}$ as is easily verified. Thus J favors large states over EP.

Why choose one stability criterion rather than another? Why one rank-index than another? Why one divisor criterion than another? Contrast, for example, SD, EP, W and J via still other characterizations (closely related to the divisor criteria), where for simplicity it is assumed no "ties" occur. An SD apportionment for h is gotten by choosing a λ such that if $a_i = \lceil p_i/\lambda \rceil$ then $\Sigma a_i = h$; an EP apportionment for h by choosing a λ such that if $a_i = \lfloor \{p_i^2/\lambda^2 + \frac{1}{4}\}^{\frac{1}{2}} + \frac{1}{2} \rfloor$ then $\Sigma_i a_i = h$; a W apportionment h by choosing a λ such that if $a_i = \lfloor p_i/\lambda + \frac{1}{2} \rfloor$ then $\Sigma_i a = h$; and a J apportionment for h by choosing a λ such that if $a_i = \lfloor p_i/\lambda \rfloor$ then $\Sigma_i a_i = h$. Viewed in this manner EP is a most peculiar choice of method: both W and J appear to be more natural. But the essential problem with the approach is: there is no *a priori* justification for choosing one test or measure of inequality over another.

What then happened in the 1920's? Repeated attempts to reapportion were defeated. Some 42 bills shared this fate through 1928. Finally, on 18 June 1929, an "automatic" apportionment act was accepted by Congress. Broadly, it provided that the President would send to Congress, together with the apportionment population of each state based on the census figures, the apportionments for a membership equal in number to the existing number of Representatives in the House (435) obtained by (i) the method used in the preceding apportionment, (ii) the Webster method (called major fractions) and (iii) the equal proportions method. As it happened, of course, this meant that (i) and (ii) would be one and the same for the census of 1930. The major force behind the automatic apportionment act was Senator Arthur H. Vandenberg of Michigan who not only spoke in Congress but also, in 1929, addressed the nation by radio on the essential democratic need for a reapportionment based upon the census, and, to avoid a repetition of the experience of the 1920's, for the automatic provision. In fact, Vandenberg favored W over EP, and thus sided with Dr. Willcox. However, the weight of "scientific" advice to Congress supported EP. At the request of the Speaker of the House, Nicholas Longworth, the National Academy of Sciences prepared and submitted a report dated 7 February 1929 signed by lions of the mathematical community, G. A. Bliss, E. W. Brown, L. P. Eisenhart and Raymond Pearl [7]. They gave what now appears to be the traditional argument for accepting EP. First, "there are five methods of apportionment now known which are unambiguous (that is, lead to a workable solution), and should be considered at this time... In the present state of knowledge your committee regards these as the only methods of apportionment avoiding the so-called Alabama paradox which require consideration at this time." Second, "... [HM] and [W] are symmetrically situated on the list. Mathematically there is no reason for choosing between them. A similar symmetry exists for [SD] and [J] for which the defining discrepancies seem, however, more artificial than those for any one of the other three methods ... but [EP] satisfies the [relative difference test] when applied either to sizes of congressional districts or to numbers of Representatives per person." Concluding, "[EP] is preferred by the committee because ... it occupies mathematically a neutral position with respect to emphasis on larger and smaller states."

Paraphrased the argument is: (1) there are five known house-monotone methods, (2) of these, EP satisfies a test which seems to be preferable to others; and (3) EP "occupies the central position among the five methods" ([**13**], p. 103). A 1948 National Academy of Sciences Report [16], this time co-authored by Marston Morse, John von Neumann and Luther P. Eisenhart, furthered sustained EP as the best compromise and buttressed this choice with an additional argument. If the tests T_2, T_3 and T_4 are accepted as the most natural ones of the five, then EP can be measured by them as against each of the other four methods. Clearly HM is always best by T_2; EP is always best by T_3; W is always best by T_4. The authors of the 1948 report introduce the following: a method M is said to be **T-superior** to M' if for every pair of states and for any populations the measure of different T cannot be made smaller by solutions of M' than by solutions of M. They state that EP is T-superior to W for $T = T_2, T_3$; EP is T-superior to SD for $T = T_3, T_4$; EP is T-superior to HM for $T = T_3, T_4$; and EP is T-superior to J by $T = T_2, T_3$. They then conclude: "The committee is unaware of any new method which has been explicitly developed in workable detail since 1920 which goes beyond the five methods discussed above...[By the T-superior criteria] the total score in favor of EP against the other methods is decisive."

Senator Vandenberg argued otherwise. In a 2 March 1929 letter to Huntington he explained: "The basic problem is not mathematical at all ... I contend as a constitutional axiom that ... a group of individuals should have as nearly as may be the same weight in choosing Representatives in the House whether they happen to live in the large States or the small States. Doctor Willcox declares that [W] is the only method in the long run that secures this end ... I assume you will consent that I am entitled to rely upon his statements of abstract fact ... Supporting [his] view is the testimony of such men as ... Professor Charles K. Burdick, dean of the Cornell University Law School, Professor J. S. Hall, dean of the University of Chicago Law School, Professor Max Farrand, former professor of American History at Yale ... There is constitutional warrant for [W] ... I stood for [W] ... then came the unfortunate detour. Quarelling over mathematics the Senate once more permitted the basic constitutional mandate to be given another anesthetic ... The contest will be renewed in the approaching extra session ... I will frankly say to you that I am perfectly willing to treat [the choice of methods] from the standpoint of expediency and to take whichever method will best win ... a majority" ([**11**], pp. 4964–4965). Thus the compromise resulting in apportionments computed by both EP and W. The choice was agreeable for the 1930 census: the two apportionments for 435 were identical.

But, in 1940, through a quirk of time limitations written into different bills, it was impossible to fulfill the conditions of the apportionment act of 1929: the census figures could not be delivered in time. Discussion flared again. Professor Willcox, perhaps piqued, but certainly unreliable in statements of abstract facts, said, on 28 February 1940, " ... if the main purpose of apportionment is to make the average population of congressional districts as nearly equal as possible, that purpose is best served by [SD]." ([**3**], p. 16.) If he meant apportionments which are T_2-stable, he was wrong, since such belong to HM. If he meant an apportionment for h solving the problem

$$\min_{a_i} \max_i |p_i/a_i - p_j/a_j| \quad \text{when} \quad \Sigma_i a_i = h, \ a_i \geqq 0 \quad \text{integer}$$

he was wrong again, for such apportionments do not specify a house-monotone method (see, for example, [17], p. 82).* On 29 February 1940 Willcox declared, before the same committee, "That is my reason for favoring the major-fractions idea, so that every fraction larger than one-half will entitle a state to an extra Representative." ([**3**], p. 37.) Almost one year later, on 27 February 1941 he stated "It is my conviction that the mathematical aspects of apportionment have been greatly exaggerated ... the first and most important reason [for rejecting EP] is the difficulty in understanding

*This same approach was advocated in Oscar R. Burt and Curtis C. Harris, Jr., "Apportionment of the U.S. House of Representatives: a minimum range, integer solution, allocation problem," *Operations Research*, 11(1963) 648–652. The fact that it admits the Alabama paradox was pointed out in E.J. Gilbert and J.A. Schatz, "An ill-conceived proposal for apportionment of the U.S. House of Representatives," *Operations Research*, 12(1964) 768–773.

it ... [Senator Vandenberg] did state that the correspondence he had had with the advocates of [EP] had given him a chronic headache." ([**4**], pp. 15 and 20.) Senator Vandenberg, in continuing hearings on H.R. 2665, declared: "When I came to the Senate 13 years ago, the question of reapportionment was a flaming issue ... the question before this committee is whether, for the sake of controlling the specific seat ... the practice of automatic reapportionment ... shall be upset ... Arkansas would not be in love with [EP] on account of [EP]. It is in love with [EP] at the moment because it involves a seat in the House. This is the vice of the situation every 10 years ... the very purpose of this automatic reapportionment law is to protect the Constitution against political appetites ... It is for these reasons that I oppose the House bill and ask for a defense of the only formula ever devised to guarantee the validity of that section of the Constitution ... " ([**4**], pp. 48–50). The Senator from Michigan should be excused his vehement defense of W; the only difference between the EP and W apportionments for 435 according to the census figures of 1940 was that EP gave Arkansas 7 and Michigan 17, whereas W gave Arkansas 6 and Michigan 18.

On 15 November 1941 President Franklin D. Roosevelt signed "An Act to Provide for Apportioning Representatives in Congress among the several States by the equal proportions method" (Public Law 291, H.R. 2665, 55 Stat 761) which also fixed the size of the House at 435.* Commenting on this event and on the previous debate concerning it, students of the Yale Law School said in 1949 ([**22**, p. 1382):
"Despite the mathematical superiority of [EP], it would be naive to assume that its continued use by Congress is assured ... Congress [in 1941] chose ... to give the extra seat to Arkansas, and thus to use [EP]. But the decision had nothing whatsoever to do with the mathematical or logical soundness of [EP]. Arkansas is usually a safe Democratic state; Michigan's normal leanings are Republican. Every Democrat in Congress, except those from Michigan, voted for [EP] ... Every Republican voted for [MF] ... There were more Democrats ... than Republicans. Thus [EP]." One can only hope that lawyers may regain some naivete, at least if they become Representatives, so that, indeed, as Representative Ernest W. Gibson of Vermont stated on 10 January 1929 ([**11**], p. 1500), "The apportionment of Representatives to the population is a mathematical problem. Then why not use a method that will stand the test under a correct mathematical formula?"

In fact, it seems that no other serious debate has arisen concerning which method should be used or how many members the House should have. Willcox, in 1952, wrote his "Last words on the apportionment problem" ([**21**]), continuing to argue that the problem is not mathematical but political *and* that SD should be adopted. There appears to have been no other challenge since then.

Incredibly, all modern contributors and commentators simply disregard the fact that *EP does not satisfy quota.* "Now it is a common misconception that in a good apportionment the actual assignment should not differ from the exact quota by more than one whole unit." ([**13**], p. 94.) "The proper apportionment ... may differ by several units from the number obtained by simple proportion." ([**7**].) Although the fact was recognized, the implicit inference from the examples illustrating it ([**13**], p. 96) is that the event is rare and at worst minor in magnitude. But, "as a proper method of apportionment must meet every conceivable variation in population no matter how fantastic" ([**17**], p. 73), consider the two examples of Table 4 which are very much in the tradition of the many lovely examples given by Huntington [**13**]. In the first example, the EP apportionment "rounds" the exact quota of state 1 up by more than $6\frac{1}{2}$; in the second, it "rounds" the exact quota of state 1 down by more than $6\frac{1}{2}$. These are fantastic artificial examples. But in Table 5 is given the census populations for the 50 states in 1960, 1970 and two hypothetical projections of populations to the year 1984 (1984 A and 1984 B). The exact quotas and equal proportions apportionments are given for each state in all three cases. In the 1984A EP apportionment four of the five largest states receive more Representatives than their upper quotas. California receives 45 while its exact quota is 42.960; New York 42 while its exact quota is 39.939; Pennsylvania 26 while its exact quota is 24.974; and Texas 25 while its exact quota is 23.952. In the 1984 B EP apportionment the reverse is true, every one of the

*In 1959, Alaska and Hawaii were admitted to the Union, each receiving one seat, thus temporarily raising the House to 437. The apportionment based on the census of 1960 reverted to a House size of 435.

State	p	$q(p, 102)$	f by EP	State	p	$q(p, 98)$	f by EP
1	60,272	61.477	68	1	68,010	66.650	60
2	1,226	1.251	1	2	1,590	1.558	1
3	1,227	1.252	1	3	1,591	1.559	1
'	'	'	'	4	1,592	1.560	2
'	'	'	'	5	1,593	1.561	2
31	1,255	1.280	1	'	'	'	'
32	1,256	1.281	2	'	'	'	'
33	1,257	1.282	2	21	1,609	1.577	2
($s = 33$)	100,000	102	102	($s = 21$)	100,000	98	98

TABLE 4.

five largest states receives less than its lower quota. In fact, examples may be constructed, using different numbers of states s and different house sizes h, to show that EP can give apportionments with delegations *arbitrarily* far off exact quota. This possibility surely makes EP unacceptable. Zecharia Chaffee, Jr., the constitutional authority, pointed out that "the preservation of a respect for the law will in the long run be best obtained by the adoption of the plan which is least likely to produce a sense of unfairness in those who are forced to obey legislation" ([**8**], pp. 1043–1044).

Notice, moreover, that both censuses of 1984 have the same total population, that in 1984 B California has a slightly higher population but *four fewer* seats by EP. Thus, in addition to all the other objections EP is very unstable: small shifts in population can lead to large shifts in the apportionment.

To Jefferson, Washington, Hamilton and other early writers on apportionment the idea of a method satisfying quota was so natural that they could not even imagine a method not having this property. As in the case of the Alabama paradox, the possibility of a non-quota method was so inconceivable that an instance had to arise before the possibility was recognized. This actually occurred in a proposed apportionment bill of 1832, and Daniel Webster at once pointed out its absurdity in his speech to the Senate on April 5, 1832: "The House is to consist of 240 members. Now, the precise portion of power, out of the whole mass presented by the number of 240, to which New York would be entitled according to her population, is 38.59; that is to say, she would be entitled to thirty-eight members, and would have a residuum or fraction; and even if a member were given her for that fraction, she would still have but thirty-nine. But the bill gives her forty ... for what is such a fortieth member given? Not for her absolute numbers, for her absolute numbers do not entitle her to thirty-nine. Not for the sake of apportioning her members to her numbers as near as may be because thirty-nine is a nearer apportionment of members to numbers than forty. But it is given, say the advocates of the bill, because the *process* which has been adopted gives it. The answer is, no such process is enjoined by the Constitution". ([**19**], pp. 105–111.) Thus Webster clearly enunciates the principle that each state is *entitled* to at least the integer part of its exact quota, but *cannot* justifiably *receive* any *more* than its upper quota; in other words, any Constitutional apportionment method ought to have the quota property.

Does any Huntington method satisfy quota?

THEOREM 2. *There exists no Huntington method satisfying quota. Of the five "known workable" methods, only one, SD, satisfies upper quota; and only one, J, satisfies lower quota.**

*It should be pointed out that C.W. Seaton, Chief Clerk of the Census Office, independently devised the Jefferson method in his letter of 1881 (referred to earlier), but presented it in a different manner. He proposed that, in common with the Hamilton method, each state should first be given $\lfloor q_i \rfloor$, the integer part of its exact quota. Then, "it is my opinion that it is not the remainders, but rather the quotients which result from dividing the populations of the States by the increased number of Representatives, which should govern the allotment." [**1**]. That he really meant J, and not a generalized Hamilton method in which a state can receive at most one extra seat in this manner, is evidenced by the fact that in applying his method Ohio received two additional seats. This is the same as J since J necessarily satisfies lower quota.

	1960	1970	1984 A	q	EP	1984 B	q	EP
Alabama	3,266,740	3,475,885	3,659,293	7.198	7	3,608,877	7.099	7
Alaska	226,167	304,067	451,884	.889	1	329,928	.649	1
Arizona	1,302,161	1,787,620	2,184,366	4.297	4	1,885,014	3.708	4
Arkansas	1,786,272	1,942,303	2,176,565	4.282	4	1,948,052	3.832	4
California	15,717,204	20,098,863	21,839,542	42.960	45	21,944,556	43.167	41
Colorado	1,753,947	2,226,771	2,664,373	5.241	5	2,410,663	4.742	5
Connecticut	2,535,234	3,050,693	3,158,612	6.213	6	3,438,575	6.764	7
Delaware	446,292	551,928	685,196	1.348	1	802,199	1.578	2
Florida	4,951,560	6,855,702	7,081,224	13.929	14	7,671,182	15.090	15
Georgia	3,943,116	4,627,306	5,112,891	10.058	10	5,053,140	9.940	10
Hawaii	632,772	784,901	993,246	1.942	2	840,834	1.654	2
Idaho	667,191	719,921	691,063	1.359	1	804,232	1.582	1
Illinois	10,081,158	11,184,320	11,947,647	23.502	24	12,290,721	24.177	23
Indiana	4,662,498	5,228,156	5,610,014	11.035	11	5,570,655	10.958	11
Iowa	2,757,537	2,846,920	3,161,153	6.218	6	2,958,171	5.819	6
Kansas	2,178,611	2,265,846	2,675,456	5.263	5	2,456,416	4.832	5
Kentucky	3,038,156	3,246,481	3,657,104	7.194	7	3,465,010	6.816	7
Louisiana	3,257,022	3,672,008	4,140,835	8.145	8	3,968,799	7.807	8
Maine	969,265	1,006,320	1,078,588	2.122	2	1,298,870	2.555	3
Maryland	3,100,689	3,953,698	4,131,001	8.126	8	3,978,966	7.827	8
Massachusetts	5,148,578	5,726,676	6,085,436	11.971	12	6,086,136	11.972	12
Michigan	7,823,194	8,937,196	9,438,773	18.567	19	9,489,634	18.667	18
Minnesota	3,413,864	3,833,173	4,129,984	8.124	8	4,003,368	7.875	8
Mississippi	2,178,141	2,233,848	2,679,798	5.271	5	2,421,339	4.763	5
Missouri	4,319,813	4,718,034	5,123,214	10.078	10	5,108,552	10.049	10
Montana	674,767	701,573	691,146	1.360	1	773,730	1.522	2
Nebraska	1,411,330	1,496,820	1,643,502	3.233	3	1,834,178	3.608	4
Nevada	285,278	492,396	686,213	1.350	1	534,291	1.051	1
New Hampshire	606,921	746,284	908,754	1.788	2	842,868	1.658	2
New Jersey	6,066,782	7,208,035	7,573,756	14.898	15	7,676,299	15.100	15
New Mexico	951,023	1,026,664	1,182,655	2.326	2	1,313,105	2.583	3
New York	16,782,304	18,338,055	20,303,765	39.939	42	19,842,029	39.031	37
No. Carolina	4,556,155	5,125,230	5,614,931	11.045	11	5,552,320	10.922	11
No. Dakota	632,446	624,181	684,688	1.347	1	755,938	1.487	2
Ohio	9,706,397	10,730,200	11,437,560	22.499	23	11,735,587	23.085	22
Oklahoma	2,328,284	2,585,486	2,675,479	5.263	5	2,901,743	5.708	6
Oregon	1,768,687	2,110,810	2,182,157	4.293	4	2,385,753	4.693	5
Pennsylvania	11,319,366	11,884,314	12,696,129	24.974	26	12,799,259	25.138	24
Rhode Island	859,488	957,798	1,131,130	2.225	2	1,316,663	2.590	3
So. Carolina	2,382,594	2,617,320	2,674,982	5.262	5	2,905,301	5.715	6
So. Dakota	680,514	673,247	686,555	1.351	1	752,887	1.481	2
Tennessee	3,567,089	3,961,060	4,133,034	8.130	8	4,031,836	7.931	8
Texas	9,579,677	11,298,787	12,176,464	23.952	25	12,228,700	24.055	23
Utah	890,627	1,067,810	1,197,568	2.356	2	1,360,383	2.675	3
Vermont	389,881	448,327	660,279	1.299	1	485,488	.955	1
Virginia	3,966,949	4,690,742	5,098,449	10.029	10	5,026,197	9.887	10
Washington	2,853,214	3,443,487	3,648,182	7.176	7	3,519,403	6.923	7
W. Virginia	1,860,421	1,763,331	1,691,133	3.327	3	1,854,004	3.647	4
Wisconsin	3,951,777	4,447,013	4,631,008	9.110	9	4,519,866	8.891	9
Wyoming	330,066	335,719	571,638	1.124	1	376,698	.741	1
TOTALS	178,559,219	204,053,325	221,138,415	435	435	221,138,415	435	435

TABLE 5.

Proof: That no Huntington method satisfies quota is obtained as a corollary to Theorem 3, which is stated and proved in Section 5. Examples show SD, HM, EP and W do not satisfy lower quota, and that HM, EP, W and J do not satisfy upper quota. In fact, the examples of Table 4 suffice (see Table 10).

Thus EP, in particular, is an unsatisfactory method of apportionment: (1) it does not satisfy quota; (2) it rests upon an arbitrary definition of measure of inequality in representation between states; and (3) it is very unstable in that small shifts in populations can produce serious differences in state delegations. Huntington's original motivation to devise a method that avoids the Alabama paradox sacrificed the essential quota property.

5. The Quota Method. The obvious question is: does there exist a house-monotone method which satisfies quota? The answer is: yes. In fact, in this section it is shown that there is, subject to a certain consistency condition, only one such method, the "quota method."

Suppose that $\boldsymbol{M}$ is a house-monotone method. Then given a solution $f \in \boldsymbol{M}$ with $f_j(\boldsymbol{p}, h-1) = a$, the state j is said to be **eligible at h** for its $(a+1)$ st seat if $a < p_j h / \Sigma_i p_i = q_j(\boldsymbol{p}, h)$. In other words j is eligible at h for its $(a+1)$ st seat if it had a seats at $h-1$, and if it can receive the hth seat without exceeding upper quota.

Let p^* and $\bar{p}$ be the populations of some two states and suppose that by some solution $f \in \boldsymbol{M}$, where $\boldsymbol{M}$ is house-monotone, the star-state is eligible at some h for its (a^*+1) st seat and the bar-state is eligible at h for its $(\bar{a}+1)$ st seat, but f gives the hth seat to the star-state. Then the star-state is said to have **weak-priority** by $\boldsymbol{M}$ over the bar-state at p^*, $\bar{p}$, a^*, and $\bar{a}$. Since both states were eligible and the star-state received the extra seat its claim to the extra seat is certainly as good as that of the bar-state. A natural requirement for any method $\boldsymbol{M}$ is that the relative claims for an extra seat between two states should depend only upon their respective populations p^* and $\bar{p}$ and current apportionments a^* and $\bar{a}$. To be precise, suppose that the star-state has weak priority by $\boldsymbol{M}$ over the bar-state at p^*, $\bar{p}$, a^*, and $\bar{a}$. Let $\boldsymbol{g} \in \boldsymbol{M}$ be a solution for some population vector $\boldsymbol{q}$, which contains a pair of states having populations p^* and $\bar{p}$, and suppose these states are, respectively, eligible for their (a^*+1) st and $(\bar{a}+1)$ st seats at h', but that $\boldsymbol{g}$ gives the h'-th seat to the bar-state rather than the star-state. Then $\boldsymbol{M}$ is said to be **consistent** if $\boldsymbol{g}^{h'-1}$, the restriction of $\boldsymbol{g}$ up to $h'-1$, has an extension by which the h'-th seat *is* given to the star-state. That is, a method $\boldsymbol{M}$ is consistent if it never switches priorities at p^*, $\bar{p}$, a^*, $\bar{a}$ unless the two states have equal claim to the extra seat. Clearly, any Huntington method is consistent, for the claim to an extra seat is determined by the rank-index $r(p, a)$ which depends only upon the population and current apportionment of any state. In fact, any Huntington method is consistent even if the condition of eligibility is dropped. Given the concern with methods satisfying quota it is important to impose the eligibility requirement since, otherwise, apportionments violating the upper quota condition could be encountered. The set of all states eligible at h will be denoted $E(h)$.

The **quota method $\boldsymbol{Q}$** is the set of all solutions f obtained recursively as follows:

$$f_i(\boldsymbol{p}, 0) = 0 \qquad 1 \leqq i \leqq s;$$

and if $a_i = f_i(\boldsymbol{p}, h)$, and $k \in E(h+1)$ is some one state satisfying $p_k/(a_k+1) \geqq p_i/(a_i+1)$ for all $i \in E(h+1)$ then

$$f_k(\boldsymbol{p}, h+1) = a_k + 1, \quad f_i(\boldsymbol{p}, h+1) = a_i \quad \text{all} \quad i \neq k.$$

It is justifiable to name $\boldsymbol{Q}$ *the* quota method because it is the *unique* method satisfying house-monotonicity, consistency, and quota in this sense: if $\boldsymbol{Q}'$ is any other set of apportionment solutions satisfying these properties then $\boldsymbol{Q}' \subset \boldsymbol{Q}$. The method is very simple to apply so examples are postponed until later in the discussion.

THEOREM 3. *$\boldsymbol{Q}$ is the unique apportionment method which is house-monotone, consistent, and satisfies quota.*

Proof: The proof is in two parts. First, it is shown that Q satisfies the three properties claimed for it; second, uniqueness is established.

(i) PROPERTIES. By definition Q is house-monotone and consistent. Moreover, since no state receives a seat without being eligible Q satisfies upper quota. Thus it is only necessary to show that Q is also lower quota. To simplify notation abbreviate $f(p,h)$ by $f(h)$, and normalize the populations letting $\bar{p}_j = p_j/(\Sigma_1^s p_i)$, for all j.

Suppose Q is not lower quota. Then, for some $p, f \in Q$ and house size h_0, there must exist a state j for which $f_j(h_0) \leqq \bar{p}_j h_0 - 1$. Since $\Sigma_i f_i(h_0) = h_0$, this implies that there exists a state l with $a_l = f_l(h_0) > \bar{p}_l h_0$, that is, whose apportionment for h_0 is at upper quota. Therefore,

$$\bar{p}_l / f_l(h_0) < 1/h_0 \leqq \bar{p}_j / (f_j(h_0) + 1). \tag{3}$$

Let h_l be the house size at which state l received its last or a_l-th seat. State l may be chosen so that h_l is largest among all states l with apportionments for h_0 at upper quota. Note that $h_l = h_0$ is impossible because j is eligible at h_0 and, by (3), would receive its $(f_j(h_0)+1)$ st seat before l received its $f_l(h_0)$ th seat. Therefore $h_l < h_0$. Let $K \neq \emptyset$ be the set of states receiving additional seats at house sizes h in the interval $h_l < h \leqq h_0$. State l cannot be eligible in this interval, so $l \notin K$. For any $k \in K$ it is impossible that $f_k(h_0) > \bar{p}_k h_0$ because then $h_k > h_l$, contradicting the choice of l. Hence

$$f_k(h_0) \leqq \bar{p}_k h_0 \quad \text{for} \quad k \in K. \tag{4}$$

But, since $f_k(h_l) < f_k(h_0)$ for all $k \in K$,

$$\bar{p}_k / (f_k(h_l) + 1) \geqq \bar{p}_k / f_k(h_0) \geqq 1/h_0 > \bar{p}_l / f_l(h_0) = \bar{p}_l / f_l(h_l) \quad \text{for} \quad k \in K.$$

This means that every $k \in K$ must have been ineligible at h_l, $K \cap E(h_l) = \emptyset$, for, otherwise, one of these states would have been given the h_l-th seat by Q. Thus,

$$f_k(h_l) = f_k(h_l - 1) \geqq \bar{p}_k h_l \quad \text{for} \quad k \in K. \tag{5}$$

In the interval $h_l < h \leqq h_0$ exactly $h_0 - h_l$ seats were awarded to the states in K, so $\Sigma_k \{f_k(h_0) - f_k(h_l)\} = h_0 - h_l$. Subtracting (5) from (4) and then summing over K

$$h_0 - h_l = \Sigma_k \{f_k(h_0) - f_k(h_l)\} \leqq \Sigma_k \bar{p}_k (h_0 - h_l).$$

But $h_0 - h_l > 0$ implies $\Sigma_k \bar{p}_k \geqq 1$, a contradiction, since K is a subset of all states, $\Sigma_1^s \bar{p}_i = 1$, $l \notin K$ and $\bar{p}_l > 0$. Therefore, Q satisfies lower quota and so Q satisfies quota. This completes the first part of the proof.

(ii) UNIQUENESS. Let Q' be any set of solutions satisfying all properties and suppose it is not contained in Q. Then there must exist a solution $f \in Q' \sim Q$ for some problem p. This means there is a house h, and a pair of states i and j, say with populations $p_i = p^*$ and $p_j = \bar{p}$ and apportionments $f_i(p,h) = a^*$, $f_j(p,h) = \bar{a}$, both eligible at $h+1$, $i, j \in E(h+1)$, and $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, but (contrary to Q) $f_j(p, h+1) = \bar{a}+1$. Since f satisfies quota, $p^*(h+1)/\Sigma_1^s p_i < a^*+1$ and $\bar{p}(h+1)/\Sigma_1^s p_i > \bar{a}$, implying

$$\bar{p}/\bar{a} > p^*/(a^*+1). \tag{6}$$

Fix the populations p^* and $\bar{p}$ and consider all choices of population vector p in which some two states have populations p^* and $\bar{p}$ respectively, and h, a^*, $\bar{a}$ and $f \in Q'$ are as assumed above. Among these choose a situation for which $a^* + \bar{a} = \lambda$ is a minimum. In other words, single out a "first" occurrence in which a solution violates the conditions of Q. We derive a contradiction from this hypothesis by induction on λ.

Suppose $\lambda = 0$. Then $p^* > \bar{p}$ but there is some solution of Q' by which a state with population $\bar{p}$ receives its first seat before a state with population p^* does. Consider, then, the problem having $t+1$ states and population vector $q = (p^*, \bar{p}, \cdots, \bar{p})$ where t is chosen to be any integer satisfying $t \geqq p^*/(p^* - \bar{p})$. For any $f \in Q'$ let h_f be the largest house for which $f_1(q, h_f) = 0$, and suppose that $h_f < t$. Since there are $t+1$ states, $f_j(q, h_f + 1) = 0$ for some j, $2 \leqq j \leqq t+1$. Therefore, states 1 and j are eligible for their first seats at $h_f + 1$ but f gives state 1 the $(h_f + 1)$st seat. Therefore, by the consistency of Q', there is an extension of f^{h_f} which, instead, gives that seat to state j. This, of course, can be repeated and so this means we can assume $h_f \geqq t$. But, then, the lower quota of state 1 at h_f is

$$p^* h_f/(p^* + t\bar{p}) \geqq p^* t/(p^* + t\bar{p}) \geqq (p^* + t\bar{p})/(p^* + t\bar{p}) = 1,$$

by choice of t, implying f is not (lower) quota at h_f, a contradiction.

Suppose, then, that $\lambda = a^* + \bar{a} > 0$. As before, form the problem $q = (p^*, \bar{p}, \cdots, \bar{p})$ having $t+1$ states where t is now any integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1) - \bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^* + t\bar{p}(a^*+1).$$

Since, $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, by assumption, t is positive. Consider a house $h' = a^* + t(\bar{a}+1)$. Then the exact quota of state 1 at h' satisfies

$$\text{(7)} \qquad \frac{p^* h'}{p^* + t\bar{p}} = \frac{p^*(a^* + t(\bar{a}+1))}{p^* + t\bar{p}} = \frac{a^* p^* + p^* t(\bar{a}+1)}{p^* + t\bar{p}} \geqq \frac{a^* p^* + p^* + t\bar{p}(a^*+1)}{p^* + t\bar{p}}$$
$$= a^* + 1.$$

For each $f \in Q'$ let h_f be the largest house such that $f_1(q, h_f) = a^*$, and among these solutions choose one f so that $h_{\bar{f}}$ is largest among the h_f. Since $\bar{f}$ is quota, (7) implies that $h_{\bar{f}} < h'$. Either (a) the exact quota of state 1 at $h_{\bar{f}} + 1$ is at least $a^* + 1$ or (b) it is less than $a^* + 1$.

(a) Suppose $p^*(h_{\bar{f}} + 1)/(p^* + t\bar{p}) \geqq a^* + 1$. By (6) this implies $\bar{p}(h_{\bar{f}} + 1)/(p^* + t\bar{p}) > \bar{a}$. $\bar{f}$ is quota so each state i, $2 \leqq i \leqq t+1$, has at least $\bar{a}$ seats at $h_{\bar{f}} + 1$; on the other hand, $h_{\bar{f}} < h' = a^* + t(\bar{a}+1)$ so at least one of these states, say j, must have at most $\bar{a}$ seats, hence exactly $\bar{a}$ seats, at $h_{\bar{f}} + 1$. This state is eligible for its $(\bar{a}+1)$-st seat at $h_{\bar{f}} + 1$, whereas, by construction, state 1 received its (a^*+1)-st seat at $h_{\bar{f}} + 1$. By consistency with the original hypothesis there must, therefore, exist an extension of $\bar{f}^{h_{\bar{f}}}$ that accords the $(h_{\bar{f}} + 1)$-st seat to state j instead of state 1. But this contradicts the choice of $\bar{f}$.

(b) Suppose $p^*(h_{\bar{f}} + 1)/(p^* + t\bar{p}) < a^* + 1$. This means state 1 is at its upper quota at $h_{\bar{f}} + 1$ so some state j, $2 \leqq j \leqq t+1$, must be at its lower quota, call it a', where $a' \leqq \bar{a}$. If $a' = \bar{a}$ then by consistency there must exist an extension of $\bar{f}^{h_{\bar{f}}}$ that accords the $(h_{\bar{f}} + 1)$-st seat to j instead of to 1, again contradicting the choice of $\bar{f}$. Therefore, $a' < \bar{a}$ whence $a^* + a' < \lambda$. Moreover, consider states 1 and j: they have a^* and a' seats, respectively, at house $h_{\bar{f}}$, are both eligible at $h_{\bar{f}} + 1$, and $\bar{f}$ gives to state 1 the $(h_{\bar{f}} + 1)$-st seat, whereas, from (6)

$$\bar{p}/(a'+1) \geqq \bar{p}/\bar{a} > p^*/(a^*+1).$$

This contradicts the inductive hypothesis on λ.

This completes the proof of uniqueness and establishes Theorem 3.

6. The Quota Method with Minimum Requirements. The preceding section considered the "pure" apportionment problem where no requirements are placed on the minimum number of representatives. However, the Constitution specifies that each State have at least one Representative and it is obvious that an apportionment method satisfying quota cannot in general meet this requirement. For example, in a house of 50 seats the 1970 exact California quota is 4.927. Certain other systems have

different minimum requirements: for example, France requires a minimum of 2 "députés" per "departément." This section broadens the formulation of the apportionment problem to explicitly include the possibility of minimum requirements different from zero. We shall show that Theorem 3 and the quota method have natural generalizations which coincide with the preceding results when the minimum requirements are zero.

In this broader view the data of the problem are the (positive integer) populations of s states $\boldsymbol{p}=(p_1,\cdots,p_s)$ and nonnegative integer **requirements** $\boldsymbol{r}=(r_1,\cdots,r_s)$, with r_i the minimum number of representatives which can be given state i in any admissible apportionment. Clearly, there are no admissible apportionments for any house size less than the **minimum house** $h^0=\Sigma_1^s r_i$. The problem is to find, for each house size $h \geqq h^0$ an **apportionment for h**: an s-tuple of integers $(a_1,\cdots,a_s)$, with $a_i \geqq r_i$ all i and $\Sigma_1^s a_i = h$. A **solution** of the apportionment problem with requirements is a function f which to every $\boldsymbol{p}$, $\boldsymbol{r}$ and $h \geqq h^0$ associates a unique apportionment for h, $a_i = f_i(\boldsymbol{p},\boldsymbol{r},h) \geqq r_i$, $1 \leqq i \leqq s$ and $\Sigma_1^s a_i = h$. An **apportionment method** with requirements is a set of apportionment solutions as here defined. **House-monotonicity** and **consistency** are as defined before (for $h \geqq h^0$, $a_i \geqq r_i$), with the **set of states eligible at** h, $E(h)$, precisely the same.

However, as was pointed out above, it is impossible, in general, to ask for solutions satisfying quota. Thus this definition needs to be modified. A very natural extension of the quota idea can be made. Given $\boldsymbol{p}=(p_1,\cdots,p_s)$, $\boldsymbol{r}=(r_1,\cdots,r_s)$ and $h \geqq \Sigma_1^s r_i = h^0$ define the **(generalized)* upper quota** $u_i = u_i(\boldsymbol{p},\boldsymbol{r},h)$ of state i to be the maximum of the previously defined upper quota and r_i,

$$u_i = \max\{r_i, \lceil p_i h/(\Sigma_1^s p_j)\rceil\}.$$

Generalizing lower quota is slightly more involved. Suppose that the exact quota $p_i h/(\Sigma_1^s p_j)$ of state i at h is less than or equal to r_i. Then state i certainly deserves no more than r_i seats, while it is required to have at least r_i seats. A fair method would, therefore, allot to i exactly r_i seats. Subtracting such seats from h there is left a smaller house which is to be allocated to the remaining states. Using this smaller house compute the exact quotas for the remaining states and give r_i to any whose exact quota is at most r_i, and so forth.

Define, then, $J_0 = J_0(h) = \{1,\cdots,s\}$ to be the set of all states, and let $h_0 = h\ (\geqq h^0)$. As suggested by the above reasoning, define also $J_1 = J_1(h) = \{i \in J_0; p_i h_0/(\Sigma_{J_0} p_j) > r_i\}$ and $h_1 = h_0 - \Sigma_{i \notin J_1} r_i$. Any state $i \in J_1$ deserves $p_i h_1/(\Sigma_{J_1} p_j)$ seats, so if this number is at most r_i then i should receive precisely r_i seats. Thus, let $J_2 = J_2(h) = \{i \in J_1; p_i h_1/(\Sigma_{J_1} p_j) > r_i\}$ and $h_2 = h_0 - \Sigma_{i \notin J_2} r_i$, and so on. This produces, for each h, a nested sequence $J_0(h) \supset J_1(h) \supset \cdots \supset J_\mu(h)$ of sets with house sizes $h = h_0 > h_1 > \cdots > h_\mu$ such that for all $i \in J_\mu(h)$, $p_i h_\mu/(\Sigma_{J_\mu} p_j) > r_i$.

It is convenient to note several relationships at this point. By definition, for all α, $0 \leqq \alpha > \mu$, $p_i h_\alpha/(\Sigma_{J_\alpha} p_j) \leqq r_i$ for $i \in J_\alpha \sim J_{\alpha+1}$.
Therefore

$$h_{\alpha+1} = h_\alpha - \Sigma_{J_\alpha \sim J_{\alpha+1}} r_i \leqq h_\alpha - \frac{\Sigma_{J_\alpha - J_{\alpha+1}} p_i h_\alpha}{\Sigma_{J_\alpha} p_j} = \frac{h_\alpha \, \Sigma_{J_{\alpha+1}} p_i}{\Sigma_{J_\alpha} p_j}$$

and so,

$$h_{\alpha+1}/\Sigma_{J_{\alpha+1}} p_j \leqq h_\alpha/\Sigma_{J_\alpha} p_j. \tag{8}$$

The set $J_\mu(h)$ is uniquely defined as a function of $\boldsymbol{p}$, $\boldsymbol{r}$ and $h\ (\geqq h^0)$ and is called the *slack set* for h. The **(generalized) lower quota** $l_i(\boldsymbol{p},\boldsymbol{r},h)$ of state i at h is defined to be

$$\begin{aligned} l_i &= \lfloor p_i(h - \Sigma_{i \notin J_\mu} r_i)/\Sigma_{J_\mu} p_j \rfloor && \text{for } i \in J_\mu \\ &= r_i && \text{for } i \notin J_\mu(h). \end{aligned}$$

*In the sequel the modifier "generalized" will be omitted wherever no confusion with the "pure" ($\boldsymbol{r}=\boldsymbol{0}$) problem can arise.

Notice that if all requirements $r_i = 0$, the generalized upper and lower quotas are the same as the ordinary upper and lower quotas (in this case the slack set $J_\mu = J_0$, the set of all states). For clarity the upper and lower quotas are computed in Table 6, for $h = 26$ in the example of Table 1.

State	p	r	$q(26)$	$l(26)$	$u(26)$
A	9061	6	9.061	8	10
B	7179	6	7.179	6	8
C	5259	5	5.259	5	6
D	3319	4	3.319	4	4
E	1182	2	1.182	2	2
	26,000	$h^0 = 23$	26		

TABLE 6.

Therefore, the vector of upper quotas $\boldsymbol{u}(26)$ is as specified, $l_E = 2$, $l_D = 4$, $J_1 = \{A, B, C\}$ and $h_1 = 20$. Thus $q'_A(20) = (9061)(20)/21499 = 8.429$, $q'_B(20) = 6.678$, and $q'_C(20) = 4.892$ implying $l_C = 5$. Finally, $J_2 = \{A, B\}$, $h_2 = 15$, $q''_A(15) = (9061)(15)/(16240) = 8.369$, and $q''_B = 6.631$, so that $J_2 = J_\mu$, and $l_A = 8$, $l_B = 6$.

A generalized apportionment method $\boldsymbol{M}$ is said to **satisfy quota** if for all $f \in \boldsymbol{M}$ and for all $\boldsymbol{p}$, $\boldsymbol{r}$, and $h \geqq h^0 = \Sigma_1^s r_i$,

$$l_i(\boldsymbol{p}, \boldsymbol{r}, h) \leqq f_i(\boldsymbol{p}, \boldsymbol{r}, h) \leqq u_i(\boldsymbol{p}, \boldsymbol{r}, h) \qquad 1 \leqq i \leqq s.$$

Thus, a generalized apportionment method satisfying quota at $h = 26$ for the data of the example above would have to yield an apportionment $\boldsymbol{f}(26)$ for 26 satisfying $\boldsymbol{l}(26) \leqq \boldsymbol{f}(26) \leqq \boldsymbol{u}(26)$.

Assume the data $\boldsymbol{p}$, $\boldsymbol{r}$ of the problem satisfy the condition

$$\text{(9)} \qquad \text{if} \quad p_i \geqq p_j \quad \text{then} \quad p_i/r_i \geqq p_j/r_j.$$

Such problems will be said to have **unbiased requirements** $\boldsymbol{r}$. In other words, if state i is larger than or equal to state j in population, then state i's minimum allocation does not advantage it over state j's minimum allocation. This seems quite natural and is, of course satisfied in the usual case where the minimum requirements are the same for all states, $r_i = r$, $1 \leqq i \leqq s$. For data satisfying (9) the **(generalized) quota method** $\boldsymbol{Q}(\boldsymbol{r})$ is defined to be the set of all apportionment solutions f obtained recursively as follows:

$$f_i(\boldsymbol{p}, \boldsymbol{r}, h^0) = r_i, \qquad 1 \leqq i \leqq s;$$

and if $a_i = f_i(\boldsymbol{p}, \boldsymbol{r}, h)$, $h \geqq h^0$, and $k \in E(h+1)$ is some one state satisfying $p_k/(a_k+1) \geqq p_i/(a_i+1)$ for all $i \in E(h+1)$ then $f_k(\boldsymbol{p}, \boldsymbol{r}, h+1) = a_k + 1$, $f_i(\boldsymbol{p}, \boldsymbol{r}, h+1) = a_i$ for all $i \neq k$, ($E(h+1)$ is the set of eligible states as defined previously). The only difference between $\boldsymbol{Q}$ and $\boldsymbol{Q}(\boldsymbol{r})$ is that the latter begins by giving, to each state i, r_i seats in a house h^0, and otherwise continues as before. Clearly $\boldsymbol{Q}$ and $\boldsymbol{Q}(\boldsymbol{0})$ are identical. The unique $\boldsymbol{Q}(\boldsymbol{r})$ solution for the above example (see Table 6) and house sizes $23 \leqq h \leqq 28$ is shown in Table 7 (here $f(h)$ abbreviates $f(\boldsymbol{p}, \boldsymbol{r}, h)$).

State	p	$f(23)$	$f(24)$	$f(25)$	$f(26)$	$f(27)$	$f(28)$
A	9061	6	7	8	8	9	10
B	7179	6	6	6	7	7	7
C	5259	5	5	5	5	5	5
D	3319	4	4	4	4	4	4
E	1182	2	2	2	2	2	2
	26,000	23	24	25	26	27	28

TABLE 7.

Again it is justifiable to baptize $\boldsymbol{Q}(\boldsymbol{r})$ **the** quota method because it is, for unbiased requirements (9), the **unique** method which is house-monotone, consistent, and satisfies quota. (For biased requirements a unique method still obtains but its definition is not quite so straightforward, for the eligible set of states at $h+1$ must be taken as $E(h+1)\cap J_\mu(h+1)$ rather than simply $E(h+1)$.) This will now be established via arguments which closely parallel those for $\boldsymbol{Q}(\boldsymbol{0})$. First, it is shown that $\boldsymbol{Q}(\boldsymbol{r})$ never gives more than r_i seats to any state i whose "adjusted exact quota" is at most r_i, (i.e., to any state not in the slack set).

LEMMA 1. *If $f\in \boldsymbol{Q}(\boldsymbol{r})$ for $\boldsymbol{r}$ unbiased then $f_i(\boldsymbol{p},\boldsymbol{r},h)=r_i$ for $i\not\in J_\mu$, where J_μ is the slack set for h.*

Proof: Given $\boldsymbol{p}$, $\boldsymbol{r}$ and $h\geqq h^0$, let $J_0\supset J_1\supset\cdots\supset J_\mu$ and $h=h_0>\cdots>h_\mu$ be defined as in the above construction. Assume, by way of contradiction, that $a_i=f_i(\boldsymbol{p},\boldsymbol{r},h)>r_i$ for some $i\not\in J_\mu$. This surely implies that i is not in the slack set $J_\mu(h')$ for any $h'\leqq h$, so it may be assumed that state i actually received the hth seat. Moreover, since $f\in\boldsymbol{Q}(\boldsymbol{r})$ is house-monotone it can be assumed that $a_i=r_i+1$. Since $i\not\in J_\mu$ there is an α, $0\leqq\alpha<\mu$ with $i\in J_\alpha\sim J_{\alpha+1}$ and

$$a_i-1=r_i\geqq(p_ih_\alpha/\Sigma_{J_\alpha}p_j). \tag{10}$$

By definition

$$h=\Sigma_{j\not\in J_\mu}r_j+\Sigma_{J_\mu}(p_jh_\mu/\Sigma_{J_\mu}p_k)$$

and, since $f\in\boldsymbol{Q}$, $a_j\geqq r_j$ for all j, so $a_i>r_i$ implies $a_k=f_k(\boldsymbol{p},\boldsymbol{r},h)<p_kh_\mu/\Sigma_{J_\mu}p_j$ for some $k\in J_\mu$. But this, in turn, implies by repeated use of (8) that for this state k

$$a_k<p_kh_\mu/\Sigma_{J_\mu}p_j\leqq p_kh_\alpha/\Sigma_{J_\alpha}p_j\leqq p_kh/\Sigma_1^sp_j. \tag{11}$$

Now (10) and (11) together yield that for the pair of states i and k, $p_i/(a_i-1)<p_k/a_k$. By (11), state k is eligible for its (a_k+1)th seat. But state i received the hth seat, and therefore, $p_i/a_i\geqq p_k/(a_k+1)$. These last two inequalities imply $p_i>p_k$. But $p_i/r_i=p_i/(a_i-1)<p_k/a_k\leqq p_k/r_k$, and this contradicts (9). This establishes Lemma 1.

THEOREM 4. *$\boldsymbol{Q}(\boldsymbol{r})$ is the unique apportionment method for unbiased requirements $\boldsymbol{r}$ which is house-monotone, consistent, and satisfies quota.*

Proof: First, it is established that $\boldsymbol{Q}(\boldsymbol{r})$ satisfies the requisite properties; second, it is shown to be unique.

(i) PROPERTIES. By definition $\boldsymbol{Q}(\boldsymbol{r})$ is house-monotone and consistent. Moreover, since no state receives a seat without being eligible $\boldsymbol{Q}(\boldsymbol{r})$ satisfies upper quota. Thus it is only necessary to show that it satisfies lower quota. To simplify notation abbreviate $\boldsymbol{f}(\boldsymbol{p},\boldsymbol{r},h)$ by $\boldsymbol{f}(h)$, and similarly for $\boldsymbol{l}$ and $\boldsymbol{u}$.

Suppose $\boldsymbol{Q}(\boldsymbol{r})$ is not lower quota. Then, for some $\boldsymbol{p}$, $\boldsymbol{r}$, $f\in\boldsymbol{Q}(\boldsymbol{r})$ and $h_0\geqq\Sigma_1^sr_i$ there must exist a state j for which $f_j(h_0)<l_j(h_0)$. By Lemma 1, $j\in J_\mu(h_0)=J_\mu$. Letting $\bar h_0=h_0-\Sigma_{i\not\in J_\mu}r_i$ and $\bar p_i=p_i/\Sigma_{J_\mu}p_k$, for each $i\in J_\mu$, this means $f_j(h_0)+1\leqq\lfloor\bar p_j\bar h_0\rfloor\leqq\bar p_j\bar h_0$. Since $\Sigma_{J_\mu}f_i(h_0)=\bar h_0$ this implies that there exists a state $l\in J_\mu$ which has more than its lower quota, that is, $a_l=f_l(h_0)>\bar p_l\bar h_0$. Thus

$$\bar p_l/f_l(h_0)<1/\bar h_0\leqq\bar p_j/(f_j(h_0)+1). \tag{12}$$

Notice also that for all $k\in J_\mu$

$$\bar p_k\bar h_0=p_k(h_0-\Sigma_{i\not\in J_\mu}r_i)/(\Sigma_{J_\mu}p_i)\leqq p_kh_0/(\Sigma_1^sp_i), \tag{13}$$

by repeated application of (8). In particular (12), and (13) with $k=j$, show that state j is eligible at h_0 for its $(f_j(h_0)+1)$st seat.

Let h_l be the house size at which state l received its last (a_lth) seat. State l may be chosen so that h_l is the largest among all states $l\in J_\mu$ which get more than their lower quota at h_0. h_l cannot equal h_0, because of (12) and the fact that j is eligible for the h_0th seat. Therefore $h_l<h_0$. Let $K\neq\varnothing$ be the set of states receiving additional seats at house sizes h, where $h_l<h\leqq h_0$. Clearly $l\not\in K$. Moreover,

by choice of l, $f_k(h_0) \leqq \bar{p}_k \bar{h}_0$ for all $k \in K$. Thus, using (13),

$$f_k(h_0) \leqq \bar{p}_k \bar{h}_0 \leqq p_k h_0 / \Sigma_1^s p_i \quad \text{for} \quad k \in K. \tag{14}$$

But, since $f_k(h_l) < f_k(h_0)$ for all $k \in K$,

$$\bar{p}_k / (f_k(h_l) + 1) \geqq \bar{p}_k / f_k(h_0) \geqq 1/\bar{h}_0 > \bar{p}_l / f_l(h_0) = \bar{p}_l / f_l(h_l).$$

This means that every $k \in K$ must have been ineligible at h_l, $K \cap E(h_l) = \varnothing$, for, otherwise, one of these states would have been given the h_lth seat by $\boldsymbol{Q}(\boldsymbol{r})$. Thus,

$$f_k(h_l) = f_k(h_l - 1) \geqq p_k h_l / \Sigma_1^s p_i \quad \text{for} \quad k \in K. \tag{15}$$

In the interval $h_l < h \leqq h_0$ exactly $h_0 - h_l$ seats were awarded to the states in K, so $\Sigma_K \{f_k(h_0) - f_k(h_l)\} = h_0 - h_l$. Subtracting (15) from (14) and summing over K

$$h_0 - h_l = \Sigma_K \{f_k(h_0) - f_k(h_l)\} \leqq \Sigma_K (p_k(h_0 - h_l)/\Sigma_1^s p_i).$$

But $h_0 - h_l > 0$ implies $\Sigma_K p_k = \Sigma_1^s p_i = 1$, a contradiction, since K is a proper subset of all states, $l \notin K$, and $p_l > 0$. Therefore, $\boldsymbol{Q}(\boldsymbol{r})$ satisfies lower quota and so satisfies quota. This completes the first part of the proof.

(ii) UNIQUENESS. Let $\boldsymbol{Q}'(\boldsymbol{r})$ be any set of solutions satisfying all properties and suppose it is not contained in $\boldsymbol{Q}(\boldsymbol{r})$. Then there must exist a solution $f \in \boldsymbol{Q}'(\boldsymbol{r}) \sim \boldsymbol{Q}(\boldsymbol{r})$ for some problem $\boldsymbol{p}, \boldsymbol{r}$. This means there is a house $h \geqq \Sigma_1^s r_i$ and a pair of states i and j, say, with populations $p_i = p^*$ and $p_j = \bar{p}$ and apportionments $f_i(\boldsymbol{p}, \boldsymbol{r}, h) = a^*$, $f_j(\boldsymbol{p}, \boldsymbol{r}, h) = \bar{a}$, with both eligible at $h+1$, $i, j \in E(h+1)$, and $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, but (contrary to $\boldsymbol{Q}(\boldsymbol{r})$) $f_j(\boldsymbol{p}, \boldsymbol{r}, h+1) = \bar{a}+1$. Among these choose a situation for which $\bar{a} + a^* = \lambda$ is a minimum. In other words, single out an occurrence in which a solution violates the conditions of $\boldsymbol{Q}(\boldsymbol{r})$ such that λ is minimum.

Either (a) $\bar{p}/\bar{a} > p^*/(a^*+1)$ or (b) $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* > \bar{p}$ or (c) $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* \leqq \bar{p}$.

Case (a): $\bar{p}/\bar{a} > p^*/(a^*+1)$. Then, just as in the uniqueness proof of Theorem 3, a contradiction is obtained.

Case (b): $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* > \bar{p}$. Choose t to be any positive integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1) - \bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^* + t\bar{p}(a^*+1)$$

and consider a problem with $t+1$ states, populations $(p^*, \bar{p}, \bar{p}-\delta, \bar{p}-\delta, \cdots, \bar{p}-\delta)$, where $0 < \delta < \bar{p}$ and δ will be specified presently, and consider the unbiased requirements $\boldsymbol{r}' = (a^*, \bar{a}, \bar{a}+1, \cdots, \bar{a}+1)$. Let h^0 be the sum of requirements, $h^0 = a^* + t(\bar{a}+1) - 1$. The exact quota of state 1 at $h^0 + 1$ is, by choice of t,

$$\frac{p^*(a^* + t(\bar{a}+1))}{p^* + t\bar{p} - (t-1)\delta} \geqq \frac{p^*a^* + p^* + t\bar{p}(a^*+1)}{p^* + t\bar{p} - (t-1)\delta} = a^* + 1,$$

so state 1 is eligible for its (a^*+1)st seat at h^0+1 for *any* $f \in \boldsymbol{Q}'(\boldsymbol{r}')$. The exact quota of state 2 satisfies

$$\frac{\bar{p}(h^0+1)}{p^* + t\bar{p} - (t-1)\delta} = \frac{\bar{p}(a^* + t(\bar{a}+1))}{p^* + t\bar{p} - (t-1)\delta} < \frac{\bar{p}(a^*+1) + t\bar{p}(\bar{a}+1)}{p^* + t\bar{p} - (t-1)\delta} < \frac{p^*(\bar{a}+1) + t\bar{p}(\bar{a}+1)}{p^* + t\bar{p} - (t-1)\delta}$$

$$= \left(\frac{p^* + t\bar{p}}{p^* + t\bar{p} - (t-1)\delta}\right)(\bar{a}+1).$$

Since t is fixed, we may therefore choose $\delta > 0$ sufficiently small so that the exact quota of state 2 is less then $\bar{a}+1$. Therefore, for any $f \in \boldsymbol{Q}'(\boldsymbol{r}')$, state 2 is eligible for its $(\bar{a}+1)$ st seat at h^0+1, and the

generalized lower quota at h^0+1 for each state $j \geqq 3$ equals its requirement, namely $\bar{a}+1$. Hence the generalized lower quota for state 1 at h^0+1 is at least a^*+1, since we have

$$\frac{p^*(a^*+\bar{a}+1)}{p^*+\bar{p}} \geqq \frac{p^*(a^*+1)+\bar{p}(a^*+1)}{p^*+\bar{p}} = a^*+1.$$

Therefore, every $f \in Q'(r')$ must give to state 1 at least a^*+1 seats (in fact, exactly a^*+1 seats) at h^0+1. But this contradicts consistency, since consistency implies, by the hypothesis, that some f gives $\bar{a}+1$ seats to state 2 and hence only a^* seats to state 1.

Case (c): $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* \leqq \bar{p}$. It is conceivable that $\bar{a} = \bar{r}$ is the minimum requirement of the bar-state. But, then, $\bar{p}/\bar{r} = \bar{p}/\bar{a} \leqq p^*/(a^*+1) < p^*/r^*$ implying $p^* > \bar{p}$ since the requirements are unbiased. Therefore $\bar{a} > \bar{r}$ and, in particular, $\bar{a} \geqq 1$.

Let $\bar{h}$ be the smallest house size at which f gives to the bar-state $\bar{a}$ seats, and b^* be the number of seats accorded the star-state at $\bar{h}$ by f. Then $b^* \leqq a^*$. Suppose, first, that $b^* < a^*$. Then $\bar{p}/\bar{a} < p^*/(b^*+1)$ implying, by induction, that the star-state is ineligible for its (b^*+1)st seat at $\bar{h}$. The bar-state, however, is eligible for its $\bar{a}$th seat at $\bar{h}$, so

$$a^* > b^* \geqq p^*\bar{h}/(\Sigma_1^s p_i) \quad \text{and} \quad \bar{a}-1 < \bar{p}\bar{h}/(\Sigma_1^s p_i)$$

together implying $\bar{p}/(\bar{a}-1) > p^*/a^*$. This inequality is incompatible with the hypothesis of Case (c), so it must be assumed that $b^* = a^*$.

If the star-state is ineligible for its $b^*+1 = a^*+1$ st seat at $\bar{h}$ the identical contradiction results, so it must be assumed that it is eligible. This implies that at $\bar{h}$ the bar-state has priority over the star-state at $\bar{p}$, p^*, $\bar{a}-1$, a^*. By induction this means $\bar{p}/\bar{a} \geqq p^*/(a^*+1)$ and, therefore, $\bar{p}/\bar{a} = p^*/(a^*+1)$.

As in Case (b), choose t to be any positive integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1)-\bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^*+t\bar{p}(a^*+1)$$

and consider a problem with $t+2$ states, populations $(\varepsilon, p^*, \bar{p}, \cdots, \bar{p})$, and requirements $(b, a^*, a^*, \cdots, a^*)$. Thus, $h^0 = b+(t+1)a^*$. Choose ε such that $0 < \varepsilon < p^*$, and let b be any integer satisfying $\varepsilon a^*/p^* < \varepsilon(\bar{a}+1)/\bar{p} < b$ and sufficiently large so that the states with population $\bar{p}$ are each eligible for $\bar{a}+1$ seats at any house $h \geqq h^0$. The requirements are unbiased, because $\varepsilon < p^* \leqq \bar{p}$ and $\varepsilon/b < p^*/a^* \leqq \bar{p}/a^*$ with $p^*/a^* = \bar{p}/a^*$ if $p^* = \bar{p}$. Let $h' = b+a^*+t(\bar{a}+1)$. For any house size h, $h^0 \leqq h < h'$, the exact quota of state 1 is less than b, and so is ineligible. For any such h, at least one of the $\bar{p}$-population states, say i, has less than $\bar{a}+1$ seats, say $a_i < \bar{a}+1$. Moreover, i is eligible. If $a_i = \bar{a}$ or $a_i = \bar{a}-1$, then, by the above, state i has priority over state 2 receiving its a^*+1st seat. If $a_i < \bar{a}-1$ then $p^*/(a^*+1) < \bar{p}/(a_i+1)$ and so by the induction hypothesis on λ, state i again has priority over state 2 receiving its a^*+1st seat. Therefore, considering successive h, $h^0 \leqq h < h'$, we can find a solution f in Q' which gives the apportionment $(b, a^*, \bar{a}+1, \cdots, \bar{a}+1)$ at h'. But this contradicts the generalized lower quota of state 2 at h', because, since state 1 is not in the slack set, the exact quota of state 2 is

$$\frac{p^*(a^*+t(\bar{a}+1))}{p^*+t\bar{p}} \geqq a^*+1.$$

This completes the proof of uniqueness.

7. Conclusion. Two basic principles emerge from the discussions surrounding apportionment from the founding of the Republic to the present day. The first principle is that any apportionment should satisfy quota. Not only does this square with common sense, but it was clearly what the architects of the Constitution had in mind when they used the phrase "apportioned ... according to their respective numbers." The discussion leading up to the adoption of the above phrase in the

Constitutional Convention illustrates this. Edmund Randolph, Delegate from Virginia, first proposed "that the rights of suffrage in the National Legislature ought to be proportioned to the quotas of contribution, or to the number of free inhabitants," ([**15a**], v. 3, p. 41). The terms "proportion" and "quota" recur repeatedly.

James Madison of Virginia enunciated the general principle that the States "ought to vote in the same proportion in which their citizens would do, if the people of all the States were collectively met" ([**15a**], v. 3, p. 385). Randolph's proposal contained the essence of the final version which stated that *both* direct taxes and Representatives should be apportioned together and by the same principle. The Convention members saw considerable justice in this. "[Mr. Read] had observed ... a backwardness in some of the members from the large states, to take their full proportion of Representatives ... He now suspects it was to avoid their due share of taxation." ([**15a**], v. 3, p. 418).The issue of whether every state should necessarily receive a Representative did not come up until later in the discussion, when Governor Morris of New Jersey pointed out that apportioning Representatives proportionally to population might mean that some states would get none: "[It] would exclude some states altogether who would not have a sufficient number to entitle them to a single Representative." ([**15a**], v. 3, p. 399). The way in which this difficulty was overcome in the Constitution was to make an *exception* to the proportionality principle in this one case. It also illustrates that the notion of quota was uppermost in the minds of those at the Convention. Certainly no scheme such as Huntington's in which every state is *entitled* to a Representative, no matter how small its population, fits with the language and intent of the Constitution.

Up until 1910, all of the apportionment constructions used, or even seriously proposed, started from the premise of satisfying quota. Hamilton's method prevailed from 1850 to 1910, but, as we have seen, it admits the Alabama paradox.

The Alabama paradox is a phenomenon of the *method* used, not of a particular solution. Since the Constitution does not speak of apportionment *solutions* or *methods*, but only of *apportionments* (for a given h), house monotonicity is not a Constitutional requirement *per se*. However, the reaction of the House when the Alabama paradox was first noticed in the 1880's is sufficient indication that such a phenomenon is politically unacceptable, as well as repugnant to both fairness and common sense. In fact, Congress showed real sophistication in considering the mathematical properties of the methods used.

Thus the second basic apportionment principle is that any acceptable apportionment method must be house-monotone. The major contribution of Willcox and Huntington was to formulate more clearly the notion of an apportionment method (in which the house size is determined in advance), and to propose methods that avoid the Alabama paradox. But in so doing they forfeited the essential, and even more basic, requirement of being quota, which is rooted in the Constitutional mandate itself.

Willcox, indeed, apparently thought that his proposal (in reality, Webster's method) was a quota method, which had the additional property that it rounded major fractions up and minor fractions down, (or, if he realized that this was not so, he did not admit it). On the other hand, while Huntington recognized that Equal Proportions (and the other four methods he considered) did not satisfy quota, he glided very quickly over this point in his work. Instead of the quota principle which takes as its standard of fairness the *exact* portion deserved by each state (i.e., the exact quotas) Huntington adopted a different principle, namely that of pairwise comparisons between states. The difficulty with comparing states by pairs, and adjusting their delegations accordingly, is that when we step back and look at the whole picture we find that the resulting solution may be very far removed from the overall standard of fairness, namely the exact quotas. The pairwise comparison leads, for example, to such absurdities as a state deserving *exactly* an integer number of seats, whereas EP gives it some other number. Huntington's own examples illustrate this. Consider the following one, ([**13**], Example 6) in which *all* five Huntington methods give state B something different than 44

Representatives, which is its exact due, and compare this with the quota solution.

State	Population	Exact Quota	SD	HM	Apportionments EP	W	J	Q
A	5117	51.1700	51	51	51	51	52	52
B	4400	44.0000	43	43	43	43	45	44
C	162	1.6200	2	2	2	2	1	2
D	161	1.6100	2	2	2	2	1	1
E	160	1.6000	2	2	2	2	1	1
Total	10000	100.0000	100	100	100	100	100	100

TABLE 8.

Actually, one can go further and use Huntington's own type of reasoning to argue against EP or any other method that does not satisfy quota. Given normalized populations p_i (i.e., $\Sigma_i p_i = 1$), a house h, and apportionment $(a_1, \cdots, a_s)$ for h, we may say that state i has a *surplus* if $a_i > p_i h$ and a *deficit* if $a_i < p_i h$. In general, in any apportionment, some states will have surpluses and others will have deficits. Consider an EP apportionment which is not quota, say state 1 is above quota, $a_1 > \lceil p_1 h \rceil$. In particular, state 1 has a surplus. Then some state, say state 2, must have a deficit. Comparing states 1 and 2, a transfer of a Representative from state 1 to state 2, leaves state 1 with a surplus and reduces the deficit of state 2 (or, possibly gives it a surplus). Clearly, any such transfer should be made. Thus, if the exact quota is considered to be the true measure of how much each state deserves, then EP does not necessarily give the best or most stable solution even in the sense of pairwise comparisons.

A second difficulty with Huntington's approach is that there is no single natural standard by which the inequality of representation between two states can be measured. As Huntington himself admits, "There has been some disagreement, however, as to what is the most suitable way of measuring the inequality between two states" ([**14a**], p. 11). This disagreement has still not been resolved and probably never can be. Huntington, of course, advocated that the *relative* difference between the average district sizes was the best measure. But this leads to various absurdities. Consider for example, two states, state 1 having a million residents and state 2 having only one resident. Suppose that there is exactly one representative to be distributed between the two states. Then Huntington's criterion says that the situation where the one-person state gets the representative and the other million go unrepresented is just as fair and desirable as the situation where the million are represented and the one is not, because the rank index $p_i/\sqrt{a_i(a_i+1)}$ yields $+\infty$ for both states when $a_i = 0$. But this conclusion is patently absurd. To cite a second example, the rank index based on relative differences says that every state should receive one representative before any state receives two, no matter how different in size the states may be. This conveniently meets the Constitutional requirement that each state receive at least one representative, but it does not correspond to any reasonable notion of fair division, which Huntington's EP method purports to be. Thus if 50 representatives are to be apportioned among 50 states, whose populations are $(10^8, 1, 1, \cdots, 1)$ then the unique EP solution is $(1, 1, \cdots, 1)$. But this means that 49 people out of a population of over a hundred million have 98% of the representation, and, if direct taxes were still assessed, these same 49 would each pay in taxes an amount equal to that paid by one hundred million! This is inherently unreasonable. Moreover, it does not correspond with the intent of the Constitution, since the phrase "but each state shall have at least one representative" was evidently meant as an *exception* to whatever method of proportional allocation was used.

Marshaling the facts against the method of equal proportions we see: (i) it violates the most intuitively basic property of all, satisfying quota; (ii) it depends upon an arbitrary, *ad hoc* measure of inequality of representation between states; and (iii) it appears to be in disagreement with the stated intent of the framers of the Constitution.

						Apportionments		
State	Population	Exact Quota	SD	HM	EP	W	J	Q
A	9061	9.0610	9	9	9	9	10	10
B	7179	7.1790	7	7	7	8	7	7
C	5259	5.2590	5	5	6	5	5	5
D	3319	3.3190	3	4	3	3	3	3
E	1182	1.1820	2	1	1	1	1	1
Total	26000	26.0000	26	26	26	26	26	26

TABLE 9.

State	Population	Exact Quota	SD	HM	EP	W	J	Q
1	60272	61.477	49	64	68	70	70	62
2	1226	1.251	1	1	1	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
12	1236	1.261	1	1	1	1	1	1
13	1237	1.262	2	1	1	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
25	1249	1.274	2	1	1	1	1	1
26	1250	1.275	2	1	1	1	1	2
27	1251	1.276	2	1	1	1	1	2
28	1252	1.277	2	2	1	1	1	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
31	1255	1.280	2	2	1	1	1	2
32	1256	1.281	2	2	2	1	1	2
33	1257	1.282	2	2	2	1	1	2
	100000	102.000	102	102	102	102	102	102

State	Population	Exact Quota	SD	HM	EP	W	J	Q
1	68010	66.650	58	58	60	64	78	67
2	1590	1.558	2	2	1	1	1	1
3	1591	1.559	2	2	1	1	1	1
4	1592	1.560	2	2	2	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
7	1595	1.563	2	2	2	1	1	1
8	1592	1.564	2	2	2	2	1	1
9	1597	1.565	2	2	2	2	1	1
10	1598	1.566	2	2	2	2	1	2
11	1599	1.567	2	2	2	2	1	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
21	1609	1.577	2	2	2	2	1	2
	100000	98.000	98	98	98	98	98	98

TABLE 10.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3266740	0.0183	7.9583	8	8	8	8	8	8
ALASKA	226167	0.0013	0.5510	1	1	1	1	1	1
ARIZONA	1302161	0.0073	3.1723	3	3	3	3	3	3
ARKANSAS	1786272	0.0100	4.3517	5	4	4	4	4	4
CALIFORNIA	15717204	0.0880	38.2897	37	38	38	38	40	39
COLORADO	1753947	0.0098	4.2729	5	4	4	4	4	4
CONNECTICUT	2535234	0.0142	6.1763	6	6	6	6	6	6
DELAWARE	446292	0.0025	1.0872	2	1	1	1	1	1
FLORIDA	4951560	0.0277	12.0628	12	12	12	12	12	12
GEORGIA	3943116	0.0221	9.6061	10	10	10	10	10	10
HAWAII	632772	0.0035	1.5415	2	2	2	2	1	1
IDAHO	667191	0.0037	1.6254	2	2	2	2	1	1
ILLINOIS	10081158	0.0565	24.5594	24	24	24	24	25	25
INDIANA	4662498	0.0261	11.3586	11	11	11	11	11	12
IOWA	2757537	0.0154	6.7178	7	7	7	7	7	7
KANSAS	2178611	0.0122	5.3075	5	5	5	5	5	5
KENTUCKY	3038156	0.0170	7.4015	7	7	7	7	7	7
LOUISIANA	3257022	0.0182	7.9346	8	8	8	8	8	8
MAINE	969265	0.0054	2.3613	3	2	2	2	2	2
MARYLAND	3100689	0.0174	7.5538	8	8	8	8	7	7
MASSACHUSETTS	5148578	0.0288	12.5428	12	12	12	13	13	13
MICHIGAN	7823194	0.0438	19.0586	18	19	19	19	20	20
MINNESOTA	3413864	0.0191	8.3167	8	8	8	8	8	8
MISSISSIPPI	2178141	0.0122	5.3063	5	5	5	5	5	5
MISSOURI	4319813	0.0242	10.5238	10	10	10	10	11	11
MONTANA	674767	0.0038	1.6438	2	2	2	2	1	1
NEBRASKA	1411330	0.0079	3.4382	4	3	3	3	3	3
NEVADA	285278	0.0016	0.6950	1	1	1	1	1	1
NEW HAMPSHIRE	606921	0.0034	1.4786	2	2	2	1	1	1
NEW JERSEY	6066782	0.0340	14.7797	14	15	15	15	15	15
NEW MEXICO	951023	0.0053	2.3168	3	2	2	2	2	2
NEW YORK	16782304	0.0940	40.8845	39	41	41	41	42	41
NORTH CAROLINA	4556155	0.0255	11.0996	11	11	11	11	11	11
NORTH DAKOTA	632446	0.0035	1.5407	2	2	2	2	1	1
OHIO	9706397	0.0544	23.6464	23	24	24	24	24	24
OKLAHOMA	2328284	0.0130	5.6721	6	6	6	6	5	6
OREGON	1768687	0.0099	4.3088	5	4	4	4	4	4
PENNSYLVANIA	11319366	0.0634	27.5758	26	27	27	27	28	28
RHODE ISLAND	859488	0.0048	2.0939	2	2	2	2	2	2
SOUTH CAROLINA	2382594	0.0133	5.8044	6	6	6	6	6	6
SOUTH DAKOTA	680514	0.0038	1.6578	2	2	2	2	1	1
TENNESSEE	3567089	0.0200	8.6900	9	9	9	9	9	9
TEXAS	9579677	0.0536	23.3377	22	23	23	23	24	24
UTAH	890627	0.0050	2.1697	3	2	2	2	2	2
VERMONT	389881	0.0022	0.9498	1	1	1	1	1	1
VIRGINIA	3966949	0.0222	9.6641	10	10	10	10	10	10
WASHINGTON	2853214	0.0160	6.9509	7	7	7	7	7	7
WEST VIRGINIA	1860421	0.0104	4.5323	5	5	5	5	4	4
WISCONSIN	3951777	0.0221	9.6272	10	10	10	10	10	10
WYOMING	330066	0.0018	0.8041	1	1	1	1	1	1
TOTAL	178559219	1.0000	435.0000	435	435	435	435	435	435

TABLE 11. 1960 Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3475885	0.0170	7.4099	8	7	7	7	7	7
ALASKA	304067	0.0015	0.6482	1	1	1	1	1	1
ARIZONA	1787620	0.0088	3.8108	4	4	4	4	4	4
ARKANSAS	1942303	0.0095	4.1406	4	4	4	4	4	4
CALIFORNIA	20098863	0.0985	42.8466	41	42	43	43	44	43
COLORADO	2226771	0.0109	4.7470	5	5	5	5	4	5
CONNECTICUT	3050693	0.0150	6.5035	7	6	6	7	6	6
DELAWARE	551928	0.0027	1.1766	2	1	1	1	1	1
FLORIDA	6855702	0.0336	14.6150	14	15	15	15	15	15
GEORGIA	4627306	0.0227	9.8645	10	10	10	10	10	10
HAWAII	784901	0.0038	1.6732	2	2	2	2	1	1
IDAHO	719921	0.0035	1.5347	2	2	2	2	1	1
ILLINOIS	11184320	0.0548	23.8427	23	24	24	24	25	24
INDIANA	5228156	0.0256	11.1454	11	11	11	11	11	11
IOWA	2846920	0.0140	6.0691	6	6	6	6	6	6
KANSAS	2265846	0.0111	4.8303	5	5	5	5	5	5
KENTUCKY	3246481	0.0159	6.9208	7	7	7	7	7	7
LOUISIANA	3672008	0.0180	7.8280	8	8	8	8	8	8
MAINE	1006320	0.0049	2.1453	3	2	2	2	2	2
MARYLAND	3953698	0.0194	8.4285	8	8	8	8	8	8
MASSACHUSETTS	5726676	0.0281	12.2081	12	12	12	12	12	13
MICHIGAN	8937196	0.0438	19.0523	19	19	19	19	20	19
MINNESOTA	3833173	0.0188	8.1715	8	8	8	8	8	8
MISSISSIPPI	2233648	0.0109	4.7621	5	5	5	5	5	5
MISSOURI	4718034	0.0231	10.0579	10	10	10	10	10	10
MONTANA	701573	0.0034	1.4956	2	2	2	1	1	1
NEBRASKA	1496820	0.0073	3.1909	4	3	3	3	3	3
NEVADA	492396	0.0024	1.0497	1	1	1	1	1	1
NEW HAMPSHIRE	746284	0.0037	1.5909	2	2	2	2	1	1
NEW JERSEY	7209035	0.0353	15.3661	15	15	15	15	16	16
NEW MEXICO	1028664	0.0050	2.1886	3	2	2	2	2	2
NEW YORK	18338055	0.0899	39.0930	37	39	39	39	41	40
NORTH CAROLINA	5125230	0.0251	10.9259	11	11	11	11	11	11
NORTH DAKOTA	624181	0.0031	1.3306	2	1	1	1	1	1
OHIO	10730200	0.0526	22.8746	22	23	23	23	24	23
OKLAHOMA	2585486	0.0127	5.5117	6	6	6	6	5	5
OREGON	2110810	0.0103	4.4998	5	5	4	5	4	4
PENNSYLVANIA	11884314	0.0582	25.3349	24	25	25	25	26	26
RHODE ISLAND	957798	0.0047	2.0418	2	2	2	2	2	2
SOUTH CAROLINA	2617328	0.0128	5.5796	6	6	6	6	5	5
SOUTH DAKOTA	673247	0.0033	1.4352	2	2	2	1	1	1
TENNESSEE	3961060	0.0194	8.4442	8	8	8	8	8	9
TEXAS	11298787	0.0554	24.0867	23	24	24	24	25	25
UTAH	1067810	0.0052	2.2764	3	2	2	2	2	2
VERMONT	448327	0.0022	0.9557	1	1	1	1	1	1
VIRGINIA	4690742	0.0230	9.9997	10	10	10	10	10	10
WASHINGTON	3443487	0.0169	7.3408	7	7	7	7	7	7
WEST VIRGINIA	1763331	0.0086	3.7591	4	4	4	4	3	4
WISCONSIN	4447013	0.0218	9.4801	9	9	9	9	9	10
WYOMING	335719	0.0016	0.7157	1	1	1	1	1	1
TOTAL	204053325	1.0000	435.0000	435	435	435	435	435	435

TABLE 12. 1970 Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3659293	0.0165	7.1982	7	7	7	7	7	7
ALASKA	451884	0.0020	0.8889	1	1	1	1	1	1
ARIZONA	2184366	0.0099	4.2969	5	4	4	4	4	4
ARKANSAS	2176565	0.0098	4.2815	5	4	4	4	4	4
CALIFORNIA	21839542	0.0988	42.9604	41	43	45	45	45	43
COLORADO	2664373	0.0120	5.2411	5	5	5	5	5	5
CONNECTICUT	3158612	0.0143	6.2133	6	6	6	6	6	6
DELAWARE	685196	0.0031	1.3478	2	2	1	1	1	1
FLORIDA	7081224	0.0320	13.9294	14	14	14	14	14	14
GEORGIA	5112891	0.0231	10.0575	10	10	10	10	10	11
HAWAII	993246	0.0045	1.9538	2	2	2	2	2	2
IDAHO	691063	0.0031	1.3594	2	2	1	1	1	1
ILLINOIS	11947647	0.0540	23.5021	23	24	24	24	25	24
INDIANA	5610014	0.0254	11.0354	11	11	11	11	11	12
IOWA	3161153	0.0143	6.2183	6	6	6	6	6	6
KANSAS	2675456	0.0121	5.2629	5	5	5	5	5	5
KENTUCKY	3657104	0.0165	7.1939	7	7	7	7	7	7
LOUISIANA	4140835	0.0187	8.1454	8	8	8	8	8	9
MAINE	1078588	0.0049	2.1217	2	2	2	2	2	2
MARYLAND	4131001	0.0187	8.1261	8	8	8	8	8	8
MASSACHUSETTS	6085436	0.0275	11.9706	12	12	12	12	12	12
MICHIGAN	9438773	0.0427	18.5669	18	19	19	19	19	19
MINNESOTA	4129984	0.0187	8.1241	8	8	8	8	8	8
MISSISSIPPI	2679798	0.0121	5.2714	5	5	5	5	5	5
MISSOURI	5123214	0.0232	10.0778	10	10	10	10	10	11
MONTANA	691146	0.0031	1.3595	2	2	1	1	1	1
NEBRASKA	1643502	0.0074	3.2329	4	3	3	3	3	3
NEVADA	686213	0.0031	1.3498	2	2	1	1	1	1
NEW HAMPSHIRE	908754	0.0041	1.7876	2	2	2	2	1	1
NEW JERSEY	7573756	0.0342	14.8963	14	15	15	15	15	15
NEW MEXICO	1182655	0.0053	2.3264	3	2	2	2	2	2
NEW YORK	20303765	0.0918	39.9394	38	40	42	42	42	40
NORTH CAROLINA	5614931	0.0254	11.0451	11	11	11	11	11	12
NORTH DAKOTA	684688	0.0031	1.3468	2	2	1	1	1	1
OHIO	11437560	0.0517	22.4987	22	23	23	23	23	23
OKLAHOMA	2675479	0.0121	5.2629	5	5	5	5	5	5
OREGON	2182157	0.0099	4.2925	5	4	4	4	4	4
PENNSYLVANIA	12696129	0.0574	24.9745	24	25	26	26	26	25
RHODE ISLAND	1131130	0.0051	2.2250	3	2	2	2	2	2
SOUTH CAROLINA	2674982	0.0121	5.2619	5	5	5	5	5	5
SOUTH DAKOTA	686555	0.0031	1.3505	2	2	1	1	1	1
TENNESSEE	4133034	0.0187	8.1301	8	8	8	8	8	9
TEXAS	12176464	0.0551	23.9522	23	24	25	25	25	24
UTAH	1197568	0.0054	2.3557	3	2	2	2	2	2
VERMONT	660279	0.0030	1.2988	2	1	1	1	1	1
VIRGINIA	5098449	0.0231	10.0291	10	10	10	10	10	10
WASHINGTON	3648182	0.0165	7.1763	7	7	7	7	7	7
WEST VIRGINIA	1691133	0.0076	3.3266	4	3	3	3	3	3
WISCONSIN	4631008	0.0209	9.1096	9	9	9	9	9	10
WYOMING	571638	0.0026	1.1245	2	1	1	1	1	1
TOTAL	221138415	1.0000	435.0000	435	435	435	435	435	435

TABLE 13. 1984A Projected Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3608877	0.0163	7.0990	7	7	7	7	7	7
ALASKA	329928	0.0015	0.6490	1	1	1	1	1	1
ARIZONA	1885014	0.0085	3.7080	4	4	4	4	3	4
ARKANSAS	1948052	0.0088	3.8320	4	4	4	4	4	4
CALIFORNIA	21944556	0.0992	43.1670	41	41	41	42	45	44
COLORADO	2410663	0.0109	4.7420	5	5	5	5	4	5
CONNECTICUT	3438575	0.0155	6.7640	7	7	7	7	7	7
DELAWARE	802199	0.0036	1.5780	2	2	2	2	1	1
FLORIDA	7671182	0.0347	15.0899	15	15	15	15	15	15
GEORGIA	5053140	0.0229	9.9400	10	10	10	10	10	10
HAWAII	840834	0.0038	1.6540	2	2	2	2	1	1
IDAHO	804232	0.0036	1.5820	2	2	2	2	1	1
ILLINOIS	12290721	0.0556	24.1770	23	23	23	24	25	25
INDIANA	5570655	0.0252	10.9580	11	11	11	11	11	11
IOWA	2958171	0.0134	5.8190	6	6	6	6	6	6
KANSAS	2456416	0.0111	4.8320	5	5	5	5	5	5
KENTUCKY	3465010	0.0157	6.8160	7	7	7	7	7	7
LOUISIANA	3968799	0.0179	7.8070	8	8	8	8	8	8
MAINE	1298870	0.0059	2.5550	3	3	3	2	2	2
MARYLAND	3978966	0.0180	7.8270	8	8	8	8	8	8
MASSACHUSETTS	6086136	0.0275	11.9720	12	12	12	12	12	12
MICHIGAN	9489634	0.0429	18.6670	18	18	18	18	19	19
MINNESOTA	4003368	0.0181	7.8750	8	8	8	8	8	8
MISSISSIPPI	2421339	0.0109	4.7630	5	5	5	5	5	5
MISSOURI	5108552	0.0231	10.0490	10	10	10	10	10	10
MONTANA	773730	0.0035	1.5220	2	2	2	1	1	1
NEBRASKA	1834178	0.0083	3.6080	4	4	4	4	3	3
NEVADA	534291	0.0024	1.0510	1	1	1	1	1	1
NEW HAMPSHIRE	842868	0.0038	1.6580	2	2	2	2	1	1
NEW JERSEY	7676299	0.0347	15.1000	15	15	15	15	15	15
NEW MEXICO	1313105	0.0059	2.5830	3	3	3	3	2	2
NEW YORK	19842029	0.0897	39.0311	37	37	37	38	41	40
NORTH CAROLINA	5552320	0.0251	10.9219	11	11	11	11	11	11
NORTH DAKOTA	755938	0.0034	1.4870	2	2	2	1	1	1
OHIO	11735587	0.0531	23.0850	22	22	22	22	24	23
OKLAHOMA	2901743	0.0131	5.7080	6	6	6	6	5	6
OREGON	2385753	0.0108	4.6930	5	5	5	5	4	5
PENNSYLVANIA	12779259	0.0578	25.1380	24	24	24	25	26	26
RHODE ISLAND	1316663	0.0060	2.5900	3	3	3	3	2	2
SOUTH CAROLINA	2905301	0.0131	5.7150	6	6	6	6	6	6
SOUTH DAKOTA	752887	0.0034	1.4810	2	2	2	1	1	1
TENNESSEE	4031836	0.0182	7.9310	8	8	8	8	8	8
TEXAS	12228700	0.0553	24.0550	23	23	23	23	25	24
UTAH	1360383	0.0062	2.6760	3	3	3	3	2	2
VERMONT	485488	0.0022	0.9550	1	1	1	1	1	1
VIRGINIA	5026197	0.0227	9.8870	10	10	10	10	10	10
WASHINGTON	3519403	0.0159	6.9230	7	7	7	7	7	7
WEST VIRGINIA	1854004	0.0084	3.6470	4	4	4	4	3	3
WISCONSIN	4519866	0.0204	8.8910	9	9	9	9	9	9
WYOMING	376698	0.0017	0.7410	1	1	1	1	1	1
TOTAL	221138415	1.0000	435.0000	435	435	435	435	435	435

TABLE 14. 1984B Projected Populations and Apportionments.

In contrast, the virtue of the quota method is that it unites the two basic apportionment principles — house-monotonicity and quota — into a single method. It replaces Huntington's artificial "measures of inequality" with a more fundamental criterion of fairness, the exact quota, and does this without introducing the Alabama paradox. Moreover, subject to the mathematical property of consistency, which is common to all Huntington methods, it is the only apportionment method with these two properties.

8. Appendix. On pages 724-728 are given apportionments (for particular house sizes) for all examples cited in this paper and others found by the "five modern workable" (Huntington) methods and by the quota method with $r_i = 1$ for all i. Included are: the example first considered (Table 9), the two examples showing how far from quota EP solutions may be (Table 10); and the unique apportionments for populations of the 1960 census (Table 11), the 1970 census* (Table 12), and the projected censuses of 1984A (Table 13), and 1984B (Table 14).

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References

1. *Apportionment Among the Several States*, House of Representatives, 56th Congress, 2nd Session, Report No. 2130, 20 December 1900.

2. *Apportionment of Representatives*, Report from the Committee on the Census, submitted by E. D. Crumpacker, House of Representatives, 61st Congress, 3rd Session, Report No. 1911, 13 January 1911.

3. *Apportionment of Representatives in Congress*, Hearings before the Committee on the Census, House of Representatives, 76th Congress, 3rd Session, 27, 28, 29 February and 1, 5 March 1940.

4. *Apportionment of Representatives in Congress*, Hearings before a subcommittee of the Committee on Commerce, United States Senate, 77th Congress, 1st Session, on H. R. 2665, 27, 28 February and 1 March 1941.

5. M. L. Balinski, Integer programming: Methods, Uses, Computation, Management Science, 12 (1965) 252–313.

6. M. L. Balinski and H. P. Young, A New Method for Congressional Apportionment, Proc. Nat. Acad. Sci. U. S. A., 71 (November 1974) No. 11.

7. G. A. Bliss, E. W. Brown, L. P. Eisenhart and Raymond Pearl, Report to the President of the National Academy of Sciences, 9 February 1929, (in [3], pp. 70–71; also in [11], pp. 4966–4967).

8. Zechariah Chaffee, Jr., Congressional Reapportionment, Harvard Law Review, XLII (1929) 1015–1047.

9. ———, Reapportionment of the House of Representatives under the 1950 Census, Cornell Law Quarterly, 36 (1950–51) 643–665.

10. *Congressional Record*, 56th Congress, 2nd Session, Vol. XXXIV, Washington, Government Printing Office.

11. *Congressional Record*, 70th Congress, 2nd Session, Vol. 70, 1929.

12. Alexander Hamilton, The Papers of Alexander Hamilton, Vol. XI (February 1792–June 1792), Harold C. Syrett, editor, Columbia University Press, New York, 1966, pp. 228–230.

13. E. V. Huntington, The Apportionment of Representatives in Congress, Trans. Amer. Math. Soc., 30 (1928) 85–110.

14. ———, The Mathematical Theory of the Apportionment of Representatives, Proc. Nat. Acad. Sci. U.S.A., 7 (1921) 123–127.

14a. ———, Methods of Apportionment in Congress, Senate Document No. 304, 76th Congress, 3rd Session, 7 October 1940, U. S. Government Printing Office, Washington, D. C.

15. Thomas Jefferson, The Works of Thomas Jefferson, vol. VI, Paul Leicester Ford, editor, G. P. Putnam's Sons, New York, 1904, pp. 460–471.

15a. James Madison, Writings of James Madison, Gaillard Hunt, editor, G. P. Putnam, New York, 1902.

*In 1970, for the first time, parts of the overseas population of U.S. citizens were allocated to their "home" states and included in the populations of those states for the purpose of apportionment. Thus, the figures for 1970 given in Table 3 are "apportionment populations" rather than resident populations. It is interesting to note that this change affects the EP solution: if resident populations were used for apportionment, Connecticut would receive 7 (rather than 6) seats, and Oklahoma 5 (rather than 6) seats.

16. Marston Morse, John von Neumann and Luther P. Eisenhart, Report to the President of the National Academy of Sciences, 28 May 1948.

17. Laurence F. Schmeckebier, Congressional Apportionment, The Brookings Institution, Washington, D. C. 1941.

18. George Washington, The Writings of George Washington, Vol. 32 (March 10, 1792–June 30, 1793), John C. Fitzpatrick, editor, United States Government Printing Office, Washington, D. C. 1939, pp. 16–17.

19. Daniel Webster, The Writings and Speeches of Daniel Webster, vol. VI, National Edition, Little, Brown & Company, Boston Mass., 1903, pp. 102–123.

20. Walter F. Willcox, The Apportionment of Representatives, The American Economic Review, vol. VI, No. 1, Supplement (March 1916) 3–16.

21. ———, Last Words on the Apportionment Problem, Legislative Reapportionment, vol. 17, No. 2 (1952) issue of *Law and Contemporary Problems*, pp. 290–302.

22. Comment, Apportionment of the House of Representatives, Yale Law Journal, 58 (1949) 1360–1386.

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POLYNOMIAL CALCULUS WITH D-LIKE OPERATORS

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1. Introduction. The newcomer to numerical analysis is usually impressed on finding an unexpected formal analogy [1] between the Taylor and Newton series expansions:

$$f(x) \sim \sum_{k=0}^{\infty} (D^k f)(a) \frac{(x-a)^k}{k!}$$

$$f(x) \sim \sum_{k=0}^{\infty} (\Delta^k f)(a) \frac{(x-a)^{(k)}}{k!}.$$

We say unexpected because the **difference operator**

$$(\Delta f)(x) = f(x+1) - f(x)$$

and its associated **factorial polynomials**

$$\frac{x^{(j)}}{j!} = \frac{x(x-1)\cdots(x-j+1)}{j!}$$

seem somewhat removed from their counterparts in the differential calculus. True, the formal identity

$$\Delta = e^D - 1 = D + \frac{D^2}{2!} + \cdots + \frac{D^k}{k!} + \cdots$$

and the **Stirling numbers** $s(i,j)$ defined by $x^{(j)} = \sum_{i=0}^{j} s(i,j) x^i$ do help to provide a connection. And yet, the question surely arises as to whether these two "expansion systems" are merely isolated curiosities, or instead, singular but typical examples from a family of such systems. Naturally, we wish to infer that it is the latter.

In order to establish an appropriate setting for the investigation, we first seek to extract the common features of the linear operators D and Δ (and their associated polynomials $x^j/j!$, $x^{(j)}/j!$, resp.) on the space P^∞ of all polynomials. In both cases, the associated polynomials form a **simple basis** (there being just one polynomial for each degree) and the operators are linear and strictly **unit-degree-decreasing** (abbreviated u.d.d. with linearity understood) over P^∞. And so, we begin with a brief analysis of these u.d.d. or "derivative-like" operators and their expansion capability relative to various simple bases. We are then led to impose a succession of familiar differential properties, leading ultimately to a characterization of the derivative among all u.d.d. operators.

Though some of our results might have been anticipated, we feel that the investigation does provide new insights as to the essential uniqueness of the classical differential calculus. Assuming this is so, we have tried to make our presentation self-contained and reasonably inclusive.

2. Expansion systems. In any linear function space, a sequence $\langle L_i \rangle$ of linear functionals and a companion sequence $\langle g_j \rangle$ of functions are said to be **bi-orthonormal** [2] if

$$L_i g_j = \delta_{ij}.$$

Perhaps the most common situations of this kind occur in the inner-product spaces where the functionals L_i take the form $L_i f = (f, g)$. Then the bi-orthonormality accounts for the familiar orthogonal expansions. But the same is seen to be true for the generalized Taylor-Newton expansions:

THEOREM 1. *Let L be a u.d.d. operator and $\langle g_j \rangle$ a simple basis in P^∞. Then every $f \in P^\infty$ has a representation*

$$f(x) = \sum_{k=0}^{\infty} (L^k f)(a) g_k(x)$$

iff the functionals $L_i f = (L^i f)(a)$ and the polynomials $g_j(x)$ are bi-orthonormal.

Proof. Assuming these representations exist, we would have in particular

$$g_j(x) = \sum_{i=0}^{\infty} (L^i g_j)(a) g_i(x).$$

And because $\langle g_j \rangle$ is a basis it would follow that

$$L_i g_j = (L^i g_j)(a) = \delta_{ij}.$$

Conversely, if we assume the bi-orthonormality and write

$$f(x) = \sum_{k=0}^{\infty} a_k g_k(x)$$

for any $f \in P^\infty$, then applying the functional L_j yields

$$(L^j f)(a) = \sum_{k=0}^{\infty} a_k (L^j g_k)(a) = a_j$$

so that f has the representation claimed. □

As a consequence, L and $\langle g_j \rangle$ are said to constitute an *expansion system* (at a) provided that $(L^i g_j)(a) = \delta_{ij}$. An **expansion system** (at 0) will be called a **Maclaurin system**. A more explicit characterization of these expansion systems is useful in the sequel, namely

THEOREM 2. *L and $\langle g_j \rangle$ constitute an expansion system (at a) iff*
(i) $g_0 = 1$
(ii) $g_j(a) = 0 \qquad (j > 0)$.
(iii) $L g_j = g_{j-1} \qquad (j > 0)$.

Proof. First suppose that all these conditions are met. Then

$$(L^i g_j)(a) = \begin{Bmatrix} g_{j-i}(a) = 0 & (i < j) \\ g_0(a) = 1 & (i = j) \\ (L^{i-j} g_0)(a) = 0 & (i > j) \end{Bmatrix} = \delta_{ij}.$$

Conversely, if L and $\langle g_j \rangle$ form an expansion system (at a) we have
(i) $\qquad g_0 \equiv g_0(a) = (L^0 g_0)(a) = \delta_{00} = 1,$
(ii) $\qquad g_j(a) = (L^0 g_j)(a) = \delta_{0j} = 0 \qquad (j > 0),$

and finally, considering Theorem 1

(iii)
$$(Lg_j)(x) = \sum_{k=0}^{\infty}(L^k(Lg_j))(a)g_k(x)$$
$$= \sum_{k=0}^{\infty}(L^{k+1}g_j)(a)g_k(x) = g_{j-1}(x). \qquad \square$$

COROLLARY. *Let $\langle g_j \rangle$ be a simple basis satisfying* (i), (ii). *Then there is a unique u.d.d. operator L such that L and $\langle g_j \rangle$ constitute an expansion system (at a). Conversely, to each u.d.d. operator L and each real number a, there corresponds a unique simple basis $\langle g_j \rangle$ for which L and $\langle g_j \rangle$ is an expansion system (at a).*

In particular, each u.d.d. operator L may be identified with that (unique) simple basis for which the pair constitute a Maclaurin system. And in considering the resulting infinite matrix representation, it should be clear that any such L will have the same "Jordan canonical form" as the derivative. We obtain somewhat more information in this same vein if we define the **Stirling transformations** $S: P^\infty \to P^\infty$ to be those linear operators with the following properties:

(I) $S(1) = 1$,

(II) *P_0 and each P^n are S-invariant subspaces,*

(III) *S is invertible.*

Here, P^n is the space of all polynomials of degree at most n, and P_0 represents the subspace of polynomials which vanish at the origin. Then, just as the factorial polynomials may be viewed as a change of basis via the matrix $s(i,j)$ of Stirling numbers, we have the following generalization:

THEOREM 3. *Let M and $\langle h_j \rangle$ be a Maclaurin system and S a Stirling transformation. If*

$$g = Sh \qquad L = SMS^{-1}$$

then L and $\langle g_j \rangle$ comprise another Maclaurin system. Conversely, if the two Maclaurin systems are given, there exists a Stirling transformation S such that

$$g = Sh \qquad L = SMS^{-1}.$$

Proof. First we observe that L is u.d.d. and $g_n = Sh_n$ has degree $\leqq n$ by (II). But if g_n had degree $< n$ we would contradict (III). So $\langle g_j \rangle$ is again a simple basis, and moreover,

(i) $g_0 = Sh_0 = S(1) = 1$ using (I),

(ii) $g_j(0) = Sh_j(0) = 0$ by (II) $(j > 0)$.

(iii) $Lg_j = SMS^{-1}g_j = SMh_j = Sh_{j-1} = g_{j-1}$,

so that according to Theorem 2, L and $\langle g_j \rangle$ constitute a new Maclaurin system.

Conversely, if L, $\langle g_j \rangle$ and M, $\langle h_j \rangle$ satisfy the conditions of Theorem 2, we have only to define S by the change of basis $g = Sh$. Then the invertibility of S and the invariance of each P^n is assured, and furthermore

(I) $S(1) = Sh_0 = g_0 = 1$,

(II) $f(0) = 0$

$$\Rightarrow \quad (Sf)(0) = \sum_{k=0}^{\infty}(M^k f)(0)(Sh_k)(0)$$
$$= (M^0 f)(0)g_0(0) = f(0) = 0$$

so that S is a Stirling transformation. Finally, for any $f \in P^\infty$ we use its Maclaurin representation relative to M, $\langle h_j \rangle$ to obtain

$$(S^{-1}LS)(f) = \sum_k (M^k f)(0)(S^{-1}LS)h_k = \sum_k (M^k f)(0)h_{k-1} = M(f). \quad \square$$

COROLLARY. *The only Maclaurin systems L, $\langle g_n\rangle$ are*

$$g_n = S(x^n/n!) \qquad L = SDS^{-1}$$

for Stirling transformations S.

3. Taylor operators. We have seen that each u.d.d. operator L may be identified with a unique simple basis $\langle g_j\rangle$ so that the pair comprise a Maclaurin system. As we move to another point of expansion, say $a \neq 0$, there will again exist a unique simple basis resulting in an expansion system (at a). But these new polynomials, in general, will bear little relationship to the sequence $\langle g_j\rangle$. In order to achieve the desired universality for the expansion basis, we find that the commutativity property of the following definition is needed.

The u.d.d. operator L is called a **Taylor operator** if

$$LT_a = T_aL$$

for every real number a. Here T_a is the translation

$$(T_af)(x) = f(x-a).$$

Considering the universality requirement as exemplified by the Taylor-Newton expansion, our above definition is justified by the following result.

THEOREM 4. *Let L be a u.d.d. operator and $\langle g_j\rangle$ the unique simple basis for which L, $\langle g_j\rangle$ constitute a Maclaurin system. Then L, $\langle g_j(x-a)\rangle$ is an expansion system (at each a) iff L is a Taylor operator.*

Proof. First suppose that $LT_a = T_aL$. Then considering the simple basis $\langle g_j(x-a)\rangle$ one has
(i) $g_0(x-a) = g_0(x) \equiv 1$,
(ii) $g_j(x-a)(a) = g_j(0) = 0 \qquad (j>0)$,
(iii) $Lg_j(x-a) = LT_ag_j(x) = T_aLg_j(x) = T_ag_{j-1}(x) = g_{j-1}(x-a)$
so that L, $\langle g_j(x-a)\rangle$ is an expansion system (at a), by virtue of Theorem 2.

On the other hand, if the usual properties (i), (ii), (iii) hold relative to L and $\langle g_j(x-a)\rangle$, then for any $f \in P^\infty$ we may consider its Maclaurin expansion

$$f(x) = \sum_{k=0}^{\infty}(L^kf)(0)g_k(x)$$

and use (iii) to obtain

$$\begin{aligned}(LT_af)(x) &= \sum_{k=0}^{\infty}(L^kf)(0)(LT_ag_k)(x)\\ &= \sum_{k=0}^{\infty}(L^kf)(0)g_{k-1}(x-a)\\ &= T_a\sum_{k=0}^{\infty}(L^kf)(0)g_{k-1}(x) = (T_aLf)(x). \quad \square\end{aligned}$$

Note that this result establishes a relationship for Taylor operators which is analogous to the connection between Maclaurin and Taylor series for D. Consequently, it seems that we have come a step closer to the derivative by our introduction of the Taylor operators (but we still admit the difference operator Δ). As for seeing just how close, it remains for us to provide an explicit characterization of the Taylor operators. This could perhaps be accomplished by several means. One could try to examine the nature of their associated Maclaurin polynomials $\langle g_j\rangle$. But instead, we have found a direct characterization of the operators themselves to be more illuminating. For this purpose, we need the following lemma.

LEMMA. *Let L be a u.d.d. operator having the matrix (λ_{ij}) relative to the basis $\langle x^j/j!\rangle$. Then L is a Taylor operator iff*

$$\lambda_{n,n+k} = \lambda_{0,k} \qquad (k \geqq 0).$$

Proof. Let all matrices be expressed relative to the basis $\langle x^j/j!\rangle$. Then, because L is u.d.d., the matrix (λ_{ij}) is upper-triangular. The same is true for the matrix $(\tau_{ij}) = (\tau_{ij}(a))$ of the translation T_a. In fact,

$$\tau_{ij} = \begin{cases} 0 & (i < j) \\ \dfrac{(-a)^{j-i}}{(j-i)!} & (i \leqq j) \end{cases}$$

as found from the binomial theorem. It follows that both of the product matrices

$$(\lambda_{ij})(\tau_{ij}) = (\rho_{ij}) \qquad (\tau_{ij})(\lambda_{ij}) = (\sigma_{ij})$$

are upper-triangular, and in addition, have zero diagonal. Otherwise, the entries in these product matrices are given by the finite sums:

$$\rho_{ij} = \sum_{k=i+1}^{j} \lambda_{ik}\tau_{kj} \qquad \sigma_{ij} = \sum_{k=i}^{j-i} \tau_{ik}\lambda_{kj}$$

for $i < j$.

Now if L is a Taylor operator, these product matrices must agree for every a. But this gives the identity

$$\sum_{k=i+1}^{j} \lambda_{ik}\frac{(-a)^{j-k}}{(j-k)!} = \sum_{k=i}^{j-1} \lambda_{kj}\frac{(-a)^{k-i}}{(k-i)!}$$

Equating like powers of a, we obtain $\lambda_{i,j-r} = \lambda_{i+r,j}$, $(i < j)$. And this means that the entries along all super-diagonals are constant, as claimed. Since our arguments are obviously reversible, the result follows. □

THEOREM 5. *L is a Taylor operator iff it has the form*

$$L = \sum_{k=1}^{\infty} \lambda_k D^k$$

for a sequence $\langle \lambda_k \rangle$ with $\lambda_1 \neq 0$.

Proof. First we observe that the infinite sum when applied to any $f \in P^\infty$ is actually finite, so that questions of convergence do not arise. Now assume that L is any Taylor operator and use the ordinary Taylor expansion

$$f(x) = \sum_{j=0}^{\infty} (D^j f)(0)\frac{x^j}{j!}$$

for $f \in P^\infty$. Applying the operator L and using the notation of the lemma, we have

$$\begin{aligned} (Lf)(x) &= \sum_{j=o}^{\infty} (D^j f)(0) L\frac{x^j}{j!} \\ &= \sum_{j=1}^{\infty} (D^j f)(0) \sum_{i=0}^{j-1} \lambda_{ij}\frac{x^i}{i!} \end{aligned}$$

$$= \sum_{j=1}^{\infty}(D^j f)(0)\sum_{i=0}^{j-1}\lambda_{0,j}\frac{x^i}{i!}$$

$$= \sum_{k=1}^{\infty} l_{0k}\sum_{j=k}^{\infty}(D^j f)(0)\frac{x^{j-k}}{(j-k)!}$$

$$= \sum_{k=1}^{\infty}\lambda_{0k}(D^k f)(x) = \left(\sum_{k=1}^{\infty}\lambda_{0k}D^k f\right)(x). \quad \square$$

Of course, if we take $\lambda_1 = 1$ and all other $\lambda_k = 0$ we obtain the derivative, whereas the choice $\lambda_k = 1/k!$ gives the difference operator. But more importantly, one sees in general the intimate connection of these Taylor operators with the derivative.

4. Rolle operators. Surely one of the most important results in the differential calculus is the mean value theorem, or equivalently, Rolle's theorem. We now propose to study the more general u.d.d. operators having this same property. Our original interest in this property was motivated by considerations which are explained in the following concluding section.

We will say that the u.d.d. operator L is a **Rolle operator** if we have

$$f(a) = f(b) = 0 \Rightarrow (Lf)(\xi) = 0 \qquad (\text{some } \xi \in (a,b))$$

for every $f \in P^\infty$. Of course D is a Rolle operator. And one would hope to discover other such operators and to characterize them as was done for the Taylor operators in the previous section. But the following theorem shows that D is (essentially) the only Rolle operator!

THEOREM 6. *Every Rolle operator is a non-zero multiple of D.*

Proof. Let L be a Rolle operator. First we show that the unique sequence $\langle g_j\rangle$ making L, $\langle g_j\rangle$ a Maclaurin system must have the form $g_j = \alpha_j x^j$. Certainly this is true for $j = 0, 1$. But suppose, on the contrary, that not all g_i are of this form, and let n be the least integer for which

$$g_n(x) = x^k p(x)$$

with $p(0) \neq 0$ and $i \leqq k < n$. Then we set

$$f(x) = g_n(x) - x^k p(a)$$

with a as yet unspecified. We have $f(0) = f(a) = 0$, whereas

$$(Lf)(x) = x^{k-1}(\alpha_{n-1}x^{n-k} - \frac{\alpha_{k-1}}{\alpha_k}p(a)).$$

Since $p(0) \neq 0$, we can now choose $a \neq 0$ so that both factors in $(Lf)(x)$ are of one sign for all x between 0 and a, thus contradicting that L is a Rolle operator. We must therefore conclude that $g_n(x) = \alpha_n x^n$ for all n.

We know that $\alpha_0 = 1$. And it is clear that L, $\langle g_j\rangle$ is a Maclaurin system with L Rolle if and only if the same is true for cL, $\langle g_j/c^j\rangle$. Thus we may continue as if $\alpha_1 = 1$ also. Introducing the quotients $\beta_k = \alpha_{k-1}/\alpha_k$ we have

$$Lx^k = \beta_k x^{k-1}.$$

This makes $\beta_1 = 1$ and we intend to show that $\beta_m = m$ for all $m \geqq 1$. Proceeding inductively, we suppose that $\beta_j = j$ for $j = 1, 2, \cdots, m-1$ and set

$$h(x) = (x-a_1)(x-a_2)\cdots(x-a_m) \qquad (a_1 < a_2 < \cdots < a_m)$$
$$= x^m - (a_1 + a_2 + \cdots + a_m)x^{m-1} + \cdots.$$

Since L is a Rolle operator and h is zero m times on $[a_1, a_m], L^{m-1}h$ should be zero once on (a_1, q_m). But our inductive assumption gives

$$(L^{m-1}h)(x) = \beta_m(m-1)!x - (a_1 + a_2 + \cdots + a_m)(m-1)!$$

and this is zero only at

$$x_0 = \frac{a_1 + a_2 + \cdots + a_m}{\beta_m}.$$

But we can have $x_0 \in (a_1, a_m)$ for all choices of $a_1, a_2, \cdots, a_m$ only if $\beta_m = m$. Thus, $\beta_j = j$ for all $j > 0$, and the g_j for this (normalized) L are

$$g_j = x^j / j!$$

Consequently, in the general case, $L = cD$. □

Another important property of differentiation is the so-called chain rule. In view of its importance in applications of the calculus, and in the sake of completeness, we now wish to investigate the class of operators satisfying such a rule. Thus, we shall say that the u.d.d. operator L is a **chain operator** provided that

$$(L(f \circ g))(x) = (Lf)(x))(Lg)(x)$$

for all $f, g \in P^\infty$. As with Rolle operators, the "class" of chain operators turns out to be rather limited. In fact, as the next theorem shows, the chain rule completely characterizes differentiation among u.d.d. operators. But first, we need a lemma.

LEMMA. *The only Taylor-chain operator is D.*

Proof. Considering the representation of Theorem 5, we may write

$$L = \sum \lambda_i D^i$$

with $\lambda_1 \neq 0$. In fact, $\lambda_1 = 1$, for if we take $f(x) = g(x) = x$ in the chain rule, we find that $\lambda_1 = \lambda_1^2$. Now suppose that there is some other non-zero coefficient λ_k so that

$$L = D + \lambda_k D^k + \cdots \qquad (k \geqq 2).$$

We have only to take $f(x) = x^2$ and $g(x) = x^k$ in the chain rule to arrive at a contradiction:

$$(L(f \circ g))(x) = 2kx^{2k-1} + \lambda_k \frac{(2k)!}{k!} x^k + \cdots$$

$$(Lf)(g(x))(Lg)(x) = 2kx^{2k-1} + 2\lambda_k k! x^k + \cdots.$$

(When $k = 2$ the last line must be modified slightly, but the argument is still the same.) Thus we conclude that $\lambda_k = 0$ for $k > 1$, and finally, $L = D$. □

THEOREM 7. *The only chain operator* (*among u.d.d. operator*) *is D.*

Proof. We need only verify that every chain operator L is a Taylor operator. Let $\langle g_j \rangle$ be the sequence of polynomials for which $L, \langle g_j \rangle$ is a Maclaurin system. Then $g_0 = 1$ and $g_1(x) = cx$. But choosing $f = g = g_1$ in the chain rule gives $c = 1$. It following that

$$(LT_a f)(x) = Lf(x-a) = (Lf)(x-a)L(x-a)$$

$$= (Lf)(x-a) = (T_a Lf)(x)$$

for any $f \in P^\infty$. So L is a Taylor operator, and by the lemma, $L = D$. □

5. Concluding remarks. Having begun with rather weak assumptions for linear operators on P^∞ sharing certain features of differentiation, we have imposed a sequence of "derivative-like"

properties, leading ultimately to D itself. We believe that the intermediate characterizations of the Taylor and Rolle operators are quite interesting in their own right. Overall, the investigation seems to provide for a new appreciation of the essential uniqueness of the classical differential calculus.

The question of possible extensions of these operators to larger spaces must naturally arise, particularly in connection with the concept of an expansion system. The usual Taylor theorem is phrased in the space C^n and introduces the idea of a remainder. And so, a suitable generalization of the Taylor remainders would be needed for the u.d.d. operators. But our result concerning Rolle operators suggests that the class of operators having "derivative-like" remainders on C^n may be rather limited, thus accounting for our restriction to polynomials in this study.

Added in proof. Since this article was accepted for publication, it has come to the authors' attention that certain of these ideas have appeared in a paper by G. C. Rota, D. Kahaner and O. Odlyzko, *On the Foundations of Combinatorial Theory, VIII, Finite Operator Calculus*, Jour. of Math. Anal. and Appl., 42 (1973).

References

1. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, New York, 1956.
2. P. J. Davis, Interpolation and Approximation, Blaisdell, New York, 1963.

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ADDENDUM TO:

"EULER AND THE ZETA FUNCTION"

A. G. HOWSON

Readers of Raymond Ayoub's article (this MONTHLY, 81 (1974) 1067–1085) may be amused to learn that one of the questions set to candidates for the first London University Matriculation Examination (in 1838), an examination for students of 19 years or under who wished to enter the university, was

"Find the sums to infinity of the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots,\quad \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots."$$

There is no indication how the examiner intended the question to be solved; the examination syllabus, which did not include the calculus, referred only to "arithmetical and geometrical progressions" and "arithmetic and algebra." It can, however, be inferred from a previous question

"Prove that

$$\text{Nap}^n \log x = (x - x^{-1}) - 1/2(x^2 - x^{-2}) + 1/3(x^3 - x^{-3}) - \cdots"$$

that a cavalier treatment of infinite series was not only tolerated but actively encouraged.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

Reply to Query 14. This Query asked for sources of mathematical films other than Modern Learning Aids or International Film Board. Educational Solutions, Inc., 85th Ave. New York, N.Y. 10011, has 3 series of films available and a pamphlet can be obtained from them describing these. The Association of Teachers of Mathematics of Great Britain has also published a list of films available in that country. It may be obtained by writing to Mr. D. G. Tahta, Association of Teachers of Mathematics, Market Street, Chambers, Nelson, Lancashire, England. Mr. T. J. Fletcher, Department of Education and Science, Mowden Hall, Staindrop Road, Darlington, County Durham, England, has a private card file of many mathematics films.

Reply to Query 16. This Query asked for precise sources of Mertens' First Theorem. Many readers wrote in to point out that this could be found in J. Reine Angew. Math., 78 (1874) 48–49.

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

ROUND METRIC SPACES

MELVYN B. NATHANSON

Let X be a metrizable topological space with at least two points, and let d be a metric for X. For $x \in X$ and $r > 0$, let $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$ denote the open ball around x of radius r, and let $B_d[x, r] = \{y \in X \mid d(x, y) \leqq r\}$ denote the closed ball around x of radius r. If $A \subset X$, let $\bar{A}$ denote the closure of A in X. Then $\overline{B_d(x, r)} \subset B_d[x, r]$. We call d a **round metric** for X if $\overline{B_d(x, r)} = B_d[x, r]$ for all $x \in X$ and $r > 0$. Clearly, the metric d is round if and only if $y \in \overline{B_d(x, d(x, y))}$ for all $x, y \in X$ with $x \neq y$. A metrizable space X is a **round metric space** if there exists a round metric for X. In this note we present some elementary results on round metric spaces.

First, some examples. If L is a real or complex linear space with norm $\| \|$, then $d(x, y) = \|x - y\|$ is a round metric for L. In particular, the Euclidean spaces R^n are round in their usual metrics. The disconnected subspace $X_1 = (0, 1) \cup (2, 3)$ of R is round in the usual metric inherited from R. But

subspaces of round metric spaces are not always round. For example, $X_2 = [0,1] \cup [2,3]$ is not round in its usual metric, since $[0,1] = \overline{B(1,1)} \neq B[1,1] = [0,1] \cup \{2\}$. In fact, Theorem 1 below shows that no metric equivalent to the usual metric on X_2 can be round. Similarly, every discrete space is metrizable, but no metric d on a discrete space is round, since $\overline{B_d(x,r)} = B_d(x,r)$ for all $x \in X$ and $r > 0$. The unit circle $T = \{e^{it} \mid 0 \leqq t \leqq 2\pi\}$ is round in the usual metric on the plane. Define a new metric d on T by

$$d(e^{is}, e^{it}) = \begin{cases} |e^{is} - e^{it}| & \text{if } 0 \leqq s, t \leqq \pi, \\ |e^{is} - \cos t| & \text{if } 0 \leqq s \leqq \pi \text{ and } \pi \leqq t \leqq 2\pi, \\ |\cos s - \cos t| & \text{if } \pi \leqq s, t \leqq 2\pi. \end{cases}$$

Then d is equivalent to the usual metric on T, but d is not round, because

$$\overline{B_d(e^{i3\pi/2}, 1)} = \{e^{it} \mid \pi \leqq t \leqq 2\pi\} \neq B_d[e^{i3\pi/2}, 1] = T.$$

Let A be a discrete valuation ring with quotient field K and valuation v, and choose $\alpha \in (0,1)$. Then $d(x,y) = \alpha^{v(x-y)}$ is a metric on K. In the metric space K, every open ball is closed, since $B_d(x,\alpha^n) = B_d[x,\alpha^{n+1}]$, but d is not a round metric since $\overline{B_d(x,\alpha^n)} = B_d[x,\alpha^{n+1}] \neq B_d[x,\alpha^n]$.

THEOREM 1. *Let $X = A \cup K$ be a metrizable space, where A and K are nonempty, disjoint, closed sets, and K is compact. Then no metric for X is round.*

Proof. Let d be a metric for X. Choose $a \in A$. Since K is compact, there exists $k \in K$ such that $\operatorname{dist}(a,K) = \inf\{d(a,x) \mid x \in K\} = d(a,k)$. Then $B_d(a, d(a,k)) \subseteq A$. Since A is closed, $\overline{B_d(a,d(a,k))} \subseteq A$, and so $k \notin \overline{B_d(a,d(a,k))}$. Therefore, d is not a round metric for X.

It follows from Theorem 1 that if a metric space X has an isolated point, then no metric for X is round.

THEOREM 2. *Let $(X;d)$ and (Y,e) be metric spaces without isolated points, and let $f: X \to Y$ be a surjection such that for $x,y,z \in X$, if $d(x,z) < d(x,y)$, then $e(f(x),f(z)) < e(f(x),f(y))$. If d is a round metric for X, then e is a round metric for Y.*

Proof. The function f is continuous. For let $x \in X$, and let $\{y_n\}$ be a sequence in X such that $d(y_n,x) > d(y_{n+1},x)$ for all $n \geqq 1$, and such that $\{y_n\}$ converges to x. Then $e(f(y_n),f(x)) > e(f(y_{n+1}),f(x))$ for all $n \geqq 1$. Let $s = \inf\{e(f(y_n),f(x))\}$. Suppose $s > 0$. Since $f(x)$ is not an isolated point of Y, there exists $f(z) \in B_e(f(x),s)$ with $z \neq x$. Then $e(f(z),f(x)) < s \leqq e(f(y_n),f(x))$ for all $n \geqq 1$. Therefore, $0 < d(z,x) \leqq d(y_n,x)$ for all $n \geqq 1$. But this is impossible, since $\{d(y_n,x)\}$ decreases to zero. Therefore, $s = 0$, the sequence $\{f(y_n)\}$ converges to $f(x)$, and the function f is continuous. (In fact, f is a homeomorphism of X onto Y.)

Let $f(x), f(y) \in Y$ with $x \neq y$. If d is a round metric for X, then $y \in \overline{B_d(x,d(x,y))}$, and so there is a sequence $\{z_n\}$ in X such that $d(z_n,x) < d(y,x)$ and $\{z_n\}$ converges to y. Then $e(f(z_n),f(x)) < e(f(y),f(x))$, and so $f(z_n) \in B_e(f(x),e(f(x),f(y)))$. Since f is continuous, $\{f(z_n)\}$ converges to $f(y)$, and so $f(y) \in \overline{B_e(f(x),e(f(x),f(y)))}$. Therefore, (Y,e) is a round metric space.

COROLLARY 1. *Let d be a round metric for X. If e is a metric on X equivalent to d such that $e(x,z) < e(x,y)$ whenever $d(x,z) < d(x,y)$, then e is also a round metric for X.*

Proof. Let $f: (X,d) \to (X,e)$ be the identity map.

COROLLARY 2. *Let (X,d) be a round metric space. Then there exists an equivalent metric e on X which is bounded and round.*

Proof. Let $e(x,y) = d(x,y)/(1 + d(x,y))$. The metric e is equivalent to d on X, and bounded. If $d(x,z) < d(x,y)$, then $e(x,z) < e(x,y)$, and, by Corollary 1, the metric e is round.

Corollary 2 of Theorem 2 contrasts with the following result.

THEOREM 3. *Let (X, d) be a metric space. Then there exists an equivalent metric e on X that is bounded but not round.*

Proof. Let a and b be distinct points of X, and let r satisfy $0 < r < d(a,b)$. Define the metric e on X by $e(x,y) = \min(r, d(x,y))$. Then e is a bounded metric equivalent to d. Moreover, $B_d(a,r) = B_e(a,r)$, and so $\overline{B_d(a,r)} = \overline{B_e(a,r)}$. Since $d(a,b) > r$, it follows that $b \not\in \overline{B_d(a,r)}$ and also $e(a,b) = r$. Therefore, $b \not\in \overline{B_d(a,r)} = \overline{B_e(a,r)} = \overline{B_e(a, e(a,b))}$, and so e is not a round metric for X.

The next two theorems give conditions for roundness.

THEOREM 4. *Let (X_k, d_k) be a countable family of metric spaces such that diam $(X_k) < \infty$ for all but finitely many k. The product space $X = \Pi_k X_k$ is metrizable. If $x = (x_k), y = (y_k) \in X$, define*

$$d(x,y) = \sum_{k=1}^{\infty} \frac{d_k(x_k, y_k)}{\lambda_k 2^k},$$

where $\lambda_k = 1$ if diam $(X_k) = \infty$ and $\lambda_k =$ diam (X_k) if diam $(X_k) < \infty$. Then d is a metric for X. The metric d is round for X if and only if the metric d_k is round for X_k for all k.

Proof. It is well known that X is metrizable, and that d is a metric for X [2]. Suppose that (X_k, d_k) is round for all k. Let $x = (x_k)$, $y = (y_k) \in X$ with $x \neq y$. Then $x_i \neq y_i$ for some i. Since $y_i \in \overline{B_{d_i}(x_i, d_i(x_i, y_i))}$, there is a sequence $\{z_i^{(n)}\} \subseteq B_{d_i}(x_i, d_i(x_i, y_i))$ which converges to y_i. Define the sequence $\{z^{(n)}\}$ in X by

$$z_k^{(n)} = \begin{cases} y_k & \text{if} \quad k \neq i \\ z_i^{(n)} & \text{if} \quad k = i. \end{cases}$$

Then $d(z^{(n)}, y) = d_i(z_i^{(n)}, y_i)/\lambda_i 2^i$, and so $\{z^{(n)}\}$ converges to y in X. Moreover,

$$\begin{aligned} d(z^{(n)}, x) &= \sum_{k=1}^{\infty} \frac{d_k(z_k^{(n)}, x_k)}{\lambda_k 2^k} = \sum_{\substack{k=1 \\ k \neq i}}^{\infty} \frac{d_k(y_k, x_k)}{\lambda_k 2^k} + \frac{d_i(z_i^{(n)}, x_i)}{\lambda_i 2^i} \\ &= d(x,y) - \frac{1}{\lambda_i 2^i}(d_i(y_i, x_i) - d_i(z_i^{(n)}, x_i)) < d(x,y), \end{aligned}$$

and so $\{z^{(n)}\} \subseteq B_d(x, d(x,y))$. Therefore, $y \in \overline{B_d(x, d(x,y))}$ for all x, $y \in X$ with $x \neq y$, and so d is a round metric for X.

Conversely, suppose that d is a round metric for X. Let $x_i, y_i \in X_i$ with $x_i \neq y_i$. Choose $w_k \in X_k$ for $k \neq i$, and let $x = (x_k)$, $y = (y_k) \in X$ be defined by

$$x_k = \begin{cases} w_k & k \neq i \\ x_i & k = i, \end{cases} \qquad y_k = \begin{cases} w_k & k \neq i \\ y_i & k = i. \end{cases}$$

Then $d(x,y) = d_i(x_i, y_i)/\lambda_i 2^i$. Let $\{z^{(n)}\} \subseteq B_d(x, d(x,y))$ such that the sequence $\{z^{(n)}\}$ converges to y. Then $z_i^{(n)}$ converges to y_i. Moreover,

$$\begin{aligned} \frac{d_i(x_i, z_i^{(n)})}{\lambda_i 2^i} &\leqq \sum_{k=1}^{\infty} \frac{d_k(x_k, z_k^{(n)})}{\lambda_k 2^k} = d(x, z^{(n)}) \\ &< d(x,y) = \frac{d_i(x_i, y_i)}{\lambda_i 2^i}, \end{aligned}$$

and so $d_i(x_i, z_i^{(n)}) < d_i(x_i, y_i)$. Therefore, $\{z_i^{(n)}\} \subseteq B_{d_i}(x_i, d_i(x_i, y_i))$, and so $y_i \in \overline{B_{d_i}(x_i, d_i(x_i, y_i))}$. Therefore, d_i is a round metric for X_i.

COROLLARY. *A countable product of round metric spaces is round.*

Proof. By Corollary 2 of Theorem 2, every round metric space has an equivalent bounded round metric.

The following result was stated by Artémiadis [1] in the special case $\alpha = 1/2$.

THEOREM 5. *Let* $\alpha \in (0,1)$. *If* (X,d) *is a metric space such that for any* $x, y \in X$ *there exists* $z \in X$ *such that* $d(z,x) = (1-\alpha)\, d(x,y)$ *and* $d(z,y) = \alpha d(x,y)$, *then* (X,d) *is round.*

Proof. Let $x, y \in X$ with $x \neq y$. Define a sequence $\{z_k\}$ in X inductively by: $z_0 = x$, and z_k satisfies $d(z_k, z_{k-1}) = (1-\alpha)\, d(z_{k-1}, y)$ and $d(z_k, y) = \alpha d(z_{k-1}, y)$ for $k \geqq 1$. Then $d(z_k, y) = \alpha^k d(x,y)$, and so the sequence $\{z_k\}$ converges to y. Also,

$$\begin{aligned} d(z_k, x) = d(z_k, z_0) &\leqq \sum_{i=0}^{k-1} d(z_{i+1}, z_i) \\ &= \sum_{i=0}^{k-1} (1-\alpha)\, d(z_i, y) = \sum_{i=0}^{k-1} (1-\alpha)\, \alpha^i d(x,y) \\ &= (1-\alpha^k)\, d(x,y) < d(x,y), \end{aligned}$$

and so $z_k \in B_d(x, d(x,y))$. Therefore, $y \in \overline{B_d(x, d(x,y))}$, and (X,d) is round.

References

1. N. Artémiadis, A remark on metric spaces, Nederl. Akad. Wetensch. Proc. Ser. A, 68 = Indag. Math., 27 (1965) 316–318.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

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EXTENSIONS OF THE WEIERSTRASS PRODUCT INEQUALITIES II

M. S. KLAMKIN

In a previous note [1], it was shown that

$$\Pi(1+A_i) \geqq (n+1)^n \Pi A_i, \tag{1}$$

$$\Pi(1-A_i) \geqq (n-1)^n \Pi A_i, \tag{2}$$

where $A_i \geqq 0$ $(i = 1,2,\cdots,n)$ and $\Sigma A_i = 1$. Under the same conditions, we now show that

$$\frac{\Pi(1+A_i)}{(n+1)^n} \geqq \frac{\Pi(1-A_i)}{(n-1)^n} \tag{3}$$

with equality iff $A_i = 1/n$. Our proof is similar to one of the more elegant proofs [2] of the arithmetic-geometric mean inequality. We consider another sequence $\{A'_i\}$ such that

$$A'_1 = A,\ A'_2 = A_1 + A_2 - A,\ A'_r = A_r \text{ for } r > 2,$$

where A_1, A_2 and A denote the smallest term, the largest term and the arithmetic mean $1/n$ of the sequence $\{A_i\}$, respectively. Then,

$$\Pi \frac{1+A_i}{1-A_i} \geqq \Pi \frac{1+A'_i}{1-A'_i}$$

since the latter is equivalent to

$$\frac{(1+A_1)(1+A_2)}{(1-A_1)(1-A_2)} \geqq \frac{(1+A)(1+A_1+A_2-A)}{(1-A)(1-A_1-A_2+A)}$$

or $(A_2 - A)(A - A_1) \geqq 0$. On repeating this transformation of the sequence $\{A_i\}$ at most $n-1$ times, we arrive at a sequence in which all the terms are equal to A, which then completes the proof.

A similar inequality had been given previously by Ky Fan [3, p. 363] but here the conditions on the A_i are tighter and relaxed on the ΣA_i, i.e.,

$$\Pi(1-A_i) \geqq \left\{\frac{n-\Sigma A_i}{\Sigma A_i}\right\}^n \Pi A_i \quad \text{for} \quad 0 < A_i \leqq 1/2. \tag{4}$$

Finally, (2) provides a simple solution to a previous proposed problem and solution by J. Berkes and C. Bindschelder, respectively, in Elem. Math., 14 (1959) 132 (also see [3, p. 214]) i.e., if n is a positive integer and $x_i > 0$ for $i = 1, 2, \cdots, n$, then

$$\sum \frac{1}{1+x_i} \geqq n-1 \Rightarrow \Pi \frac{1}{x_i} \geqq (n-1)^n. \tag{5}$$

First note that we can replace the $\geqq$ sign by an equal sign in $\Sigma 1/(1+x_i) \geqq n-1$. Then simply let $A_i = x_i/(1+x_i)$. For other proofs of (5) and related results, see [4].

References

1. M. S. Klamkin, D. J. Newman, Extensions of the Weierstrass product inequalities, Math. Mag., 43 (1970) 137–140.

2. G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, London, 1934, p. 20.

3. D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970.

4. G. Kalajdžić, Two elementary inequalities, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 419 (1973) 53–56.

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A HELLY-TYPE THEOREM FOR LINEAR COLLINEATIONS

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We shall prove a theorem concerning families of linear transformations of a pappian plane. The result was suggested by Helly's Theorem on convex sets (see [1]). Basic definitions and theorems concerning projective and affine planes can be found in Dembowski [2]. In particular we shall require the following results:

(1) If the product of two central collineations of a projective plane is central, and if the centers are collinear, then either the centers or the axes are identical. This follows immediately from ([2], p. 120, 7).

(2) If γ is a linear collineation of a pappian projective plane which fixes three distinct points of the line l, then γ is central with axis l.

For γ must fix every point on l, by the so-called Fundamental Theorem of Projective Geometry, ([2], p. 160, 8) and hence is central with axis l, ([2], p. 30, 9.)

DEFINITION. A collection of transformations $\{\alpha_i\}$ *agree* on a point P if $\alpha_i(P) = P'$ for every i.

They *agree* on a line p if $\alpha_i(p) = p'$ for every i. They *strongly agree* on a point P if they agree on every line incident with P. They *strongly agree* on a line p if they agree on every point incident with p.

THEOREM. *Let $\mathscr{A} = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ be a finite collection of linear collineations of a pappian projective plane π which agree on a line p. If each three of the transformations agree on a point of π-p, then they all agree on a point P of π. If there is no such P in π-p, then the agreement on P is strong and $\alpha_i^{-1}\alpha_j$ are central collineations, all having the unique center $P \in p$.*

Proof. The theorem will be proved by induction on n. The theorem is true for $n = 2$. Assume that $n \geqq 3$ and that the theorem holds for all m, with $2 \leqq m < n$. Let $\mathscr{A}_i = \{\alpha_0, \hat{\alpha}_i \cdots, \alpha_n\}$, $i = 0, 1, 2, \cdots, n$, where $\hat{\alpha}_i$ indicates that α_i is missing from $\mathscr{A}_i$. Let $\mathscr{P}_i$ be the set of points in π on which the members of $\mathscr{A}_i$ agree. By induction hypothesis $\mathscr{P}_i \neq \emptyset$, $i = 0, 1, \cdots, n$. If α_i are not distinct, the theorem is valid; in what follows we will assume they are distinct, or equivalently for $i \neq j$, $\alpha_i^{-1}\alpha_j$ is not the identity.

We note that if for some $i \neq j$, $\mathscr{P}_i \cap \mathscr{P}_j \neq \emptyset$, then the theorem is valid. If for some k, $\mathscr{P}_k \cap (\pi\text{-}p) = \emptyset$, then by the induction hypothesis all the central collineations $\alpha_i^{-1}\alpha_j$, $i \neq j \neq k$, $i = 0, 1, \cdots, n$; $j = 0, 1, \cdots, n$ have the same center $\mathscr{P}_k$. Also if for some $l, l \neq k$, $\mathscr{P}_l \cap (\pi\text{-}p) = \emptyset$, then all the central collineations $\alpha_i^{-1}\alpha_j$, $i \neq j \neq l$, $i = 0, 1, \cdots, n$; $j = 0, 1, \cdots, n$, will have the same center $\mathscr{P}_l$. Since $n \geqq 3$, then $\mathscr{P}_k = \mathscr{P}_l$ and the theorem is valid. Hence it will be assumed that exactly one $\mathscr{P}_k$, call it $\mathscr{P}_0$, may fail to have a point in π-p. Now the $P_i \in \mathscr{P}_i \cap (\pi\text{-}p)$, $i = 1, 2, \cdots, n$ may be chosen as distinct points.

We prove the theorem by examining the two cases of $\mathscr{P}_0$.

CASE 1. $\mathscr{P}_0$ has a point P_0 in π-p. The set of points $\mathscr{L} = \{P_0, P_1, \cdots, P_n\}$, which may be taken as distinct, are all in π-p and they are, with the possible exception of P_i and P_j, fixed points of $\alpha_i^{-1}\alpha_j$. Since $n \geqq 3$, each $\alpha_i^{-1}\alpha_j$, $i \neq j$, has at least two fixed points in $\mathscr{L}$ and hence a fixed line l_{ij}; $\alpha_i^{-1}\alpha_j$ also fixes p and consequently fixes a third point $l_{ij} \cap p$ on l_{ij}. Therefore, by the result (2) above every point of l_{ij} is fixed and $\alpha_i^{-1}\alpha_j$ is central with axis l_{ij} and a center on p. We note that all points of $\mathscr{L}$ with the possible exception of P_i and P_j belong to $l_{ij} (i \neq j)$. If $P_i \in l_{ij}$, then all α_i agree on P_i, and if $P_i, P_j \in l_{ij}$, then all α_i strongly agree on l_{ij}.

CASE 1a. Some three points of $\mathscr{L}$ are collinear. It will seen that in this case all points of $\mathscr{L}$ are collinear. Assume, with possible renaming, that P_1, P_2 and P_3 are collinear on l. $P_1, P_3 \in l_{20}$ and $P_2, P_3 \in l_{01}$ imply that $l = l_{20} = l_{01}$, and $\alpha_2^{-1}\alpha_0, \alpha_0^{-1}\alpha_1$ both have the same axis l. Their product $(\alpha_2^{-1}\alpha_1) = (\alpha_2^{-1}\alpha_0)(\alpha_0^{-1}\alpha_1)$ also has same axis, and $l = l_{21}$.
Since $P_2, P_1 \in l$, hence l_{21} contains all points of $\mathscr{L}$, and α_i strongly agree on a line $l \neq p$.

CASE 1b. No three points of $\mathscr{L}$ are collinear. We note that this can only occur if $n = 3$. For if $n > 3$, P_1, P_2 and P_3 are collinear on the axis l_{04}.

Let 0 be the centre of $(\alpha_0^{-1}\alpha_1)$, and consider $(\alpha_0^{-1}\alpha_1) = (\alpha_0^{-1}\alpha_i)(\alpha_i^{-1}\alpha_1)$, $i = 2, 3$. The axes of the central collineations inside the parentheses are distinct, and their centers are collinear on p, hence by result (1) above, it follows that their centers must coincide with 0. This holds for each i, and consequently all α_i strongly agree on 0; 0 is unique, because $\alpha_i^{-1}\alpha_j$ is not the identity for $i \neq j$. This proves the theorem for Case 1.

CASE 2. $\mathscr{P}_0$ has no point in π-p. In this case $\mathscr{P}_0$ is a unique point $P_0 \in p$, and by induction all $\alpha_i^{-1}\alpha_j$, $i \neq 0, j \neq 0$, in particular $\alpha_2^{-1}\alpha_1$, are central with center P_0. To complete the proof it is sufficient to show that $\alpha_0^{-1}\alpha_1$ is also central with center P_0.

$\alpha_2^{-1}\alpha_0$ and $\alpha_0^{-1}\alpha_1$ are central with centers on p, because they have fixed points P_1, P_3 and P_2, P_3 respectively in π-p. Also, their axes P_1P_3 and P_2P_3 must be different because, as in Case 1a, the collinearity of P_1, P_2 and P_3 implies that all α_i agree at a point of π-p, contradicting Case 2. This also shows that Case 2 can occur only if $n = 3$.

Central collineations in the parentheses, $(\alpha_2^{-1}\alpha_1)=(\alpha_2^{-1}\alpha_0)(\alpha_0^{-1}\alpha_1)$ have collinear centers, and the ones in the right hand side have different axes, hence by result (1) above, they must have the same center P_0. Therefore, $\alpha_0^{-1}\alpha_1$ is also central with center P_0, and so are all $\alpha_i^{-1}\alpha_j$. The uniqueness of P_0 has already been established. This completes the proof of the theorem.

COROLLARY 1. *Let $\{\alpha_i\}$ be a finite collection of linear affine transformations of a pappian affine plane π. If each three transformations agree on a point of π, then they all agree either on a point of π, or on every line of a pencil of parallel lines. In the latter case, $\alpha_i^{-1}\alpha_j$, $i\neq j$, are perspective affinities, all having the same direction.*

In the case of isometries in a Euclidean plane, $\alpha_i^{-1}\alpha_j$, $i\neq j$, become line reflections. But $\alpha_0^{-1}\alpha_j, =(\alpha_0^{-1}\alpha_2)(\alpha_2^{-1}\alpha_1)$ shows that the product of two line reflections is a line reflection, which is impossible. In this case we simply have:

COROLLARY 2. *In a finite collection of isometries of a Euclidean plane, if each three isometries agree on a point, then they all agree on a point.*

References

1. L. Danzer, B. Grunbaum, V. Klee, Helly's theorem and its relatives, Proc. of Symposia in Pure Math. Vol. VII, Convexity, American Mathematical Society, 1963.
2. P. Dembowski, Finite Geometries, Springer-Verlag, New York, 1968.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

EXTENSIONS OF SEMILATTICES

M. A. ARBIB AND E. G. MANES

In our study of nondeterministic automata [1] we have formulated what appears to be an unsolved problem in the theory of semilattices.

Let X be a finite set. By a **composite algebra** (A,ψ) we mean a sup-semilattic A (including the empty supremum 0) equipped with a sup-preserving operator $\psi_x\colon A\to A$ for each x in X.

By an **extension** of (A,ψ) we mean a composite algebra (A',ψ') such that A is a sub sup-semilattice of A' and $\psi'_x(a)=\psi_x(a)$ for all a in A.

PROBLEM 1. *Given a finite composite algebra, find an extension with the minimal number of join-irreducibles.*

For our applications in automata theory, the problem is somewhat more specific: let X^* be the free monoid on X and let Y be a finite sup-semilattice. Let Y^{X^*} be the composite algebra with coordinatewise suprema and $\psi_x(f)=fL_x$ where $L_x(w)=xw$.

PROBLEM 2. *Let $(A, \psi) \subset Y^{X^*}$ be the join-closure of the X^*-closure of a single element f of Y^{X^*}, and assume A is finite, Find an extension of A having the minimal number of join-irreducibles.*

AN EXAMPLE [2, 10.6] AND ELUCIDATION. Let $Y = 2^{\{a,b,c,d\}}$, let $X = \{x\}$ and let f: $X^* \to Y$ be the sequence $a/b/c/d//ab/ac/ad//$, i.e., $f(x^0) = \{a\}, f(x^5) = \{a, c\}$ etc., and // indicates a return point for cycling, e.g., $f(x^8) = \{a, c\}$. Writing fw for fL_w we have:

$$fx = b/c/d//ab/ac/ad//$$
$$fx^2 = c/d//ab/ac/ad//$$
$$fx^3 = d//ab/ac/ad//$$
$$fx^4 = //ab/ac/ad// = fx^7 = \cdots$$
$$fx^5 = //ac/ad/ab// = fx^8 = \cdots$$
$$fx^6 = //ad/ab/ac// = fx^9 = \cdots$$

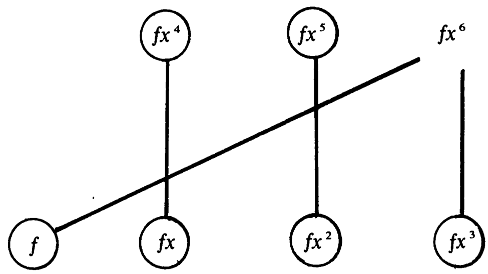

FIG. 1

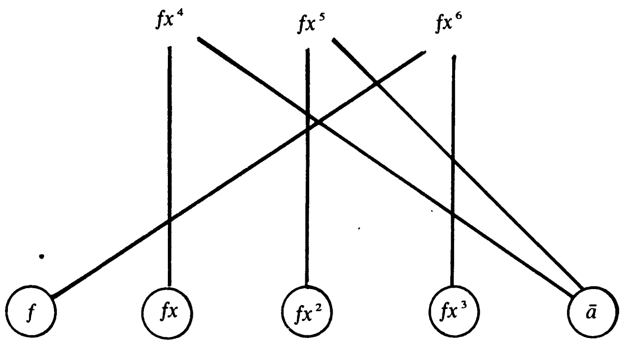

FIG. 2

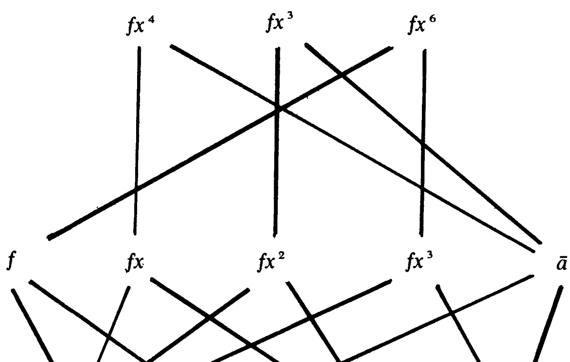

FIG. 3

The inclusion relationships between $f, \cdots, fx^6$ are shown in Figure 1. Here fx^6 *is* the union of f and fx^3. The circled items are the join-irreducibles of the composite subalgebra A generated by f. Adjoining the function $\bar{a}: X^* \to Y$ which is constantly $a, \{f, \cdots, fx^6, \bar{a}\}$ is X^*-closed and we have that fx^4 is the union of fx and $\bar{a}$, whereas fx^5 is the union of fx^2 and $\bar{a}$. As shown in Figure 2, the join-closure A' has only the five circled join-irreducibles. It is easy to show (With *ad hoc* methods) that no extension of A can have fewer than five join-irreducibles. Figure 3 depicts a *semilattice* containing A' as a sub semilattice which has only four join-irreducibles; (as usual, we have not shown extraneous suprema).

References

1. M. A. Arbib and E. G. Manes, Fuzzy morphisms in automata theory, in Proceedings of the First International Symposium: Category Theory Applied to Computation and Control, E. G. Manes (ed.), Springer-Verlag, Lecture Notes in Computer Science, 25 (1975) 80–86.

2. H. Ehrig, K.-D. Kiermeier, H.-J. Kreowski and W. Kühnel, Universal Theory of Automata, Teubner, 1974.

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ARE DISJOINT CONVEX COVERINGS OF A CONVEX SET DIAMETRICALLY OPTIMAL?

G. T. SALLEE

Let X be any subset of E^d, d-dimensional Euclidean space. The **diameter** of X, $\operatorname{diam}(X) = \sup\{d(x, y): x, y \in X\}$ where $d(x, y)$ is the Euclidean distance between x and y. We say $X_1, \cdots, X_n$ is a **(disjoint)** n**-covering** of X if $X \subseteq \cup X_i$ (and if the X_i are disjoint on interiors). A covering is **convex** if each of the X_i is convex.

The title question may now be precisely formulated as follows:

Suppose K is any compact, convex subset of E^d and $X_1, \cdots, X_n$ is an n-covering of K. Does there exist a disjoint, convex n-covering $K_1, \cdots, K_n$ such that diam $(K_i) \leqq$ max (diam (X_i))?

The question has been settled affirmatively in the plane by Lenz [2] who established a somewhat stronger result.

THEOREM (LENZ). *If K is any plane set which can be covered by n convex sets of diameter $\leqq \delta$, then K can be covered by n disjoint convex sets of diameter $\leqq \delta$.*

In higher dimensions, however, the convexity of K is needed. An easy example of this in E^3 is obtained if K is the union of a ball B of diameter 1 and a set S of constant width 1 containing a regular tetrahedron of side 1 which have a small intersection near one vertex of S. (See [**3**, p. 81] for a construction for S.) Then bd $B \cap$ bd S does not lie in a plane and it is easily seen that K has no disjoint, convex n-covering for any n.

A more general variant of the problem above may be posed by considering r-fold coverings of K. Here, $X_1, \cdots, X_n$ is an **[exact]** r**-fold,** n**-covering** of K if $K \subseteq \cup X_i$ and each point of K lies in at least r of the X_i [exactly r of the X_i]. Then we may ask:

Suppose K is any compact, convex subset of E^d and $X_1, \cdots, X_n$ is an r-fold, n-covering of K. Does there exist an exact, convex r-fold, n-covering $K_1, \cdots, K_n$ such that diam $(K_i) \leqq$ max (diam (X_i))?

The example given above can be modified slightly by taking r copies each of B and S to see that

the convexity of K is also needed in this problem if an exact r-fold n-covering is to be found.

I am grateful to the referee for suggesting this second question.

Finally, let $X_1, \cdots, X_n$ be an n-covering of the compact convex set K, and put $\sigma(K; X_1, \cdots, X_n) = \max(\operatorname{diam} X_i)$. Let $\sigma_n(K) = \min \sigma(K; X_1, \cdots, X_n)$ where the minimum is taken over all n-coverings of K, and let $\sigma(d, n) = \max \sigma_n(K)$ where K ranges over all d-dimensional sets of diameter 1. A number of authors have discussed the problem of determining the exact bounds for $\sigma(d, n)$ or $\sigma(K, n)$ for various K, d and n (see [1] for a good summary of known results).

If the title problem were settled affirmatively for all compact, convex sets K, it would simplify the investigation not only of the problems mentioned above (which is how Lenz used his result), but might also assist in settling Borsuk's conjecture.

References

1. B. Grünbaum, Borsuk's problem and related questions, Proc. Symp. Pure Math., 7 (Convexity), Amer. Math. Soc., Providence, (1963) 271–284; MR 27 # 4134.
2. H. Lenz, Über die Bedeckung ebener Punktmengen durch solche kleineren Durchmessers, Arch. Math., 7 (1956) 34–40; MR 17 # 888.
3. I. M. Yaglom and V. G. Boltyanskii, Convex Figures, Holt, Rinehart and Winston, New York, 1961.

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CLASSROOM NOTES

EDITED BY RICHARD A. BRUALDI

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THE GREEN'S FUNCTION AND DETERMINING EQUATIONS

DAVID A. SÁNCHEZ

In an introductory course on boundary value problems for ordinary differential equations one usually does not have the time to examine the case of periodic boundary conditions, i.e., the existence of periodic solutions. Since such a course traditionally only considers linear equations, the discussion of nonlinear equations and oscillation theory would also be a rather lengthy excursion from the principal subject matter.

However, some insights and a feeling for some of the qualitative problems of nonlinear oscillations can be easily developed in a boundary value problem course via the generalized Green's function $G(t,s)$ for the operator $L = d^2/dt^2$. One imposes the periodic boundary conditions $x(0) = x(T)$, $\dot{x}(0) = \dot{x}(T)$ and proceeds to construct the Green's function

$$G(t,s) = \frac{1}{T}\left(\frac{1}{2}|t-s|^2 - \frac{T}{2}|t-s| + \frac{T^2}{12}\right).$$

The construction may be found for instance, in the book by K. Yosida [3, pp. 73–76].

One now observes that a solution $x(t)$ of the relation

$$x(t) = \int_0^T G(t,s)\,\phi(s)\,ds, \quad \phi(t+T) = \phi(t),$$

is a T-periodic function satisfying

$$\ddot{x}(t)+\phi(t)=\frac{1}{T}\int_0^T \phi(s)\,ds.$$

This is to be expected since the differential equation $\ddot{x}+\phi(t)=0$ will not have a T-periodic solution, unless the integral average value of ϕ is zero.

The above remarks can now be applied to give some insights into the existence of periodic solutions of nonlinear second order ordinary differential equations. From the preceding argument it follows that if $x(t)$ is a T-periodic function satisfying the nonlinear integral equation

$$x(t)=\int_0^T G(t,s)\ f(s,x(s),\ \dot{x}(s))\,ds,$$

where $f(t+T,\cdot,\cdot)=f(t,\cdot,\cdot)$, then it also satisfies

$$\begin{aligned} x(t)-\int_0^T G_1(t,s)\ f(s,x(s),\ \dot{x}(s))\,ds &= \frac{T^2}{12}\left[\frac{1}{T}\int_0^T f(s,x(s),\ \dot{x}(s))\,ds\right] \\ &= \frac{T^2}{12}\left[\ddot{x}(t)+f(t,x(t),\dot{x}(t))\right], \end{aligned}$$

where

$$G_1(t,s)=\frac{1}{T}\left(\frac{1}{2}|t-s|^2-\frac{T}{2}|t-s|\right).$$

Hence if $x(t)$ is a T-periodic solution of $\ddot{x}+f(t,x,\dot{x})=0$, then the relation

$$x(t)-\int_0^T G_1(t,s)\ f(s,x(s),\ \dot{x}(s))\,ds=0 \tag{1}$$

is satisfied.

The last result can be used to obtain the equations which must be satisfied by an approximate solution $x_0(t)=a_1\cos\omega t+b_1\sin\omega t$ of the differential equation—the so-called determining equations. One observes that

$$\int_0^T G_1(t,s)\begin{Bmatrix} A\ \sin\,k\omega s \\ A\ \cos\,k\omega s\end{Bmatrix}\,ds=\frac{A}{(k\omega)^2}\begin{Bmatrix}\sin\,k\omega t \\ \cos\,k\omega t\end{Bmatrix},$$

where $T=2\pi/\omega, k$ is a positive integer, so if

$$f(t,x_0(t),\ \dot{x}_0(t))=\sum_{k=1}^{n}(A_k\cos k\omega t+B_k\sin k\omega t),$$

where A_k and B_k will depend on parameters associated with f, then letting $x(t)=x_0(t)$ in relation (1) and equating coefficients gives the equations

$$\omega^2 a_1=A_1,\ \ \omega^2 b_1=B_1.$$

These are the determining equations, and under suitable hypotheses, a periodic solution will lie in a neighborhood of $x_0(t)$. For a discussion of these matters see for instance the book by J. Hale [1], especially Chapter 7, where numerous examples are worked out.

Example. For the van der Pol equation $f(t,x,\dot{x})=\varepsilon(x^2-1)\dot{x}+x,\ \ \varepsilon>0,$ and if $x_0(t)=a_1\cos\omega t+b_1\sin\omega t$, then $T=2\pi/\omega$ and one obtains

$$\begin{aligned}\omega^2 a_1 &= A_1=\varepsilon\omega b_1[\tfrac{1}{4}(a_1^2+b_1^2)-1]+a_1\\ \omega^2 b_1 &= B_1=-\varepsilon\omega a_1[\tfrac{1}{4}(a_1^2+b_1^2)-1]+b_1.\end{aligned}$$

Letting $\omega^2 = 1 + \varepsilon\beta$ then $\varepsilon\omega = \varepsilon + O(\varepsilon)$ and equating first order terms in ε gives

$$\beta a_1 = b_1[\tfrac{1}{4}(a_1^2 + b_1^2) - 1]$$

$$\beta b_1 = -a_1[\tfrac{1}{4}(a_1^2 + b_1^2) - 1]$$

which has a solution $\beta = 0$, $a_1^2 + b_1^2 = 4$. This leads to the well-known result that for $\varepsilon \ll 1$ a periodic solution of the van der Pol equation is closely approximated by $x(t) = 2\cos(\omega t + \alpha)$ where $\omega = 1 + O(\varepsilon^2)$.

One may apply the technique to the Duffing equation where $f(t, x, \dot{x}) = x + \beta x^3 - F\cos\omega t$ and obtain the familiar frequency response curves $\omega^2 = 1 + \frac{3}{4}\beta A^2 - F/A$ where $x_0(t) = A\cos\omega t$; see for instance [2, Ch. 4]. In summary, using the generalized Green's function one can naturally digress from the onerous computation of eigenvalues and eigenfunctions to give a pleasant interlude in the world of nonlinear oscillation theory.

Research partially supported by NSF Grant No. 27973.

References

1. J. K. Hale, Oscillations in Nonlinear Systems, McGraw-Hill, New York, 1963.

2. W. D. Lakin and D. A. Sánchez, Topics in Ordinary Differential Equations, A Potpourri, Prindle, Weber, and Schmidt, Boston, 1970.

3. K. Yosida, Lectures on Differential and Integral Equations, Wiley-Interscience, New York, 1960.

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EXAMPLES OF ULTRAMETRICS

J. E. VAUGHAN

In this note, we give a simple way to construct an ultrametric d for the rational numbers $\mathbf{Q}$, such that the metric topology on $\mathbf{Q}$ induced by d is the same as the usual topology on $\mathbf{Q}$ as a subspace of the real line $\mathbf{R}$. We believe that this construction is accessible to a student who has begun to study abstract metric spaces and the topology induced by a metric. We also shall mention some other examples of ultrametrics found in the literature, and discuss a little of the theory behind this concept.

1. Much of one's early intuition concerning metric spaces naturally is based on subspaces of the Euclidean spaces with the usual Euclidean metric. On the other hand, the axioms of a metric space are weak enough to allow properties which are not possessed by the Euclidean metrics. A good set of exercises to illustrate this point, and to show how misleading intuition based on Euclidean metrics can be, is given in [**1**; IX, §2, Exercises 4 and 5] and [**2**; 3.2.6, 3.8 Exercise 4, 3.14 Exercise 1]. These exercises list the unusual properties of ultrametrics (also called non-archimedean metrics). A metric d on a set X is called an *ultrametric* if it satisfies the following stronger form of the triangle axiom. For every x, y, and z in X,

$$d(x, z) \leqq \max\{d(x, y), d(y, z)\}.$$

For example, we mention one "odd" property of ultrametrics: If two spherical neighborhoods have a point in common, then one of them is contained in the other.

2. The discussion of the properties of ultrametrics in [**1**] and [**2**] is incomplete in our opinion, because the examples of ultrametrics given there seem too complicated to be fully appreciated by a

Anyone who has worked through the construction in §2 should have no trouble in proving that B implies A. To prove that A implies B, one need only set $\mathcal{B}_i$ equal to the collection of all ρ-spherical neighborhoods of radius $1/(i+1)$. The reader might enjoy showing that B and C are also equivalent. This would furnish a proof of J. de Groot's theorem which is slightly different from the proofs given in [3; p. 293], [4], [5], and [8; p. 361] in that it defines an ultrametric directly on the space X rather than embedding X into a universal ultrametric space such as Baire's space.

De Groot's theorem has analogues in higher dimensions where the metric which is constructed is often called J. Nagata's n-dimensional metric. In the case that $n = 0$, Nagata's metric is an ultrametric [9; p. 150], and his construction of that metric, which is set up to handle the general n-dimensional case, simplifies greatly in the case $n = 0$ to (essentially) the construction given in §2. Like the ultrametric, Nagata's metric has some unusual properties. For example, a 1-dimensional Nagata's metric ρ which was constructed on $\mathbf{R}$ in [11], induces the usual topology on $\mathbf{R}$, but has the property that for certain $\varepsilon > 0$ and certain x in R, the set $\{y \mid \rho(x, y) = \varepsilon\}$ contains an entire interval of real numbers.

4. We now restate the example of an ultrametric used in [1: IX, 2, Exercise 5] which was given by A. F. Monna in [7, p. 476]. If x and y are points in a metric space (X, d) and $\varepsilon > 0$, we say that there is an *ε-chain from x to y* provided there is a finite set of points $\{x_i \mid i \leqq n\}$ in X such that $x_0 = x$, $x_n = y$, and $d(x_i, x_{i+1}) < \varepsilon$ for all $i \leqq n-1$. Define $d_0(x, y) = \inf\{\varepsilon \mid$ there is an ε-chain from x to $y\}$. Now d_0 is a pseudometric on X. That is, d_0 satisfies all the axioms of a metric, except that $d_0(x, y) = 0$ does not have to imply that $x = y$. Let $\mathcal{R}$ be the equivalence relation on X defined by $x\mathcal{R}y$ if and only if $d_0(x, y) = 0$. We shall denote the equivalence class of x by $[x]$, and the set of all equivalence classes by $X/\mathcal{R}$. An ultrametric ρ on $X/\mathcal{R}$ can be defined by $\rho([x],[y]) = d_0(x, y)$. If we apply the above process to $\mathbf{Q}$ and $\mathbf{P}$, we find that $\mathbf{Q}/\mathcal{R}$ and $\mathbf{P}/\mathcal{R}$ consist of only one point, but if we apply this process to $\mathbf{K}$, then $\mathbf{K}/\mathcal{R} = \mathbf{K}$, and the ultrametric ρ is essentially the one defined by the construction in §2. Further results on this ultrametric are given in [7].

References

1. N. Bourbaki, Elements of Mathematics, General Topology, Part 2, Addison-Wesley, Reading, Mass. 1966.
2. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
3. R. Engelking, Outline of General Topology, North-Holland, Amsterdam, 1968.
4. J. de Groot, Non-Archimedean metrics in topology, Proc. Amer. Math. Soc., 7 (1956) 948–953.
5. F. Hausdorff, Über inner Abbildungen, Fund. Math., 23 (1934) 279–291.
6. W. Hurewicz and H. Wallman, Dimension Theory, Princeton, 1948.
7. A. F. Monna, Remarques sur les métriques non-archimédiennes, Proc. Kon. Ned. Ak. Wet. Amsterdam, 53 (1950) 470–481 and 625–637.
8. K. Morita, Normal families and dimension theory for metric spaces, Math. Ann., 128 (1954) 350–362.
9. J. Nagata, Modern Dimension Theory, New York, 1965.
10. D. D. Rothman, A nearly discrete metric, this MONTHLY, 81 (1974) 1018.
11. J. E. Vaughan and B. R. Wenner, Note on a theorem of J. Nagata, Comp. Math., 21 (1969) 4–6.

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THE CHROMATIC POLYNOMIAL OF A COMPLETE r-PARTITE GRAPH

RENU LASKAR AND W. R. HARE

This paper is a generalization of the result in [2]. For each integer $r \geqq 2$, an *r-partite graph* is one whose vertex set can be partitioned into r sets, $V_1, \cdots, V_r$, such that every edge of the graph joins vertices in some V_i and V_j, $i \neq j$. It is called *complete* if every vertex in V_i is joined by an edge to

every vertex in V_j for every choice of i and j, $i \neq j$. Denote the complete r-partite graph in which V_i has p_i vertices $(1 \leqq i \leqq r)$ by $K_{p_1, \cdots, p_r}$.

Denote by $f(G, t)$ the number of distinct colorings of the labelled graph G using t or fewer colors. The resulting function is a polynomial in t, and is called the *chromatic polynomial* of G. In general, it is difficult to evaluate $f(G, t)$; however, we offer an argument for an explicit calculation of $f(K_{p_1, \cdots, p_r}, t)$. The proof is inductive, and gives

$$f(K_{p_1, \cdots, p_r}, t) = \sum_{t_r=1}^{p_r} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r} S(p_i, t_i)(t)_{t_1 + \cdots + t_r}$$

where $S(p_i, t_i)$ and $(t)_{t_1 + \cdots + t_r}$ are defined below.

The result of [2] proves the formula for $r = 2$. Assume the formula valid for the case of an $(r-1)$-partite graph, and consider the graph $K_{p_1, \cdots, p_r}$. Suppose $K_{p_1, \cdots, p_r}$ is partitioned into its vertex sets $V_1, \cdots, V_r$ as described earlier. From the t colors available let us choose some t_r of them, $1 \leqq t_r \leqq p_r$. Let there be $c(p_r, t, t_r)$ ways to color the p_r vertices of V_r using exactly a set of t_r colors. There are $t - t_r$ colors remaining and, by our inductive hypothesis, the remaining vertices in $V_1 \cup \cdots \cup V_{r-1}$ may be colored in

$$f(K_{p_1, \cdots, p_{r-1}}, t - t_r) = \sum_{t_{r-1}=1}^{p_{r-1}} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r-1} S(p_i, t_i)(t - t_r)_{t_1 + \cdots + t_{r-1}}$$

ways. (Here the notation $(a)_b$ denotes the "falling factorial" $a(a-1) \cdots (a-b+1)$.) Thus, there are

$$\begin{aligned} f(K_{p_1, \cdots, p_r}, t) &= \sum_{t_r=1}^{p_r} c(p_r, t, t_r) f(K_{p_1, \cdots, p_{r-1}}, t - t_r) \\ &= \sum_{t_r=1}^{p_r} c(p_r, t, t_r) \left(\sum_{t_{r-1}=1}^{p_{r-1}} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r-1} S(p_i, t_i)(t - t_r)_{t_1 + \cdots + t_{r-1}} \right) \end{aligned}$$

ways to color $K_{p_1, \cdots, p_r}$ with the t colors.

In [2] Swenson shows that $c(p, t, r) = S(p, t)(t)_r$, where

$$S(p, t) = \frac{1}{t!} \sum_{i=0}^{t} (-1)^i \binom{t}{i} (t - i)^p$$

is a Stirling number of the second kind. Substituting this back into the expression for $f(K_{p_1, \cdots, p_r}, t)$,

$$\begin{aligned} f(K_{p_1, \cdots, p_r}, t) &= \sum_{t_r=1}^{p_r} S(p_r, t_r)(t)_{t_r} \left(\sum_{t_{r-1}=1}^{p_{r-1}} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r-1} S(p_i, t_i)(t - t_r)_{t_1 + \cdots + t_{r-1}} \right) \\ &= \sum_{t_r=1}^{p_r} \sum_{t_{r-1}=1}^{p_{r-1}} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r} S(p_i, t_i)(t - t_r)_{t_1 + \cdots + t_{r-1}}(t)_{t_r} \\ &= \sum_{t_r=1}^{p_r} \cdots \sum_{t_1=1}^{p_1} \prod_{i=1}^{r} S(p_i, t_i)(t)_{t_1 + \cdots + t_r}, \end{aligned}$$

since $(t - t_r)_{t_1 + \cdots + t_{r-1}}(t)_{t_r} = (t)_{t_1 + \cdots + t_r}$.

This final expression exhibits the invariance under any permutation Π of $p_1, \cdots, p_r$ which must be present since $K_{p_1, \cdots, p_r} \cong K_{\Pi(p_1), \cdots, \Pi(p_r)}$. Also, it shows that the polynomial is of degree $\Sigma_{i=1}^{r} p_i$. Since the highest power of t occurs when $t_i = p_i$, $1 \leqq i \leqq r$, and since $S(p_i, p_i) = 1$, it is seen that the leading coefficient is 1.

We note, in conclusion, that the complete graph, K_r, on r vertices can be viewed as $K_{1, \cdots, 1}$, where

there are r 1's. Our formula "reduces" (with $p_1 = \cdots = p_r = 1$ and $S(1,1) = 1$) to $(t)_r$, a well-known result.

References

1. C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, New York, 1968.
2. J. R. Swenson, The chromatic polynomial of a complete bipartite graph, this MONTHLY, 80(1973) 797–798.

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

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A SMALL COLLEGE COOPERATIVE SEMINAR

D. F. BAILEY AND E. T. HILL

Several articles in this MONTHLY have reminded us that mathematics teachers at small colleges must make a special effort to remain mathematicians. (See [1], [3], [4].) In an attempt to "keep alive" as mathematicians, the members of our department and members of the mathematics department of a nearby college have been conducting a weekly seminar with unusually satisfactory results. Both schools are liberal arts colleges of approximately one thousand students and both departments have three and a fraction full-time faculty members. The campuses are only twenty miles apart and meetings occur at alternate colleges on alternate weeks.

The involvement of two colleges provides several benefits. Obviously the "cross fertilization" of ideas on general college problems is an important asset. Also we effectively double the number of mathematicians with whom we have regular contact; for example, each school has only one algebraist but they have the chance to "talk algebra" once a week. Perhaps most important is the fact that the involvement of persons outside our department provides needed impetus to meet regularly. It is tempting to allow internal duties such as pre-registration or committee meetings to pre-empt a departmental seminar; this does not happen when persons outside the department are involved. There is also a subtle, but very real, pressure to do a better job.

No one individual is in charge of our seminar. We usually operate by having someone volunteer to act as "lecturer" for a few weeks and then someone else takes a turn. Topics to be covered are arrived at by mutual consent. During our first year we read *Elementary Theory of Numbers* by Harriet Griffin and worked essentially all of the exercises. This may sound a bit tame, but bear in mind that our objective is not, nor can it reasonably be, to work on any frontier. We are all narrowly trained and yet we work in positions where great breadth rather than great depth is required. The decision was made that we were all equally ignorant of number theory; Griffin seemed accessible and proved to be a good choice. The book is rather traditionally written and we found many results which we had known before as, perhaps, special cases in group theory, and many profitable exchanges were made. Finally, while we do not have students participating, Griffin equipped us with many fascinating results which our students were able to appreciate and understand.

In the Fall of '72 we settled down to a semester of lectures by one of the group who attended the

N.S.F. Summer Institute in Category Theory. (A fringe benefit of a seminar such as ours is that an application to attend an institute should be strengthened by the fact that a forum already exists for the returning participant to enlighten his colleagues.) The semester went well and we all saw a great deal of mathematics we had not seen before. However, knowing that one person will do all the lectures on a certain topic weakens the seminar. The press of other duties makes it easy for most of the group to just listen. We all know that mathematics is not a spectator sport, but the knowledge that someone else will do it encourages sloth.

We are presently reading *Linear Algebra and Geometry* by J. A. Murtha and E. R. Willard. This delightful and well-written book, like Griffin's, provides us with review of many topics we covered in our graduate study and the review comes in a context which touches the topics we teach every day. As an example, the treatment of determinants has caused us to think rather precisely again about topics on which we had admittedly become somewhat stale.

We recommend this scheme to all our colleagues in the small colleges. If there is not another mathematics department close by, consider working with another department on your campus. One might read Yan's little book [5] on input-output analysis with the economics department or perhaps the biologists would like to look at some examples in Maki and Thompson [2]. The possibilities are great and if you try it you may find, as we have, that seminar day is the high point of the week.

References

1. H. Flanders, A survival kit for the college mathematician, this MONTHLY, 78 (1971) 291–296.
2. D. P. Maki and M. Thompson, Mathematical Models and Applications, Prentice-Hall, Englewood Cliffs, N.J., 1973.
3. B. B. Peterson, Survival for mathematicians or mathematics?, this MONTHLY, 79 (1972) 70–76.
4. D. W. Western, The stimulation of a mathematics staff—a report, this MONTHLY, 79 (1972) 512–518.
5. C. Yan, Introduction to Input-Output Economics, Holt, Rinehart and Winston, New York, 1969.

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PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before December 31, 1975.

E2546. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Let n be a positive integer and let S be a set of n-tuples of nonnegative integers with the property that if $(a_1,\cdots,a_n)\in S$ and if $0\leqq b_i\leqq a_i$ for $i=1,2,\cdots,n$, then $(b_1,\cdots,b_n)\in S$. Let $H(m)$ be the number of elements of S whose coordinates sum to m. Prove that $H(m)$ is a polynomial in m for m sufficiently large. (E.g., if S is finite, then $H(m)=0$ for m sufficiently large.)

E 2547. *Proposed by T. S. Bolis, State University College of New York at Oneonta*

Let p and q be positive numbers with $p+q=1$. Show that for all x,

$$pe^{x/p}+qe^{-x/q}\leqq e^{x^2/8p^2q^2}.$$

E 2548. *Proposed by Murray S. Klamkin, University of Waterloo*

Let $A_0,A_1,\cdots,A_n$ be distinct points of n-space which lie in a hyperplane. Suppose that these points are parallel projected into another hyperplane and that their images are $B_0,B_1,\cdots,B_n$ respectively. Prove that for any $r=0,1,\cdots,n$, the volumes of the simplexes spanned by $A_0,A_1,\cdots,A_r,B_{r+1},B_{r+2},\cdots,B_n$ and by $B_0,B_1,\cdots,B_r,A_{r+1},A_{r+2},\cdots,A_n$ are equal.

E 2549. *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Let G be a connected graph with $2k$ vertices of odd degree. It is well known that G can be covered by a *k-part Euler path*, i.e., a union of k edge-disjoint paths having no repeated edges. (When $k=0$ or $k=1$, we get an *Euler circuit.*) When can G be covered by a single path with at most $k-1$ repeated edges?

E 2550. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let $q>1$ and let n be a natural number. Show that the polynomial

$$P(x)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{x^k-1}{q^k-1}$$

has the property that for $k=1,2,\cdots,n$

$$(-1)^{k+1}P^{(k)}(x)>0 \qquad \text{if } x\leqq q.$$

E 2551. *Proposed by Hugh L. Montgomery, University of Michigan*

Let $r_1,\cdots,r_n$ be real numbers such that $-1\leqq r_i\leqq 1$ for $i=1,2,\cdots,n$ and such that $r_1+\cdots+r_n=0$. It is easy to see that there is a permutation π of $\{1,\cdots,n\}$ with the property that all of the partial sums

$$S_k(\pi)=\sum_{i=1}^{k}r_{\pi(i)} \qquad (k=1,2,\cdots,n)$$

lie in the interval $[-1,1]$. Strengthen this as follows: Show that there exists a permutation π such that $\max_k S_k(\pi)-\min_k S_k(\pi)<2-n^{-1}$. Show also that if the right-hand side of the above is replaced by $2-4n^{-1}$, then the assertion is false for certain arbitrarily large n.

SOLUTIONS OF ELEMENTARY PROBLEMS

$x^n + x + 1$ is Usually Reducible mod 2

E 2482 [1974, 660]. *Proposed by H. D. Ruderman, Hunter College Campus School*

In the ring of polynomials with coefficients from the integers mod 2, let n_k be the number of positive integers h not exceeding k with the property that $f_h(x) = x^h + x + 1$ is reducible. Show that

$$\liminf_{k\to\infty} \frac{n_k}{k} > \frac{1}{2}.$$

Editor's comment. In the original printing the word reducible was incorrectly given as irreducible. The correct form, as shown above, was given by the proposer, but someone "irred." All of the solvers below correctly interpreted the problem, so it was not felt necessary to print a correction.

Editor's composite of solutions by Leonard Carlitz, Duke University, and A. F. Long, University of North Carolina at Greensboro, (jointly), O. P. Lossers, Technological University, Eindhoven, the Netherlands, J. Peter Matelski, Columbia University, and the proposer. Since x^2+x+1 divides x^3+1, it also divides $x^{3t}+1$ and hence also $x^{3t+2}+x+1 = x^2(x^{3t}+1)+(x^2+x+1)$. Thus

A. $x^h + x + 1$ is reducible if $h = 3t+2$, $t \geqq 1$.

Similarly since x^3+x+1 divides x^7+1, we have

B. $x^h + x + 1$ is reducible if $h = 7t+3$, $t \geqq 1$.

In general, if $x^n + x + 1$ is irreducible, then it divides $x^N + 1$, were $N = 2^n - 1$. Since x^4+x+1, x^6+x+1, and x^7+x+1 are irreducible, similar arguments yield

C. $x^h + x + 1$ is reducible if $h = 15t+4$ $t \geqq 1$;

D. $x^h + x + 1$ is reducible if $h = 63t+6$, $t \geqq 1$;

E. $x^h + x + 1$ is reducible if $h = 127t+7$, $t \geqq 1$.

It can also be established that x^3+x^2+1 divides both x^5+x+1 and x^7+1, so that it divides $x^{7t+5}+x+1 = x^5(x^{7t}+1)+(x^5+x+1)$, giving

F. $x^h + x + 1$ is reducible if $h = 7t+5$, $t \geqq 0$.

Finally it can be verified that $x^5+x^4+x^3+x+1$ divides both $x^{13}+x+1$ and $x^{31}+1$, so that

G. $x^h + x + 1$ is reducible if $h = 31t+13$, $t \geqq 0$.

Now using A, B, C, and F, we see that there are precisely 60 equivalence classes (mod 105) for which $x^h + x + 1$ is reducible: 2, 3, 4, 5, 8, 10, 11, 12, 14, 17, 19, 20, 23, 24, 26, 29, 31, 32, 33, 34, 35, 38, 40, 41, 44, 45, 47, 49, 50, 52, 53, 54, 56, 59, 61, 62, 64, 65, 66, 68, 71, 73, 74, 75, 77, 79, 80, 82, 83, 86, 87, 89, 92, 94, 95, 96, 98, 101, 103, 104. These give 180 classes (mod 315); from D, we get 5 more classes (mod 315) of reducible polynomials: 6, 69, 132, 195, and 258, and it can be verified that there is no overlap between these 5 classes and the 180 classes previously determined. Thus, from A, B, C, D, and F we have that there are (at least) 185 classes (mod 315) for which $x^h + x + 1$ is reducible, and so $\liminf n_k/k \geqq 185/315 = 0.5873$.

Now considering E and G and noting that 315, 31, and 127 are pairwise relatively prime, we can use inclusion-exclusion to establish that (at least) 748,755 classes (mod 1,240,155) give reducible polynomials, so that

$$\liminf_{k\to\infty} \frac{n_k}{k} \geqq \frac{748755}{1240155} = 0.6038.$$

Editor's comment. None of the solvers had all seven classes of reducible polynomials given above, so a composite solution was formed, so as to give the best estimate for the lim inf. Carlitz and Long had A, B, C, and D for a lower bound of $158/315 = 0.5016$. Lossers and Matelski (independently) had A, B, and F for a lower bound of $11/21 = 0.5238$, and the proposer had A, B, C, and G for a lower bound of $109/217 = 0.5023$.

Eric Rosenthal, Princeton University, supplied the following refences: Neal Zierler and John Brillhart, *On primitive trinomials (mod 2)*, I and II, Information and Control 13 (1968), 541–554, and 14 (1969), 566–569, and Neal Zierler, *On* $x^n + x + 1$ *over* GF(2), Information and Control 16 (1970), 502–505. In their 1968 paper, Zierler and Brillhart tabulate all irreducible polynomials of the form $x^n + x^k + 1$ for $1 \leqq k < n \leqq 1000$, and in his 1970 paper, Zierler tabulates all irreducible polynomials of the form $x^n + x + 1$ for n less than 30,000. Suprisingly, perhaps, there are but 33 such irreducible polynomials, the list beginning with 2, 3, 4, 6, 7, 9, 15, 22, 28, 30, 46, ⋯ and ending with 28, 713. This numerical evidence strongly suggests that the lim inf should be unity, a result conjectured by Carlitz and Long. The problem of the precise lim inf should be considered as still open.

An Inequality with Many Verifications

E 2483 [1974, 660]. *Proposed by M. S. Klamkin, University of Waterloo*

Let x be nonnegative and let m, n be integers with $m \geqq n \geqq 1$. Prove that

$$(m+n)(1+x^m) \geqq 2n\,\frac{1-x^{m+n}}{1-x^n}.$$

I. *Solution by Allen Stenger, Student, Pennsylvania State University.* First, we make several observations: If $x = 0$, the inequality reduces to $m + n \geqq 2n$, which is certainly true, so that we can assume that x is positive. Also, the right-hand side of the inequality is undefined if $x = 1$; however, if we let $x \to 1$, the right-hand side becomes $2(m+n)$, so that in the limit, the inequality is actually an equality. Moreover, substituting $1/x$ for x in the inequality yields the same inequality after multiplying by x^m. Thus, it suffices to prove the inequality for $0 < x < 1$. Finally, if $m = n$, there is obvious equality, so that we can assume that $m > n$. We shall show that with $0 < x < 1$ and $m > n$, the inequality is strict.

Make the substitutions $y = x^n$ and $r = m/n > 1$; the problem is to show

$$(r+1)(1+y^r) > \frac{2(1-y^{r+1})}{1-y}$$

for $0 < y < 1$. Rearranging this gives

$$(*) \qquad \frac{1}{2}(1-y)(1+y^r) > \frac{1-y^{r+1}}{r+1}.$$

The right-hand side of (*) is $\int_y^1 t^r dt$, and the left-hand side is the area of the trapezoid with vertices $(y,0), (y, y^r), (1,1)$, and $(1,0)$. This area approximates the value of the integral (the "trapezoidal rule" with one trapezoid) and since the graph of $f(t) = t^r$ is strictly concave upwards (because $r > 1$), this area is actually greater than the value of the integral. Hence (*) is established, and so is the desired inequality. (Note that the above proof still holds if m and n are any positive real numbers with $m > n$.—Ed.)

II. *Solution by Paul Zwier, Calvin College.* As above, we can assume that $m > n$ and that $0 < x < 1$. (There is again no reason why m and n cannot be arbitrary positive real numbers with $m > n$.—Ed.) Let

$$F(x) = \frac{(1-x^n)(1+x^m)}{1-x^{m+n}}.$$

Writing $f(x) = (1-x^n)(1+x^m)$ and $g(x) = 1 - x^{m+n}$, we see that

$$F(x) = \frac{f(x)-f(1)}{g(x)-g(1)},$$

so, by Cauchy's Extended Mean Value Theorem, there is a $t \in (x,1)$ such that

$$F(x) = \frac{(1-x^n)(1+x^m)}{1-x^{m+n}} = \frac{f'(t)}{g'(f)} = \frac{nt^{-m} + m + n - mt^{-n}}{m+n}.$$

If we can show that for all $t \in (0,1)$

$$nt^{-m} + m + n - mt^{-n} > 2n, \tag{1}$$

we shall be done. But clearly (1) is equivalent to

$$\phi(t) \equiv (m-n)t^m - mt^{m-n} + n > 0 \tag{2}$$

for $0 < t < 1$. Since $\phi'(t) = m(m-n)t^{m-n-1}(t^n - 1) < 0$ for $0 < t < 1$, the function ϕ is strictly decreasing on $(0,1)$, and since $\phi(1) = 0$, equation (2) is established.

III. *Solution by Robert K. Meany, Iowa State University.* As in Solution I, we can assume that $m > n \geqq 1$ and $0 < x < 1$. The given inequality is equivalent to $P(x) > 0$, where

$$\begin{aligned} P(x) &= (m+n)(1+x^m)(1-x^n) - 2n(1-x^{m+n}) \\ &= (m-n) - (m+n)x^n + (m+n)x^m - (m-n)x^{m+n}. \end{aligned}$$

By the Rule of Signs of Descartes, $P(x)$ has either one or three positive zeros, each zero being counted according to its multiplicity. But it is easy to verify that $P(1) = P'(1) = P''(1) = 0$, so that 1 is a zero of multiplicity three, and there are no other positive zeros. Since $P(0) = m - n > 0$, the continuity of $P(x)$ assures that $P(x) > 0$ for $0 < x < 1$. Note also that we have $P(x) < 0$ for $x > 1$ directly, since $P(x) < 0$ for large x.

IV. *Solution by the proposer.* As above, we can assume that $0 < x < 1$ and $m > n$, and we shall show that the inequality is strict. Also, we shall not restrict m and n to be integers, but shall allow them to be any positive real numbers $m > n > 0$. Now let $m + n = r$ and $m - n = s$ so that $r > s > 0$ and let $t = x^{1/2}$ so that $0 < t < 1$. Rearranging the desired inequality we get

$$\frac{1-t^{2r}}{rt^r} > \frac{1-t^{2s}}{st^s}.$$

On letting $t = e^{-y}$, we get

$$\frac{\sinh ry}{ry} > \frac{\sinh sy}{sy},$$

where now $0 < y < \infty$. But $x^{-1}\sinh x$ has a power series expansion with only positive coefficients, so that it is a strictly increasing function on $(0,\infty)$.

V. *Solution by J. M. Brown, Western Illinois University.* The inequality is an equality if $m = n$, so suppose that $m > n$. Let

$$f(x) = (m-n)x^{m+n} + (m+n)(x^n - x^m) + (n-m);$$

we shall show that $f(x)$ is negative if $0 \leqq x < 1$ and positive if $x > 1$, and this will be sufficient to verify the inequality. Differentiating, we get

$$f'(x) = (m+n)x^{n-1}g(x), \qquad g(x) = (m-n)x^m - mx^{m-n} + n.$$

From this, we see that $f'(x)$ and $g(x)$ have the same sign for positive values of x. Since

$$g'(x) = m(m-n)x^{m-n-1}(x^n - 1),$$

we see that $g(x)$ is strictly decreasing on $[0,1]$ and strictly increasing on $[1,\infty)$, with a minimum value $g(1) = 0$ at $x = 1$. Since $g(0) = n > 0$, it follows that $g(x) > 0$ for all positive $x \neq 1$. Since $f'(x)$ has the same sign as $g(x)$, we see that $f(x)$ is strictly increasing. The verification is complete when we note that $f(0) = n - m < 0$ and $f(1) = 0$. (Note that the proof is still valid if m and n are allowed to be arbitrary real numbers satisfying $m > n > 0$.—Ed.)

Also solved by B. V. Bechdolt, Jr., & C. E. Sticklen, Paul Chauveheid (Belgium), R. J. Evans, T. H. Foregger,

Robert Heller, Hans Kappus (Germany), I. G. Kastanas (Greece), B. G. Klein, Benjamin Lepson, P. W. Lindstrom, O. P. Lossers (Netherlands), Carolyn MacDonald, L. E. Mattics, M. R. Modak (India), William Nuesslein, B. Papaioannou, O. G. Ruehr, Steven Russ, St. Olaf College Students, Ken Schilling, Michael Skalsky, Wolfe Snow, Skip Thompson, John Tung, John Williams (Australia), D. L. Wright, P. H. Young, Larry Zimmermann, and the proposer.

Editor's comment. Assume that $m > n$ are positive real numbers and that $x \neq 1$ is nonnegative. We have seen that

$$\text{(A)} \qquad (m+n)(1+x^m) > \frac{2n(1-x^{m+n})}{1-x^n}.$$

It is not hard to see that the following inequality is actually equivalent to (A); let us call it the *dual* of (A):

$$\text{(B)} \qquad \frac{2m(1-x^{m+n})}{1-x^m} > (m+n)(1+x^n).$$

(Both (A) and (B) can be rearranged so as to assert that

$$(m-n)-(m+n)x^n+(m+n)x^m-(m-n)x^{m+n}$$

is positive if $0 \leqq x < 1$ and negative if $x > 1$.)

Lepson shows that

$$\text{(C)} \qquad \frac{2m(1-x^{m+n})}{1-x^n} > (m+n)(1+x^m).$$

In the same way, the dual of this inequality is

$$\text{(D)} \qquad (m+n)(1+x^n) > \frac{2n(1-x^{m+n})}{1-x^m}.$$

(Both (C) and (D) are equivalent to the statement that

$$(m-n)+(m+n)x^n-(m+n)x^m-(m-n)x^{m+n}$$

is positive if $0 \leqq x < 1$ and negative if $x > 1$.)

These inequalities can be displayed conveniently by the following diagram, where the arrows run from the larger quantities to the smaller:

$$\begin{tikzcd}
\frac{2m(1-x^{m+n})}{1-x^n} \arrow[r] \arrow[d] & (m+n)(1+x^m) \arrow[r] \arrow[d, shift left=2, "x>1"] & \frac{2n(1-x^{m+n})}{1-x^n} \\
\frac{2m(1-x^{m+n})}{1-x^m} \arrow[r] & (m+n)(1+x^n) \arrow[r] \arrow[u, shift left=2, "0\leqq x<1"] & \frac{2n(1-x^{m+n})}{1-x^m} \arrow[u]
\end{tikzcd}.$$

(The inequality in the center of the diagram can run either way, according as $x > 1$ or $0 \leqq x < 1$.) Note that in the case $0 \leqq x < 1$, there is actually a chain of five inequalities, and that in the limit as $x \to 1^-$, the middle three equalities of the chain approach equality.

The proposer comments that the special case of his inequality corresponding to $n = 1$ and $m = 2p + 1$ appears as Problem 4.8 in D. S. Mitrinović, *Elementary Inequalities*, Noordhoff, Groningen, 1964, p. 95, and as Equation 3.2.4 in D. S. Mitrinović and P. M. Vasić, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 198. He observes that similar results by V. I. Levin (3.2.12), by I. Beck (3.2.11), and by J. M. Wilson (3.2.27), which appear in the latter reference can be extended by analogous methods.

A Harmonic Limit

E 2484 [1974, 660]. *Proposed by Hal Forsey, California State University at San Francisco*

Let S_k be the kth partial sum of the harmonic series. Define k_n to be the least integer k such that $S_k \geqq n$. (For example, $k_1 = 1$ and $k_2 = 4$.) Find

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n}.$$

I. *Solution by Kenneth Schilling, University of California at Davis.* Write $S(k) \equiv S_k$. It is clear that $0 \leqq S(k_n) - n < k_n^{-1} \to 0$, so from the well-known fact that $S(k_n) - \log k_n \to \gamma$ (Euler's constant), we conclude that $n - \log k_n \to \gamma$. This implies that $n + 1 - \log k_{n+1} \to \gamma$, and subtracting these two limits gives $1 - \log(k_{n+1}/k_n) \to 0$. Hence $k_{n+1}/k_n \to e$.

II. *Solution by David S. Rubin, University of North Carolina at Chapel Hill.* We know that

$$\log\left(\frac{k_{n+1}}{k_n}\right) = \int_{k_n}^{k_{n+1}} \frac{1}{x}\, dx.$$

Estimating the integral by an upper sum gives

$$\int_{k_n}^{k_{n+1}} \frac{1}{x}\, dx < \sum_{j=k_n}^{k_{n+1}-1} j^{-1} = S(k_{n+1} - 1) - S(k_n - 1)$$

$$= S(k_{n+1} - 1) - S(kn) + k_n^{-1} < (n+1) - n + k_n^{-1},$$

whereas estimating it by a lower sum gives

$$\int_{k_n}^{k_{n+1}} \frac{1}{x}\, dx > \sum_{j=k_n+1}^{k_{n+1}} j^{-1} = S(k_{n+1}) - S(k_n)$$

$$= S(k_{n+1}) - S(k_n - 1) - k_n^{-1} > (n+1) - n - k_n^{-1}.$$

(The end inequalities follow by definition, since $S(k_{n+1}) \geqq n + 1 > S(k_{n+1} - 1)$ and $S(k_n) \geqq n > S(k_n - 1)$.) From these two estimates we have

$$1 - k_n^{-1} < \log\left(\frac{k_{n+1}}{k_n}\right) < 1 + k_n^{-1},$$

and since $k_n \to \infty$, it follows immediately that $\log(k_{n+1}/k_n) \to 1$, that is, $k_{n+1}/k_n \to e$.

III. *Solution by R. P. Boas, Northwestern University.* It is known that k_n differs from $e^{n-\gamma}$ by less than one , from which it follows that $k_{n+1}/k_n \to e$. (See R. P. Boas and J. W. Wrench, *Partial sums of the harmonic series*, this MONTHLY 78 (1971), 864–870.)

IV. *Comment by I. Philip Scalisi, Bridgewater (Mass.) State College.* The sequence $\{k_n\} = \{1, 4, 11, 31, 83, 227, 616, \cdots\}$ is recorded as sequence number 1385 in N. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973.

Also solved by K. F. Andersen, Günter Bach (Germany), Anders Bager (Denmark), M. T. Bird, P. S. Bruckman, Benjamin Burrell, John Christopher, George Crofts, R. J. Evans, E. W. Ewing, Michael Goldberg, W. E. Gould, M. B. Gregory, Ellen Hertz, G. A. Heuer, Carl Hurd, Dennis Jesperson, Hans Kappus (Switzerland), Emil Knapp, R. Kopp, Peter Lindstrom, O. P. Lossers (Netherlands), Arthur Marshall, L. E. Mattics, M. R. Murty & V. K. Murty, R. B. Nelson, D. F. Neu, M. R. Railkar (India), Simeon Reich (Israel), Eric Rosenthal, H. D. Ruderman, O. G. Ruehr, Steven Russ, T. Šalát (Czechoslovakia), I. P. Scalisi, M. I. Shamos, Michael Skalsky, Paul Smith, St. Olaf College Students, Skip Thompson, E. H. Umberger, University of San Francisco Problems Group, Barney Weiss, John Williams (Australia), C. N. Winton, Jeanne Wright, P. H. Young, and the proposer. Also Paul Chauveheid (Belgium).

Editor's comment. The reference to Boas and Wrench was supplied also by Bach, Christopher, Heuer, Reich, Rosenthal, and Thompson. Boas and Wrench actually show more than is cited in Solution III: they show that $[e^{n-\gamma}] \leqq k_n \leqq [e^{n-\gamma}] + 1$ if $n > 1$. This result was obtained earlier by L. Comtet in his solution to Problem 5346 [1965, 1136; 1967, 209]; the reference to this problem was supplied by Bach, Reich, and Rosenthal.

The careful reader will note that our k_n is defined by $S(k_n) \geqq n > S(k_n - 1)$, whereas the corresponding quantity in the two references cited above is defined by $S(k_n) > n \geqq S(k_n - 1)$. Wright and Boas remind us that (for $n > 1$), these two definitions are equivalent, since for $k > 1$, S_k is never an integer. This classical result can be found in G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. II, Springer-Verlag, 1964, Problem VIII.250, p. 159, with solution on p. 380. It also appeared as Problem E 46 [1933, 360; 1934, 48], and was reprinted in the *Otto Dunkel Memorial Problem Book*.

From the inequalities $[e^{n-\gamma}] \leqq k_n \leqq [e^{n-\gamma}] + 1$, it is seen that k_n is specified as one of two possible integers. An open question is whether or not k_n is always the nearest integer to $e^{n-\gamma}$, i.e., whether $k_n = [e^{n-\gamma} + \frac{1}{2}]$. This is the substance of Problem 5989* [1974, 910]. See Boas and Wrench, *op. cit.*, for further details. (Note that $k_1 = 2$ must be taken for these results to hold.)

Triangles from Random Integers

E 2485 [1974, 660]. *Proposed by R. H. Eddy, Memorial University of Newfoundland*

Three numbers are chosen at random (without replacement) from the first n natural numbers. What is the probability that they can be the sides of a triangle?

Solution by W. R. McEwen, University of Minnesota, Duluth. Let N_n be the number of possible triangles whose sides are positive integers not exceeding n, and let M_n be the number of such triangles whose longest side is precisely n. It is easy to see that M_n is equal to the number of lattice points in the interior of the triangle bounded by the lines $x = y$, $x + y = n$, and $y = n$. (*Proof*: the point (i, j) is interior to this triangle if and only if $0 < i < j < n$ and $i + j > n$ which means the triangle with sides i, j, n is of the required type.) If $j = n - 1$ (the largest permissible value of j), then i can have any of the $n - 3$ values $2, 3, \cdots, n - 2$; because of the triangular arrangement of the points, every time j is reduced by 1, the number of permissible values of i is reduced by 2. Thus

$$M_{2k} = 1 + 3 + \cdots + (2k - 3) = (k - 1)^2$$

$$M_{2k-1} = 2 + 4 + \cdots + (2k - 4) = (k - 1)(k - 2).$$

Now using the fact that $N_n = \sum_{m=4}^{n} M_m$, we see that

$$N_{2k} = \sum_{j=2}^{k} (M_{2j-1} + M_{2j}) = \sum_{j=2}^{k} (2(j - 1)^2 - (j - 1))$$

$$= k(k - 1)(4k - 5)/6,$$

and therefore

$$N_{2k-1} = N_{2k} - M_{2k} = (k - 1)(k - 2)(4k - 3)/6.$$

Since the required probability is $P_n = \binom{n}{3}^{-1} N_n$, we have

$$P_n = \frac{2n - 5}{4(n - 1)} \qquad \text{if } n \text{ is even, and}$$

$$P_n = \frac{(n - 3)(2n - 1)}{4n(n - 2)} \qquad \text{if } n \text{ is odd.}$$

Also solved by R. L. Andrews, Günter Bach (Germany), G. E. Bergey, D. R. Beuerman, Janice Blazina, D. M. Bloom, L. C. Bourburgh, D. M. Bressoud, D. S. Browder, P. S. Bruckman, John Christopher, W. D. Clewell, J. D. Emerson, R. J. Evans, J. D. Faires, David Farnsworth & Robert Braun, R. S. Fisk, G. T. Frey, W. W. Funkenbusch, Michael Goldberg, W. E. Gould, Sylvan Greene, Carl Hurd, Dennis Jesperson, E. E. Keese & J. S. Evans, N. J. Kuenzi & Bob Prielipp, S. H. Kulkarni (India), Harry Lass, P. W. Lindstrom, R. B. Nelsen, R. D. Nelson (England), L. O. Olson, R. L. Poss, D. M. Rosenblum, D. S. Rubin, H. D. Ruderman, Steven Russ, P. J. Ryan, F. G. Schmitt, Jr., M. I. Shamos, Joseph Silverman, Jean Simutis, Michael Skalsky, J. Slattery (England), Paul Smith (Canada), Wolfe Snow, R. E. Spaulding, Selig Starr, Eric Sturley, E. H. Umberger, University of San Francisco Problem Group, G. W. Valk, M. R. Vitale, D. L. Wright, and the proposer. Seventeen incorrect or incomplete solutions were received.

Editor's comment. Bergey notes the following related problem from the Stanford University Competitive Examination (this MONTHLY 80 (1973), p. 632): The three sides of a triangle are of lengths l, m, and n, respectively. The numbers l, m, and n are positive integers such that $l \leqq m \leqq n$. Find a general law for the number of triangles of the described kind.

Shamos observes that if we count also degenerate triangles (i.e., "triangles" for which the triangle inequality is an equality), indicating this by a prime on all quantities, then $N'_n = N_{n+1}$ for $n \geqq 3$ and hence

$$P'_n = \frac{2n + 1}{4(n - 1)} \qquad n \text{ even}$$

$$P'_n = \frac{(2n - 3)(n + 1)}{4n(n - 2)} \qquad n \text{ odd.}$$

He also notes that N_n satisfies the following interesting relations: $N_{n+2} = N_n + \binom{n}{2}$ and $N_n + N_{n+1} = \binom{n}{3}$. This latter formula is of interest because it says there is a one-to-one correspondence between triples (i, j, k) with $1 \leqq i < j < k \leqq n+1$ which do form triangles and triples (i', j', k') with $1 \leqq i' < j' < k' \leqq n$ which do not. However, there does not seem to be a natural correspondence between these two sets of triples.

Shamos further observes the following curious result. If we go through a similar analysis, but count also isosceles triangles (i.e., choosing the sides "with replacement"), indicating this by a double prime, then

$$N''_n = \begin{cases} n(n+2)(2n+5)/24 & n \text{ even} \\ (n+1)(n+3)(2n+1)/24 & n \text{ odd.} \end{cases}$$

Contrasting this with the formulas for N_n,

$$N_n = \begin{cases} n(n-2)(2n-5)/24 & n \text{ even} \\ (n-1)(n-3)(2n-1) & n \text{ odd,} \end{cases}$$

we see that N''_n can be obtained from N_n by simply changing all of the minus signs to plus. Further, it is known that the number of triples that can be selected from the first n natural numbers is $\binom{n+2}{3}$, when order is disregarded. (e.g., $(1,2,1)$ and $(1,1,2)$ are not distinguished). Thus if we use these triples equally weighted as our sample space then the probability P''_n that i, j, k can be the sides of a triangle is

$$P''_n = \begin{cases} \dfrac{2n+5}{4(n+1)} & n \text{ even} \\[2ex] \dfrac{(2n+1)(n+3)}{4n(n+2)} & n \text{ odd,} \end{cases}$$

and these numbers can be obtained from the P_n by again changing all of the minus signs to plus signs.

The reader has seen that there are different formulas for P_n, according as n is even or odd. Much ingenuity was exhibited by the solvers in combining these into a single formula. Several examples follow:

$$P_n = 1 - \frac{1}{16}\binom{n}{3}^{-1}\left\{\binom{2n}{3} - 4\left[\frac{n}{2}\right]\right\} \qquad \text{(Bloom)}$$

$$P_n = \frac{2n^3 - 9n^2 + 10n - 3\sin^2(\frac{1}{2}\pi n)}{4n(n-1)(n-2)} \qquad \text{(Bach)}$$

$$P_n = \frac{2n-7}{4n-8} + \frac{3[\frac{1}{2}n]}{2n(n-1)(n-2)} \qquad \text{(Bruckman)}$$

$$P_n = \frac{(2n-5+(-1)^n)(2n-3-2(-1)^n)}{4(n-2)(2n-1)-(-1)^n)} \qquad \text{(Smith).}$$

(The square brackets in the above indicate the greatest integer function.)

Poss investigates the following problem: Four numbers are chosen at random (without replacement) from the first n natural numbers. What is the probability that they can be the sides of a quadrilateral? He discovers that there are six different formulas for P_n, depending on which equivalent class (mod 6) n lies in, just as in the case of the triangle where there two formulas, depending on which equivalence class (mod 2) n lies in. It seems reasonable that the generalized problem for a k-gon would exhibit similar behavior, perhaps involving $(k-1)!$ different formulas, and would therefore be very complicated. (However, the limiting behavior of P_n as $n \to \infty$ is not difficult to establish as will be seen later.)

Many solvers observe that the sequence of probabilities $\{P_n\}$ is monotonically increasing and that it approaches $\frac{1}{2}$ as $n \to \infty$. This is to be expected, for in the limit, the problem is essentially the following continuous problem: Let X_1, X_2, and X_3 be chosen independently and (uniformly) at random from the interval $[0, 1]$. What is the probability that they can be the sides of a triangle? (Note that the distinctions made in the finite case between degenerate and nondegenerate and between isosceles and nonisosceles triangles are superfluous here.) The answer to this continuous problem is $\frac{1}{2}$ as can be seen from the following argument. We want

$$P[X_{(3)} \leqq X_{(1)} + X_{(2)}],$$

where $X_{(1)}$, $X_{(2)}$, and $X_{(3)}$ are the appropriate order statistics. It is convenient to consider the complementary event for then by symmetry

$$P[X_{(3)} > X_{(1)} + X_{(2)}] = 3P[X_3 > X_1 + X_2],$$

since the events $[X_3 > X_1 + X_2]$, $[X_2 > X_1 + X_3]$, and $[X_1 > X_2 + X_3]$ are pairwise disjoint. Since

$$P[X_3 > X_1 + X_2] = \int_0^1 \tfrac{1}{2}(x_3)^2 dx_3 = 1/6,$$

the result is shown. A similar argument gives the result that if $X_1, X_2, \cdots, X_k$ are chosen independently and (uniformly) at random from the interval $[0, 1]$, then the probability that they can be the sides of a k-gon is $1 - 1/(k-1)!$.

Symmetric Functions

E 2487 [1974, 776]. *Proposed by S. R. Conrad, B. R. Cardozo High School, Bayside, New York (attributed to D. J. Newman)*

Let $S_k = x_1^k + x_2^k + \cdots + x_n^k$. If $S_k = k$ for $k = 1, 2, \cdots, n$, find S_k.

Solution by O. G. Ruehr, Michigan Technological University. Define

$$f(x) = \prod_{i=1}^{n} (x - x_i) = x^n + \sum_{j=1}^{n} (-1)^j \sigma_j x^{n-j},$$

where σ_j denotes the sum of all possible products of the form $x_{i_1} x_{i_2} \cdots x_{i_j}$. Let $x = x_i$ and sum over i, taking account of the given value of the power symmetric functions to obtain the Newton formula

$$\sigma_n = (-1)^{n+1} + \frac{(-1)^n}{n} \sum_{j=1}^{n-1} (-1)^{j+1} \sigma_j (n-j).$$

Repeating this process for $xf(x)$ then yields

$$S_{n+1} = \sum_{j=1}^{n} (-1)^{j-1} \sigma_j (n - j + 1).$$

Note that the values of the σ's do not depend on the number of variables x. Thus

$$\sigma_n = (-1)^{n+1} + \frac{(-1)^n}{n} S_n,$$

where S_n here denotes the previously determined value for one less variable.

The last two equations combine to give the recurrence relation

$$S_{n+1} = \sum_{j=1}^{n} (n - j + 1)\left(1 - \frac{S_j}{j}\right), \qquad S_1 = 0,$$

which leads directly to the three-term recurrence relation

$$S_{n+1} + S_{n-1} = 1 + \frac{(2n-1)}{n} S_n, \qquad S_1 = 0, \quad S_2 = 1.$$

Now introduce the generating function

$$g(x) = \sum_{n=1}^{\infty} S_{n+1} x^n.$$

The defining relation for S_{n+1} then yields

$$(1-x)^2 g(x) = \frac{x}{1-x} - \int_0^x g(t)\, dt,$$

$$(1-x)^2 \frac{dg}{dx} - (2x - 1)g = \frac{1}{(1-x)^2}, \qquad g(0) = 0.$$

The solution of this first order linear ordinary differential equation is readily obtained as

$$g(x)=\frac{1-e^{-x/(x-1)}}{(1-x)^2}.$$

Expanding this function in its Maclaurin series, we have, finally

$$S_{n+1}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\binom{n+1}{k+1}.$$

Also solved by M. T. Bird, P. S. Bruckman, L. Carlitz, Ron Evans, Thomas Foregger & Jeff Lagarias, O. P. Lossers (Netherlands), L. E. Mattics, Ram Murty & Kumar Murty, C. H. Rasmussen, Bernd Richter (Germany), and Allen Stenger.

Editor's comments. A number of solutions gave a recurrence for S_k but not an explict formula. Foregger and Lagarias remark that one can also consider the generalization of the problem in which $S_k = a + kd$, $1 \leqq k \leqq n$.

Primitive Roots mod p Relatively Prime to $p-1$

E 2488 [1974, 776]. *Proposed by Richard Stanley, University of California, Berkeley*

Let p be an odd prime. It has been conjectured that there exists a natural number $k \leqq p-1$ which is a primitive root modulo p and which is relatively prime to $p-1$. Prove this conjecture in the special case that $p \equiv 1 \pmod 4$ and $3\phi(p-1) > p-1$, where ϕ denotes Euler's totient function.

Solution by Karl Goldberg, Bethesda, Maryland. In fact, for $p \equiv 1 \pmod 4$ there are at least $(3\phi(p-1)-(p-1))/2$ such k.

Let $S=\{1,\cdots,p-1\}$, let A be the subset of primitive roots (mod p), and let B be the subset of numbers relatively prime to $p-1$. Define $f(n)=p-n$, and note that f is a permutation of S. For $p\equiv 1 \pmod 4$, if a is a primitive root (mod p) so is $p-a$: let $(p-a)^r\equiv 1 \pmod p$; then $a^{2r}\equiv(p-a)^{2r}\equiv 1 \pmod p$ and therefore $(p-1)\mid 2r$ which, with $4\mid(p-1)$, implies that r is even; whence $a^r\equiv(p-a)^r\equiv 1 \pmod p$, and $(p-1)\mid r$ as required. We thus have $f(A)=A$. Because $p-1$ is even, every element of B is odd, whence every element of $f(B)$ is even; hence B and $f(B)$ are disjoint.

Let $|X|$ denote the number of elements in the set X; thus $|S|=p-1$, $|A|=|B|=|f(B)|=\phi(p-1)$. By inclusion-exclusion the number of elements in the union of A, B, and $f(B)$ is

$$|A|+|B|+|f(B)|-|A\cap B|-|A\cap f(B)|$$

since the terms $-|B\cap f(B)|$ and $+|A\cap B\cap f(B)|$ vanish. Now $|A\cap f(B)|=|f(A)\cap f(B)|=|f(A\cap B)|=|A\cap B|$ so that $|A\cup B\cup f(B)|=3\phi(p-1)-2|A\cap B|$. Since $|S|\geqq|A\cup B\cup f(B)|$ we have

$$2|A\cap B|\geqq 3\phi(p-1)-(p-1)$$

as desired.

This is a special case of the following result: Given subsets A, B of a finite set S, there is a permutation of S which fixes A and "disjoints" B if and only if

$$|A|\geqq 2|A\cap B|\geqq|A|+2|B|-|S|.$$

Also solved by S. C. Currier, J. E. Fisher, S.J., Peter Frankl (Hungary), J. G. Huard, Carl Hurd, L. E. Mattics, Ram Murty & Kumar Murty, Phil Tracy, the Temple University Problem Solving Group, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before December 31,1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6042*. *Proposed by F. T. Laseau, G. M. Leibowitz, C. H. Rasmussen and S. J. Sidney, University of Connecticut*

Is every C^∞ real-valued function on the line which vanishes outside [0,1] expressible as a difference of two such functions which are non-negative?

6043. *Proposed by Brian Peterson, University of California at Berkeley*

Let P be a nonempty proper subset of the primes. Consider algebraic extensions F of the rationals Q with the property

(*) every x in F has degree over Q divisible only by primes in P.

A Zorn's lemma argument shows that there exist maximal extensions satisfying (*). Is such a maximal extension unique up to isomorphism?

6044*. *Proposed by Jacques Gilles, Saint-Servais, Belgium*

Show that $\Pi(\alpha^4+\alpha+1)=83^3$, the product being taken over all the roots of the equation $\alpha^{49}=1$ (except $\alpha=1$).

6045. *Proposed by J. B. Rosser, University of Wisconsin and Rockefeller University*

Let D be a domain of the complex plane. For each fixed a let D_a be the set of z's such that both $a+z$ and $a-z$ lie in D. Choose a fixed complex α and let $f(z)$ be a function such that for each fixed a

$$f(a+z)+\alpha f(a-z)$$

is analytic in D_a.

Can one conclude that $f(z)$ is analytic throughout D? If not, give some additional weak conditions on f from which one could infer this.

6046. *Proposed by Stephen McAdam, University of Texas, Austin*

Let f and g be two nonconstant monic irreducible polynomials over the field K. Let u and v be roots of f and g respectively in some extension field of K. Suppose that over $K[v]$, the irreducible decomposition of f is $f=f_1^{e_1},\cdots,f_n^{e_n}$ while over $K[u], g$ decomposes into $g=g_1^{d_1},\cdots,g_m^{d_m}$. Then $n=m$ and, when appropriately ordered, $e_i=d_i$ and $\deg g_i/\deg f_i=\deg g/\deg f$.

6047. *Proposed by C. D. Minda, University of Cincinnati*

Let E_1 and E_2 be ellipses in the complex plane. Prove that there is a conformal mapping of the interior of E_1 onto the interior of E_2 which maps the foci of E_1 onto the foci of E_2 if and only if E_1 and E_2 have the same eccentricity. Moreover, show that if such a conformal mapping exists, then it must necessarily be of the form $az+b$ for some complex numbers a and b with $a\neq 0$.

SOLUTIONS OF ADVANCED PROBLEMS

A Probability Integral

5687 [1969, 835]. *Proposed by Z. Govindarajulu, University of Kentucky*

Prove or disprove:

$$\int_{-\infty}^{\infty} [f^2(x)/\{1-F(x)\}]\,dx = 2\sqrt{2}/\pi,$$

where $f(x)=(2\pi)^{-1/2}e^{-x^2/2}$, $-\infty<x<\infty$ and $F(x)=\int_{-\infty}^{x} f(t)dt$.

I. *Solution by Cesar Levy, Courant Institute of Mathematical Sciences, New York University.* First we transform the integral divided by $2\sqrt{2}/\pi$ into the form below. Then we evaluate the transformed integral numerically by Simpson's rule, using 200 intervals of length 0.1 for the integral, and the smaller interval length 0.001 to evaluate erf t. The result is

$$\frac{\pi}{2\sqrt{2}}\int_{-\infty}^{\infty} f^2(x)/\{1-F(x)\}dx = \int_0^{\infty} e^{-2t^2}/\{1-\operatorname{erf}^2 t\}\,dt = 1.0031995245.$$

The same result was obtained with other interval lengths. The fact that the value differs from unity disproves the conjecture.

Also solved by Louis Bauer.

II. *Note by J. B. Keller, Courant Institute of Mathematical Sciences.* (1) Two methods of verification were used to check the 10 decimal value of the integral, one by each of the solvers listed. In the above a sequence of decreasing mesh sizes, tending to zero, was used, until the approximation by Simpson's rule continued to give the same results to ten decimal places. Bauer used a refinement of the mesh with another method of quadrature, again obtaining ten decimal places for the integral which turned out to be the same as above.

III. *Comment by Marcel Neuts, Purdue University.* For what it is worth, the given integral I has the following probabilistic interpretation which differs (as I understand) from the context in which it first arose. Let X_n, $n\geqq 0$, be independent normal random variables with means zero and $EX_0^2=\sigma^2\leqq 1$, and $EX_n^2=1$, for $n\geqq 1$. Define T_σ by $T_\sigma=\min_k\{X_k>X_0\}$, then

$$P\{T_\sigma>k\}=\int_{-\infty}^{\infty}\phi_\sigma(x)\Phi^k(x)dx,$$

so that

$$H(\sigma)+ET_\sigma=\frac{1}{\sigma\sqrt{2\pi}}\int^{+\infty} e^{-x^2/2\sigma^2}[1-\Phi(x)]^{-1}dx.$$

For $\sigma=\sqrt{2}/2$, we obtain that $H(\sqrt{2}/2)=\sqrt{\pi}I$.

Increasing Polynomials on an Ordered Field

5861* [1972, 667]. *Proposed by Michael Slater, University of Bristol, England*

Let F be an ordered field.

(a) If $p\in F[x]$; $a,b\in F$, $a<b$, and $p'(x)>0$ for $a\leqq x\leqq b$, does it follow that $p(a)<p(b)$?

(b) If Rolle's Theorem holds in F, does it follow that F is real-closed?

Solution to (a) *by G. A. Edgar, Northwestern University.* The answer is affirmative. An ordered field has characteristic 0, so F contains (an isomorphic copy of) Q, the ordered field of rational numbers. Let F_0 be the set of "finite" elements of F, i.e., elements t such that $-r<t<r$ for some $r\in Q$. Let F_1 be the set of "infinitesimal" elements of F, i.e., elements t such that $-r<t<r$ for every positive $r\in Q$. Then F_1 is a maximal ideal in F_0, and the residue class ring $S=F_0/F_1$ is an Archimedian ordered field, so there is an order-preserving embedding $f\colon S\to R$, where R is the

ordered field of real numbers. Let $g: F_0 \to S$ be the canonical projection, so that $h = fg: F_0 \to R$ is the "standard part" map, in the terminology of nonstandard analysis.

Let $p \in F[x]$, $a < b$, $p'(x) > 0$ for $a \leqq x \leqq b$. By linear changes of variable, if necessary, we may assume $a = 0$, $b = 1$, $p(0) = 0$. By dividing by the coefficient of largest absolute value, if necessary, we may assume all coefficients of p are finite, and not all coefficients are infinitesimal. Let $q \in R[x]$ be the polynomial resulting from p under h. Then the derivative q' is the polynomial resulting from p' under h. For every $r \in Q$, between 0 and 1, we have $p'(r) > 0$, so $q'(r) \geqq 0$. Since q' is continuous, $q'(r) \geqq 0$ for all $r \in R$ with $0 \leqq r \leqq 1$. Not all coefficients of p are infinitesimal, and $p(0) = 0$, so not all coefficients of p' are infinitesimal, so q' is not identically zero. Therefore $q(0) < q(1)$, so $p(0) < p(1)$.

Also solved (part (a)) by J. R. Isbell.

Note. Part (b) of the problem remains unsolved.

Infinite Complete Subgraph of a Random Graph

5933 [1973, 949; 1975, 87]. *Proposed by P. L. Renz, Wellesley College*

Let $\mathcal{G}$ be a family of random graphs constructed on a fixed countably infinite vertex set V and having the property that the probability of $\{v, v'\}$ being an edge of a graph G of $\mathcal{G}$ is p, with $0 \leqq p \leqq 1$, for each distinct pair of vertices v and v' in V. What is the probability that a graph G in $\mathcal{G}$ contains an infinite complete subgraph?

II. *Comment by C. W. Henson, New Mexico State University.* The idea of a random graph on a fixed, countably infinite vertex set V occurs in a paper by P. Erdös and A. Renyi, *Asymmetric graphs*, Acta. Math. Acad. Sci. Hungar., 14 (1963), pp. 295–315, in which it is shown that, with probability 1, such a random graph satisfies the property:

(A) If F_1, F_2 are disjoint finite subsets of V, then there exists a vertex in V which is connected by an edge to each member of F_1 and to no member of F_2.

This observation is used to prove that almost all graphs on V have a nontrivial automorphism.

It is easy to see that any graph on V which satisfies (A) must contain a vertex-subgraph isomorphic to each countable graph. Moreover, as observed in my paper, *A family of countable homogeneous graphs*, Pacific J. Math., 38 (1971), pp. 69–83, any two graphs on V which satisfy (A) must be isomorphic. Thus there is a fixed graph G on V such that almost all graphs on V are isomorphic to G. (Of course, all of this assumes that the probability p that a given edge occurs satisfies $0 < p < 1$.)

Independent Normal Distributions

5942 [1973, 1147; 1975, 186]. *Proposed by D. M. Bloom, Brooklyn College*

Let X_1, X_2, X_3 be independent random variables such that $E(X_1) > E(X_2) > E(X_3)$. Assume that the X_i have normal distributions with a common variance. Prove or disprove: If $P(X_1 > X_2) = K$ and $P(X_2 > X_3) = L$, then $P(X_1 > X_3)$ is greater than M where M is defined by

$$\frac{K}{1-K} \cdot \frac{L}{1-L} = \frac{M}{1-M}.$$

II. *Solution by G. P. Steck, Sandia Laboratories.* The proposition is true. It is also easy to derive an upper bound on $P(X_1 > X_3)$ also based on M.

Let U and V be normally distributed random variables with means μ and ν, respectively, and

common variance σ^2 and let G denote the standard normal cumulative distribution function. Then

$$P(U>V)=\int_{-\infty}^{\infty} P\Big(\frac{V-\mu}{\sigma}<u\Big)G'(u)du=\int_{-\infty}^{\infty} G\Big(u+\frac{\mu-\nu}{\sigma}\Big)G'(u)du=G\Big(\frac{\mu-\nu}{\sigma\sqrt{2}}\Big).$$

Let $P(X_1>X_3)=Q$. We are given that $(\mu_1-\mu_2)/\sigma\sqrt{2}=G^{-1}(K)>0$, $(\mu_2-\mu_3)/\sigma\sqrt{2}=G^{-1}(L)>0$ and it is easy to see that

$$\frac{\mu_1-\mu_3}{\sigma\sqrt{2}}\equiv G^{-1}(Q)=G^{-1}(K)+G^{-1}(L). \tag{1}$$

We are also given a quantity M defined by

$$\log\frac{M}{1-M}\equiv\log\frac{K}{1-K}+\log\frac{L}{1-L}, \tag{2}$$

(all logarithms are to base e) and must prove $Q>M$ for $K,\ L>\frac{1}{2}$.

Note that (1) is a statement about the percent points of the standard normal distribution and (2) is a statement about the percent points of the standard logistic distribution defined by $F(x)=1/(1+e^{-x})$, $-\infty<x<\infty$. Hence (2) can be rewritten as $F^{-1}(M)=F^{-1}(K)+F^{-1}(L)$ and the requirement $Q>M$ becomes $G[G^{-1}(K)+G^{-1}(L)]>F[F^{-1}(K)+F^{-1}(L)]$ for $K,\ L>\frac{1}{2}$. Letting $f(x)=F^{-1}[G(x)]$ and $x=G^{-1}(K)$ and $y=G^{-1}(L)$ we see that we must prove

$$f(x+y)>f(x)+f(y)\quad\text{for}\quad x,y>0. \tag{3}$$

Since $f(0)=0$, this will follow if f is star shaped, that is $f(x)/x$ increasing, but we shall prove something stronger—namely that f is convex—by showing $f''>0$. Since $f(x)=\log[G(x)/\{1-G(x)\}]$, we have $f'=G'/[G(1-G)]$ and $f''=\{G'/[G(1-G)]^2\}\{G'(2G-1)-xG(1-G)\}$. But the second curly bracketed expression is positive for $0<x<\infty$ by Lemma 1 of R. F. Tate (*Ann. Math. Stat.*, **24**, 132–134.) Since we have $f(0)=0$ and $f'>0$, convexity implies $f(x)/x$ increasing, and the proposition is proved.

The upper bound which results when we use convexity, i.e., replace (3) by $f(\frac{1}{2}[x+y])\leqq\frac{1}{2}[f(x)+f(y)]$, is

$$Q\leqq G[2G^{-1}F(\tfrac{1}{2}F^{-1}(M))]$$

with equality if and only if $K=L$.

The convexity of $F^{-1}G$ is also equivalent to $\nu(x)-\nu(-x)\geqq x$ where $R(x)=1/\nu(x)=(1-G)/G'$ is Mill's ratio. This gives, together with an upper bound due to M. R. Sampford (*Ann. Math. Stat.* **24**, 130–132),

$$x\leqq\nu(x)-\nu(-x)\leqq 2x.$$

To see the accuracy of these bounds on Q, consider the following numerical example. If $\mu_1=3$, $\mu_2=2$, $\mu_3=3/2$ and $\sigma=1/\sqrt{2}$, we have

$K=G(1)$	$=.84134475$	$F^{-1}(K)=$	1.66826789
$L=G(1/2)$	$=.69146246$	$F^{-1}(L)=$	$.80696534$
$M=F(2.475\cdots)$	$=.92238724$	$F^{-1}(M)=$	2.47523324
$Q=G(3/2)$	$=.93319280$	$F(\frac{1}{2}F^{-1}(M))=$	$.77564888$
		$G[2G^{-1}(.77564888)]=$	$.934711.$

This gives $.922387<Q\leqq.934711$.

Completeness Criterion of Orthonormal Systems in L^2

5957 [1974, 176]. *Proposed by Dietrich Marsal, Hannover, Germany*

If the sequence of partial sums $\phi_n(x)$

$$f_n(x) = \phi_1(x)\int_a^x \phi_1(t)dt + \cdots + \phi_n(x)\int_a^x \phi_n(t)dt \quad (n = 1,2,\cdots)$$

of the orthonormal system $\{\phi\} \in L^2[a,b]$ is uniformly bounded and convergent almost everywhere, then $\{\phi\}$ is complete if and only if $\lim_{n\to\infty} f_n(x) = \frac{1}{2}$ on $a \leqq x \leqq b$ almost everywhere.

Solution by Paul Chauveheid, Vielsalm, Belgium. We will use the following criterion: the orthonormal sequence (ϕ_m) is complete in $L^2(a,b)$ if and only if

$$(*) \qquad \sum_{m=1}^{\infty}\left[\int_a^x \phi_m(t)\,dt\right]^2 = x - a$$

for all $x \in I = [a,b]$. This statement follows from the fact that an orthonormal sequence in a Hilbert space is complete if and only the Parseval formula holds for any element of some total subset A (this is easily checked by means of the Bessel inequality, cf. H. G. Garnir, *Fonctions de Variables réelles*, vol. 2, Gauthier-Villars, 1965, ex. on p. 451; here A is the family of the characteristic functions of the intervals $[a,x]$ with $x \in I$).

Writing

$$g_m(x) = \phi_m(x)\int_a^x \phi_m(t)dt,$$

we have

$$G_m(y) = \int_a^y g_m(x)dx = \tfrac{1}{2}\left[\int_a^y \phi_m(x)dx\right]^2$$

for any $y \in I$. Now we have $f_m \to f$ in $L^1(a,b)$; by the dominated convergence theorem

$$\int_a^y f(x)dx = \tfrac{1}{2}\sum_{m=1}^{\infty}\left[\int_a^y \phi_m(x)dx\right]^2$$

for all $y \in I$.

If $f = \frac{1}{2}$ almost everywhere, we deduce the equality (*) and the sequence $\{\phi_m\}$ is complete.

If we assume $\{\phi_m\}$ to be complete, f is almost everywhere the derivative of

$$\int_a^y f(x)dx = \sum_{m=1}^{\infty} G_m(y);$$

hence we have $f = \Sigma G_m' = \Sigma g_m$, since this last series converges in $L^1(a,b)$, and $f = \frac{1}{2}$ almost everywhere.

It should be noted that, in the case of complex-valued functions, the stated condition becomes

$$\sum_{m=1}^{\infty} \operatorname{Re}\left[\phi_m(x)\int_a^x \overline{\phi_m(t)}dt\right] = \tfrac{1}{2} \quad \text{a.e.}$$

Also solved by Finbarr Holland (England), L. Kuipers, O. P. Lossers (Netherlands), and the proposer. One solution with signature which the editor cannot decipher.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.

Probability. By Marcel F. Neuts. Allyn and Bacon, Boston, Massachusetts, 1973, xii + 555 pp. \$12.95. (Telegraphic Review, February 1975.)

This text is a well-written introduction to probability theory for undergraduate and beginning graduate students in mathematics, statistics and related sciences. In addition to covering the traditional topics, it contains an in-depth treatment of the laws of large numbers and central limit theorems and good introductions to renewal theory and discrete Markov Chains. Little formal background beyond advanced calculus is required. However, a familiarity with the basic properties of complex numbers is a desirable prerequisite for the sections dealing with characteristic and generating functions. The entire text can be covered in two quarters.

One outstanding feature of this text is its exceptionally clear exposition of the Kolmogorov extension theorem. Models for random events typically specify the probabilities of only a relatively small class of events. Simple coin tossing, for example, can be extended in a unique way to a model which assigns probabilities to complex events such as the equalization of the numbers of heads and tails infinitely often in an infinite sequence of independent tosses. Although my students (most of whom were undergraduates) found the general concept difficult, they were fully able to appreciate its use in the development of product measure. In fact, Neuts covers this material so well that I can almost agree with his view that a good understanding of independence is impossible without the extension theorem.

Many interesting and challenging problems (most of which can even be done by instructors who are not probabilists) appear, some of them with solutions. However, I found it necessary to supplement the text with easy problems of the type which serve to acquaint the student with new ideas and definitions without unduly frustrating her. Topics such as order statistics, runs, the Glivenko-Cantelli theorem and the convergence of the Student's t to the normal distribution, make the book particularly well-suited for students of statistics. Although stopping times do not appear and difference equations come only at the end of the book, my only serious regret in the author's selection of subject matter is his omission of conditional expectations.

This is an important book in which Neuts has aimed high. A series of starred sections, which may be omitted without loss of continuity, contain more advanced topics, such as proofs of the strong law of large numbers assuming only the existence of a mean, Liapunov's central limit theorem and the equivalence of Blackwell's and Smith's renewal theorems. The starred sections which I covered were the least successful parts of the course. The inherent difficulty, in some cases, lies in covering sophisticated material without sophisticated machinery. The Kolmogorov Maximal Inequality, for example, is stated without assuming identical distributions and proved without the martingale techniques which clarify the essentially simple idea of the theorem. Students who do not possess sufficient mathematical maturity (which includes most who have not gone beyond advanced calculus) will find the resulting proof difficult to follow. I shall, however, certainly use the text again. It contains a wealth of material which can be selected within the level and needs of most students.

PAUL I. NELSON, Bucknell University

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P. *Lecture Notes in Mathematics-383: Séminaire Bourbaki Vol. 1972-73, Exposés 418-435.* Springer-Verlag, 1974, iv + 334 pp, $12.30 (P).

GENERAL. *The Computer People.* Anne Denny Angus. Transatlantic Arts, 1970, 244 pp, $9.50. Account of a British journalist's safari among the machines and people of Computerland. Of no value except to reassure the computer-shy, since there is no explanation of how a computer works. PJC

GENERAL, S*, L**. *Mathematics on Vacation.* Joseph S. Madachy. Scribner's, 1966, 251 pp, $4.95 (P). Paperback edition of original 1966 publication. Puzzles and problems, many from the defunct *Recreational Mathematics Magazine* (1960-1964). Geometric dissections, chessboard placements, flexagons, magic and antimagic squares, alphametics, digital recreations, and "conglomerate." PJC

GENERAL, P?. *A Handbook for the Technical and Scientific Secretary.* George Freedman. Dover, 1974, xiv + 300 pp, $3.50 (P). Reprint of 1967 guide. First section presents a general introduction to "what the technical secretary should know about (physical) science and engineering." Mathematics coverage stops at calculus. Second section discusses techniques, and includes an extensive glossary of technical terms. Of limited usefulness to a mathematics secretary. RSK

GENERAL, P. *Proceedings of the 1974 Army Numerical Analysis Conference.* US Army Research Office, Durham, N.C., 1974, xix + 601 pp, $15.25 (P). Available from Tech. Info. Serv., 5285 Port Royal Rd., Springfield, VA 22161 for a mailing fee of $2.75.

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1972, V. 27.* AMS, 1975, iii + 270 pp, $30.70.

GENERAL, T*(13-14: 1), S. *The Nature of Modern Mathematics.* Karl J. Smith. Brooks/Cole, 1973, xiii + 466 pp, $10.95. Math appreciation for liberal arts students. No prerequisites. Lively format, captivating problems. Student asked to discover nature of counting, logic, number theory, probability. Will work with or without computer. LH

GENERAL, T(13-14: 1). *Topics in Contemporary Mathematics.* Jack R. Britton, Ignacio Bello. Har-Row, 1975, xv + 487 pp, $13.95. Mathematics for liberal arts students. Sets, logic, statistics and probability, algebra. •Biographical sketches of prominent mathematicians and suggested further reading at the end of each chapter. PJM

GENERAL, T(13-14: 1). *Mathematics: A Survey of Its Foundations.* Fredric N. Misner. Canfield Pr, 1975, xii + 436 pp, $11.95. A text in mathematics for liberal arts students, not a book on foundations. Includes some very introductory material on development of real numbers and logic, but also covers algebra, number theory, probability and statistics and computers. An interesting gimmick is the interspersing of selections from books and articles about mathematics ranging from the OED's definition of mathematics to a selection from Hardy's *A Mathematician's Apology*. At first glance the exercises seem to be too short. Good index and excellent suggestions for further reading. PJM

GENERAL. *Technology Mathematics Handbook.* Jan J. Tuma. McGraw, 1975, xiv + 370 pp, $15.95. A wide range of facts about arithmetic, algebra, plane geometry, space geometry, plane trigonometry, plane analytic geometry, differential calculus, sequences and series, integral calculus, matrices and determinants, scalars and vectors, elementary numerical procedures, units. Appendices of numerical tables and unit conversion tables. RBK

BASIC, T(13: 1, 2). *Elementary Algebra, Structure and Use, Second Edition.* Raymond A. Barnett. McGraw, 1975, xiv + 287 pp, $10.50. Simplified version (with more examples) of the first edition (TR February 1969). Suitable for self-paced course. FLW

BASIC, T(13: 1, 2). *An Algebra Primer: Abecedarian Mathematics for College Students.* Ronald D. Ferguson, Raymond W. Tebbetts, Kenneth D. Reeves. Macmillan, 1975, xii + 561 pp, $9.95. Basic algebra through quadratic equations plus an appendix on flow charts. FLW

BASIC, T(13: 1). *Arithmetic: Basic Skills for College Students.* Robert M. Pickrell. Hamilton, 1975, viii + 358 pp, $8.95 (P). For self study or for use in a self-paced course. FLW

BASIC, T(13). *Arithmetic for College Students, Second Edition.* D. Franklin Wright. Heath, 1975, xi + 307 pp, $10.95. A well written book, with abundant exercises and examples. Covers place value arithmetic, factorization, rationals, decimals, percent, square root, the metric system, some algebraic equations with appendices on ancient numeration and molecular systems. Instructor's Guide is available, as is a companion semi-programmed text. TAV

BASIC, S. *Arithmetic.* Jack Barker, James Rogers, James van Dyke. Saunders, 1975, xi + 357 pp, $8.50 (P); *Instructor's Manual*, 56 pp, free (P). A "write in" text for students who need to review whole numbers, decimals, percent and ratio. LAS

BASIC, T(13: 1). *Modern Intermediate Algebra, Second Edition.* Margaret F. Willerding. Wiley, 1975, xii + 411 pp, $11.95. Substantial changes from the first edition (TR, February 1970) have been made. The text consists of concise explanations with lots of worked examples and problems. Topics include real numbers, exponents, polynomials, rational expressions, linear equations, relations, quadratic functions, systems of equations, complex numbers, sequences and log functions. CEC

BASIC, T(13: 1). *Fundamentals of Algebra.* M. Wiles Keller. GLP, 1975, xi + 369 pp, $7.95 (P). A workbook type text beginning with first principles and including linear equations, relations, factoring, rational functions, systems of equations, exponents and logs, quadratic equations, inequalities and progressions. Explanations are very thorough and there are many exercises and several achievement tests to be worked in the book. CEC

BASIC, T(13: 1, 2). *Arithmetic, A Modern Approach, Second Edition.* Mervin L. Keedy, Marvin L. Bittinger. A-W, 1975, xxi + 454 pp, $8.95 (P); *Introductory Algebra, A Modern Approach, Second Edition,* x + 467 pp, $8.95 (P); *Intermediate Algebra, A Modern Approach, Second Edition,* xix + 587 pp, $9.95 (P). Second editions of 3 soft-cover workbooks with good format for remedial work. (First editions TR: October 1971, February 1972, and March 1972, respectively.) Explanations and examples with problems in the margins and further exercises after each section with answers in the back. Each chapter has a test and pre-test. LLK

PRECALCULUS, T(13: 1). *Fundamentals of Trigonometry.* M. Wiles Keller. GLP, 1975, ix + 239 pp, $6.95 (P). A thorough workbook which includes sections on relations, exponential functions, logarithmic functions, and complex numbers, along with a standard dose of trigonometry. Lots of drill problems to be worked in the text. Suitable for independent study or classroom use. CEC

PRECALCULUS, T(13: 1). *Elementary Functions: Algebra and Analytic Geometry.* Gus Klentos, Joseph Newmyer, Jr. Merrill, 1975, vii + 439 pp, $10.95 (P). A paperback study guide for algebra and functions. Presentation is in frames and very minimal. There are 33 units with definitions, examples and exercises. Practice tests are given after each 5 units. LLK

PRECALCULUS, T*(13: 1). *Plane Trigonometry.* Bernard J. Rice, Jerry D. Strange. Prindle, 1975, viii + 280 pp, $8.95. A very fine trigonometry text. Traditional triangle trigonometry with good choice of topics and modern applications. LLK

PRECALCULUS, T(13: 2). *Fundamentals of Algebra and Trigonometry, Third Edition.* Earl W. Swokowski. Prindle, 1975, xii + 532 pp, $12.50. Revisions (second edition, TR, February 1972) vary from addition of more worked examples in some chapters to extensively rewritten material in others. LLK

EDUCATION, T(15-16: 1), S, L. *Wahrscheinlichkeitsrechnung.* W. Walser. Teubner, Stuttgart, 1975, 164 pp, DM 15,80 (P). An introduction to probabilistic ideas for high school teachers. Emphasizes pedagogical methods and understanding of concepts rather than theorems and development of advanced theory. JAS

HISTORY, L. *A Biographical Dictionary of Scientists, Second Edition.* Ed: Trevor I. Williams. Halsted Pr, 1974, xv + 641 pp, $17.95. Just over 1000 brief nontechnical (typically, 200-500 words) biographies encompassing science, medicine, technology and mathematics. Second edition adds 9 biographies (at the end--not in alphabetic position) and a chronological table of births and deaths. Only deceased scientists are eligible for inclusion, but neither Lebesgue nor von Neumann made it.LAS

HISTORY, P**, L**. *Thomas Reid's Inquiry: The Geometry of Visibles and The Case for Realism.* Norman Daniels. Burt Franklin, 1974, xix + 147 pp, $12.95. In *Inquiry into the Human Mind* (1764), Thomas Reid asks "what geometry would be led to if we really did abstract geometry solely from vision? He argues that the goemetry of (monocular) vision is not Euclid's geometry but another geometry (Riemannian) which, he argues, is just as consistent as Euclid's geometry although incompatible with it" (p. v., Foreward by Hilary Putnam). Although Reid stated a number of theorems of this geometry, he never realized the implications of his discovery (see *Philosophy of Science,* 39 (1972) 219-234). Author Daniels, from whose 1970 Ph.D. dissertation the book is taken, elaborates Reid's philosophical position in its historical setting. PJC

HISTORY, P, L*. *Notice sur les Travaux Scientifiques.* Élie Cartan. Gauthier-Villars, 1974, 128 pp, 24F (P). Chronology of Cartan's life (1869-1951) and chronological list of his works. Reprint from his *Selecta* (Gauthier-Villars, 1939) of a lengthy summary and survey of his work to 1931, written by himself and revised for the *Selecta,* accompanied by the essay "Le parallélisme absolu et la théorie unitaire du champ." PJC

HISTORY, P, L*. *Condorcet, From Natural Philosophy to Social Mathematics.* Keith Michael Baker. U of Chicago Pr, 1975, xiv + 538 pp, $22. A principal precursor to the social science of Comte, Quetelet, Laplace and Cournot, Condorcet was, according to Robespierre,a "great mathematician in the eyes of men of letters, and a distinguished man of letters in the eyes of the mathematicians." Historian Baker has written a broad, detailed analysis of Condorcet's mental universe, accepting "as given" Condorcet's mathematical demonstrations. "Historians of mathematics, should they read this book, will have particular cause for disappointment." (Preface, p. xi.) LAS

HISTORY, P. *Leonard Euler Briefwechsel. Opera Omnia Series Quarta A: Commercium Epistolicum, Volume 1.* Leonard Euler. Birkhäuser, 1975, xviii + 666 pp, $60. First volume of Series IV A of Euler's exchanges of letters, summarizing in German each of the 2850 which will appear in the remaining 6 volumes. Represents joint work of USSR Academy of Science and Swiss Society of Natural Sciences. Beautiful color frontispiece portrait. (Series I, *Opera Mathematica,* is available complete in 30 volumes.) PJC

FOUNDATIONS, P, *Les Fondements des Mathématiques: De la Géométrie d'Euclide à la Relativité générale et à l'Intuitionisme.* D.F. Gonseth. Blanchard, 1974, xiv + 243 pp, (P). A reprinting of the 1926 original offering history, meta-geometry, and philosophy for the mathematically prepared reader. JAS

FOUNDATIONS, T(18), P*, *Elementary Induction on Abstract Structures.* Yiannis N. Moschovakis. Stud. in Logic and Found. of Math., V. 77. North-Holland, 1974, x + 218 pp, $17.50. Pursuing Gandy's contention that "the key notion of abstract recursion theory should be that of an inductive definition" (p. 4), this book generalizes the classical theory of hyperarithmetic relations on the integers to inductively definable relations on arbitrary structures. Much recent work is easily subsumed in this reorganization and exhibited in the form of exercises. PJC

FOUNDATIONS, P, *Studies in Algebraic Logic.* Ed: Aubert Daigneault. Stud. in Math., V. 9. MAA, 1974, vii + 207 pp, $10. Four papers (by W. Craig, D. Monk, H. Rasiowa, and G. Reyes) on contemporary results in cylindric and polyadic algebras (Boolean algebras with additional structure), on the relations between Post algebras and various logics, and on recent investigations in category theory centered on the concept of a topos. LAS

FOUNDATIONS, P, *Logic in Algebraic Form: Three Languages and Theories.* William Craig. Stud. in Logic and Found. of Math., V. 72. North-Holland, 1974, viii + 204 pp, $19.20. First-order logic with equality is successively translated into three algebraic languages, and each of them generates an equational theory proved to be sound and complete. The result is that validity in first-order logic is reduced to derivability in each equational theory. "The value of this reduction has yet to be demonstrated" (p. 3), but there are some advantages. Mostly material unpublished previously. PJC

FOUNDATIONS, T(16-18: 1, 2), *Elementare Logik und Mengenlehre I.* Arnold Oberschelp. Bibliographisches Inst, 1974, 254 pp, (P). A text designed to give advanced mathematics students who do not seek to be specialists in logic a substantial survey of logic and formal languages without model theory or set theories (possibly this latter will appear in another volume). JAS

FOUNDATIONS, P*, L*, *L.E.J. Brouwer Collected Works: Volume 1: Philosophy and Foundations of Mathematics.* Ed: A. Heyting. North-Holland, 1975, xv + 628 pp, $96.25. A meticulously edited and cross-referenced collection of the major part of Brouwer's foundational work. Dutch originals appear here (usually for the first time) in English translation; German and French papers remain in the original. Includes an extensive bibliography of others' papers which directly relate to Brouwer's work. Far more than a mere collection of reprints, this volume itself makes a major contribution to the foundations of intuitionistic mathematics. LAS

FOUNDATIONS, P, L, *An Algebraic Approach to Non-Classical Logics.* Helena Rasiowa. Stud. in Logic and Found. of Math., V. 78. North-Holland, 1974, xv + 403 pp, $30.80. Formulates an algebraic approach to a wider class of logics than treated in Rasiowa and Sikorski's *The Mathematics of Metamathematics.* Provides basic theorems for this class, using the basic concept of implicative algebra, and concentrates on examples of non-classical logics not treated in the earlier book. Emphasis on propositional calculi, with a supplement on algebraic treatment of predicate calculi. PJC

FOUNDATIONS, P, *General Theory of Knowledge.* Moritz Schlick. Trans: Albert E. Blumberg. Lib. of Exact Philo., V. 11. Springer-Verlag, 1974, xxvi + 410 pp, $32.80. First English translation of *Allgemeine Erkenntnislehre* originally published in 1918. Schlick (1882-1936) was founder of the Vienna Circle of logical positivism; the work analyzes the nature of scientific knowledge, touching tangentially on mathematics. PJC

ALGEBRA, P, *Localization of Noncommutative Rings.* Jonathan S. Golan. Pure and Appl. Math., V. 30. Dekker, 1975, 346 pp, $22.75. A survey of recent results concentrating on material not covered in the notes of Lambek and Stenstrom. Extensive discussion of various torsion theories; e.g., faithful, stable, semisimple, exact, perfect, prime, semiprime, and of their use in localization; primary decomposition; topologies on the space of all torsion theories. Many bibliography entries. No exercises. SG

ALGEBRA, P, *Divisor Theory in Module Categories.* W.V. Vasconcelos. Math. Stud., V. 14. North-Holland, 1974, ix + 120 pp, $7.75 (P). Divisor theory for various categories of modules; e.g., finitely generated torsion modules, spherical modules, and modules of finite injective dimension. Includes background material on local algebra and homology of local rings. No exercises. SG

ALGEBRA, P, *Two Papers: H-Coextensions of Monoids and the Structure of a Band of Groups.* Jonathan Leech. Memoirs No. 157. AMS, 1975, vii + 95 pp, $3.30 (P).

ALGEBRA, P, *Lecture Notes in Mathematics-407: Cohomologie Cristalline des Schémas de Caractéristique $p > 0$.* Pierre Berthelot. Springer-Verlag, 1974, 604 pp, $18.10 (P). Studies in the p-adic cohomology of "nice" schemes. A nice scheme is called a crystal. PJM

ALGEBRA, P, *Lecture Notes in Mathematics-420: Category Seminar.* Ed: Gregory M. Kelly. Springer-Verlag, 1974, vi + 375 pp, $13.20 (P). Ten papers by three authors from the 1972-73 Sydney category theory seminar. LAS

ALGEBRA, T(17-18: 1, 2), P, *Algèbre non commutative.* G. Renault. Gauthier-Villars, 1975, ix + 181 pp, 65F (P). A concise, up-to-date reference on non-commutative algebra. After a short introduction on categories and functors, including the Morita theorems, most of the standard topics (radicals, Noetherian rings, etc.) appear. The text is typescript; underlining of theorems occasionally makes for difficult reading. The French is easy. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-391: Formal Category Theory: Adjointness for 2-Categories.* John W. Gray. Springer-Verlag, 1974, xii + 282 pp, $9.90 (P). Part I of a projected 3-volume work on 2-categories as a context for category theory. The category CAT of all small categories is a 2-category since functors and natural transformations interact the way they do. PJM

ALGEBRA, P. *On Infinite Sharply Multiply Transitive Groups.* William Kerby. Vandenhoeck & Ruprecht, 1974, 70 pp, DM 18 (P). An exposition of recent results on infinite 2-transitive and 3-transitive groups. Consideration of related problems concerning near fields and near-domains and of connections with geometric incidence structures. SG

ALGEBRA, P. *Report of Algebra Group.* Pure and Appl. Math., No. 41. Queen's U, 1974, ii + 306 pp, $9.50 (P). Second report, this one on the 1973-74 activities--theses, speakers, courses, seminars--of the Queen's University algebra group. LAS

FINITE MATHEMATICS, S, P, L. *Combinatoire: Graphes et Algèbre.* Ed: M. Barbut. Gauthier-Villars, 1973, 178 pp, 38F (P). Six different independent essays each with bibliography: order structures and classification, trees, Boolean algebras and rings, monoids and groups, simplicial objects, and scales of measurement. Despite intended audience of humanists and social scientists, no use is made of motivating examples, and there is little reference to what applications these topics may have. PJC

FINITE MATHEMATICS, T(13-14). *Finite Mathematics, Second Edition.* Francis H. Hildebrand, Cheryl G. Johnson. Prindle, 1975, vii + 535 pp, $12.95. Second edition with a new chapter on sets, and more exercises. Topics: Logic, set theory, vectors and matrices, probability, statistics, linear programming and game theory. Application areas from "Academia" to "Stock Market" with a flow chart of what chapters they occur in on the end-papers. PJM

REAL ANALYSIS, T(17-18: 1), P. *Einführung in die Theorie der Distributionen.* Wolfgang Walter. Bibliographisches Inst, 1974, viii + 211 pp, (P). The basic theory of distributions including Fourier series and transformations. Does not include material on general topological vector spaces. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-409: Fonctions de Plusieurs Variables Complexes.* Ed: Francois Norguet. Springer-Verlag, 1974, xiii + 612 pp, $19.30 (P). Lectures of the "Séminaire Francois Norguet" from October 1970 to December 1973. JAS

COMPLEX ANALYSIS, S(15-17), L. *Aufgaben und Lösungen zur Funktionentheorie I.* L.I. Volkovyskii, G.L. Lunts, I.G. Aramanovich. Bibliographisches Inst, 1973, 168 pp, (P). A translation of *A Collection of Problems on Complex Analysis,* Pergamon, 1965. JAS

COMPLEX ANALYSIS, T(15-16: 1). *An Introduction to Complex Analysis.* Peter L. Walker. Halsted Pr, 1974, viii + 141 pp, $11.95. Another text for a first course in complex analysis with very little to distinguish it from a host of others. TAV

DIFFERENTIAL EQUATIONS, P. *Mathematical Aspects of Finite Elements in Partial Differential Equations.* Ed: Carl de Boor. Acad Pr, 1974, ix + 420 pp, $17. Proceedings of a 1974 Madison conference containing 13 papers on questions related to the use of the finite element method in numerical solutions of partial differential equations. SG

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-415: Ordinary and Partial Differential Equations.* Ed: B.D. Sleeman, I.M. Michael. Springer-Verlag, 1974, xvii + 447 pp, $15.20 (P). Proceedings of conference held in Dundee, Scotland in March 1974 covering a wide range of topics in differential equations. SG

DIFFERENTIAL EQUATIONS, P. *Partial Differential Equations, Second Edition.* F. John. Appl. Math. Sci., V. 1. Springer-Verlag, 1975, viii + 250 pp, $9.50 (P). Corrected printing of first edition with problems added. See extended review of first edition published in November 1972. SG

DIFFERENTIAL EQUATIONS. *Ecuaciones Diferenciales Parciales Elípticas.* Laurent Schwartz. Universidad Nacional de Colombia, 1973, iii + 98 pp, (P). Every elliptic operator is hypoelliptic, but the converse fails. Differential operators on fibre spaces. PJC

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-430: Constructive and Computational Methods for Differential and Integral Equations.* Ed: D.L. Colton, R.P. Gilbert. Springer-Verlag, 1974, vii + 476 pp, $15.20 (P). 19 papers from a February, 1974 symposium at Bloomington, Indiana. LAS

NUMERICAL ANALYSIS, P. *Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations.* Ed: A.K. Aziz. Acad Pr, 1975, ix + 369 pp, $18.50. Proceedings of a June, 1974 symposium held at U. Maryland: three survey lectures, nine invited papers plus abstracts of contributed talks. LAS

NUMERICAL ANALYSIS, T(16-17: 1), S, P. *Computer Solutions of Ordinary Differential Equations, The Initial Value Problem.* L.F. Shampine, M.K. Gordon. Freeman, 1975, x + 318 pp, $13.95. A self-contained treatment of the initial value problem, this text avoids the cookbook type approach. Focuses on effective methods, mainly, the Adams and modified Adams methods. Details the practical aspects and implementations of the algorithms. Includes well-developed and lab-tested FORTRAN codes. A useful multi-function book. I-CH

FUNCTIONAL ANALYSIS, P. *Unitary Dilations of Hilbert Space Operations and Related Topics.* Béla Sz.-Nagy. CBMS Reg. Conf. in Math., No. 19. AMS, 1974, viii + 54 pp, $4 (P). Lectures at a June 1971 Regional Conference based upon the author's (with C. Foias) *Harmonic Analysis of Operators on Hilbert Space.* Includes connections between invariant subspaces and factorization of the characteristic function, and the elements of the functional calculus of contractions, among other topics. RBK

FUNCTIONAL ANALYSIS, P. *Fourier Analysis on Local Fields.* M.H. Taibleson. Math. Notes, V. 15. Princeton U Pr, 1975, xii + 294 pp, $7 (P). An exposition of basic facts about harmonic analysis on local fields and n-dimensional vector spaces over these fields, with emphasis on the analogy with the Euclidean case as to form of statements, manner of proof, and variety of applications. RSK

FUNCTIONAL ANALYSIS, S(17-18), P. *Topological Vector Spaces.* A. Grothendieck. Trans: Orlando Chaljub. Gordon, 1973, x + 245 pp, $19.50; $10.50 (P). Lecture notes from 1954 course on locally convex spaces. No discussion, just definition-theorem-proof. No bibliography. No index. LH

FUNCTIONAL ANALYSIS, S(18), P. *Lecture Notes in Mathematics-374: Differential Calculus in Topological Linear Spaces.* Sadayuki Yamamuro. Springer-Verlag, 1974, iv + 179 pp, $7.40 (P). Considers weakest and strongest derivatives in topological linear spaces which yield Fréchet derivative when spaces are normed. Differentiability of inverses, seminorms. S-category theory in such spaces. LH

FUNCTIONAL ANALYSIS, T(17-18: 1), P. *Integration and Harmonic Analysis on Compact Groups.* R. E. Edwards. Notes on Pure Math., No. 5. Australian Natl U, 1970, viii + 222 pp, $3.50 (P). A synthesis of two sets of lecture notes: "The Riesz representation theorem" and "Harmonic analysis on compact groups." Well worth having. RBK

FUNCTIONAL ANALYSIS, T(18: 2), P. *Functional Analysis.* Kôsaku Yosida. Grund. der math. Wissenschaften, B. 123. Springer-Verlag, 1974, xi + 496 pp, $28.30. Two new sections on non-linear evolution of equations are added in the fourth edition of this classic work. Previous editions have not been superseded. RBK

OPTIMIZATION, S(15-17), P, L*. *Studies in Optimization.* Ed: G.B. Dantzig, B.C. Eaves. Stud. in Math., V. 10. MAA, 1974, viii + 180 pp, $10. Eight papers on aspects of mathematical programming and optimal algorithms by A.W. Tucker, H.W. Kuhn, G.B. Dantzig, H. Scarf and others of equal OR fame. Five of the eight are reprints, some being nearly 15 years old. The introduction contains a brief exhortation on the future role of optimization in the undergraduate curriculum. LAS

OPTIMIZATION, S(15-16), P, L. *On the Height of a Railway Bridge.* J. van Daal, F. van Doeland. Rotterdam U Pr, 1974, xiii + 124 pp, $19. A unique detailed case study of a real OR problem concerning the optimal height of an operable railway bridge in Dordrecht, Holland, including the complete FORTRAN simulation program. The delicate analysis combines such variables as waiting time of ships, the costs of taking a possible detour, the additional energy required to accelerate trains to a higher elevation... . LAS

OPTIMIZATION, T(15-16: 2), S. *Optimization Techniques in Operations Research.* B.D. Sivazlian, L.E. Stanfel. P-H, 1975, x + 502 pp, $19.75. Introduces major optimization methods of OR. Includes computational aspects. Classical methods, PERT, transportation and assignment, searching, programming (linear, geometric, dynamic). Good problems, no solutions. LH

ANALYSIS, T?(15), S. *Calculus of Variations.* J.W. Craggs. Prob. Solvers, No. 9. Allen & Unwin (U.S. Distr: Crane, Russak), 1973, vii + 80 pp, $3.75 (P). The subject is too large for adequate treatment in so few pages. Mostly problems, exercises. Could be a useful supplement for a text with few examples or exercises. The material is too thinly spread to be used as a primary source. TAV

ANALYSIS, S?, *Fourier Series and Boundary-Value Problems.* W.E. Williams. Prob. Solvers, No. 12. Allen & Unwin (U.S. Distr: Crane, Russak), 1973, 86 pp, $4.75 (P). Possibly of value as a source of examples and exercises. The treatment of Fourier series and integrals and the techniques for their use in boundary value problems are too sparse as an only encounter. At this price, it is of doubtful worth. TAV

ANALYSIS, P. *Lecture Notes in Mathematics-419: Topics in Analysis.* Ed: Olli Lehto, I.S. Louhivaara, Rolf Nevanlinna. Springer-Verlag, 1974, xiii + 392 pp, $14.40 (P). 42 papers from a 1970 colloquium in Jyväskylä, Finland. LAS

ANALYSIS, P. *Linear Operators and Approximation II.* Ed: P.L. Butzer, B. Szökefalvi-Nagy. Int. Ser. Num. Math., V. 25. Birkhäuser, 1974, xix + 585 pp, $28. Papers from Oberwolfach, April 1974. Concludes with a section on unsolved problems. LAS

ANALYSIS, T(16-17: 1), S. *Asymptotic Analysis.* H.A. Lauwerier. Math Centre Tracts, No. 54. Math Centrum, 1974, i + 145 pp, Dfl. 16 (P). A completely rewritten version of an out-of-print earlier tract, this book contains new chapters on fractional series, the confluent hypergeometric functions, and the asymptotic behavior of Cauchy integrals. Much attention is given to the asymptotic expansion of functions which can be expressed as integrals (esp. Laplace integral) to which the saddle point technique can be applied. With many worked-out examples, this tract is a good systematic treatment of asymptotics, an art that can be mastered only by studying a great number of special cases. I-CH

ANALYSIS, T?(14-15: 1), S. *Fourier Series.* N.W. Gowar, J.E. Baker. Crane, Russak, 1974, viii + 139 pp, $12. Assumes elementary calculus and is aimed at engineering or science students: methods of computing Fourier series, some standard applications in physics, and an indication of the general theory of orthogonal functions. Includes a meagre collection of exercises with solutions. Not enough material for a semester course. CEC

ANALYSIS, P. *Lecture Notes in Mathematics-404: Théorie du Potential et Analyse Harmonique.* Ed: Jacques Faraut. Springer-Verlag, 1974, 245 pp, $10.30 (P). Proceedings of the conference at Strasbourg, May 1973. JAS

ANALYSIS, T(17-18: 1, 2). *Differentialformen.* Henri Cartan. Bibliographisches Inst, 1974, 250 pp, (P). A translation of the French original (Hermann, 1967). JAS

ANALYSIS, P. *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups.* Loren Argabright, Jesús Gil de Lamadrid. Memoirs No. 145. AMS, 1974, vi + 53 pp, $2.80 (P). The theory of transformable measures is developed to provide a unifying framework for the Fourier transform and generalizations of results in harmonic analysis. JAS

ANALYSIS, P, L. *$SL_2(R)$.* Serge Lang. A-W, 1975, xvi + 428 pp, $19.50. An exposition of the infinite dimensional representations of $SL_2(R)$ and of the group modulo a discrete subgroup. Methods and notation are suggestive of the approach in the higher dimensional case. Includes Faddeev's spectral decomposition of the Laplace operator on the upper half plane. RBK

GEOMETRY, P. *Lecture Notes in Mathematics-411: Hypergraph Seminar.* Ed: Claude Berge, Dijen Ray-Chaudhuri. Springer-Verlag, 1974, 287 pp, $11.50 (P). Papers from Ohio State U., 1972. An attempt at unifying graphs, matroids and finite geometries. LAS

GEOMETRY, S(17-18), P. *Alternierende Differentialformen.* Harald Holmann, Hansklaus Rummler. Bibliographisches Inst, 1972, 257 pp, (P). Differential forms and their integration theory on manifolds, prefaced by necessary material on manifolds and tangent bundles in the role of consumers of the theory of multi-linear algebra. JAS

GEOMETRY, P. *Lecture Notes in Mathematics-412: Classification of Algebraic Varieties and Compact Complex Manifolds.* Ed: H. Popp. Springer-Verlag, 1974, 333 pp, $12.30 (P). Reports on the state of the art derived from talks given at the University of Mannheim. JAS

GEOMETRY, P. *Vorlesungen über Riemannsche Flächen.* Robert C. Gunning. Bibliographisches Inst, 1972, 276 pp, 9,90 DM (P). A German translation of *Lectures on Riemann Surfaces,* Princeton U Pr, 1966. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-428: Algebraic and Geometrical Methods in Topology.* Ed: L.F. McAuley. Springer-Verlag, 1974, xi + 280 pp, $11.50 (P). Twelve papers from the supplementary program of a CBMS regional conference at Binghamton, October 1973. LAS

TOPOLOGY, T(18: 1), P. *Braids, Links, and Mapping Class Groups.* Joan S. Birman. Annals of Math. Stud., No. 82. Princeton U Pr, 1975, ix + 228 pp, $8.50 (P). Based on lectures at Princeton in 1971-72 and recorded by James Cannon. Self-contained for persons familiar with basic homotopy theory. Many figures. Extensive references to the literature. Contents: braid groups; braids and links; Magnus representations; mapping class groups; plats and links; research problems; bibliography; index. RBK

TOPOLOGY, T(16-18: 2). *Topologie.* Horst Schubert. B.G. Teubner. 3rd Edition, 1971, 328 pp, 49 DM (P). An unaltered reprint of the second edition which was a corrected edition of the 1964 original with an updated bibliography. This book still offers a nearly unique text covering point set topology, homotopy, and (singular) homology theory. JAS

TOPOLOGY, T(18: 2), P. *Topologie Algebrica: Omologie, Omotopie, Spatii de acoperire.* R. Miron, I. Pop. Editura Academiei Romania, 1974, 341 pp, Lei 25. A substantial treatment of the basics of modern algebraic topology: category theory, homology (Čech, singular, simplicial), homotopy, deRahm cohomology theory, and covering spaces. JAS

TOPOLOGY, P. *The Category of H-modules over a Spectrum.* Jack Palmer Sanders. Memoirs No. 141. AMS, 1974, iii + 136 pp, $3.50 (P). A revision of a thesis written under the direction of E.E. Floyd. The construction of a representing spectrum for certain homology theories requires the assumption of a number of homotopy commutativity conditions which are incorporated in the definition of an H-module over a ring spectrum. These conditions apply to certain mapping cone spectra over assorted Thom spectra. JAS

TOPOLOGY, S(17-18), P. *Theorie der Punktmengen.* Herbert Meschkowski, Ingrid Ahrens. Bibliographisches Inst, 1974, v + 175 pp, (P). A study of subsets of the plane: Hausdorff-metric, Minkowski addition and subtraction, integrals of set valued functions. A last chapter "The Isoperimetric Property of the Sphere" studies surface areas of convex subsets of R^n. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-418: Localization in Group Theory and Homotopy Theory and Related Topics.* Ed: Peter Hilton. Springer-Verlag, 1974, vi + 171 pp, $8.20 (P). Papers from a March, 1974 symposium at Battelle Seattle. LAS

TOPOLOGY, P. *Three Lectures on Local Monodromy.* Lê Dũng Tráng. Lect. Notes Ser., No. 43. Aarhus U, 1974, 29 pp, (P). Technical results on hypersurface singularities. PJM

TOPOLOGY, P. *Notes on Bundle Theory.* J. Alexander Lees. Lect. Notes Ser., No. 42. Aarhus U, 1974, 83 pp, (P). Standard approach to theory of fibre bundles. No tables of contents. PJM

TOPOLOGY, P. *Espaces Vectoriels Topologiques Preordonnés.* Michel Duhoux. Inst. de Math., Université Catholique de Louvain, 1974, iv + 188 pp, (P). A study of real topological vector spaces with a preorder. Aims towards functional analysis. PJM

TOPOLOGY, P. *Lecture Notes in Mathematics-422: Algèbres Connexes et Homologie des Espaces de Lacets.* Jean-Michel Lemaire. Springer-Verlag, 1974, xiv + 134 pp, $9.40 (P). The author's thesis on homology of loop spaces, with particular interest in the loop space of the mapping cone for a map between suspensions. PJM

TOPOLOGY, P. *Equivariant Singular Homology and Cohomology I.* Sören Illman. Memoirs No. 156. AMS, 1975, ii + 74 pp, $3.10 (P). A G-space is a topological space together with a continuous action by a topological group G. An equivariant map is a map f with $f(gx) = g(f(x))$. This memoir constructs homology and cohomology theories for the category of G-spaces and equivariant maps. PJM

TOPOLOGY, P*, *Localization of Nilpotent Groups and Spaces*. Peter Hilton, Guido Mislin, Joe Roitberg. Math. Stud., V. 15. North-Holland, 1975, x + 156 pp, $10.95 (P). Every abelian group is a Z-module. Localization of abelian groups handles the question of filling in the blank: "Every____ abelian group is a Z_p-module." This technique extends to nilpotent groups. PJM

TOPOLOGY, P, *Index Theorems of Atiyah-Bott-Patodi and Curvature Invariants*. Ravindra S. Kulkarni. Pr U Montreal, 1975, 86 pp, $5 (P). An exposition of the index theorems based on lectures by Bott. The point of view is geometric, based on the idea of "curvature invariant." The notes are fairly self-contained and provide a useful introduction to the index theorems. SG

TOPOLOGY, P, *Piecewise Linear Concordances and Isotopies*. Kenneth C. Millett. Memoirs No. 153. AMS, 1974, vi + 73 pp, $3.10 (P).

PROBABILITY, P, *Martingale Inequalities: Seminar Notes on Recent Progress*. Adriano M. Garsia. Benjamin, 1973, viii + 184 pp, $14, $6.50 (P). "The object...(is) to present the recent work of C. Fefferman on functions of bounded mean oscillation and its relation to some work of Burkholder, Davis and Gundy on martingales." Appendices, bibliography. Photocopied from typed original; the hardcover edition is nearly 20¢ per page of text! TAV

PROBABILITY, T(18: 1), P, *Almost Sure Convergence*. William F. Stout. Prob. and Math. Stat., No. 24. Acad Pr, 1974, x + 381 pp, $24.50. Treats almost sure convergence of partial sums of random variables, almost sure stability of partial sums and of weighted partial sums, law of the iterated logarithm for partial sums, and recurrence of partial sums, in a theorem-proof format. Good bibliography. RSK

STATISTICS, T?(15), *Statistics 1: Probability and Probability Distributions*. A.K. Shahani, P.K. Nandi. Prob. Solvers, No. 11. Allen & Unwin (U.S. Distr: Crane, Russak), 1973, 92 pp, $3.75 (P). Too short for any practical value. A collection of problems with very little textual material. Most reasonable texts have at least as many problems with substantially better explanation of concepts.TAV

STATISTICS, T(15-17: 1, 2), S, P, *Statistical Theory of the Analysis of Experimental Designs*. J. Ogawa. Statistics, V. 8. Dekker, 1974, vi + 465 pp, $23.75. Analyses of variance, factorial designs, theory of block designs (interblock and intrablock analyses), and randomization of partially balanced incomplete block designs. No exercises. FLW

STATISTICS, T(13: 1), *Introduction to Statistical Analysis, A Modern Computational Approach*. Irving Allen Dodes. Hayden, 1974, 193 pp, $6.20 (P); $9.95. Presupposes intermediate algebra. Cookbook statistics and some data sets to use with calculators or computers. FLW

STATISTICS, S(13-16), *Statistics for Comparative Studies*. M. Hills. Halsted Pr, 1974, viii + 194 pp, $7.95 (P). After a summary of basic statistical notions, large sample methods are discussed and illustrated with examples from medical studies. No exercises. FLW

STATISTICS, T(13-14: 1), *Statistical Inference: Basic Concepts*. Richard B. Ellis. P-H, 1975, xiv + 258 pp, $9.95. Presupposes only high school algebra. Traditional topics. Informal style. Some interesting examples. No Bayesian methods. FLW

STATISTICS, P, L*, *Handbook of Tables for Probability and Statistics, Second Edition*. Ed: William H. Beyer. CRC Pr, 1968, xiv + 642 pp, $25. Second (1974) printing of the second (1968) edition. An "extensive collection of relatively standard statistical tables," together with brief explanations of their use. Begins with an expository section covering a variety of topics in probability and statistics, and concludes with a collection of miscellaneous mathematical tables. RSK

STATISTICS, T(16-17: 1), S, P, L, *Analysis of Variance in Complex Experimental Designs*. Harold R. Lindman. Freeman, 1974, xi + 352 pp, $15. Presupposes a "relatively advanced, noncalculus course in statistics." Presents many different analyses of variance techniques along with numerical examples. FLW

STATISTICS, T(13: 1), *Elementary Statistics Through Problem Solving*. Frank J. Avenoso, Phillip M. Cheifetz. Williams & Wilkins, 1974, viii + 184 pp, $7.50 (P). A non-technical introduction to some of the standard topics. Very little on probability or on the distinctions needed to make statistical inference meaningful. FLW

STATISTICS, T(15-17: 1, 2), *Probability and Mathematical Statistics*. Lester D. Taylor. Har-Row, 1974, xiv + 346 pp, $14.95. Presupposes calculus and basic linear algebra. Traditional development of probability and then estimation, hypothesis testing, regression, and order statistics. A final chapter indicates the nature of Bayesian methods. No exercises. There is a study guide (unseen by this reviewer) by R. Kushlev that provides examples. FLW

STATISTICS, T(13-16: 1, 2), S, P, L, *An Introduction to Medical Statistics*. H.O. Lancaster. Wiley, 1974, xiv + 305 pp, $18.95. The first two-thirds of this book presupposes only elementary mathematics and considers descriptive statistics, population dynamics, statistical aspects of genetics, and the testing of hypotheses. Later chapters consider more distribution theory and various special problems of medical statistics. Extensive bibliography. FLW

STATISTICS, T(13-14: 1, 2), *Statistics: Methods and Analyses, Second Edition*. Lincoln L. Chao. McGraw, 1974, xii + 556 pp, $12.95. Presupposes only high school algebra and yet attempts to give a rather theoretical treatment of the standard topics including time series and some Bayesian methods. FLW

STATISTICS, T(13-16: 1, 2), S, P, L, *Statistical Theory of Sample Survey Design and Analysis*. H.S. Konijn. North-Holland, 1973, xv + 429 pp, $34.60. A thorough treatment that presupposes an elementary introduction to probability and statistics. FLW

STATISTICS, P. *Progress in Statistics.* Ed: J. Gani, K. Sarkadi, I. Vincze. North-Holland, 1974, $69.25 set. *V. I:* 453 pp; *V. II:* 455 pp. 81 papers from an August 1972 Budapest conference, including both theoretical and applied work. Authors represent 28 different countries. LAS

STATISTICS, P. *Applied Statistics.* Ed: R.P. Gupta. North-Holland, 1975, x + 416 pp, $30.95. Proceedings of a conference at Dalhousie, May 1974. LAS

STATISTICS, P. *Proceedings of the Nineteenth Conference on the Design of Experiments.* U.S. Army Research Office, Durham, N.C., 1974, xv + 590 pp, $13.75 (P). Six invited and several dozen shorter contributed papers from an October 1973 conference. LAS

COMPUTER SCIENCE, T(14-16), S, L. *The Science of Computing.* Loren P. Meissner. Wadsworth, 1974, 354 pp, $12.95. A well-motivated introduction to non-numeric computation. Language independent. Generally high-level, although it considers cells and pointers. Needs supplementary manuals. RWN

COMPUTER SCIENCE, S(15), L. *Program Style, Design, Efficiency, Debugging, and Testing.* Dennie van Tassel. P-H, 1974, xii + 256 pp, $10.50. A timely book. General machine-independent principles of programming plus language-dependent techniques. Appendices on Program Rewriting, A Fortran Execution Time Estimator, and Optimization of Tape Operations. RWN

COMPUTER SCIENCE, S(14). *Assembly Language Basics: An Annotated Program Book.* Irving Allen Dodes. Hayden, 1975, 102 pp, $6.25 (P). 14 sample programs illustrating BAL (Basic Assembly Language for the IBM/360). Stresses usage of the system, I/O and arithmetic. RWN

COMPUTER SCIENCE, T(13: 1). *Exploring the World of Data Processing.* Claude J. DeRossi. Reston, 1975, xv + 256 pp, $9.95. An easy-to-read introduction to numerous topics in data processing including history, I/O, internal representations, hardware, a survey of languages, operations and applications. RWN

COMPUTER SCIENCE, T(13-16), S. *Computers in Perspective.* William W. Cotterman. Wadsworth, 1974, 227 pp, $7.95. Computer systems, data processing, programming, applications, economic and social impact. RWN

COMPUTER SCIENCE, T(15-16). *Study and Compilation of Computer Languages.* Y. Wallach. Gordon, 1974, x + 614 pp, $45. Rather expensive as a text; incomplete as a reference. Teaches PL/1 as a second language. Introductory compiling techniques. List processing. A late chapter on Algol. RWN

COMPUTER SCIENCE, T(14). *Assembler Language Programming: System/360 and 370.* Sharon K. Tuggle. SRA, 1975, xii + 512 pp, $9.95 (P). Fundamentals, control, assembly, JCL, Macros. RWN

COMPUTER SCIENCE, T(13: 1), S. *Principles of Business Data Processing, Second Edition.* V. Thomas Dock, Edward Essick. SRA, 1974, xvii + 395 pp, $9.95; *Instructor's Guide,* 99 pp, $2 (P); *Study Guide,* vi + 141 pp, $3.95 (P). Elementary description of data processing for business. Not mathematical. No problems. LH

COMPUTER SCIENCE, P. *Hybridrechnen.* Manfred Feilmeier. Int. Ser. Num. Math., V. 2. Birkhäuser, 1974, 304 pp, $14. A treatise on the hardware and software involved in combined analog and digital computing. A presentation of the theory with plenty of attention to practical matters. JAS

COMPUTER SCIENCE, T, S, P, L. *Information Systems: Technology, Economics, Applications.* Chris Mader, Robert Hagin. SRA, 1974, ix + 405 pp, $11.95. Technology and economics of computers used in organizations. Designed for managers, systems professionals, and students, this book introduced the reader to systems concepts, uses, costs and benefits, including reference to specific equipment. No mathematics as such, but possibly useful to mathematicians charged with choosing the campus computer. PJC

COMPUTER SCIENCE, T?(18: 1), P. *Syntactic Methods in Pattern Recognition.* K.S. Fu. Math. in Sci. and Eng., V. 112. Acad Pr, 1974, xi + 295 pp, $23.50. Survey of research in pattern recognition by means of the syntactic (structural) approach, which attempts to draw an analogy between the structure of patterns and the syntax of a language. Major topics: use of formal languages, use of stochastic languages, grammatical inference. Includes among the 9 appendices brief surveys on syntactic recognition of chromosome patterns, of two-dimensional mathematical expressions, of Chinese characters and of spoken words, each with its own bibliography. PJC

COMPUTER SCIENCE, S(4-8). *The Story of Computers.* Donald D. Spencer. Abacus, 1975, 64 pp, $4.95 (P); $6.95. A simple pictorial discussion of the history of computers, the component parts of a computer system, and how computers are used. Overpriced, even for juvenile literature. PJC

COMPUTER SCIENCE, T(13-14: 1). *Analyse Binaire.* R.L. Vallée. Masson, 1970. *Tome I: Théorie et Applications aux Circuits Combinatoires,* 151 pp, 70F (P); *Tome II: Clef des Automates Numériques,* 187 pp, 80F (P). Applied Boolean algebra: simplification of Boolean functions, design of logical circuits, coding functions, memory functions, flip-flops and their functions, generating functions, and sequential systems. Worked exercises for each chapter, but no bibliography or references to the literature. PJC

COMPUTER SCIENCE, S(13). *Introduction to Digital Computer Plotting.* T.C. Smith, Y.C. Pao. Gordon, 1973, 86 pp, $9.95 (P). A manual for programming a hard copy plotter in FORTRAN. Straight lines and curves; labeling; scaling the axes; graph plotting; applications (in engineering graphics). RBK

Reviewers Whose Initials Appear Above

Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steve Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to The Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Professor I. H. Rose, Herbert H. Lehmann College, CUNY, represented the Association at the inauguration of Dr. George Bugliarello as President of the Polytechnic Institute of New York on March 13, 1975.

Dr. Jonathan Golan, University of Haifa, has been promoted to Senior Lecturer.

Professor R. T. Gregory, University of Texas, Austin, has been appointed Professor of Mathematics and Head of the Computer Science Department at the University of Tennessee, Knoxville.

Associate Professor Charles Heuer, Chairman of the Mathematics Department at Concordia College, has been promoted to Professor.

Dr. Don Hill, Florida A&M University, has been appointed Assistant to the Dean of the College of Science and Technology.

Associate Professor R. A. Horn, Chairman of the Department of Mathematical Sciences at the Johns Hopkins University, has been promoted to Professor.

Professor Lawrence Kuipers, Southern Illinois University, has retired with the title of Professor Emeritus. He has been awarded the "Leo Kaplan Award for Research and Lectureship" by the SIU Sigma Xi Chapter.

Assistant Professor Michael Levitan, Villanova University, has been promoted to Associate Professor.

Professor Louis F. Tolle, St. John's University, New York, retired on June 30, 1974, with the title of Professor Emeritus.

Reverend Joseph Windolph, Chairman of the Mathematics Department at Quincy College, has been promoted from Associate Professor to Professor.

Dean Emeritus Edwin B. Allen, Rensselaer Polytechnic Institute, died in October 1974 at the age of 76. He was a member of the Association for forty years.

Mr. Claude W. Anderson, University of California, Berkeley, died on October 27, 1974. He was a member of the Association for nine years.

Professor Emeritus Jasper O. Hassler, University of Oklahoma, died on December 22, 1974, at the age of 90. He was a Charter Member of the Association.

Dr. Irvin L. Lynn, Cooper Union, died on January 19, 1975, at the age of 53. He was a member of the Association for six years.

Dr. Reuben R. McDaniel, Ettrick, Virginia, died on January 19, 1975, at the age of 72. He was a member of the Association for twenty-one years.

Professor John H. Raymond, OSB, St. Martin's College, died on December 7, 1974, at the age of 63. He was a member of the Association for twenty-one years.

Professor Emeritus James B. Scarborough, U. S. Naval Academy, died on December 29, 1974, at the age of 89. He was a member of the Association for fifty-one years.

Dr. Joseph W. Weihe, Sandia Corporation, died on May 7, 1974, at the age of 52. He was a member of the Association for nineteen years.

BERTRAND RUSSELL MEMORIAL LOGIC CONFERENCE —£ 200 ESSAY PRIZE

A prize of £ 200 is offered for an essay which examines in detail some aspect of the relationship between mathematics and the development of social or economic conditions. The essay should be of general interest to mathematicians and should include a consideration of current mathematical practice.

Essays should be submitted by February 1, 1976. Prospective entrants should read the further particulars which are available from Dr. A. Slomson, School of Mathematics, The University, Leeds LS2 9JT, England.

Organizing committee: J. L. Bell (London School of Economics); M. A. Dickmann (CNRS, Paris); M. Machover (Chelsea College); G. Priest (St. Andrews University); A. B. Slomson (Leeds University); Y. Suzuki (Sussex University); G. M. Wilmers (Manchester University).

COOPERATIVE COLLEGE REGISTRY

The Cooperative College Registry (One Dupont Circle, Washington, D. C. 20036) reports that position listings in Mathematics and Computer Science received as of February 28, 1975, were twice those on the same date last year. The increase in listings, combined with fewer available candidates in these disciplines, documents improved job opportunities according to CCR statistics. The ratio of registrants to vacancies (R/V Ratio) in Mathematics and Computer Science is 1:2. Last year the ratio was 1.5:1. These ratios are based on the number of candidates whose highest earned degree is in Math or Computer Science. Many registrants with varied backgrounds express interest and receive referrals in more than one teaching or administrative area. Similarly, schools frequently express an interest in interdisciplinary ability.

In Computer Science approximately one third of the listings are open only to those persons at the doctoral level. While Programming and Systems Management predominate in the Computer Science listings, six Computer Center Directorships have been listed as of February 28. It has been noted that experience in working with computer systems and certain types of application areas such as CAI is frequently preferred by administrators listing positions. In mathematics four fifths of the positions require that candidates must have completed the doctorate by September. A wide range of specializations has been noted, but statistics and applied math are the most frequently mentioned areas.Positions have been listed for appointment at all levels including several chairmanships.

CCR, a national non-profit educational organization, serves institutions of higher education which range in size from small private colleges to multicampus university systems. Listings are received throughout the year though most specify September appointment.

HOWARD UNIVERSITY — THE ELBERT F. COX SCHOLARSHIP FUND

The Department of Mathematics of Howard University announces with pride the establishment of The Elbert F. Cox Scholarship Fund for undergraduate mathematics majors. Dr. Elbert F. Cox (1895–1969) was the first Black to receive a doctor of philosophy degree in mathematics (Cornell, 1925). Dr. Cox was a member of the Howard mathematics department from 1929 through 1966 and was chairman of the department from 1947–61.

HOUSTON JOURNAL OF MATHEMATICS

The Houston Journal of Mathematics is a new international journal that will appear quarterly. The scope of the journal is intended to be broad, covering all areas of mathematics. There is no restriction on the length of papers.

The members of the Editorial Board are RH Bing, D. G. Bourgin, Jutta Hausen (Associate Managing Editor), G. G. Johnson (Managing Editor), J. A. Johnson (Associate Managing Editor), Bjarni Jonsson, Andrzej Lelek, J. S. Mac Nerney, J. W. Neuberger, B. H. Neumann and Jurgen Schmidt.

Starting January 1976 each volume will consist of four issues of about two hundred pages each. Publication of Volume I consisting of one issue only is planned for October 1975. The subscription price is $35 per volume ($40 if sent outside the United States and Canada). Subscriptions received prior to September 1975 will receive all issues published in 1975 and 1976 for $35.

For further information write to G. G. Johnson, Managing Editor, Houston Journal of Mathematics, Department of Mathematics, University of Houston, Houston, Texas 77004 (U. S. A.).

THE 1975–76 SABBATICAL EXCHANGE INFORMATION SERVICE

For the past two years, the MAA Sabbatical Exchange Information Service (SEIS) has provided information to assist faculty members in universities and colleges (both two- and four-year) in arranging what might be called "no-cost sabbaticals." Reaction from participants has been enthusiastic, but SEIS remains under-used. Since the value of such an information exchange to its users increases with the number of listings, we are making an effort to broaden the visibility and hence the scope of SEIS. We ask readers not only to consider using SEIS, but also to remind colleagues of its existence. It costs nothing to be listed in SEIS, and the results can be a year of expanded horizons at little or no cost to faculty member or institution. For the benefit of readers not familiar with the concept and operation of SEIS, we reprint here the entire announcement published originally in 1973 and, in expanded form, in 1974.

Many MAA members are in institutions not offering faculty members a program of sabbatical leaves. Even in

institutions with sabbatical leave programs individual faculty members often find themselves ineligible for such a leave at a time when the desire for one is strongest. We all recognize the rejuvenating effect of an occasional change of scene, even if it does not involve release from teaching duties. The Association therefore suggests that an occasional exchange between two faculty members of similar interests, training, and experience at different institutions could be of great benefit to the individuals and also to their institutions. The individuals and institutions all stand to gain from the refreshment of exchanged ideas and insights.

It is often possible for two such faculty members to trade identities, so to speak, for a year. Such an exchange might involve trading teaching responsibilites, living quarters, and some departmental responsibilties. The extent of the exchange would depend on the individual circumstances. It is suggested, however, that salaries should not be exchanged or even discussed. Each faculty member would remain on the payroll of his permanent institution and receive all of his normal fringe benefits. Financially, his institution would not recognize the exchange at all.

A type of exchange that might be attractive in view of today's employment market is the following: Occasionally, a university with a research-oriented graduate program employs one of its own new Ph.D.'s in a postdoctoral position for a fixed term, usually one or two years. The young mathematician may be attracted to such a temporary position because of the opportunity for one or two more years in a research center. However, having received his Ph.D. he may be more concerned about his credentials and preparation for a more permanent position, which may not be primarily in research. Even though he anticipates a career in a smaller teaching-oriented institution, the new Ph.D may remain in such a postdoctoral position while waiting for the right opportunity to materialize. Under these circumstances, the young mathematician might be happier at a smaller teaching-oriented institution for this transition year. At the same time an established mathematics faculty member at such a teaching-oriented institution frequently feels the desire to return to a research center for a year of mathematical refreshment.

These two individuals can both benefit from an exchange with each other. The new Ph.D. obtains valuable teaching experience enhancing his credentials for a permanent position in a similar institution. The teaching-oriented institution benefits from the stimulating enthusiasm of a young mathematician fresh from several years in a research center. The older colleague benefits from a return to the well, and he undoubtedly contributes much to the teaching staff of the university from his years of classroom experience.

For these reasons we urge young mathematicians who anticipate occupying such postdoctoral positions in 1976, and who might be interested in an exchange of the type described above, to list themselves in SEIS. We also urge mathematicians in research centers who counsel prospective and recent Ph.D.'s to have on hand copies of the SEIS list in case such an exchange should be suggested. Finally, we urge faculty members in smaller colleges who would be interested in an exchange involving a return to a research center to list themselves in SEIS.

The MAA proposes to become involved only to the extent of assisting in bringing together like-minded mathematics faculty members who are interested in an exchange. The information exchange will be accomplished by the annual publication by the Association in December of a list containing the names, addresses, and other pertinent information about members of the Association interested in arranging a "Sabbatical Exchange" with a colleague in another institution. This list will be sent free of charge to all those on the list and to any other MAA member who requests it.

Members interested in being listed in December 1975 should write to "SEIS, The Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036," enclosing the following information about themselves:

1. Name
2. Institution
3. Department
4. Address
5. Rank
6. Major field of interest
7. Highest earned degree
8. Names of from one to five courses recently taught
9. Normal teaching load
10. Section of country preferred for visit: Northeast, Southeast, Northcentral, Southcentral, Northwest, Southwest
11. Period for which exchange is desired, e.g., all of the academic year 1976–77, or the first two quarters of 1976–77, or the second semester of 1976–77, etc.

Communications must reach the Washington office by November 17, 1975, for inclusion in the December 1975 list.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE MICHIGAN SECTION

The annual Spring meeting of the Michigan Section of the MAA was held at Central Michigan University, Mt. Pleasant, Michigan, on May 3 and 4, 1974. There were approximately 130 people in attendance. This was the fourth time that the Michigan Section had used the two day format, and the results were again encouraging.

Professor George Feeman, Oakland University, Chairman of the Section, presided at the business meeting. Reports on the High School visiting-lectureship program, the Summer Conference at Northern Michigan University, the Michigan Mathematics Prize Competition, as well as the Secretary-Treasurer's report were presented. The revised by-laws were approved except for the portions dealing with the specific Junior College representation on the officers board or the Executive Committee, for which a committee will be appointed to report at the next meeting. Officers elected for the coming year were: Professor Yousef Alavi, Western Michigan University, Chairman; Professor C. B. Stortz, Northern Michigan University, Vice-Chairman; and Professor Delia Koo, Eastern Michigan University, Secretary-Treasurer.

The program included invited speakers, contributed papers, student papers, two panel discussions, luncheons, and a dinner banquet. The Principal Speaker was Professor J. S. Frame of Michigan State University who presented a one-hour lecture of Friday and a one-hour lecture on Saturday. Other hour addresses were "Permutation Modesl of Set Theory", by Professor Andreas Blass, University of Michigan, and "A Basic Result, from an Unlikely Source", by Professor W. J. Walkoe, Jr., Grand Valley State Colleges.

The Banquet Address was given by Professor John Conway of Cambridge University who spoke on "Life is Universal."

The following contributed papers were presented:

A geometric approach to optimization in economics, by C. J. Titus, University of Michigan.
Non-arithmetic permutations, by T. E. Elsner, General Motors Institute.
Symmetries of the Cayley group table, by J. D. O'Neill, University of Detroit.
Some rotary mathematics, by David Nash, General Motors Research Laboratories.
Group theory and counting, by C. E. Ealy, Northern Michigan University.
Embedding quasigroups and Latin squares, by Douglas Smith, Central Michigan University.
Steiner triple systems and generalizations, by M. L. Eggen, Central Michigan University.
Simpson's rule revisited, by John Empoliti (and John Gibson), Alma College.
Computer graphics and their use in teaching mathematics, by Larry Edison, Alma College.
Elementary calculus: an approach using infinitesimals, by W. P. Francis, Michigan Technological University.

The following student papers were presented:
An application of discriminant analysis, by Richard Gurnee, Oakland University.
Fibonacci numbers, by John Leonard, Alma College.
A course in computerized combinatorics, by Mary Zimmerman, Western Michigan University.
On absolutely pseudo-prime numbers, by Neil Stefling, Kalamazoo, Michigan.
The sum of k-th powers of the first n integers, by David Levinson, University of Michigan.

There were also two panel discussions:

Symposium on Modular Instruction. Moderator: George Van Zwalenberg, Calvin College. Participants: H. B. Anderson, Michigan Technological University; Carl Arendsen, Grand Valley State Colleges; D. L. Ross, Washtenaw Community College; D. P. Wegener, Central Michigan University.

Symposium on the Two-Year College Curriculum. Moderator: Thomas Jefferson Smith, Kalamazoo College. Participants: W. E. Lakey, Central Michigan University; Newell Remington, Delta College; William Tschirthart, Macomb County Community College.

YOUSEF ALAVI, *Secretary-Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The forty-ninth annual meeting of the Philadelphia Section of the MAA was held at Swarthmore College on November 23, 1974. Chairman Joerg Mayer presided over the sessions that were attended by 151 persons.

The following papers were presented:
Bernoulli processes after the isomorphism theorem, by James England, Swarthmore College.

Are closed surfaces rigid? by Herman Gluck, University of Pennsylvania.
In search of a modern understanding of differentials, by F. Cunningham, Jr., Bryn Mawr College.

A special session for students heard the following twenty-minute papers:
The development of Newton's calculus, by Jody Branse, Swarthmore College.
Self-complementary graphs and their degree sequences, by Paul Edelman, Swarthmore College.
Gaussian integers vs. quotients of integer rings, by Thomas Sheehan, Villanova University.
A mathematical model of wheeling between electric utility firms, by John Ries, University of Delaware
A survey of symbolic programming for the non-computer scientist, by Rance De Long, Moravian Colleg
Flexagons, by Adam Rosenberg, Princeton University.

The program closed with a short movie:
Rotating polyhedral forms: M. C. Escher with a twist, produced by Doris Schattschneider, Moravian Colleg
Brief reports were given by the following newly chosen special activites chairpersons: "Applied mathem tics", Wallace Growney, Susquehanna University; "Community colleges", Louis Hoelzle, Bucks Coun Community College; "Newsletter," Dorothy Wolfe, Widener College; "Visiting lecturers," Jerry King, Lehi University.

The top performer from the Section in the 1973 Putnam Competition was Mark Anderson, Lehigh Universit He was recognized and awarded a one year membership in the MAA. Honorable mention citations were al made to D. S. Shucker and Leslie Hogben of Swarthmore College.

Section officers for 1975 are: J. W. P. Mayer, Lebanon Valley College, Chairman; E. A. Klotz, Swarthmo College, Vice-Chairman; P. E. Bedient, Franklin and Marshall College, Secretary-Treasurer. Newly electe members of the Executive Committee are A. E. Bragg, Delaware State College, and Nicholas Grant, Philadelph School District.

P. E. BEDIENT, *Secretary-Treasurer*

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual meeting of the Northern California Section of the MAA was held on the campus of Menl College, on Saturday, February 8, 1975. The meeting was held jointly with the Northern California Section o SIAM. One hundred twenty-five persons were in attendance to hear a program that featured Henry Pollak, Be Telephone Laboratories, President of the MAA, who spoke on "A Recent Example of Applied Mathematics. The following papers were presented:

Computers, confusion and complacency, by H. Peckham, Gavilan College.
What happened — And now what? (A personalized view of mathematics in the last thirty years), by R Gaskell, Naval Postgraduate School.
Star theory, by R. L. Woodriff, Menlo College.
Mathematics of discovery, by Lenore Blum, Mills College.
Discrete symmetry — Groups in the plane, by Frederick Luttman, Sonoma State College.
Hyperbolic numbers, by Herbert Holden, SRI.
The condition number of a matrix grows with dimension, by Alan Shorb, Naval Postgraduate School.
Singular points of algebraic curves defined parametrically, by H. E. Fettis, Mountain View, California.

At the annual business meeting presided over by Chairman Donald Albers, Menlo College, a special gift and tribute were accorded Professor Henry Alder for his long and devoted service to the Northern California Section and as Secretary, MAA. The presentation was made by Professor Harold Bacon, Stanford University (Emeritus).

Further items included a report on the visiting lecturer program (L. Klosinski, Santa Clara) and the election of the following section officers: Chairman: Kenneth Rebman, Hayward State University; Vice-Chairman: David Barnette, U. C. Davis; Program Chairman: Donald Albers, Menlo College; Secretary-Treasurer: Newman Fisher, San Francisco State University. Craig Comstock, Naval Postgraduate School, served as Program Chairman of the meeting and was assisted by Kenneth Rebman, Hayward State University and Susann Shaw, San Francisco State University.

NEWMAN FISHER, *Secretary-Treasurer*

MARCH MEETING OF THE FLORIDA SECTION

The Eighth Annual Spring Meeting of the Florida Section of the MAA was held on March 7 and 8, 1975, at Manatee Junior College in Bradenton, Florida.

Six invited addresses were presented as follows: "The Line Reconstruction Conjecture for Graphs," Professor Frank Harary, University of Michigan; "How to Choose a Wife," Professor Leonard Gillman, University of Texas; "A Fast Switch," Professor Jacob Schwartz, Courant Institute of Mathematical Sciences;

"What your College Administration needs to know about a Mathematics Program," Professor R. C. Fisher, Florida International University; "Volumes of Parallelotopes," Professor Herman Meyer, University of Miami; and "Mathematics in Africa," Professor Don Hill, Florida A & M University.

In conjunction with the meeting there was a State Articulation Conference which began with a report on progress in articulation and an open discussion on current questions. The following talks were presented to the Conference: "Personalized Self Instruction $\cup$ Open Enrollment=Success," Bob Alwin, St. Petersburg Junior College; "General Education Mathematics: Give them the Business," Ignacio Bello, Hillsborough Community College; "The Role of Applications in Two Year College Mathematics Courses," Joe Dorsett, St. Petersburg Junior College; "Decimal Expansion of Rational Numbers," Kenneth Pothoven, University of South Florida; and "Computer Instruction to Nonmathematicians at UNF," Bill Caldwell, University of North Florida.

A Saturday morning session was sponsored by Pi Mu Epsilon, the Mathematics Honorary Fraternity. The following talks were presented: "Double-Suspension Problem," B. S. Trace, Jr., Florida Beta Chapter, Florida State University; "Transformations," Webster Snapp, Florida Eta Chapter, University of North Florida; and "Hypergeometric Functions," E. D. Baker, III, Florida Epsilon Chapter, University of South Florida.

The following papers were presented to the section:

1. *Creating algebraic systems from combinatorial designs*, by Jane W. DiPaola, Florida Atlantic University.
2. *Duplication rates and sampling for matching pairs*, by R. B. Levow, Florida Atlantic University.
3. *Structured programming*, by John Grant, University of Florida.
4. *(m,n) rings*, by J. J. Leeson, University of North Florida.
5. *Not all topologically simple semigroups are simple*, by Kermit Sigmon, University of Florida.
6. *Math majors at Penn State: Their academic performance*, by Bertha Mather, Pennsylvania State University, Retired.
7. *Teaching the polynomials*, by Alan Wayne, Pasco-Hernando Community College.
8. *Motivating existence-uniqueness theorems for applications-oriented students*, by Dave Snider, University of South Florida.
9. *Mathematical sciences and urban emergency services*, by Anthony Shershin, Florida International University.
10. *Padé approximants to the exponential*, by Edward Saff, University of South Florida.
11. *A characterization of isometries on C_0*, by Soo Bong Chae, New College.
12. *On a new method for studying the stability of Volterra integral equations*, by William Zigler, University of South Florida.
13. *A modified method of successive integrations*, by Albert Rust, University of South Florida.
14. *Holomorphic vector fields and exact differentials*, by Annette Sinclair, Purdue University.
15. *Development of the elementary complex functions*, by D. L. Sherry, University of West Florida.

The luncheon-business meeting was held Saturday, March 8, 1975. Chairman Ralph McWilliams presided at the meeting. Committee reports were presented and Professor A. W. Goodman of the University of South Florida was elected chairman-elect; Professors D. M. Hill of Florida A & M University and L. R. Wirak of Lake Sumpter Community College, were elected as Vice-Chairmen.

F. L. CLEAVER, *Secretary*

MARCH MEETING OF THE SOUTHEASTERN SECTION

The fifty-fourth annual meeting of the Southeastern Section was held on March 21 and 22, 1975, at the University of South Alabama in Mobile, Alabama. A total of 225 persons attended the meeting, including 177 members of the MAA. The local arrangements were handled by R. G. Vinson and D. G. Belanger.

Three invited addresses were given: W. R. Mann (Section Lecturer) of the University of North Carolina, Chapel Hill, "Some Aspects of Nonlinear Mathematics"; R. C. Buck of the University of Wisconsin, Madison, on "A Role for Speculation," and H. O. Pollak (President of the MAA) of Bell Telephone Laboratories on "What Industry Wants Mathematicians to Know, and How they Want them to Know it." There was a Symposium on Mathematics in the Two-Year Colleges, featuring S. H. Brown of Auburn University and R. F. Drennen of Jefferson State Junior College in a discussion of "In-Service Training in Mathematics for Junior College Faculty."

There were seven sessions for contributed papers. Presiders for the general sessions were J. H. Wahab (Chairman of the Section), R. G. Vinson, D. L. Hunter, and J. R. Wesson; and for the special sessions were Catherine C. Aust, J. V. Herod, M. C. Wicht, D. S. Nymann, J. C. Wiener, B. Farmer, and E. V. Haynsworth. Two MAA films were shown on Friday evening.

Officers elected for 1975–6 were: Chairman, J. R. Wesson, Vanderbilt University; Chairman-Elect, Emilie V. Haynsworth, Auburn University; Vice Chairman, Roy Dobyns, Clayton Junior College; Section Lecturer, J. V. Brawley, Clemson University, and J. D. Neff, Georgia Institute of Technology, was re-elected Secretary-Treasurer for the second term.

At the business meeting, it was announced that P. N. Strenski of Armstrong State College was a Putnam Fellow and the Winner of the $25 prize given by the Section to the student in a Section school who scores highest in the Putnam Examination.

The following contributed papers were presented:

1. *An alternate proof of a theorem on normal matrices*, by Lynne H. Moorer, Tennessee Technological University.
2. *Finite unions of ideals and modules*, by Philip Quartararo, Jr., Southern University, and H. S. Butts, Louisiana State University.
3. *The identification of all two-by-two and three-by-three matric semigroups with same zero pattern*, by T. F. Mulcrone, S. J., Spring Hill College.
4. *A source of counterexamples for properties of direct products*, by R. R. Appleson, Georgia Institute of Technology.
5. *Algebras in the theory of elementary particles*, by Andrew Sobczyk, Clemson University.
6. *Goals and procedures in hypothesis testing*, by F. C. Barnett, University of West Florida.
7. *The effectiveness of a competency-based algebra course*, by W. E. Haver, University of Tennessee, Knoxville.
8. *Mathematics and the underprepared student*, by S. A. Doblin, University of Southern Mississippi.
9. *Diagnostic and mastery testing in remedial mathematics*, by Barbara J. Webster, Gainesville Junior College.
10. *Generators for evolution systems II*, by S. D. Purdom, Georgia Institute of Technology.
11. *Polynomials with preassigned values and multiplicities at their branch points*, by M. L. Marx, Vanderbilt University.
12. *Point-determining graphs*, by D. P. Geoffroy, University of South Carolina.
13. *A characterization of Δ-representable sets*, by J. I. Moore, Jr., University of South Carolina.
14. *Three new separation axioms for topological spaces*, by J. A. Bond, Jr., Macon Junior College.
15. *Lipschitz condition in topological vector spaces*, by T. W. Goodman, Catawba College.
16. *Using pocket calculators in teaching undergraduate mathematics*, by J. H. Carruth, University of Tennessee, Knoxville.
17. *Computer assistance in elementary differential equations*, by W. R. Boland, T. G. Proctor, and H. H. Suber, Clemson University.
18. *A visual demonstration of the Fourier integral approximation of functions via computer terminal output*, by L. L. Long, Tennessee Technological University.
19. *Using the computer as a classroom demonstration device*, by J. R. Smith, Appalachian State University.
20. *A zero-g liquid film problem by Lagrange multipliers*, by Wesley Darbro, Marshall Space Flight Center, Huntsville.
21. *Fibonacci representations and Wythoff's nim*, by Robert Silber, North Carolina State University.
22. *An optimal control model in human capital theory*, by F. A. Massey and Paula E. Stephen, Georgia State University.
23. *A problem of restricted partitions, II*, by V.R.R. Uppuluri and J. A. Carpenter, Union Carbide Corporation, Oak Ridge.
24. *A personalized system of instruction with constraints*, by Mildred B. Johnson, Jacksonville State University.
25. *Placing the responsibility for learning mathematics on the student by using different testing contingencies*, by Robert Catanzano, Berry College.
26. *Taking college calculus to the high school*, by R. L. Curd, Appalachian State University.
27. *Some formulas for computing f(A)*, by Ben Caldwell, Auburn University.
28. *Hexagon form of the fifteen puzzle*, by R. W. Gibson, Auburn University.

J. D. NEFF, *Secretary-Treasurer*

THE 1975 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The 36th annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 6, 1975. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship and is administered by The Mathematical Association of America. Colleges and

universities in Canada and the United States may register eligible undergraduates. Registration forms will be mailed to institutions that participated in the 35th competition by September 22, 1975. Other institutions that wish to enter contestants should request these forms from Dr. A. P. Hillman, Director; William Lowell Putnam Mathematical Competition; 709 Solano Dr., S. E.; Albuquerque, New Mexico 87108. Completed registrations must be received by the director no later than October 28, 1975.

Further details are given in the Announcement Brochure that will be mailed with the registration material. Reports of previous competitions, including examination questions and outlines of solutions, are in past issues of this MONTHLY; the most recent of these reports were in the December 1974, November 1973, February 1973, and August-September 1971 issues.

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1975 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Emil Grosswald and I. J. Schoenberg, were announced by President Henry O. Pollak at the business meeting of the Association on August 19, 1975, at Western Michigan University. The recipients of the Ford Awards for articles published in 1974 were the following:

R. Ayoub, *Euler and the Zeta Function,* MONTHLY, 81 (1974), 1067–1086.

J. Callahan, *Singularities and Plane Maps,* MONTHLY, 81 (1974), 211–240.

D. E. Knuth, *Computer Science and its Relation to Mathematics,* MONTHLY, 81 (1974), 323–343.

J. C. C. Nitsche, *Plateau's Problems and Their Modern Ramifications,* MONTHLY, 81 (1974), 945–968.

S. K. Stein, *Algebraic Tiling,* MONTHLY, 81 (1974), 445–462.

L. Zalcman, *Real Proofs of Complex Theorems,* MONTHLY, 81 (1974), 115-137.

DAVID P. ROSELLE, *Secretary*

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association representing the Sections indicated:

ALLEGHENY MOUNTAIN, Earle F. Myers, University of Pittsburgh.
INDIANA, Maynard Mansfield, Purdue University, Ft. Wayne.
KENTUCKY, Donald B. Coleman, University of Kentucky.
METROPOLITAN NEW YORK, William Zlot, New York University.
NEBRASKA, Mildred L. Gross, Doane College.
NORTHERN CALIFORNIA, Gerald L. Alexanderson, University of Santa Clara.
OKLAHOMA-ARKANSAS, John M. Jobe, Oklahoma State University.
ROCKY MOUNTAIN, Forrest Fisch, University of Northern Colorado.
WISCONSIN, Phillip R. Bender, Marquette University.

The highest percentage of voters was 40.6%, occurring in the Oklahoma-Arkansas Section. The Kentucky Section was the runner-up with 39.6%.

A. B. WILLCOX, *Executive Director*

CORRECTION: The third sentence of paragraph 8 on page 566 of the May 1975 issue of this MONTHLY should be corrected to read: "Professor Davis recalled two of his teachers from the time he was an undergraduate at City College in New York."

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Valparaiso University, Valparaiso, November 15, 1975.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13-14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D.C November 22, 1975.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State College, Springfield, April 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY

NORTH CENTRAL, Southwest Minnesota State College, Marshall, October 25, 1975.

NORTHEASTERN, Simmons College, Boston, Mass., November 29, 1975.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

OHIO

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Ark., March 26–27, 1976.

PACIFIC NORTHWEST, Second Saturday in June. Deadline for papers 6 wks. bef. mtg.

PHILADELPHIA, Franklin and Marshall College, Lancaster, November 22, 1975.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, May 1–2, 1976.

SEAWAY, State University College, Cortland, N.Y., October 31-November 1, 1975.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, Spring 1976.

SOUTHERN CALIFORNIA, First or Second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, N. M. April 1976.

TEXAS, Texas A & M University, College Station, first or second weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 20–22 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION, California State University, San Francisco, October 18, 1975.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976

OPERATIONS RESEARCH SOCIETY OF AMERICA, MGM Grand Hotel, Las Vegas, November 17–19, 1975.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Sheraton-Palace Hotel, San Francisco, December 3–5, 1975 (SIAM-SIGNUM 1975 Fall Meeting).

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A complete collection of examination questions and solutions from the first decade of this national high school mathematics competition. No mathematics beyond intermediate algebra required. Elementary procedures are consistently used in solutions, but more sophisticated alternatives are included where appropriate.

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by P. J. Davis
How to work with numbers, big and small, and understand some of their less obvious properties.

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How mathematics have transformed the almost mystic concept of infinity into a precise tool essential in all branches of mathematics.

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Concerned with isometries (distance-preserving transformations), a completely different way of looking at familiar geometrical facts that supplies the students with methods by which powerful problem solving techniques may be developed.

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by Carl D. Olds
Shows how rational fractions can be expanded into continued fractions and introduces such diverse topics as the solutions of linear Diophantine equations, expansion of irrational numbers into infinite continued fractions and rational approximation to irrational numbers.

GRAPHS AND THEIR USES (NML-10)
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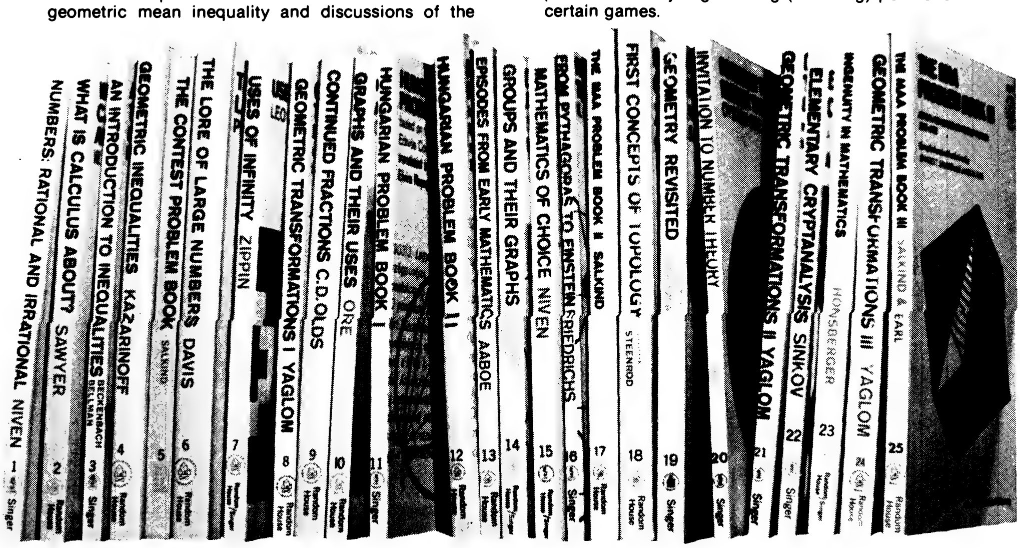

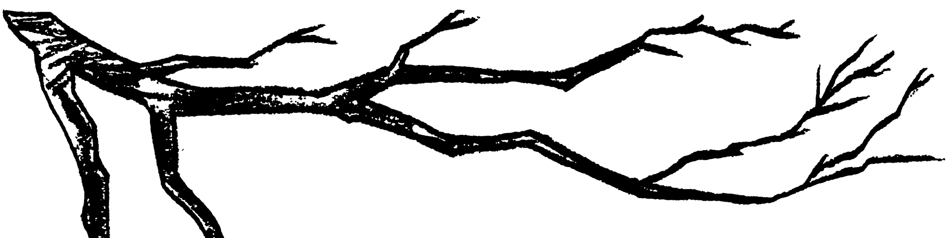

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Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

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ON THE APPLICATIONS OF MOBIUS INVERSION IN COMBINATORIAL ANALYSIS

E. A. BENDER AND J. R. GOLDMAN

1. Introduction. Inversion of a finite series is one of the most useful tools in combinatorics and probability. The classical inclusion-exclusion principle is a special case (Feller (1968), Ryser (1963)). Although many inversion problems can be phrased in terms of inclusion-exclusion, the framework often seems artificial. Frequently a "natural" ordering of the objects being studied is possible. This is the gestalt of the technique of Möbius inversion.

Möbius inversion is an overcounting-undercounting, or sieving, procedure. We keep track of the over and undercount by indexing with the elements of a partially ordered set which classically was the subsets of a finite set. The Möbius inversion formula of number theory as given in Hardy and Wright (1960) indexes functions with the set of positive integers under the divisibility order. This latter formula lends its name to the general subject.

The principle of inclusion-exclusion, which after all is not a very deep statement, was investigated by several 19th century mathematicians and perhaps stated most clearly by Poincaré. It has been rediscovered many times in varying degrees of generality. A fairly complete development of this principle together with a history and development of classical applications in probability theory is given in the monograph of Fréchet (1940, 1943).

The statement of the general Möbius inversion formula was first given independently by Weisner (1935) and Philip Hall (1936); both authors were motivated by group theory problems. Neither author seems to have been aware of the combinatorial implications of his work and neither developed the theory of Möbius functions. In a fundamental paper on Möbius functions, Rota (1964) showed the importance of this theory in combinatorial mathematics and gave a deep treatment of it. He noted the relation between such topics as inclusion-exclusion, classical number theoretic Möbius inversion, coloring problems and flows in networks. Since then, under the strong influence of Rota, the theory of Möbius inversion and related topics has become an active area of combinatorics.

Here we present many applications of Möbius inversion in combinatorics with emphasis on recent results. This paper complements Rota's original paper (1964) (also to be referred to as Foundations I) in that Rota developed the theory of the Möbius function as related to the structure of the ordering. Foundations I contains an extensive bibliography. We have not reproduced this here but we have attempted to bring it up to date.

We begin with a series of examples to motivate the framework of Möbius inversion.

Example 1. *Finite Series.* Let $f(n)$ be a function on the positive integers (i.e., a series $f(1), f(2), f(3), \cdots$) and let $g(n) = \Sigma_{m \leq n} f(m)$. We invert the sum, i.e., express $f(n)$ in terms of g; the answer is obviously

$$f(n) = g(n) - g(n-1). \tag{1}$$

Example 2. *Inclusion-Exclusion Principles.* Given a set $S = \{s_1, s_2, \cdots, s_k\}$ and a collection of properties $P = \{p_1, p_2, \cdots, p_n\}$. A property p_i is defined by stating which elements have it and which do not (hence a property is subset of S; viz., those elements which satisfy it). For any collection T of properties, $T \subseteq P$, let $N_{\geqq}(T)$ (read "N sub $\geqq$ of T") be the number of elements of S which satisfy every property in T *and possibly others.* Let $N_{=}(T)$ be the number of elements which satisfy exactly the properties in T *and no others.* Clearly

$$N_{\geqq}(T) = \sum_{X \supseteq T} N_{=}(X),$$

for every element which satisfies at least all properties in T satisfies exactly some set X of

properties where $X \supseteq T$. Our problem is to solve for $N_=(T)$ in terms of the function $N_\geq(X)$. Frequently we want $N_=(\varnothing)$, the number of elements satisfying no properties.

Example 3. *Classical Möbius Inversion.* The following problem from number theory motivates some of our general terminology and results. Let $f(n)$ be a function defined on the positive integers and define

$$h(n) = \sum_{k|n} f(k)$$

where "$k \mid n$" is read "k divides n" and the summation is therefore over all integral divisors of n. We wish to invert the sum, i.e., solve for $f(n)$ in terms of h. The problem is solved in many elementary number theory texts. See, for example, Hardy and Wright (1960). We shall derive it as a special case of a more general theory.

Example 4. *Spanning Sets of a Vector Space.* How many subsets of $V_n(q)$, the n-dimensional vector space over a field of q elements, span the whole space? For any subspace U of $V_n(q)$ let $N_=(U)$ be the number of subsets of vectors of V_n which span U. Let $N_\leq(U)$ be the number of sets spanning U or a subspace of U. Then we have $N_\leq(U) = \Sigma_{V \subseteq U} N_=(V)$ where the sum is over all subspaces of U. Our problem is to solve for $N_=(U)$ in terms of $N_\leq(U)$ and set $U = V_n(q)$.

Our four examples have a number of common ideas which we abstract in the following table:

		Example 1	Example 2	Example 3	Example 4
(i)	A set S	Positive Integers	Subsets of P	Positive Integers	Subspaces of $V_n(q)$
(ii)	An "order" relation	$\leq$	$\supseteq$ (Set inclusion)	$\mid$ divisibility	is a subspace of $(\leq)$
(iii)	A given function on S	$f(n)$	$N_=(T)$	$f(n)$	$N_=(U)$
(iv)	A summation function	$g(n) = \sum_{m \leq n} f(m)$	$N_\geq(T) = \sum_{X \supseteq T} N_=(X)$	$h(n) = \sum_{k|n} f(k)$	$N_\leq(U) = \sum_{V \leq U} N_=(V)$

In each case we want to invert a system of linear equations, i.e., solve for the given function in terms of the summation function. The summation function is with respect to a given "ordering." This ordering generalizes the usual notion of order for integers or real numbers.

To study the inversion problem in its proper generality we now review the theory of "order" relations or, as they are more commonly known, "partially ordered sets."

2. Partially ordered sets.

DEFINITION: *A partially ordered set (POS)* $\Sigma = (S, \leqq)$ is a pair consisting of a set S and a binary relation $\leqq$ on S, satisfying the following properties:

(a) (reflexive) $x \leqq x$ for all $x \in S$,
(1) (b) (transitive) if $x \leqq y$ and $y \leqq z$ then $x \leqq z$,
(c) (anti-symmetric) if $x \leqq y$ and $y \leqq x$ then $x = y$.

We read "$x \leqq y$" as "x is less than or equal to y." Partially ordered sets are also called *ordered*

sets. The notation and terminology of ordered sets is similar to that for ordinary inequality, e.g., $x < y$ means $x \leqq y$ and $x \neq y$, and $x \not\leqq y$ means $x \leqq y$ is not true.

What distinguishes ordered sets from ordinary inequality is that elements may be "incomparable." x and y are *incomparable* if $x \leqq y$ is false and $y \leqq x$ is also false. If for every two elements x, y either $x \leqq y$ or $y \leqq x$ is true, then $\Sigma = (S, \leqq)$ is called a *linearly ordered set* or a *chain*.

Example 1. (a) *Integers with ordinary ordering*: Let S be the positive integers Z^+ or all integers Z with the usual ordering ($a \leqq b$ if and only if $b - a$ is positive). $\Sigma = (S, \leqq)$ is linearly ordered.

(b) Let S be the integers between 1 and n with the ordinary ordering. $(S, \leqq)$ is a linearly ordered set.

(c) *Subsets of a set* (Boolean algebra) (see Example 1.2): Let T be a set and S the collection 2^T of subsets of T. If $A, B \subseteq T$, then $A \leqq B$ iff $A \subseteq B$ (A is a subset of B). $(S, \leqq)$ is not a linearly ordered set, e.g., any two 1 element subsets are incomparable. This ordered set is often called the "subsets of T ordered by *inclusion*."

(d) *Integers under divisibility* (see Example 1.3): Let S be the positive integers and let $a \leqq b$ iff $a \mid b$ (a divides b). Let Δ denote this *POS*.

(e) *Divisors of n*: Let S be all divisors of the integer n and let $a \leqq b$ mean $a \mid b$ as in the previous example. This *POS* will be denoted by Δ_n or $D(n)$.

(f) *Subspaces of a vector space* (see Example 1.4): Let S be the set of subspaces of a vector space and let $\leqq$ mean "is a subset of."

(g) In general, given any "mathematical system," the "sub-systems" ordered by inclusion, form a partially ordered set, e.g., subgroups of a group.

DEFINITION: An *interval* $[x, y]$ is the set of all elements "between" x and y, i.e., $[x, y] = \{z \in S \mid x \leqq z \leqq y\}$. However, by an abuse of language, we sometimes use $[x, y]$ to denote the induced sub-*POS*. $(\{z \in S \mid x \leq z \leq y\}, \leq)$. A partially ordered set is *locally finite* if every interval has a finite number of elements.

Example 2. (a) The real numbers with the usual ordering is not locally finite.

(b) The *POS* of finite subsets of any set T is locally finite.

DEFINITION: Two partially ordered sets are *isomorphic* if they differ only by a labeling of their elements and ordering relation; more formally, $(S, \leqq)$ is *isomorphic* to $(S', \leqq')$, written $(S, \leqq) \cong (S', \leqq')$, if and only if there is a one-one onto map $\phi: S \to S'$ such that $x \leqq y$ if and only if $\phi(x) \leqq' \phi(y)$.

Example 3. *Subsets* (continued). Let $B(T_n)$ be the subsets of T_n ordered by inclusion, where $|T_n| = n$, and let S_n be the set of all n-tuples of zeros and ones with $a \leqq b$ meaning $a_i \leqq b_i$ for each of the n components of a and b. Let $\Sigma_n = (S_n, \leqq)$. We claim that $B(T_n) \cong \Sigma_n$: Let $t_1, \cdots, t_n$ be a listing of of elements of T_n. If $X \subseteq T_n$, define $\phi(X) = x = (x_1, \cdots, x_n) \in S_n$, where

$$x_i = \begin{cases} 0 & \text{if } t_i \notin X \\ 1 & \text{if } t_i \in X \end{cases}.$$

It is easy to see that ϕ is an isomorphism.

3. Möbius inversion. We can now formulate and solve the general inversion problem discussed in Section 1. Proofs are given in Foundations I.

THEOREM 1: MÖBIUS INVERSION FORMULA I. *Let $N_=(x)$ be a real valued function defined for all x in a locally finite partially ordered set $(S, \leqq)$ and assume there is an element $m \in S$ such that $N_=(x) = 0$ when $x \not\leqq m$. Define $N_\geqq(x)$ by*

$$N_\geqq(x) = \sum_{y : y \geqq x} N_=(y). \tag{1a}$$

Then

(1b) $$N_=(x) = \sum_{y:y \geqq x} \mu(x,y) N_{\geqq}(y),$$

where $\mu(x,y)$, *the Möbius function of* $(S, \leqq)$, *is an integer valued function of two variables on* S *defined by* $\mu(x,z) = 0$ *when* $x \not\leqq z$ *and, when* $x \leqq z$, *by*

(2) $$\sum_{y:x \leqq y \leqq z} \mu(x,y) = \delta(x,z).$$

($\delta(x,z)$, *the Kronecker delta, is given by* $\delta(x,x) = 1$, $\delta(x,z) = 0$ *if* $x \neq z$.)

NOTE: The condition $N_=(x) = 0$ when $x \not\leqq m$ assures that all sums in our theorem are finite. Conditions under which infinite sums are allowed remains an open question (see Hille (1937)).

NOTE: It is not necessary to restrict $N_=(x)$ to real valued functions.

THEOREM 2: MÖBIUS INVERSION FORMULA II. *Let* $(S, \leqq)$ *be a locally finite partially ordered set. Let* $N_=(x)$ *be defined for all* $x \in S$ *and let there be an* $l \in S$ *such that* $N_=(x) = 0$ *when* $x \not\geqq l$. *Define*

(3a) $$N_{\leq}(x) = \sum_{y:y \leq x} N_=(y).$$

Then

(3b) $$N_=(x) = \sum_{y:y \leq x} \mu(y,x) N_{\leq}(y),$$

where μ *is defined by* (2).

COROLLARY 1. *The Möbius function* μ *of a locally finite POS can be computed recursively by either of the formulae*

(4a) $$\mu(x,z) = -\sum_{y:x \leqq y < z} \mu(x,y), \qquad x < z,$$

(4b) $$\mu(x,z) = -\sum_{y:x < y \leqq z} \mu(y,z), \qquad x < z,$$

together with $\mu(x,x) = 1$.

COROLLARY 2. *If* $x \leqq y \leqq z \leqq w$ *in a locally finite partially ordered set* Σ, *then* $\mu(y,z)$ *in* Σ *equals* $\mu(y,z)$ *in* $[x,w]$. (The "surroundings" don't matter — only the interval on which you want μ.)

COROLLARY 3. *If* Σ *and* Σ' *are isomorphic P.O.S's with Möbius functions* μ *and* μ' *and if* $[x,y] \cong [x',y']$, *then* $\mu(x,y) = \mu'(x',y')$.

Example 1. *Integers* (continued). If S is the set of integers with the usual ordering, the Möbius function is given by $\mu(n,n) = 1$, $\mu(n,n+1) = -1$, and $\mu(n,k) = 0$ otherwise. This follows immediately since we have already solved the inversion problem in equation (1.1) and we need only compare the coefficients of the terms in (1.1) with those of the general inversion formula in equation (3b). This is the method of undetermined coefficients. The Möbius function can also be derived from formula (4a) or (4b).

Direct products. Our main approach to computing Möbius functions will be to construct complicated *POS*'s from simple ones, compute μ for the simple sets by undetermined coefficients,

and use these results to compute μ for complicated sets. Our construction tool is the "direct product." Other more sophisticated approaches are found in Rota (1964).

DEFINITION: Let $\Sigma_1 = (S_1, \leqq_1)$ and $\Sigma_2 = (S_2, \leqq_2)$ be *POS*'s. The *direct product* $\Sigma = \Sigma_1 \times \Sigma_2$ of Σ_1 and Σ_2 is the *POS* $(S, \leqq)$, where

(i) $S = S_1 \times S_2 = \{(a,b) | a \in S_1, b \in S_2\}$,

(ii) $a \leqq b$ in Σ if and only if $a_1 \leqq_1 b_1$ and $a_2 \leqq_2 b_2$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

THEOREM 3: PRODUCT THEOREM. *If Σ_1 has Möbius function μ_1 and Σ_2 has Möbius function μ_2, then the Möbius function μ of $\Sigma_1 \times \Sigma_2$ is given by*

$$\mu((x_1,x_2),(y_1,y_2)) = \mu_1(x_1,y_1)\mu_2(x_2,y_2). \tag{5}$$

Example 2: *Inclusion-Exclusion, Subsets.* By Example 2.3 the Boolean algebra $B(T_n)$ is isomorphic to Σ_n, the set of n-tuples of 0's and 1's. But $\Sigma_n \cong \Sigma_1 \times \overset{n \text{ times}}{\cdots} \times \Sigma_1$. For Σ_1 and $y \geqq x$ we have $\mu(x,y) = (-1)^{y-x}$ since the only possibilities are $x = y$ or $x = 0$, $y = 1$. Under the isomorphism $B(T_n) \cong \Sigma_1 \times \cdots \times \Sigma_1$, let $x \leftrightarrow (x_1, \cdots, x_n)$ and $y \leftrightarrow (y_1, \cdots, y_n)$, then

$$\mu(x,y) = \mu((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \prod_{i=1}^{n} \mu(x_i,y_i) = (-1)^{\Sigma y_i - \Sigma x_i} = (-1)^{|y|-|x|}, \tag{6}$$

where $|y|$ is the number of elements in y. Substituting into equation (1b) we get

$$N_=(x) = \sum_{y \geqq x} (-1)^{|y|-|x|} N_{\geqq}(y),$$

the basic Inclusion-Exclusion Principle.

Example 3: *Divisors, Classical Möbius Inversion.* (See Examples 1.3 and 2.1e.) By the Unique Factorization Theorem $D(n) \cong D(p_1^{\alpha_1}) \times \cdots \times D(p_s^{\alpha_s})$. Hence it suffices to compute μ on $D(p^\alpha)$. We have already done this because $D(p^\alpha)$ is the chain $1|p|p^2 \cdots |p^\alpha$, which is isomorphic to the integers treated in Example 1. Hence

$$\mu(p^i, p^j) = \begin{cases} 1 & \text{if} \quad i = j \\ -1 & \text{if} \quad j - i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By the product theorem

$$\mu\left(\prod_{i=0}^{s} p_i^{a_i}, \prod_{i=0}^{s} p_i^{b_i}\right) = \begin{cases} (-1)^{\Sigma(b_i - a_i)} & \text{if } b_i - a_i = 0 \text{ or } 1 \text{ for all } i, \\ 0 \text{ if } b_i - a_i > 1 \text{ for some } i. \end{cases}$$

Thus

$$\mu(a,b) = \mu(1, b/a) \equiv \mu(b/a), \tag{7}$$

where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if a square divides } n. \end{cases} \tag{8}$$

This is the classical Möbius function. Since $D(n)$ is the interval $[1,n]$ of Δ, we have computed μ for Δ. We could have deduced (7) directly by observing $[a,b] \cong [1, b/a]$ by the unique factorization

theorem. Equation (3b) becomes

$$N_{=}(x)=\sum_{y\mid x}\mu(y,x)N_{\leq}(y)=\sum_{y\mid x}\mu\left(\frac{x}{y}\right)N_{\leq}(y) \tag{9a}$$

as expected and (1b) gives

$$N_{=}(x)=\sum_{y:x\mid y}\mu(x,y)N_{\geq}(y)=\sum_{k=1}^{\infty}\mu(k)N_{\geq}(kx), \tag{9b}$$

a somewhat less known type of inversion using the classical Möbius function (Hardy and Wright, 1960).

We now present two applications of the former result.

(a) *The Euler phi-function.* The ϕ-function, $\phi(n)$, is the number of positive integers x not exceeding n which are prime to n; i.e., $\gcd(n,x)=1$.

Let $N_{=}(n)=\phi(n)$. To compute $N_{\leq}(n)$ we break up the set $[n]=\{1,2,\cdots,n\}$ according to the gcd with n, i.e., let $S_d=\{i\in[n]\mid \gcd(i,n)=d\}$. The S_d are mutually disjoint and their union is $[n]$. Hence $n=\Sigma_{d\mid n}|S_d|$. But $i\in S_d$ iff $i=kd$, where $k\leqq i$ and $\gcd(k,n/d)=1$. Hence $|S_d|=\phi(n/d)$ and $n=\Sigma_{d\mid n}\phi(n/d)=\Sigma_{d'\mid n}\phi(d')=N_{\leqq}(n)$. By Möbius inversion

$$\phi(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)d=n-\frac{n}{p_1}-\frac{n}{p_2}\cdots+\frac{n}{p_1p_2}\cdots, \tag{10}$$

since $\mu(n/d)$ is non-zero only if n/d is a product of distinct primes. Thus

$$\phi(n)=n\prod_{p\mid n}\left(1-\frac{1}{p}\right),$$

where the product ranges over all primes p dividing n.

(b) *Counting Necklaces.* Suppose we have k different colors of beads in unlimited supply, how many n bead necklaces can be formed? We must specify precisely when two necklaces are the same. Every necklace has a front and back, but shifting the beads circularly (bead at $i\to$ location $i+1$) does not change the necklace.

If we shift n beads circularly, we discover they eventually return to the initial color configuration after, say, d shifts where $d\mid n$. The *period* is the smallest number of shifts required for a return. Since

$$\text{RWBRWB}\to\text{BRWBRW}\to\text{WBRWBR}\to\text{RWBRWB},$$

this string has period 3. Suppose we have an n long string of period d. Including itself, it has d shifts all of which give the same necklace when the ends are joined. Furthermore, these are the only strings that give this necklace. Our circular problem is reduced to a linear one:

$$\#\text{ necklaces of length } n=\sum_{d\mid n}\frac{1}{d}(\#\text{ strings of period } d),$$

where we have omitted the length on the right hand side since the initial d beads determine the string. Clearly

$$\#\text{ of strings of length } n=\sum_{d\mid n}\#\text{ of strings of period } d.$$

The left side is clearly k^n since there are k colors of beads. Möbius inversion gives

$$\#\text{ of strings of period } d=\sum_{x\mid d}\mu\left(\frac{d}{x}\right)k^x.$$

Hence

$$\# \text{ of necklaces of length } n = \sum_{d|n} \frac{1}{d} \sum_{x|d} \mu\left(\frac{d}{x}\right) k^x = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) k^d,$$

where the simplification involves the use of (10).

Example 4: *Convex polytopes.* A very detailed and beautiful study of convex polytopes is given by Grünbaum (1967).

A d-dimensional convex polytope is a bounded d-dimensional set of points in a Euclidean space which can be given as the intersection of half-spaces (i.e., all points on one side of a hyper-plane). For example the triangle in Figure 1 is the intersection of the three indicated half planes — determined by the lines (hyper-planes) a, b, c. We call $\overline{123}$ a 2–face (i.e., 2–dimensional face) of the polytope, $\overline{12}, \overline{23}, \overline{31}$ the 1–faces and $1, 2, 3$ the 0–faces. In higher dimensions the notion of a face can be defined in terms of supporting hyperplanes (Grünbaum, 1967). A polytope can also be thought of as the convex closure of a finite set of points in n-space R^n.

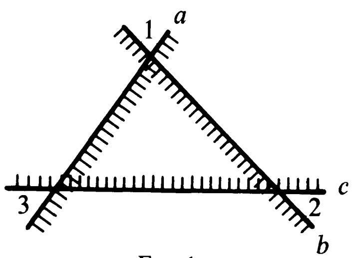

FIG. 1

Let P be a d-dimensional polytope and $\mathscr{F}_P$ the *POS* of faces of P ordered by inclusion, including the empty face $\varnothing$ of dimension -1 and the face P. For any $x \in \mathscr{F}_P$

$$[\varnothing, x] \cong \mathscr{F}_x. \tag{11}$$

Let $f_k(x)$ be the number of k-dimensional faces containing x. The generalized Euler relation states $\Sigma_j (-1)^{d-j} f_j(x) = \delta(x, P)$, where $\delta(x, P)$ is the Knonecker delta (Grünbaum, 1967). We can rewrite this as

$$\sum_{y : y \geqq x} (-1)^{d(P)-d(y)} = \delta(x, P), \tag{12}$$

where $d(y)$ is the dimension of y. From (12) and (2) we have $\mu(y, P) = (-1)^{d(P)-d(y)}$ and then by (11)

$$\mu(x, y) = (-1)^{d(y)-d(x)}. \tag{13}$$

This suggests that a homology theory might be defined for *POS*'s with μ related to the Euler characteristic. This has been started by Rota (1964, 1971).

A k-dimensional simplex has $k+1$ vertices and every subset of $j+1$ vertices determines a j-dimensional face. A simplicial polytope is a polytope P in which every face, except possibly P, is a simplex, e.g., triangles, octahedrons and tetrahedrons, are simplicial polytopes. While Euler's relation (equivalently, equation (13)) is the only relation satisfied by the faces of a general polytope, we might expect other relations for a simplicial polytope. In this case, $x \leqq y < P$ implies that $[x, y]$ is isomorphic to the *POS* of subsets of a $d(y) - d(x)$ element set. By using (4a) to sum $\mu(x, P)$ over all $x \geqq w$ with $d(x) = j < d(P)$ we get

$$(-1)^{d(P)-1} f_j(w) = \sum_k (-1)^k \binom{k - d(w)}{j - d(w)} f_k(w). \tag{14}$$

If we put $w = \varnothing$, then $d(w) = -1$ and we get the Dehn-Sommerville equations (Grünbaum, 1967).

Example 5: *Map coloring.* A map is a planar graph: a (finite) collection of connected, bounded regions in the plane whose boundaries are smooth curves. Two countries sharing a segment of a curve (more than a point) are *adjacent.* If the countries are colored so that no two adjacent countries are the same color, the result is a proper coloring. Let G be a map and let $M_G(\lambda)$ be the number of proper colorings. A *submap* of G is obtained by erasing boundaries between countries. Any map can be colored in $\lambda^{|G|}$ ways where $|G|$ *is the number of countries in G.* Any such coloring is proper for precisely one submap of G. (Just erase those boundaries between countries of the same color.) The relation "is a submap of" makes the submaps of G into an ordered set and

$$\lambda^{|G|} = \sum_{x \leqq G} M_x(\lambda).$$

Since $[0, y]$ is isomorphic to the ordered set of submaps of y, we have

$$\lambda^{|y|} = \sum_{x \in y} M_x(\lambda)$$

Hence if we set $N_=(x) = M_x(\lambda)$, then $N_{\leqq}(y) = \lambda^{|y|}$. By Möbius inverting and setting $y = G$, we obtain

$$M_G(\lambda) = \sum_{x \leqq G} \lambda^{|x|} \mu(x, G).$$

For obvious reasons, $M_G(\lambda)$ is called the *chromatic polynomial* of G. Computation of $M_G(\lambda)$ is difficult when we have no easy way to compute μ. The chromatic polynomials were introduced as a tool for attacking the 4–color problem by Birkhoff and Lewis (1946). Other references include Whitney (1932) who derives a formula by μ-inversion over Boolean algebras, Rota (1964), Wilf (1969), and Read (1968) who has a very nice introduction to the properties of chromatic polynomials. Redoing some of Read's proofs by using properties of the μ-function makes a good exercise.

By introducing the dual graph to a map, where the operation of erasing boundaries is replaced by contracting edges, the general problem of properly coloring the vertices of an arbitrary graph can be treated just as we have done for maps (Rota, 1964).

Using Möbius inversion as a key tool Crapo and Rota (1971) embed the four color problem and the study of chromatic polynomials into a more general problem namely the *critical problem* for combinatorial geometries. This problem, which is one of finding minimal sets of separating hyperplanes for sets of points in finite projective spaces, includes as special cases problems in coding theory and Segre's results characterizing sets of independent points in projective space, (Dowling, 1971).

4. Partitions of a set. An (unordered) *partition* of a finite n element set S_n is a collection $\{\pi_1, \pi_2, \cdots\}$ of non-empty mutually disjoint subsets of S_n whose union is S_n, i.e., $\pi_1 \cap \pi_j = \varnothing$ if $i \neq j$ and $\cup_i \pi_i = S_n$. For instance $\{\{1,3\},\{2\}\}$ is a partition of $\{1,2,3\}$. The sets π_i are called the *blocks* of the partition.

Let $S(n,k)$ denote the number of partitions of S_n into k blocks. The $S(n,k)$ are called *Stirling numbers of the second kind.* The numbers $B_n = \Sigma_{k=1}^n S(n,k)$ are called *Bell numbers.*

Although the study of Stirling and Bell numbers presents some difficulties, (Rota, 1964) the number of partitions having exactly b_i blocks of size i, $i = 1, 2, \cdots$ is easily derived. A partition of this sort is said to be of *type* $\vec{b}$. Thses are easily counted by enumerating permutations of the given set in two ways giving

$$\# \text{ of partitions of type } \vec{b} = \frac{(\Sigma i b_i)!}{\Pi b_i!(i!)^{b_i}}. \tag{1}$$

Let P be the set of all partitions of S and let $\pi = \{\pi_1, \pi_2, \cdots\}$ and $\sigma = \{\sigma_1, \sigma_2, \cdots\}$ lie in P. We

call π a *refinement* of σ if every block π_i of π is contained in some block σ_j of σ. Another was of thinking of this is to say that π is a refinement of σ if every block σ_i is gotten by merging blocks π_i. We make P into an ordered set $\Pi(S_n) = \Pi_n = (P, \leqq)$ by defining $\pi \leqq \sigma$ to mean π is a *refinement of* σ. *Π_n is called the POS of partitions of S_n ordered by refinement.*

We will compute $\mu(\pi,\sigma)$ for π, $\sigma \in P$ following Frucht and Rota (1965). By Corollary 3.2 it suffices to study $[\pi,\sigma]$. Since σ is gotten by merging blocks of π, the individual elements of the blocks of π are not essential; e.g., in saying that $\{\{1,2\},\{3\},\{4\}\}$ is a refinement of $\{\{1,2\},\{3,4\}\}$ the elements 1 and 2 are really inessential and we could write that $\{\{2\},\{3\},\{4\}\}$ is a refinement of $\{\{2\},\{3,4\}\}$. Hence in studying μ on the interval $[\pi,\sigma]$ we need only consider those π whose blocks contain one element, i.e., those π which are complete refinements of a set. We use 0 *to denote a complete refinement.* Thus we restrict ourselves to intervals of the form $[0,\sigma]$.

Let $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ (the σ_i are the blocks). Since every refinement of σ consists of some partition of each of the σ_i, we can regard a refinement ρ of σ as an ordered k-tuple $(\rho_1, \cdots, \rho_k)$ where ρ_i is a refinement of σ_i. Thus $[0,\sigma] \cong [0_1,\sigma_1] \times \cdots \times [0,\sigma_k]$. By the product theorem it suffices to consider μ on $[0,w]$ where w consists of one block. The partition consisting of one block is usually written as 1. Let 1 have n elements and write $\mu_n = \mu(0,1)$. We have shown that

(2) $$\mu(\pi,\sigma) = \prod_{i=1}^{m} \mu_{n_i} \quad \text{for all} \quad \pi, \sigma \in P,$$

where $\sigma = \{\sigma_1, \cdots, \sigma_m\}$ and $\pi = \{\pi_1, \cdots, \pi_n\}$ and σ_i is the union of exactly n_i of the π_j's.

We now compute $\mu(0,1)$ by the method of undetermined coefficients. To do this we relate partitions to functions.

Let S_n be an n-set and X an arbitrary set with x elements. We associate with any function $f\colon S_n \to X$ a partition of S_n as follows: the blocks of the partition are the inverse images of the elements of X. This partition is called the *kernel* or co-image of the function. Different functions may have the same kernels. Kernels and their generalization form the basis for a combinatorial interpretation of finite differences (Mullin-Rota, 1970).

Let $N_=(\pi)$ be the number of functions from S_n to X whose kernel is π and let $N_\geq(\pi)$ be the number of functions whose kernel is $\geq \pi$ (in the ordering of Π_n). We have

(3) $$N_\geqq(\pi) = \sum_{\sigma:\sigma \geqq \pi} N_=(\sigma).$$

By Möbius inversion

(4) $$N_=(\pi) = \sum_{\sigma:\sigma\geqq\pi} \mu(\pi,\sigma) N_\geqq(\sigma).$$

Setting $\pi = 0$, we get

(5) $$N_=(0) = \sum_{\sigma\in\Pi_n} \mu(0,\sigma) N_\geq(\sigma).$$

$N_=(0)$ is the number of one to one functions since the inverse image of every point must be a point. Hence $N_=(0) = x(x-1)\cdots(x-n+1) = (x)_n$. Suppose σ has $r(\sigma)$ blocks. Then a function is counted by $N_\geq(\sigma)$ if it maps all elements in the same block of σ into one point. Different blocks can map into the same point since the kernel need only be $\geqq \sigma$. Hence $N_\geq(\sigma) = x^{r(\sigma)}$. Substituting in (5) yields

(6) $$x(x-1)\cdots(x-n+1) = \sum_{\sigma} \mu(0,\sigma) x^{r(\sigma)}.$$

Since this relation is true for infinitely many values of x it is a polynomial identity. This is an important combinatorial technique for deriving polynomial identities.

Clearly $r(\sigma)=1$ if and only if $\sigma=1$, the partition with one block. Equating coefficients of x on both sides of (6) we get

$$\mu_n=(-1)^{n-1}(n-1)! \tag{7}$$

Substituting this result into (2) we see that

$$\mu(\pi,\sigma)=\prod_i(-1)^{n_i-1}(n_i-1)!=(-1)^{r(\pi)-r(\sigma)}\prod_i(n_i-1)! \tag{8}$$

where the ith block of σ (for some fixed order) is the union of exactly n_i blocks of π.

Equation (6) gives some more information. Let $s(n,k)=\sum_{\sigma:r(\sigma)=k}\mu(0,\sigma)$. Then by (6)

$$(x)_n=\sum_{k=1}^{n}s(n,k)x^k. \tag{9}$$

The $s(n,k)$ are called *Stirling numbers of the first kind.* We have provided here a combinatorial interpretation of $s(n,k)$, due to Rota (1964), as a sum of values of a Möbius function.

Example 1: *Waring's Formula For Symmetric Functions.* In computing the Möbius function for partitions we derived equation (5) expressing 1–1 functions in terms of all functions. By repeating this argument in terms of a "generating function" associated with each function we are led to symmetric functions.

Let $S_n=\{1,2,\cdots,n\}$ and let $X=\{x_1,\cdots,x_l\}$, where $l\geq n$ and $x_1,\cdots,x_l$ are independent variables. To each function $F\colon S_n\to X$ with kernel σ associate the monomial generating function

$$g(F)=x_1^{|F^{-1}(x_1)|}x_2^{|F^{-1}(x_2)|}\cdots x_l^{|F^{-1}(x_l)|}.$$

The monomial has $r(\sigma)$ non-trivial factors and degree n. If F is a set of functions, the *generating function* $g(F)$, given by $g(F)=\sum_{f\in F}g(f)$, is a polynomial in several variables.

We now mimic the argument mentioned. Let $N_=(\sigma)$ be the generating function for the set of all functions from S_n to X with kernel σ. Then $N_\geq(\sigma)$ is the generating function for the set of all functions from S_n to X with kernel $\geq\sigma$. Möbius inverting and setting $\sigma=0$ we obtain

$$N_=(0)=\sum_\pi\mu(0,\pi)N_\geq(\pi). \tag{10}$$

Now $N_=(0)$, the generating function of the set of all one-one functions is clearly

$$N_=(0)=\sum_{i_1,i_2,\cdots,i_n}x_{i_1}\cdots x_{i_n}$$

where the sum is over all sets of n distinct indices from $\{1,2,\cdots,l\}$. By definition this is $n!$ times the elementary symmetric function of degree n in l variables, denoted by a_n. We now show that

$$N_\geq(\pi)=(x_1+x_2+\cdots+x_l)^{b_1}(x_1^2+x_2^2+\cdots+x_l^2)^{b_2}\cdots(x_1^l+x_2^l+\cdots+x_l^l)^{b_l} \tag{11}$$

where π has b_i blocks of size i, and a given factor $(x_1^i+x_2^i+\cdots+x_l^i)$ corresponds to a specific block of size i. Each term in the expansion of the right hand side of (11) corresponds to a choice of images for each of the blocks. Hence we obtain the generating function for the set of all functions which are constant on the blocks of π. But this generating function is precisely $N_\geq(\pi)$ since prescribing the same image for two different blocks is equivalent to merging them in the kernel.

In the theory of symmetric functions, $x_1^i+x_2^i+\cdots=s_i$ is called the power sum symmetric function. If we substitute (11) in (10) and collect terms according to the type of π we will obtain *Waring's formula*:

$$a_n = \frac{1}{n!}\sum_{\pi} \mu(0,\pi) s_1^{b_1(\pi)} s_2^{b_2(\pi)} \cdots$$

$$= \sum_{b} \frac{1}{\prod_i i!^{b_i} b_i!} (-1)^{n-\Sigma b_i} \prod_i (b_i - 1)! s_i^{b_i}$$

by (1) and (8) where $\vec{b}$ ranges over all types; i.e., $b_1 + 2b_2 + \cdots = n$. Hence

$$a_n = \sum_{b} (-1)^n \prod_i \left(\frac{-s_i}{i!}\right)^{b_i} \Big/ b_i, \qquad b_1 + 2b_2 + \cdots = n. \tag{12}$$

See Solomon and McEliece (1966, Section 7) for a generalization.

Doubilet (1972) has developed the basic theory of symmetric function by this approach.

Example 2: *Connected graphs.* We wish to count c_n, the number of connected labeled graphs on n vertices. Let S be the vertex set and $\Pi(S)$ the lattice of partitions of S. The number of loopless labeled graphs on n vertices is $2^{\binom{n}{2}}$ since we may choose any collection of pairs of vertices for edges. Let $N_=(\pi)$ be the number of labeled graphs such that each block of π labels a connected component, i.e., those graphs whose components induce the partition π on the vertices. Then $c_n = N_=(1)$ and we have

$$2^{\binom{n}{2}} = N_{\leq}(1) = \sum_{\pi} N_=(\pi).$$

We can compute $N_{\leq}(\pi)$. This counts all labeled graphs in which different blocks of π label distinct sets of components. Hence

$$N_{\leq}(\pi) = \prod_i 2^{\binom{i}{2} b_i} \tag{13}$$

where π is of type $\vec{b}$. By Möbius inversion $N_=(1)$, the number of connected graphs, is

$$N_=(1) = \sum_{\pi} \mu(\pi, 1) N_{\leq}(\pi)$$

which by (8) and (13)

$$= \sum_{\pi} (-1)^{\Sigma b_i - 1} \left(\sum b_i - 1\right)! \prod_i 2^{\binom{i}{2} b_i}$$

and by (1)

$$= n! \sum_{b : \Sigma i b_i = n} (-1)^{\Sigma b_i - 1} \left(\sum b_i - 1\right)! \prod_i \frac{2^{\binom{i}{2} b_i}}{b_i!(i!)^{b_i}}.$$

This formula is equivalent to the generating function equation $C(x) = \ln G(x)$ or $G(x) = \exp C(x)$. For a discussion of exponential formulas for generating functions of the form $A(x) = \exp B(x)$ see Doubilet, Rota and Stanley (1973) for a Möbius approach and Bender-Goldman (1971) for another approach.

5. Vector Spaces. Let $V_n(q)$ be an n dimensional vector space over the field of q elements. We partially order the subspaces of $V_n(q)$ by inclusion: if U and V are subspaces of $V_n(q)$, then $U \leqq V$ iff U is a subspace of V. The resulting *POS* is denoted by $L(V_n(q))$. It is a "geometric lattice" (Crapo-Rota, 1971), because $L(V_n(q))$ is the lattice of subspaces of a projective space.

The study of subspaces of a finite vector space has deep analogies with the study of subsets of a finite set (Goldman-Rota, 1969, 1970). The relation is probably deeper than just an analogy but as yet there is no explanation for this. Some of these analogies are discussed in this section.

Just as $\binom{n}{k}$ counts the k-subsets of an n-set, we introduce the Gaussian coefficient $\binom{n}{k}_q$ $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is also used in the literature) defined by $\binom{n}{k}_q = \#\, k$-dimensional subspaces of $V_n(q)$. To derive $\binom{n}{k}_q$ we reason by analogy with binomial coefficients:

$$\binom{n}{k} = \frac{\#\text{ sequences of } k \text{ distinct elements in an } n\text{-set}}{\#\text{ sequences of } k \text{ distinct elements in a } k\text{-set}}$$

(1)

$$\binom{n}{k}_q = \frac{\#\text{ sequences of } k \text{ independent vectors in } V_n(q)}{\#\text{ sequences of } k \text{ independent vectors in } V_k(q)}.$$

Let us compute the numerator of (1). We can choose the first vector in $(q^n - 1)$ ways (the number of non-zero vectors in $V_n(q)$). The vector chosen generates q vectors, viz. all multiples of it; hence we may choose the second vector in $(q^n - q)$ ways. The two vectors now chosen generate, by linear combinations, q^2 vectors; hence we may choose the third vector in $(q^n - q^2)$ ways. Continuing this argument we have

$$\begin{aligned}&\#\text{ sequences of } k \text{ independent vectors in } V_n(q)\\ &= (q^n-1)(q^n-q)(q^n-q^2)(q^n-q^3)\cdots(q^n-q^{k-1}).\end{aligned}$$

When $n = k$ we obtain the denominator in (1). Hence

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}. \tag{2}$$

If in (2) we regard the right side as a function in the variable q and hence $\binom{n}{k}_q$ as a function defined by (2), then we have

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k}. \tag{3}$$

This is the first manifestation of the relation between subspaces and subsets which is as mysterious as it is fascinating. *In some sense* "a set is a vector space over a field with one element"; a concept which needs a definition. Unfortunately no good one is known.

The relation (3) is a strong heuristic guide to guessing relations over vector spaces in analogy with those over sets. It also provides a check on the correctness of vector space formulas by letting $q \to 1$. For example,

$$\binom{n}{k}_q = \binom{n}{n-k}_q \quad\text{and}\quad \binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q. \tag{4}$$

When $q \to 1$ we obtain well-known binomial coefficient identities. Both of these identities can be derived immediately from (2) by algebraic manipulation. Combinatorial proofs are a bit more involved (Goldman-Rota (1970)).

To compute the Möbius function $\mu(x, y)$ for $L(V_n(q))$ we show first that the structure of $[x, y]$ depends only on $d(y) - d(x)$ where d denotes dimension. To see this, let $v_1, \cdots, v_{d(y)}$ be a basis for y such that the first $d(x)$ elements are a basis for x. Define

$$f(v_i) = 0 \quad\text{if}\quad i \leqq d(x) \quad\text{and}\quad f(v_i) = v_i \quad\text{otherwise.}$$

Using this map it can be shown that $[x, y] \cong [0, z]$ for some z where $d(z) = d(y) - d(x)$. In fact we

can take z to be the quotient space y/x. Since all k-dimensional spaces over fields with q elements are isomorphic, we need only compute $\mu(0, V_n(q))$ for all n. The Möbius function is

$$\mu_n = \mu(0, V_n(q)) = (-1)^n q^{\binom{n}{2}}. \tag{5}$$

In (Rota, 1964) this result is derived from more general theorems on Möbius functions. With the tools presently at our disposal, it could be done by induction. We present here a proof by the method of undetermined coefficients: we count the number of one-one linear transformations from $V_n(q)$ into a vector space X with x vectors in two ways.

For every subspace $U \in L(V_n(q))$ let $N_=(U)$ be the number of linear transformations $f: V_n \to X$ whose null space is U, i.e., $f^{-1}(0) = U$. Then $N_\geq(U)$ is the number of linear maps from V_n to X whose null space contains U. By Möbius inversion

$$N_=(U) = \sum_{W \geqq U} \mu(U, W) N_\geq(W),$$

and with $U = 0$

$$N_=(0) = \sum_{W \in L(V_n)} \mu(0, W) N_\geq(W). \tag{6}$$

By definition $N_=(0)$ is the number of linear maps whose null space is 0, i.e., the number of one-one linear maps. Such a map is specified by giving a list of n independent vectors — the image of an ordered basis for $V_n(q)$. By the argument used to derive formula (2), we see that the number of one-one maps from V_n into X is given by $(x-1)(x-q)\cdots(x-q^{n-1})$.

We now compute $N_\geq(W)$. A linear map has null space containing W if it maps W onto 0 and does anything at all with the rest of the vectors. Hence, if $b_1, b_2, \cdots, b_n$ is a basis for V_n where $b_1, \cdots, b_{d(W)}$ is a basis for W, we must map $b_1, \cdots, b_{d(W)}$ onto 0 and the other $n - d(W)$ basis vectors onto any vectors in X. Thus $N_\geq(W) = x^{n-d(W)}$. Substituting in (6) we obtain

$$(x-1)(x-q)\cdots(x-q^{n-1}) = \sum_W \mu_{d(W)} x^{n-d(W)} = \sum_{k=0}^{n} \binom{n}{k}_q \mu_k x^{n-k}. \tag{7}$$

Since this identity is true for infinitely many values of x, it is a polynomial identity. Equating the constant terms on both sides gives

$$\mu_n = (-1)(-q)\cdots(-q^{n-1}) = (-1)^n q^{\binom{n}{2}}$$

which proves (5). As $q \to 1$ we have $\mu_n = (-1)^n$, the Möbius function for Boolean algebras.

Example 1: *q-identities.* Now that we have proved (5), we can rewrite (7) as

$$\prod_{k=0}^{n-1} (x - q^k) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^{n-k}. \tag{8}$$

A typical trick is to set $x = z/y$ and multiply by y^n to clear fractions. This introduces another variable:

$$\prod_{k=0}^{n-1} (z - yq^k) = \sum_{k=0}^{n} \binom{n}{k}_q (-y)^k q^{\binom{k}{2}} z^{n-k}. \tag{9}$$

One often obtains such q-binomial identities by counting in two ways. This can be done by considering vector spaces or by considering partitions of a number. To see how the latter enters write

$$\binom{n+k}{k}_q = \sum_{i=0}^{\infty} a_{nk}(i) q^i.$$

Then $a_{nk}(i)$ is the number of partitions of i into at most k parts each of size at most n (MacMahon,

1916). In "sum equals product" identities like (8), q often appears to a quadratic power. Here we have $q^{\binom{k}{2}}$. Cubic and higher powers apparently never do appear. This is not understood, but may be associated with the formula for μ_n.

Since (9) is true for infinitely many values of q, it is a polynomial identity. Setting $z = 1$ and choosing $|q| < 1$ we obtain as $n \to \infty$

$$\prod_{k=0}^{\infty} (1 - yq^k) = \sum_{k=0}^{\infty} (-y)^k q^{\binom{k}{2}} (1-q) \cdots (1-q^k).$$

By considering convergence in the q-adic norm (van der Waerden (1953)), many limits for q-identities can be simplified (Goldman-Rota, 1970).

We now present an example where inversion over subspaces is a natural tool.

Example 2: *Spanning Subsets.* Continuing our discussion of Example 1.4, we say by convention that $\varnothing$ spans nothing and $\{0\}$ spans the 0-dimensional subspace. Since every non-empty subset spans some subspace we see that $N_{\leq}(U)$ is given by

$$N_{\leq}(U) = 2^{q^{d(U)}} - 1.$$

Möbius invert to get $N_{=}(V_n(q))$:

$$\text{(10)} \qquad \#\ \text{spanning subsets of } V_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q \mu_k (2^{q^{n-k}} - 1).$$

When $q \to 1$, we might reasonably expect one spanning subset, but (10) reduces to

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (2-1) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0.$$

For (10) to be "right" at $q = 1$, we need to replace $2^{q^{n-k}}$ by 2^{n-k} at $q = 1$. This happens in

$$\text{(11)} \qquad \sum_{k=0}^{n} \binom{n}{k}_q \mu_k (2^{\binom{n-k}{1}_q} - 1).$$

In fact (11) counts spanning sets for projective spaces whereas (10) refers to affine spaces. This illustrates a limitation of the $q \to 1$ idea. In order to use this idea our formula must refer to objects counted in projective and not affine space, i.e., it must be in terms of objects in $L(V_n)$, the subspaces, and not contain any direct reference to the vectors in the space. Vectors are affine points whereas projective points are 1-dimensional subspaces.

P. Hall (1934) and Weisner (1935, 1935a) applied Möbius inversion to p-group enumeration problems. Since the value of μ for $L(V_n(p))$ enters and $p \mid \mu_k$ when $k \neq 0$, congruences modulo p are obtained.

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References

1. E. A. Bender and J. R. Goldman, Enumerative uses of generating functions, Indiana University Math. J., 20 (1971) 753–765.

2. G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, Trans. Amer. Math. Soc., 60 (1946) 355–451.

3. H. Crapo and G. C. Rota, On the Foundations of Combinatorial Theory: Combinatorial Geometries, M. I. T. Press, 1971.

4. P. Doubilet, On the foundations of combinatorial theory VII. Symmetric functions through the theory of distribution and occupancy, Studies in Applied Math., (51) 4 (1972) 377–396.

5. P. Doubilet, G. C. Rota, and R. P. Stanley, The idea of generating function, Proc. Sixth Berkeley Symp. Math. Stat. and Prob., vol. 2, University of California Press, 1973, pp. 267–318.

6. T. Dowling, Codes, packing, and the critical problem, Atti Convegno Geometria Combinatoria Sue Applicazioni, Perugia, (1971).

7. W. Feller, An Introduction to Probability Theorey and its Applications, 3rd ed., Wiley, New York, 1968, chap. 4.

8. H. J. Ryser, Combinatorial Mathematics, MAA, Carus Monograph No. 14, 1963.

9. M. Frechet, Les Probabilités associées à un système d'évènement, compatible et dépendants, Actualités Sci. Indust., (1940 and 1943) 859 and 942. Paris, Hermann.

10. R. Frucht and G. C. Rota, Polinomios de Bell y Particiones de Conjunto Finitos, Scientia, 130 (1966) 67–74.

11. R. Gilman, A combinatorial identity with applications to representation theory, Illinois J. Math., 17 (1973) 347–351.

12. J. R. Goldman and G. C. Rota, The Number of Subspaces of a Vector Space, Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 75–83, edited by W. T. Tutte.

13. ———, On the foundations of combinatorial theory IV, Finite vector spaces and Eulerian generating functions, Studies in Appl. Math., 49 (1970) 239–258.

14. B. Grünbaum, Convex Polytopes, Interscience, New York, 1967.

15. P. Hall, A contribution to the theory of groups of prime-power order, Proc. London Math. Soc., 36 (1934) 24–80.

16. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford University Press, New York, 1960, Theorem 270.

17. E. Hille, The inversion problem of Möbius, Duke Math. J., 3 (1937) 549–568.

18. P. A. MacMahon, Combinatory Analysis, vols. I, II, Cambridge University Press, New York, 1916. Reprinted as one volume by Chelsea, New York, Sect. 241.

19. R. Mullin and G.-C. Rota, On the Foundations of Combinatorial Theory, III: Theory of Binomial Enumeration, Graph Theory and its Applications, Academic Press, New York 1970, pp. 167–213, B. Harris, ed.

20. R. C. Read, An introduction to chromatic polynomials, J. Combinatorial Theory, 4 (1968) 52–71.

21. G.-C. Rota, On the foundations of combinatorial theory I, Theory of Möbius functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2 (1964) 340–368.

22. ———, On the Combinatorics of the Euler Characteristic, Studies in Pure Mathematics, Academic Press, New York, 1971, pp. 221–233. (L. Mirsky, ed.)

23. ———, The number of partitions of a set, this MONTHLY, 71 (1964) 498–504.

24. G. Solomon and R. McEliece, Weights of cyclic codes, J. Combinatorial Theory, 1 (1966) 459–475.

25. B. L. van der Waerden, Modern Algebra, vol. I, Ungar, New York, 1953, pp. 235–236. (Trans. from German by F. Blum)

26. L. Weisner, Abstract theory of inversion of finite series, Trans. Amer. Math. Soc., 38 (1935) 474–484.

27. ———, Some properties of prime-power groups, Trans. Amer. Math. Soc., 38 (1935a) 485–492.

28. H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc., 38 (1932) 572–579.

29. H. S. Wilf, The Möbius Function In Combinatorial Analysis and Chromatic Graph Theory, Proof Techniques in Graph Theory, (F. Harary, ed.), Academic Press, New York, 1969, pp. 179–188.

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CORRECTION TO "AN ELEMENTARY PROOF OF THE KRONECKER-WEBER THEOREM"

M. J. GREENBERG

Joe L. Mott informed me that the argument for the case $m > 1$ in Lemma 4, Volume 81, (1974) 606, is incorrect. What is correct is that V_i is the unique subgroup of G of index λ, where i is the smallest index such that $V_i \neq G$; this can be proved using the transitivity of the different and Hilbert's formula — see P. Ribenboim, *Algebraic Numbers*, Wiley-Interscience, New York, 1972, p. 235.

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A COMPUTER-ASSISTED STUDY OF PURSUIT IN A PLANE

FEREDOON BEHROOZI AND RALPH GAGNON

Abstract. The motion of an arbitrary set of points in a plane chasing one another in cyclic pursuit is studied. Given a set of labeled points $1, 2, 3, \cdots, n$, with coordinates $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, it is arranged for point (x_i, y_i) to chase point (x_{i+1}, y_{i+1}) along the instantaneous line joining them. To close the cycle, point (x_n, y_n) chases point (x_1, y_1). All points move with the same speed. The motion just described is called motion in "forward time." If the sign of the time derivative operator is changed, with the result that point (x_i, y_i) runs away from point (x_{i+1}, y_{i+1}), the motion is called motion in "reverse time." Motion in both forward and reverse time is studied with the aid of a computer program. In forward time, it is observed that ultimately, in almost all cases tried, the points become collinear, with large maximum-to-minimum distance ratio, regardless of the shape of the initial configuration. In reverse time, it is observed that the points ultimately arrange themselves in a star pattern, with the star vertex angle less than some number in the vicinity of ninety degrees. By an analysis based on the stability of the equations of motion, the exact value of the maximum star angle is computed, as a function of the number of points.

1. Introduction. Arthur Bernhart in a series of four papers gives an excellent historical background to the problem of pursuit [1], [2], [3], [4]. In general, the trajectory of a point A which always moves along its instantaneous line of sight to another point B, which in turn may be chasing a third point C, etc., is to be found. Usually, all the points are assumed to move with the same constant speed. If the set of points are finite in number and are in motion such that each one follows the next in a cyclic pattern, the motion is known as cyclic pursuit. When the initial configuration is a plane then the motion remains in this plane, and the problem is one of pursuit in a plane.

Naturally, the simplest cases to deal with are those in which the points form regular plane configurations such as regular polygons. These cases have been treated by many workers [5], [6]. It can readily be shown that all regular polygons stay regular and collapse to the centroid of the system in the course of time.

Next in complexity are nonregular plane figures which remain similar to the initial configuration during the course of motion. While it is not possible in general to achieve this result with any arbitrary plane figure, one can readily accomplish this in the case of three points. What is required is to alter the speeds of two of the points with respect to the third so that the approach speed of any two points is proportional to the initial distance between them. It is shown [6] that in this case the triangle collapses to one of the two Brocard points [7] of the initial triangle. Furthermore, Bernhart [2] has shown that by selecting appropriate point speeds, certain nonregular convex polygons can be constructed such that they remain similar to themselves throughout the course of motion, and collapse to a predetermined point.

The general problem of cyclic pursuit in a plane when the points have arbitrary initial positions and chase each other with constant and equal speeds has received little attention in the literature. Here we describe a computer-assisted study of the problem and discuss some new results in the light of an analysis based on the stability of the equations of motion.

2. Computational procedure. Basically, we enter the initial positions of the points $A, B, \cdots,$ into the computer and instruct it to change the positions in small steps to simulate the cyclic chase. To achieve this, a vector version of the fourth-order Runge-Kutta formula is used to calculate the new position of a point after each step. For simplicity and without loss of generality, we assume the speed of each point to be unity, so that the step size in time and in distance is numerically the same.

The step size is variable and is adjusted at every step to equal a preselected fraction of the minimum separation between any point and its target. (The word "target" will be used to signify the point which a given point chases.) Normally this fraction is set equal to 1/10 in our runs. We found that a smaller fraction merely increases the computation time with no significant change in the results. Also, the maximum separation between any point and its target is calculated, and in the event that the step size multiplied by 10^5 drops below this maximum separation, a warning to beware

of round-off errors is printed. Thus when the largest separation between any two paired points exceeds the smallest separation in the system by a factor of 10^4, the run is discontinued. This usually happens long before the round-off errors become significant. (However, there do exist certain ill-conditioned configurations in which a small perturbation in initial conditions results in a large difference in final conditions. The computer does not automatically detect these configurations.)

IBM 1130 extended precision option is used, which gives about nine-figure accuracy. In order to reduce the influence of round-off errors further, from time to time coordinate values are transformed by translation of the origin to the centroid of the system.

We have taken advantage of the sense switches available on the console of the 1130. With their aid we can terminate a run without losing the final values, change step size, delete the graph for faster computation, and cause the computer to perform a variety of other useful tricks. In particular, we may command the computer to effect a time reversal, which corresponds to having the points run away from their targets. This option was originally incorporated as a test of the accuracy of the computations. (An accurate computation would have the property that after the initial configuration has shrunk by several orders of magnitude, a time reversal would restore the original configuration.) However, we have found many interesting results with our time reversal option, surpassing our original intention.

3. Results. Our survey of the available literature had led us to speculate that any convex plane configuration would remain obviously so as it collapses to a point with the passage of time. This view was esthetically pleasing and in line with what happens in the case of regular polygons. Therefore we were a little surprised to find that this was not the case. Instead we found, in almost every case which we tried, that the configuration approached the intermediate stage of a line, with very large

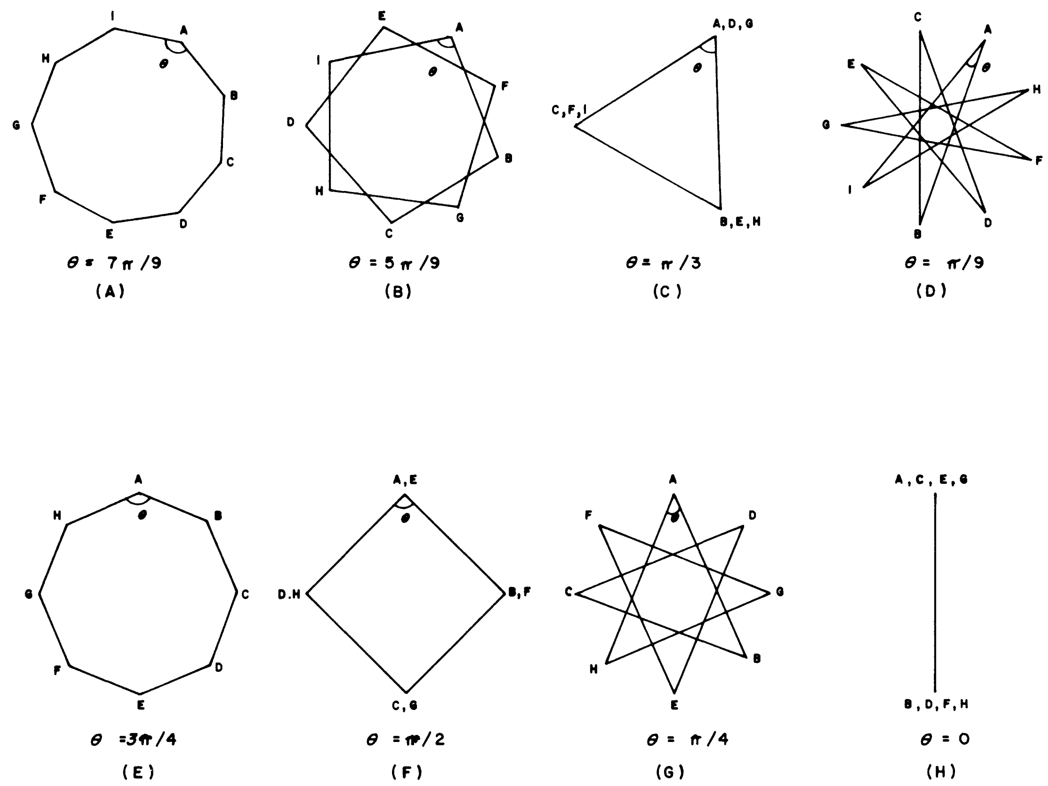


FIG. 1. The possible regular configurations for polygons of 9 and 8 sides.

maximum-to-minimum side ratio, as it collapsed to a point (see Figure 5A). In fact, even many regular polygons turn out to be unstable in forward time, i.e., a small perturbation on the regularity grows in time causing the polygon ultimately to collapse to a line. We suspect that the maximum-to-minimum side ratio becomes infinite as the configuration collapses to a point, although, of course, this cannot be decided on the basis of the computer results.

We then decided to investigate the motion in reverse time. (This corresponds to the case of the points running away from their targets.) Here we were surprised to find that no matter what initial configurations we entered into the computer, the final results had a regularity indicative of some underlying order. In particular, we found that if the number of points, n, was odd, the initial configuration evolved into a regular center-symmetric star-shaped figure. But when n was even, the final configuration was observed to be a line, with the even-index points located at one end and their targets on the other. The maximum-to-minimum side ratio of this line is unity, and thus its character is entirely different from the line which appears when a configuration collapses to a point.

We will digress here to briefly state the regular (i.e., center symmetric) configurations which can be constructed. If the number of sides n is even, the number of configurations is equal to $n/2$, with vertex angles θ equal to $(n-2)\pi/n$, $(n-4)\pi/n, \cdots, 0$. If the number of sides is odd, then the number of configurations is equal to $(n-1)/2$, with angles $(n-2)\pi/n$, $(n-4)\pi/n, \cdots, \pi/n$. Examples of the

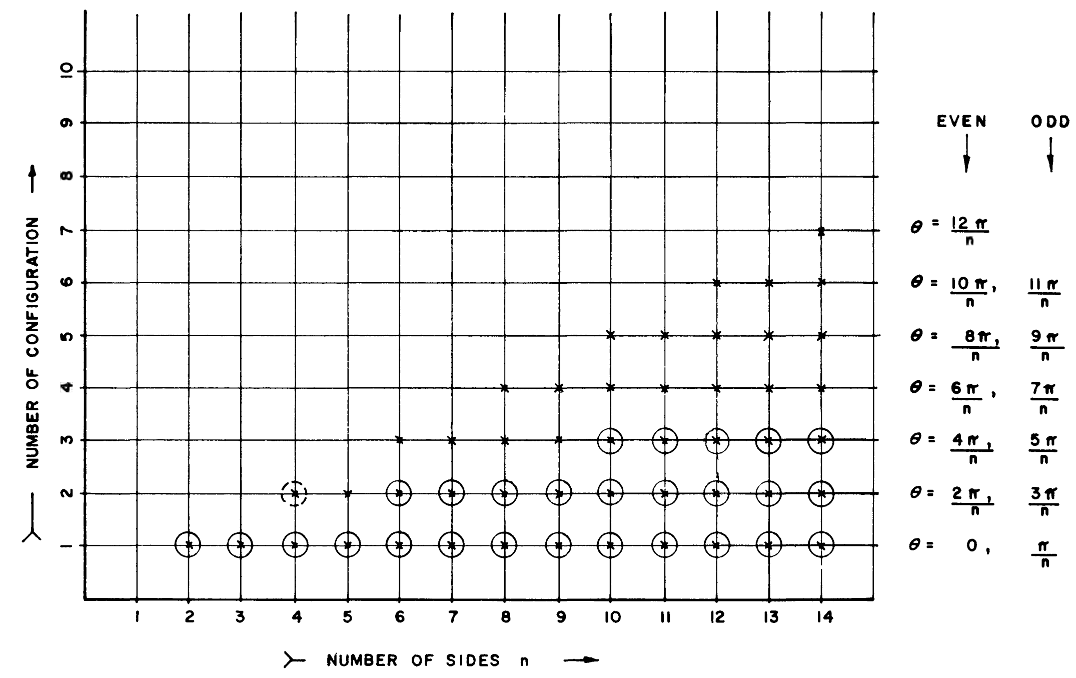


FIG. 2. A diagram of configurations versus number of sides. A cross represents a configuration and a circle indicates stability in reverse time.

two cases are shown in Figure 1. Figure 2 graphically shows the possible regular figures as a function of n. For a given n, the number of crosses in the corresponding column is equal to the number of configurations. Circled crosses represent configurations which are stable in reverse time.

Returning to our discussion of the computer calculations in reverse time, we obtained the following results. For arbitrary input data and for $n = 3$, the final configuration is an equilateral triangle. A theoretical exception is the case where the initial points form a straight line. In this case, the points remain on a line. However, a line configuration is ill-conditioned, meaning that the final configuration is very sensitive to small changes in the initial conditions. A small perturbation sends the line into a final configuration of an equilateral triangle.

For some initial configurations of $n = 4$, the computer showed a marked vacillation before finally settling for a line. For other initial conditions, the final state of a line was achieved more quickly. In a few special cases, the final state was a square. For $n = 5$, the final configuration was always a star with $\theta = \pi/5$. The regular pentagon with $\theta = 3\pi/5$ was not stable and a small perturbation was enough to send it into the star configuration.

When n is greater than 5, the results were as follows. For even n, we obtained various star configurations which depended on the initial conditions, but most often the final figure was a line corresponding to the case of zero vertex angle. In particular, the final configuration for random input data was invariably a line. For odd n, we obtained different star patterns depending on the initial conditions, but most often the final figure was the slimmest star, corresponding to a vertex angle equal to π/n.

But in all cases the final stars had a vertex angle satisfying $\theta < \pi/2$, whether n was even or odd. For example, with $n = 9$, as shown in Figure 1, the regular polygon with $\theta = 7\pi/9$ is followed by a first star with $\theta = 5\pi/9$ which is greater than $\pi/2$. This first star turns out to be unstable, and a small perturbation sends it to the $\theta = \pi/3$ case. Thus we observed that for a given n, not all possible regular configurations are stable in reverse time. In particular, the regular polygons (except for $n = 2, 3$, and possibly 4) are all unstable in reverse time. Furthermore, there is a preference for those star configurations which have the smallest possible vertex angle. For even n, this angle is zero, and for odd n it is π/n.

4. Analysis. In what follows the term "regular polygon" is used for both the convex and the star-shaped regular figures discussed in the preceding section.

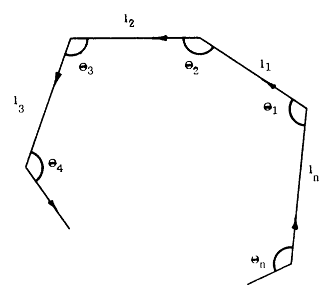


FIG. 3.

Consider the general polygon with variables defined as in Figure 3. Assuming that the ith point chases the $(i + 1)$th point along the line joining them with unit velocity, the polygon's evolution in time is governed by the following equations:

$$-\dot{l}_i = 1 + \cos\theta_{i+1} \tag{1}$$

$$-\dot{\theta}_i = (\sin\theta_{i+1})/l_i - (\sin\theta_i)/l_{i-1}. \tag{2}$$

where the dots denote differentiation with respect to time.

If the points are running away from their targets, then the equations of motion become

$$\dot{l}_i = 1 + \cos\theta_{i+1} \tag{3}$$

$$\dot{\theta}_i = (\sin\theta_{i+1})/l_i - (\sin\theta_i)/l_{i-1}. \tag{4}$$

We can think of equations (3) and (4) as representing the problem of pursuit in reverse time since they result from equations (1) and (2) by a simple time reversal. We wish to investigate whether or not regular polygons are stable in forward and reverse time.

We shall call a "state" stable if, when a small perturbation disturbs it, the system returns to its

initial state. The emphasis on small perturbations permits us to take advantage of the simplification afforded by linearizing the equations of motion. The states whose stability we wish to consider are those of permissible regular configurations as discussed in the previous section. Stability of such a configuration requires that all the angles be equal and constant in time. It also requires that the distances remain equal to each other, but not constant in time. Hence it is advantageous to switch to variables which remain constant when a configuration remains regular in time.

Assume that the regular polygon expands in reverse time from a point at $t=0$. Define

$$h_i = t/l_i. \tag{5}$$

Then, assuming θ_{i+1} constant, we may compute an equilibrium or normal value of h_i using equation (3),

$$h_i = (1+\cos\theta_{i+1})^{-1}. \tag{6}$$

Further, let us define a new variable s by

$$s = \ln t. \tag{7}$$

Then in terms of s and h_i, equation (4) may be written

$$d\theta_i/ds = h_i \sin\theta_{i+1} - h_{i-1}\sin\theta_i. \tag{8}$$

Similarly, by differentiating equation (5), using equation (3), and changing variables from t to s we obtain

$$dh_i/ds = h_i = h_i^2(1+\cos\theta_{i+1}). \tag{9}$$

Here we observe that by putting the operator $d/ds = p$ equal to zero, we get the equilibrium conditions. Equation (9) yields equation (6) for its equilibrium condition, while equation (8) together with equation (6) produce the following additional condition,

$$(\sin\theta_{i+1})/(1+\cos\theta_{i+1}) = (\sin\theta_i)/(1+\cos\theta_i). \tag{10}$$

Naturally, these equilibrium conditions are satisfied by regular polygons.

Next, following a standard procedure, we linearize equations (8) and (9) by replacing h_i by $h_i+\Delta_i$ and θ_i by $\theta_i+\phi_i$. In the following equations, h_i and θ_i are understood to be the equilibrium values, while Δ_i and ϕ_i represent small departures from these values. After deleting the high-order terms, we are left with

$$p\Delta_i = -\Delta_i + (h_i^2\sin\theta_{i+1})\phi_{i+1} \tag{11}$$

$$p\phi_i = \Delta_i \sin\theta_{i+1} - \Delta_{i-1}\sin\theta_i + (h_i\cos\theta_{i+1})\phi_{i+1} - (h_{i-1}\cos\theta_i)\phi_i. \tag{12}$$

Henceforth we shall put $\theta_i = \theta$, and $h_i = h$, where θ and h are the equilibrium values.

To determine the stability of our equilibrium states, we examine the roots of the secular equation of a $2n$ by $2n$ determinant characteristic of the $2n$ equations (11) and (12). As an example, we shall write down this determinant for the case of three points.

$$\begin{vmatrix} p+h\cos\theta & -h\cos\theta & 0 & -\sin\theta & 0 & \sin\theta \\ 0 & p+h\cos\theta & -h\cos\theta & \sin\theta & -\sin\theta & 0 \\ -h\cos\theta & 0 & p+h\cos\theta & 0 & \sin\theta & -\sin\theta \\ 0 & -h^2\sin\theta & 0 & p+1 & 0 & 0 \\ 0 & 0 & -h^2\sin\theta & 0 & p+1 & 0 \\ -h^2\sin\theta & 0 & 0 & 0 & 0 & p+1 \end{vmatrix} = 0.$$

This determinant can be simplified by the following steps. First define

$$q = p/h\cos\theta, \tag{13}$$

replace p by $(qh\cos\theta)$, and divide all terms by $(h\cos\theta)$. Next, divide the right-hand columns by $(\tan\theta)/h$ and the bottom rows by $(h\tan\theta)$ to get

$$\begin{vmatrix} q+1 & -1 & 0 & -1 & 0 & 1 \\ 0 & q+1 & -1 & 1 & -1 & 0 \\ -1 & 0 & q+1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1/k & 0 & 0 \\ 0 & 0 & -1 & 0 & 1/k & 0 \\ -1 & 0 & 0 & 0 & 0 & 1/k \end{vmatrix} = 0, \tag{14}$$

where

$$k = \tan^2\theta/[q + 1/(h\cos\theta)]. \tag{15}$$

This determinant can further be simplified by means of the partitioned-matrix formula

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{22}|\ |A_{11} - A_{12}A_{22}^{-1}A_{21}|,$$

which is valid provided that k is finite and nonzero. The use of this formula reduces the determinant to

$$\begin{vmatrix} \frac{q}{k+1}+1 & -1 & 0 \\ 0 & \frac{q}{k+1}+1 & -1 \\ -1 & 0 & \frac{q}{k+1}+1 \end{vmatrix} = 0,$$

resulting in $[1 + q/(1+k)]^3 - 1 = 0$. The secular equation for n points is the same as this, except that the exponent must be replaced by n. Thus for the general case we obtain

$$[1 + q/(1+k)]^n - 1 = 0, \tag{16}$$

which leads to

$$q = (1+k)(e^{2\pi i m/n} - 1), \tag{17}$$

where $m = 0, 1, 2, \cdots, n-1$, $i = \sqrt{-1}$, and n is the number of points.

Having evaluated the determinant, we shall now express the results in terms of our original variables. Using equations (13) and (15) in equation (17) and defining r and ψ by

$$r = (e^{2\pi i m/n} - 1) = (e^{i\psi} - 1), \tag{18}$$

we obtain the following quadratic equation for p:

$$p^2 + p(1 - rh\cos\theta) - rh(\cos\theta + h\sin^2\theta) = 0. \tag{19}$$

Thus for each r there are two values of p. Since there are n values of r, we then have a total of $2n$ values of p corresponding to the $2n$ distances and angles of the system. To insure stability, all these roots must have negative, or at least non-positive, real parts.

5. Discussion. Let us consider the roots of equation (19) in some detail. A special case, for which $r = m = 0$, applies to all regular polygons. Equation (19) then reduces to $p(p+1) = 0$, giving $p = -1$ and $p = 0$. The root $p = -1$ corresponds merely to an offset between assumed and actual time origins, while the root $p = 0$ corresponds to an offset in the angle sum (the differential equations do not demand that the angles sum to $(n-2)\pi$, even though the geometry does). Hence these roots are not relevant to the question of stability.

Another special case is that for which $r = -2$, corresponding to $\psi = \pi$, which leads to the equation

$$(1+\cos\theta)p^2 + (1+3\cos\theta)p + 2 = 0. \tag{20}$$

By either a Routh test or direct manipulation of the quadratic formula, one may conclude that the condition for roots with negative real parts is $(1+3\cos\theta) > 0$, giving $\theta < 109.47°$. The upper limit $\theta = 109.47°$ turns out to be the largest vertex angle for which root pairs with non-positive real parts are possible.

For other values of θ and $\psi = 2\pi m/n$, we solved equation (19) by means of the quadratic formula and classified the results according to whether or not all the roots had negative real parts. The results are displayed in figure 4. A regular polygon of n sides and vertex angle θ will be stable in reverse time only if an F symbol appears in the figure for each value of ψ equal to $2\pi/n$, $4\pi/n$, $6\pi/n, \cdots 2\pi(n-1)/n$. Before we had derived equation (19), we had noticed that in all cases which we had evaluated on the computer in reverse time, the distances l_i all ultimately became equal. However, as discussed in section 3, some regular polygons were stable, while others were not. Our first attempt to formulate a theory to explain the results took advantage of the equal-distance feature and incorporated it as an assumption. The resulting analysis led to the conclusion that a polygon is stable if $\cos\theta > 0$. All our experimental results at that point agreed with this criterion.

Then, in the interest of completeness, we repeated our analysis taking the distances into account. Our result was Figure 4. This graph implies the existence of polygons with angles less than $\pi/2$ which are not stable in reverse time. One such polygon is an 11-gon with $\theta = 5\pi/11$. This polygon is stable by the simple $\cos\theta > 0$ test, but is unstable according to Figure 4, which shows that for $\psi = 2\pi/11$ and $\psi = 20\pi/11$ equation (19) has roots with positive real parts.

We tested this case on the computer starting with the coordinates perturbed by about 5% from

PARAMETER PSI (DEGREES)

170	190	R	R	R	R	R	R	R	R	
160	200	R	R	R	R	R	R	R	R	
150	210	R	R	R	R	R	R	R		
140	220	R	R	R	R	R	R	R		
130	230	R	R	R	R	R	R			
120	240	R	R	R	R	R	R			
110	250	R	R	R	R	R				
100	260	R	R	R	R	R				
90	270	R	R	R	R	?				
80	280	R	R	R	R					
70	290	R	R	R						
60	300	R	R	R						
50	310	R	R							
40	320	R	R							
30	330	R								
20	340	R								
10	350	R								
		70		80		90		100		110

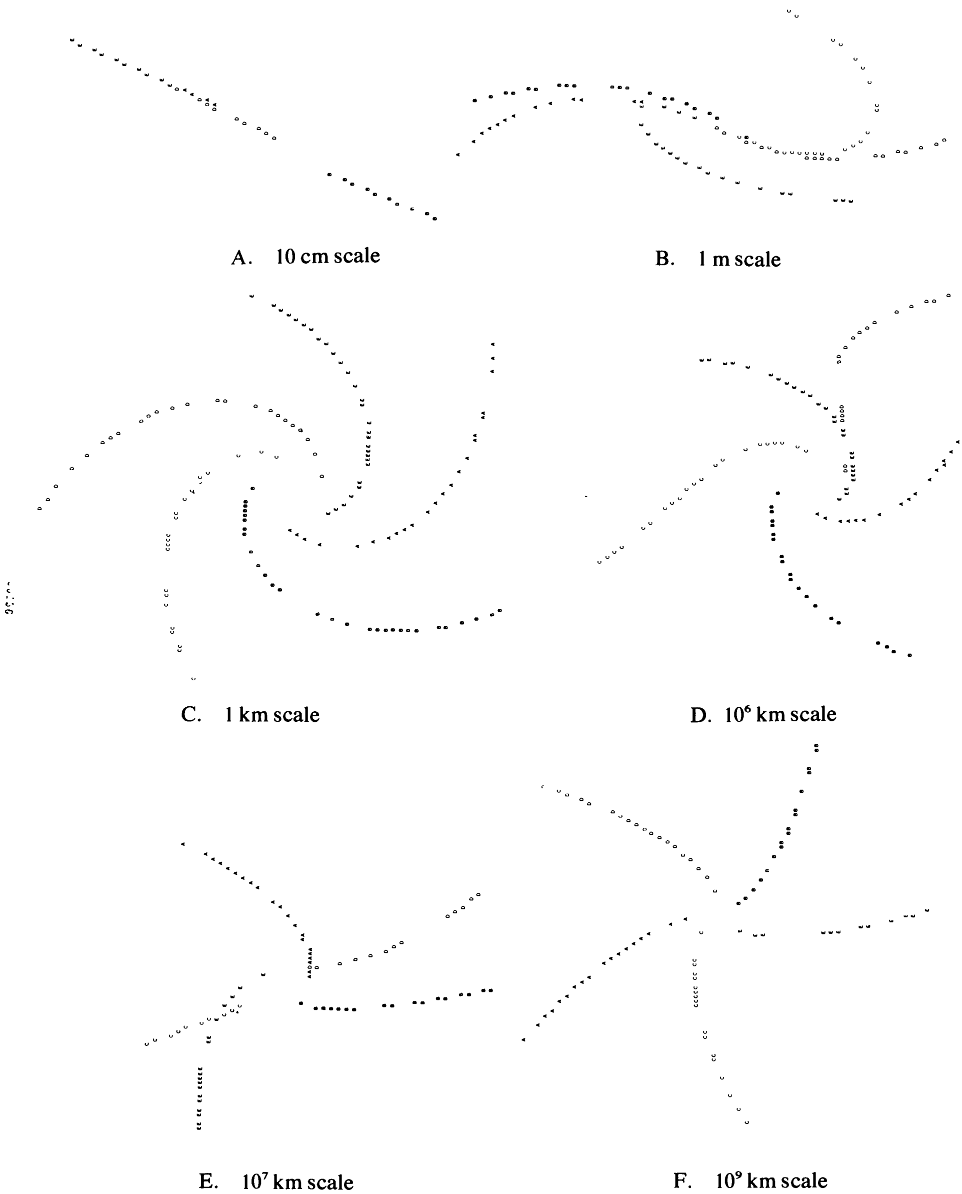
VERTEX ANGLE THETA (DEGREES)

FIG. 4. Regions of stability in reverse (**R**) time.

the regular values. We found that slowly but definitely the polygon evolved into the $\theta = 3\pi/11$ star, which is a stable polygon. The change from the $\theta = 5\pi/11$ configuration to that of $\theta = 3\pi/11$ was accomplished after the distances had increased by a factor of about 10^{72}!!

According to Figure 4, the regular triangle ($n = 3$) is clearly stable in reverse time, while the square ($n = 4$) is questionable. In fact, equation (19) yields an additional pair of zero roots for the square, placing it on a boundary between stability and instability. The computer results indicated that a perfect square remains so even after the distances have increased by 60 orders of magnitude. Furthermore, regular parallelograms evolve into perfect squares rather quickly. A randomly disturbed square (by about 10%), on the other hand, always collapses to a line as the distances grow by 60 or so orders of magnitude.

6. An illustrative example. As an illustrative example of the phenomena which we have observed, we give a calculation for five points, labeled A, B, C, D and E. Figure (5A) shows these



A. 10 cm scale

B. 1 m scale

C. 1 km scale

D. 10^6 km scale

E. 10^7 km scale

F. 10^9 km scale

FIG. 5. Evolution from line to star in reverse time.

points in the configuration of a line with large maximum-to-minimum side ratio. Choosing an arbitrary scale of distance, let us suppose that the length of Figure 5A corresponds to a distance of 10 cm; we will speak of a "scale of 10 cm."

Following the evolution of the configuration in reverse time, we come to Figure 5B, which has a scale of 1 meter. Here we see the transition from the line to the polygon. Skipping two stages, Figure 5C shows the resulting pentagon when the scale is 1 km. To a scale of 10,000 km, the graphs look much like Figure 5C.

The graph for a scale of 100,000 km (not shown) looks a bit irregular compared to its immediate predecessors. Figure 5D, with scale of 10^6 km, shows a crossing of points D and E. The pentagon has begun to evolve into a star. In Figure 5E, with a scale of 10^7 km, there are crossing of points C and E and also A and D. At a scale of 10^8 km, a star begins to take shape (not shown), which reaches its final form in Figure 5F, with a scale of 10^9 km. Additional computation in reverse time does not change the form of this star.

This sequence of graphs was not calculated in the same order as presented, because the starting conditions of Figure 5 are ill conditioned and it was desired to display an intermediate pentagon stage. Instead, a configuration of a slightly irregular pentagon (maximum-to-minimum side ratio of 1.14) was chosen as input data to a graph of scale 1000 km. The evolution of this pentagon was first calculated in forward time to Figure 5A. Then the data for the slightly irregular pentagon was re-entered into the computer and calculated in reverse time to Figure 5F.

The important features demonstrated by this calculation are the following:

(1) The ultimate configuration in reverse time is a star.

(2) The ultimate configuration in forward time is a line, with very large maximum-to-minimum side ratio.

(3) The regular (convex) pentagon is unstable in both forward and reverse time.

References

1. A. Bernhart, Curves of pursuit, Scripta Mathematica, 20(1954) 125–141.
2. ———, Curves of pursuit, II, Scripta Mathematica, 23(1957) 49–65.
3. ———, Polygons of pursuit, Scripta Mathematica, 24(1959) 23–50.
4. ———, Curves of general pursuit, Scripta Mathematica, 24(1959) 189–206.
5. A. Watton and D. W. Kydon, Analytical aspects of the *N*-bug problem, Amer. J. Phys., 37(1969) 220–221.
6. M. S. Klamkin and D. J. Newman, Cyclic pursuit or "The three bug problem," this MONTHLY, 78(1971) 631–639.
7. R. A. Johnson, Advanced Euclidean Geometry, Dover, New York, 1960, 263–270.

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A HEURISTIC PRINCIPLE IN COMPLEX FUNCTION THEORY

LAWRENCE ZALCMAN

1. Introduction. A well-known heuristic principle in the theory of functions asserts that "a family of holomorphic (meromorphic) functions which have a property P in common in a domain D is (apt to be) a normal family in D if P cannot be possessed by non-constant entire (meromorphic) functions in the finite plane" [**4**, p. 250]. In his recent retiring presidential address to the Association for Symbolic Logic [**7**] (required reading for anyone whose interests extend across professional boundaries to mathematics as an intellectual discipline), the late Professor Abraham Robinson cited the explication of this principle as one of twelve problems worthy of the attention of logicians (and, by extension, of mathematicians in general).

This paper is devoted to such an explication. To be precise, we prove a simple theorem which makes the principle rigorous and from which the standard applications of the principle follow quite routinely. Some of these applications are noted at the end, together with a slightly novel approach to the proof of Picard's big theorem.

Major credit fôr the mathematical content of this paper belongs to Christian Pommerenke. He proved a result similar to the main lemma for *normal functions* [**6**, Theorem 1]. It turns out that the same proof works in the (more general) context of normal families and even simplifies a little. Thus, this paper is perhaps viewed most properly as a public relations effort for Pommerenke's theorem.

2. Normal families. A family $\mathcal{F}$ of functions holomorphic on a domain $D \subseteq \mathbf{C}$ is said to be *normal* on D if each sequence of functions in $\mathcal{F}$ contains a subsequence which either converges (to some function not necessarily in $\mathcal{F}$) uniformly on every compact subset of D or tends uniformly to infinity on each compact subset of D. For meromorphic functions, it is advantageous to introduce the familiar *spherical*, or *chordal*, *metric*

$$\chi(z, z') = \frac{|z - z'|}{(1 + |z|^2)^{1/2}(1 + |z'|^2)^{1/2}} \qquad \chi(z, \infty) = \frac{1}{(1 + |z|^2)^{1/2}}$$

on the Riemann sphere. A family $\mathcal{F}$ of meromorphic functions on D is then said to be normal if each sequence of functions in $\mathcal{F}$ has a subsequence which converges uniformly on compacta *with respect to the spherical metric.*

The spherical distance and the ordinary Euclidean absolute value are boundedly equivalent on compact subsets of the plane. It follows that if $\{f_n\}$ converges spherically uniformly on a set S to the limit function $f \not\equiv \infty$, then the f_n actually converge to f uniformly on any compact subset of S disjoint from the poles of f. This observation (and a little thought) shows that for analytic (i.e., holomorphic) functions the two definitions of normality agree. See [**1**, pp. 210–219] for an illuminating discussion of these matters.

We shall need the following characterization of normal families, due to F. Marty.

THEOREM. *A family $\mathcal{F}$ of functions analytic or meromorphic on D is normal if and only if the functions*

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \qquad f \in \mathcal{F}$$

are uniformly bounded on each compact subset of D.

Here $f^{\#}$ is the *spherical derivative* of f, sometimes denoted by $\rho(f)$; the present notation (used in [**6**]) is better adapted for displaying the argument of the function explicitly. At the poles of f, $f^{\#}$ is defined by continuity; equivalently, one may use the relation $f^{\#} = (1/f)^{\#}$, obviously valid at regular points of f and $1/f$, to define $f^{\#}$ throughout all of D. The spherical derivative has an appealing

geometric interpretation: one has

$$f^{\#}(z) = \lim_{h \to 0} \chi(f(z+h), f(z))/|h|,$$

and if γ is an are in D, then $\int_\gamma f^{\#}(z)|dz|$ measures the length of $f(\gamma)$ on the Riemann sphere. Observe that $f^{\#}(z) = 0$ only if $f'(z) = 0$ or $f(z) = \infty$.

Various proofs of Marty's theorem exist in the literature. Marty's original, rather geometric, argument can be found in Ahlfors [**1**, pp. 218–219]; a purely analytic proof is in [**3**, pp. 158–160].

3. The main lemma. Before formulating our principle, it will be convenient to give an alternative characterization of normality, which makes explicit the relation with entire or meromorphic functions on the plane. The following lemma is perhaps best understood in the context of non-standard analysis (for which connection see [**7**, pp. 509–510]); crudely put, it says that in the absence of normality a certain kind of infinitesimal convergence must take place.

LEMMA. *A family $\mathcal{F}$ of functions meromorphic [analytic] on the unit disc Δ is not normal if and only if there exist*

(a) *a number* $0 < r < 1$
(b) *points* z_n, $|z_n| < r$
(c) *functions* $f_n \in \mathcal{F}$
(d) *numbers* $\rho_n \to 0+$

such that

$$f_n(z_n + \rho_n \zeta) \to g(\zeta) \tag{1}$$

spherically uniformly [uniformly] on compact subsets of $\mathbf{C}$, *where g is a nonconstant meromorphic [entire] function on* $\mathbf{C}$.

Proof. Suppose $\mathcal{F}$ is not normal on Δ. Then by Marty's theorem there exists a number r^*, $0 < r^* < 1$, points z_n^* in $\{z: |z| \leqq r^*\}$, and functions $f_n \in \mathcal{F}$ such that $f_n^{\#}(z_n^*) \to \infty$. Fix a number r, $r^* < r < 1$, and let

$$M_n = \max_{|z| \leqq r} \left(1 - \frac{|z|^2}{r^2}\right) f_n^{\#}(z) = \left(1 - \frac{|z_n|^2}{r^2}\right) f_n^{\#}(z_n). \tag{2}$$

The maximum exists since $f_n^{\#}$ is continuous for $|z| \leqq r$, and it is clear that $M_n \to \infty$. Setting

$$\rho_n = \frac{1}{M_n}\left(1 - \frac{|z_n|^2}{r^2}\right) = \frac{1}{f^{\#}(z_n)}, \tag{3}$$

we obtain

$$\frac{\rho_n}{r - |z_n|} = \frac{r + |z_n|}{r^2 M_n} \leqq \frac{2}{r M_n} \to 0. \tag{4}$$

Thus, the functions

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta)$$

are defined for $|\zeta| < R_n$, where $R_n = (r - |z_n|)/\rho_n \to \infty$ as $n \to \infty$. It follows from (3) that

$$g_n^{\#}(0) = \rho_n f_n^{\#}(z_n) = 1.$$

For $|\zeta| \leqq R < R_n$, $|z_n + \rho_n \zeta| < r$ so that by (2) and (3)

$$g_n^{\#}(\zeta) = \rho_n f_n^{\#}(z_n + \rho_n \zeta) \leqq \frac{\rho_n M_n}{1 - \frac{|z_n + \rho_n \zeta|^2}{r^2}} \leqq \frac{r + |z_n|}{r + |z_n| + \rho_n R} \cdot \frac{r - |z_n|}{r - |z_n| - \rho_n R}.$$

The first factor on the right is bounded by 1, while (for fixed R) the second tends to 1 as $n \to \infty$ by (4). Thus, by Marty's theorem, $\{g_n\}$ is a normal family; taking a subsequence, we may assume that the g_n converge uniformly (in the spherical metric) on compact subsets of $\mathbf{C}$ to a meromorphic function g. Finally, g is nonconstant since $g^{\#}(0) = \lim g_n^{\#}(0) = 1 \neq 0$. It is now clear that if $\mathcal{F}$ consists of analytic functions the limit function will be entire.

For the converse, suppose that $\mathcal{F}$ is normal on Δ. By Marty's theorem, there exists $M > 0$ such that

$$\max_{|z| \leqq (1+r)/2} f^{\#}(z) \leqq M$$

for all $f \in \mathcal{F}$. Suppose (1) holds and fix $\zeta \in \mathbf{C}$. For large n, $|z_n + \rho_n \zeta| \leqq (1+r)/2$, so that $\rho_n f_n^{\#}(z_n + \rho_n \zeta) \leqq \rho_n M$. Thus, for all $\zeta \in \mathbf{C}$,

$$g^{\#}(\zeta) = \lim \rho_n f_n^{\#}(z_n + \rho_n \zeta) = 0.$$

It follows that g is constant (possibly infinity).

As noted earlier, the proof of the lemma is virtually identical to Pommerenke's proof of a slightly different result. The clever trick of using a cut-off function $(1 - (|z|^2/r^2))$ can be traced back at least to Landau's proof of Bloch's theorem [**5**, p. 99].

4. A matter of principle. To formulate the heuristic principle precisely, it will be convenient to follow Robinson's idea of displaying the domain of definition of a function explicitly together with the function. Thus, we write $\langle f, D \rangle$ to denote the function f defined on the domain $D \subset \mathbf{C}$, and we distinguish between the functions $\langle f, D \rangle$ and $\langle f, D' \rangle$ if $D \neq D'$

The principle may then be stated as follows.

THEOREM. *Let P be a property (i.e., a set) of meromorphic [holomorphic] functions which satisfies the following conditions.*

(i) *If $\langle f, D \rangle \in P$ and $D' \subset D$ then $\langle f, D' \rangle \in P$.*

(ii) *If $\langle f, D \rangle \in P$ and $\phi(z) = az + b$, then $\langle f \circ \phi, \phi^{-1}(D) \rangle \in P$.*

(iii) *Let $\langle f_n, D_n \rangle \in P$, where $D_1 \subset D_2 \subset D_3 \subset \cdots$ and $D = \cup D_n$. If $f_n \to f$ spherically uniformly on compact subsets of D, then $\langle f, D \rangle \in P$.*

Suppose $\langle f, \mathbf{C} \rangle \in P$ only if f is constant. Then for any domain D the family of functions satisfying $\langle f, D \rangle \in P$ is normal on D.

The present formulation differs from Robinson's (tentative) version [**7**, p. 509] in that it applies to meromorphic as well as analytic functions. More importantly, we only require invariance with respect to linear (as opposed to general conformal) maps.

Condition (i) is only a convenience; it can be avoided by a suitable reformulation of (iii). Conditions (ii) and (iii), on the other hand, are quite essential, as the following examples show. (Actually, (ii) need hold only for $0 < a \leqq 1$, and it would be enough to require (iii) only for the case where the D_n are discs centered at the origin and $D = \mathbf{C}$.)

EXAMPLE 1. Let $\langle f, D \rangle \in P$ if and only if $D \subset \Delta = \{z; |z| < 1\}$. Then (i) and (iii) are satisfied, while (ii) does not hold. Since $\langle f, \mathbf{C} \rangle \in \mathrm{P}$ is never satisfied, P contains no entire or globally meromorphic functions at all. But the family of all analytic (or meromorphic) functions on Δ is clearly not normal (consider the sequence $\{f_n\}$, where $f_n(z) = nz$).

EXAMPLE 2. Let $\langle f, D \rangle \in P$ if and only if $D \neq \mathbf{C}$ $(D \subset \mathbf{C})$. For analytic functions, P may be phrased informally as "f is not entire." Obviously (i) and (ii) hold, but (iii) fails. Again P contains no entire functions; yet if D is any proper subdomain of $\mathbf{C}$, $\langle f, D \rangle \in P$ for all f analytic on D, and this family is not normal.

EXAMPLE 3. This is a more "natural" version of the preceding phenomenon. Let $\langle f, D\rangle \in P$ if and only if f is bounded on D, i.e., there exists a constant $M = M(f, D)$ such that $\sup_D|f(z)| \leqq M$. Conditions (i) and (ii) hold while (iii) fails. If $\langle f, \mathbf{C}\rangle \in P$, f must be constant by Liouville's theorem. But the family of all bounded analytic functions on a disc (for instance) is not normal.

A slight modification of the condition of Example 3 yields a positive result.

EXAMPLE 4. Fix $M > 0$ and let $\langle f, D\rangle \in P$ if and only if $\sup_D|f(z)| \leqq M$. Then (i)–(iii) are clearly satisfied and P contains no nonconstant entire functions. The theorem applies, and we recapture a classical sufficient condition for normality.

The proof of the theorem is hardly more than a restatement of the lemma of the preceding section, to which we refer the reader for the notation used below. Indeed, let $\mathcal{F}$ be the family of all functions on the domain which have property P. If $\mathcal{F}$ is not normal on D, Marty's condition (or the usual compactness argument) shows that it already fails to be normal on some subdisc, which (by (ii)) we may assume to be Δ. Let $R_n = (r - |z_n|)/\rho_n$; since $R_n \to \infty$, we may suppose (by taking a subsequence, if necessary) that the R_n form an increasing sequence. Set $g_n(\zeta) = f_n(z_n + \rho_n\zeta)$, $D_n = \{\zeta: |\zeta| < R_n\}$. The functions $\langle g_n, D_n\rangle$ satisfy P by (i) and (ii), so by (iii) $\langle g, \mathbf{C}\rangle$ does also. Since P contains no nonconstant functions defined on $\mathbf{C}$, this yields a contradiction. Thus, $\mathcal{F}$ must be normal on D.

5. An application. Perhaps the most celebrated criterion for normality is the following theorem, due to Paul Montel.

MONTEL'S THEOREM. *Let $\mathcal{F}$ be a family of functions meromorphic on the domain D. If there exist three points w_1, w_2, w_3 on the Riemann sphere such that $w_i \notin f(D)$ $(i = 1, 2, 3)$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Thus, Montel's theorem asserts that if each function in $\mathcal{F}$ omits the *same* three values then $\mathcal{F}$ is normal. (For families of *analytic* functions the value ∞ is always omitted, so one need require only that two finite values be omitted.) The usual proof makes use of Jacobi's elliptic modular function and is thus "nonelementary." Our principle, together with Picard's little theorem, gives an elementary proof. (Quite a different proof, also of elementary character, is in [**8**, pp. 347–350].)

Indeed, it is enough to take for P the property "either f is constant or it omits the values w_1, w_2, and w_3 on D." Conditions (i) and (ii) are at once seen to hold, while (iii) is a consequence of Hurwitz's theorem [**1**, p. 176] (or the argument principle). That any meromorphic function on $\mathbf{C}$ which satisfies P must be constant is, of course, Picard's little theorem.

Montel's theorem can be generalized in various directions. One extension, less well-known than it deserves, is the following.

EXTENDED MONTEL THEOREM. [**2**, vol. 2, p. 202] *Let $\mathcal{F}$ be a family of functions meromorphic on the domain D. Suppose that each $f \in \mathcal{F}$ omits three distinct values (which may depend on f) a, b, c on the sphere, the product of whose chordal distances $\chi(a, b)\,\chi(b, c)\,\chi(a, c)$ is bounded away from* 0 *independently of f. Then $\mathcal{F}$ is a normal family.*

For the proof, let ε be a positive lower bound for the product of the distances and take P to be the property "f omits three values a, b, c such that $\chi(a, b)\chi(b, c)\chi(a, c) \geqq \varepsilon$." By Picard's theorem, no nonconstant meromorphic function can have P. Thus, since (i) and (ii) are trivially satisfied, it remains only to show that P is preserved under uniform convergence with respect to the spherical metric.

Suppose then that $f_n \to f$ spherically uniformly on compact subsets of D and that f_n omits a_n, b_n, c_n with $\chi(a_n, b_n)\chi(b_n, c_n)\chi(a_n, c_n) \geqq \varepsilon$. We may assume f is nonconstant, for otherwise it trivially satisfies P. Since the sphere is compact, we can find points a, b, c and a subsequence (again denoted $\{f_n\}$) such that $\chi(a_n, a) \to 0$, $\chi(b_n, b) \to 0$, $\chi(c_n, c) \to 0$. By continuity, $\chi(a, b)\,\chi(b, c)$

$\chi(a,c) \geqq \varepsilon$, so we only need to prove that f never takes on the values a, b, c. Indeed, suppose $f(z_0) = a$, where $a \neq \infty$. Choose $r > 0$ such that $K = \{z : |z - z_0| \leqq r\} \subset D$ and f is analytic on K. Since f is bounded on K (and $a \neq \infty$), $f_n(z) - a_n$ converges *uniformly* on K to $f(z) - a$. The latter function is nonconstant and vanishes for $z = z_0$, so by Hurwitz's theorem $f_n(z) - a_n$ must (for large n) vanish on K. This is a contradiction. If $a = \infty$, we consider the functions $1/f$, $1/f_n$ and argue as before, using the invariance property $\chi(z, z') = \chi(1/z, 1/z')$.

6. Pedagogics. Montel's theorem is the central device in one of the standard proofs of the Big Picard Theorem: *in the neighborhood of an (isolated) essential singularity, a meromorphic function takes on every value in the Riemann sphere infinitely often with at most two exceptions.* For analytic functions, even more is true, as was proved by Gaston Julia.

JULIA'S THEOREM. *Let $f(z)$ be analytic in $D = \{z : 0 < |z| < 1\}$ with an essential singularity at* 0. *Then there exists a point $z_0 \in D$ such that, for each $\varepsilon > 0$, $f(z)$ assumes every complex value, with at most one exception, infinitely often on the union of the homothetic discs*

$$D_n = \{z : |z - z_0/2^n| < \varepsilon/2^n\}.$$

Again, the main tool in the proof is Montel's theorem. (The reader should be warned that the proofs given in [**4**, p. 259] and the first two editions of [**8**] are incomplete; a correct proof is in [**8**, pp. 351–352]).

We see no particular merit in avoiding the use of the modular function, which is at any rate required to obtain the precise values of the constants appearing in the theorems of Schottky and Landau [**2**, vol. 2, pp. 195–201]. On the other hand, it is perhaps of methodological interest that the theorems of both Picard and Julia can be given a purely elementary proof. One such development may be found in the important text of Saks and Zygmund [**8**, pp. 341–353].

An alternate program for obtaining these theorems in elementary fashion may be sketched as follows. First prove (Landau's version of) Bloch's theorem; this is a natural sequel to the lovely one-quarter theorem of Koebe and might well appear at the end of a unit on conformal mapping. Next, derive the Little Picard Theorem from Bloch's theorem. (This much is standard; cf. [**4**, pp. 384–390], [**5**, pp. 98–102], [**8**, pp. 341–346]). Prove Montel's theorem via the heuristic principle (made rigorous) and Picard's little theorem. Finally, derive the Big Picard Theorem and Julia's extension. If desired, Montel's theorem can also be used to give a very brief proof of Schottky's theorem [**4**, pp. 261–262], and Landau's theorem then follows in a couple of lines [**8**, pp. 354–355]. Instructors interested in emphasizing the importance and usefulness of normal families, who find themselves pressed for time and unwilling to tell less than the full truth about the modular function, may find the approach outlined above an attractive alternative to the existing routes.

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References

1. Lars V. Ahlfors, Complex Analysis, 2-nd ed., McGraw-Hill, New York, 1966.

2. C. Carathéodory, Theory of Functions, vols. 1 and 2, 2-nd English edition, Chelsea, New York, 1958 and 1960.

3. W. K. Hayman, Meromorphic Functions, Oxford University Press, London, 1964.

4. Einar Hille, Analytic Function Theory, vol. 2, Ginn, Boston, 1962.

5. Edmund Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2nd ed. Chelsea, New York, 1946.

6. A. J. Lohwater and Christian Pommerenke, On normal meromorphic functions, Ann. Acad. Sci. Fenn. Ser. A I, 550 (1973).

7. Abraham Robinson, Metamathematical problems, J. Symbolic Logic, 38 (1973) 500–516.

8. Stanislaw Saks and Antoni Zygmund, Analytic Functions, 3rd ed, Elsevier, New York, 1971.

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SECOND CORRECTION TO: "INNER PRODUCT SPACES"

STANLEY GUDDER AND SAMUEL HOLLAND

In a recent correction [1] the first author pointed out a gap in his paper in this MONTHLY [2]. We now fill that gap. Our argument is a simplified version of that given in [3].

THEOREM. *An inner product space V is complete if every maximal orthonormal set in V is basic.*

Proof. Suppose every maximal orthonormal set in V is basic. We show that (5) of Theorem 3.1 [2] holds. Let f be a nonzero continuous linear functional on V (if $f \equiv 0$, the result is trivial). Let $M = \{x \in V : f(x) = 0\}$. Then M is a closed subspace of V. Let $B = \{x_i : i \in I\}$ be a maximal orthonormal set in M. Extend B to a maximal orthonormal set $B \cup B_1$ of V, where $B_1 = \{y_j : j \in J\}$. Now $J \neq \emptyset$ since otherwise B would be basic in V and then $M = V$ which is a contradiction. Also $y_j \notin M$ for every $j \in J$ since B is maximal in M. Suppose $y_1, y_2 \in B_1$ and let $y = y_1 - f(y_1)y_2/f(y_2)$. Then $y \in M$ and $y \perp x_i$ for every $i \in I$. Since B is maximal in M, $y = 0$ and hence $y_1 = y_2$. It follows that $B_1 = \{y_1\}$. If $x \in V$, then

$$x = \sum \left\langle x, x_i \right\rangle x_i + \left\langle x, y_1 \right\rangle y_1.$$

Hence $f(x) = \left\langle x, y_1 \right\rangle f(y_1) = \left\langle x, \overline{f(y_1)} y_1 \right\rangle$.

References

1. S. Gudder, Correction to inner product spaces, this MONTHLY, 82 (1975) 251.
2. S. Gudder, Inner product spaces, this MONTHLY, 81 (1974) 29–36.
3. S. Holland, Jr., Hilbert spaces and von Neumann algebras, Mimeographed notes, Univ. of Mass., 1972–73, unpublished.

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

Reply to Query 17. This Query asked for a complete list of programmed text in elementary mathematics. E. G. Begle points out that a bibliography "Programmed Learning and Individually Placed Instruction" can be obtained from Hendershot Programmed Learning, 4114 Ridgewood, Bay City, Michigan 48706.

Reply to Query 18. In this Query, the addresses of firms that manufacture mathematical models for educational uses, was asked for. D. Wheeler suggests the La Pine Scientific Company, 600 S.

Knox Ave., Chicago, Ill. 60629. J. M. Malone, II suggests the Edmund Scientific Company, Edscorp Bldg., Barrington, N. J. 08007. He also suggests that models can be developed by computer graphics. Spatial Data Systems, 500 S. Fairview, Goleta, CA 93017 and Computer and Information Services, USS Engineers and Consultants, 600 Grant St., Pittsburgh, PA 15230, may be able to assist in this matter.

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

A FAMILY OF GALILEO SEQUENCES

DAVID ZEITLIN

1. Introduction. Let a_n, $n = 1,2,\cdots$, be positive integers and set $S_n = a_1 + a_2 + \cdots + a_n$. Then a_n is a standard Galileo sequence (GS) if the partial sums S_n satisfy

$$S_{2n} - S_n = pS_n, \qquad n = 1,2,\cdots, \tag{1.1}$$

where p is a fixed constant, independent of n. Equivalently, we can write (1.1) as

$$S_{2n} = qS_n, \qquad (q = p+1), \tag{1.2}$$

which implies $a_{2n-1} + a_{2n} = qa_n$, $n = 1,2,\cdots$. The first Galileo sequence, provided by Galileo himself in 1615, is the sequence of odd integers $a_n = 2n-1$, which entails $S_n = n^2$, so that $S_{2n} = 2^2S_n$, $S_{mn} = m^2S_n$, $m = 1,2,\cdots$. See the paper by May [1]. A natural generalization is to sequences with $S_n = n^k$, or $a_n = n^k - (n-1)^k \equiv a_n(k)$, where for each fixed k, $k = 1,2,\cdots$, we have $n = 1,2,\cdots$. These sequences are examined in Section 2. Another basic sequence is that of odd integers with alternating sign: $a_n = (-1)^n(2n-1)$ for which $S_n = (-1)^n n$; its generalization (see Section 3) is given by $S_n = (-1)^n n^k$, where $a_n = (-1)^n(n^k + (n-1)^k) \equiv a_n(k)$ so that $S_{2n} = 2^k(-1)^n S_n$, $S_{mn} = m^k(-1)^{(m-1)n}S_n$, $m = 1,2,\cdots$. Results for sequences whose terms are fractions are given in Section 6. Linear difference equations and Galileo sequences are discussed in Section 4. Iterated Galileo sequences are given in Section 5.

2. Polynomial solutions of (1.2). We shall write a_n as $a_n(k)$ for sequence identification, where for each fixed k, $k = 1,2,\cdots$, we have $n = 1,2,\cdots$

THEOREM 1. *Let k, m, and c be positive integers with $k \geqq 1$. The polynomial solutions of*

(2.1) $S_{2n} = 2^kS_n$,

(2.2) $S_{mn} = (m/c)^kS_{cn}, \qquad m > c \geqq 1$,

(2.3) $S_{m(cn+b)} = m^kS_{cn+b}, c+b > 0, c \geqq 1, b = 0, \pm 1, \pm 2, \cdots; m \geqq 2$, *are given by the family*

(2.4) $S_n = n^k$, *where* $a_n = a_n(k) = n^k - (n-1)^k$, $n = 1,2,\cdots$; $k = 1,2,\cdots$.

REMARKS. Note that $q = 2^k$ in (2.1), $q = (m/c)^k$ in (2.2), and $q = m^k$ in (2.3). The solution sequences are polynomials in n of degree $k-1$. For each k, $a_n(k)$, $n = 1,2,\cdots$, is a sequence of odd integers, with $a_1(k) = 1$. Since $p = q-1$, then $p = (m/c)^k - 1$ (see (2.2)) can now have fractional values. Let $M(x,k) = x^k - (x-1)^k$, $k = 1,2,\cdots$. By differentiation with respect to x, $M'(x,k) = kM(x,k-1)$, a property of Appell polynomials. Thus, the polynomial solutions $a_n(k)$ given by (2.4) are Appell polynomials with x replaced by n.

The first few values of $a_n(k)$ are given below.

TABLE 1. $a_n(k) = n^k - (n-1)^k$.

n	1	2	3	4	5	6	7
$a_n(2)$	1	3	5	7	9	11	13
$a_n(3)$	1	7	19	37	61	91	127
$a_n(4)$	1	15	65	175	369	671	1105

The following identities can be used to check Table 1 (and its extensions):

$$a_n(k+2) = (2n-1)a_n(k+1) - n(n-1)a_n(k), \qquad n, k = 1, 2, \cdots, \tag{2.5}$$

$$a_n(k) = n\,a_n(k-1) + S_{n-1}(k-1) = na_n(k-1) + \sum_{i=1}^{n-1} a_i(k-1), \qquad n, k = 1, 2, \cdots. \tag{2.6}$$

3. Solutions by sequences whose terms alternate in sign. Thus far, our solution sequences (see (2.4)) have positive terms only. A companion result to Theorem 1 is now given by

THEOREM 2. *Let k, m, and c be positive integers with $k \geqq 1$. Then solutions of*

(3.1) $S_{2n} = 2^k(-1)^n S_n, \qquad S_{mn} = m^k(-1)^{(m-1)n} S_n, \qquad m = 2, 3, \cdots,$

(3.2) $S_{mn} = (m/c)^k S_{cn}$, $m > c \geqq 1$, *with m and c of same parity,*

(3.3) $S_{m(cn+b)} = m^k S_{cn+b}$, $c + b > 0$, $c \geqq 1$, $b = 0, \pm 1, \pm 2, \cdots$; m odd, $m \geqq 3$,
are given by the family

(3.4) $S_n = (-1)^n n^k$, *where* $a_n(k) = (-1)^n(n^k + (n-1)^k)$, $n = 1, 2, \cdots$.

REMARKS. From (3.4), we note that $(-1)^n a_n(k)$ is a polynomial in n of degree k. For each k, $a_n(k)$, $n = 1, 2, \cdots$, is a sequence of odd integers with alternating signs, and $a_1(k) = -1$. Values of $a_n(k)$ for Theorem 2 are given by Table 2.

TABLE 2. $a_n(k) = (-1)^n(n^k + (n-1)^k)$.

n	1	2	3	4	5	6	7
$a_n(1)$	−1	3	−5	7	−9	11	−13
$a_n(2)$	−1	5	−13	25	−41	61	−85
$a_n(3)$	−1	9	−35	91	−189	341	−559

We note that identities (2.5) and (2.6) are also valid for Table 2.

4. Linear difference equations and Galileo sequences. In Theorem 1, for each fixed k, $S_n = n^k$ and $a_n(k) = n^k - (n-1)^k$ are solutions of linear difference equations with constant coefficients of order $k+1$ and k, respectively; and with characteristic equations $(x-1)^{k+1} = 0$ and $(x-1)^k = 0$. For $k = 2$, $S_n = n^2$ satisfies $S_{n+3} = 3S_{n+2} - 3S_{n+1} + S_n$; $a_n(2) = 2n - 1 \equiv a_n$ satisfies $a_{n+2} = 2a_{n+1} - a_n$, $n = 0, 1, \cdots$.

In Theorem 2, for each fixed k, $S_n = (-1)^n n^k$ and $a_n(k) = (-1)^n(n^k + (n-1)^k)$ are solutions of the *same* linear difference equation with constant coefficients of order $k+1$, whose characteristic equation is $(x+1)^{k+1} = 0$. For $k = 1$; $S_n = (-1)^n n$ satisfies $S_{n+2} + 2S_{n+1} + S_n = 0$, and $a_n(1) = (-1)^n(2n-1) \equiv a_n$ satisfies $a_{n+2} + 2a_{n+1} + a_n = 0$. Moreover, from (3.1), $S_{2n} = V_n S_n$, where $V_n = 2(-1)^n$. The above examples are special cases of the following general result.

THEOREM 3. *Let W_k, $k = 0, 1, \cdots$, be a sequence of integers satisfying*

$$W_{k+2} = P\,W_{k+1} + Q\,W_k, \qquad PQ \neq 0, \qquad P \text{ and } Q \text{ are integers.} \tag{4.1}$$

Set $S_0 = 0$, $S_n = \Sigma_{k=1}^n W_k$, $n = 1, 2, \cdots$. Then S_n satisfies

$$S_{n+3} = (P+1)S_{n+2} + (Q-P)S_{n+1} - QS_n. \tag{4.2}$$

If $W_1 + QW_0 = 0$, *then* S_n *also satisfies*

$$S_{n+2} = PS_{n+1} + QS_n. \tag{4.3}$$

Let U_k, *with* $U_0 = 0$, $U_1 = 1$, *be a solution of* (4.1). *If* (4.3) *is satisfied, then* $S_n = W_1 U_n$,

$$S_{2n} = V_n S_n, \qquad V_n = a^n + b^n, \qquad n = 1, 2, \cdots, \tag{4.4}$$

where a *and* b *are the roots of* $x^2 = Px + Q$.

Proof. Since $W_{k+j} = S_{k+j} - S_{k+j-1}$, (4.2) follows from (4.1); and (4.3) is obtained by summing both sides of (4.1), and then noting that $W_1 + QW_0 = 0$. Zeitlin [**2**, p. 239, (3.6)] has shown that $S_n = S_0 U_{n+1} + (S_1 - PS_0) U_n$, if S_n satisfies (4.3). But $S_0 = 0$, $S_1 = W_1$, and so $S_n = W_1 U_n$. Since $U_n = (a^n - b^n)/(a - b)$, $a \neq b$, ($U_n = na^{n-1}$, if $a = b$), then $U_{2n} = V_n U_n$. Thus, $S_{2n} = W_1 U_{2n} = V_n(W_1 U_n) = V_n S_n$.

REMARKS. The ideas of Theorem 3 can also be applied to higher order linear difference equations. If $a = b = -1$, then $V_n = 2(-1)^n$, as in the example cited above for $k = 1$ in Theorem 2. For $k = 1$ in (2.1) of Theorem 1, $a_n(1) \equiv 1$ and $S_n = n$, with $S_{2n} = 2S_n$. Note that both 1 and n satisfy $W_{n+2} = 2W_{n+1} - W_n$, where $a = b = 1$ and $V_n = 2$.

5. On iterated Galileo sequences. An interesting problem is the enumeration of Galileo sequences a_n whose sums S_n, $n = 1, 2, \cdots$, are themselves elements of a second Galileo sequence. If $S_{2n} = qS_n$ and $S^*_{2n} = q^* S^*_n$, where $S^*_n = \Sigma_{k=1}^n S_k$, the sequence a_n must satisfy

$$a_{2n-2} + 2a_{2n-1} + a_{2n} = q^* a_n \quad \text{and} \quad a_{2n-1} + a_{2n} = qa_n. \tag{5.1}$$

Since $S^*_{2n} = q^* S^*_n$ implies that $S_{2n-1} + S_{2n} = q^* S_n$, we see that $S_{2n-1} = (q^* - q) S_n$, $n \geqq 1$, and so $q^* = q + 1$, noting that $S_{2n-1} = S_n$ and $S_{2n} = qS_{2n-1}$. For $a_1 = 1$ and $q = 2, 3, \cdots$, the first eight members of the sequences a_n, S_n, and S^*_n are given by

$$\begin{aligned}
&a_n = (1, q-1, 0, q^2 - q, q - q^2, q^2 - q, 0, q^3 - q^2, \cdots),\\
(5.2)\qquad &S_n = (1, q, q, q^2, q, q^2, q^2, q^3, \cdots),\\
&S^*_n = (1, q+1, 2q+1, (q+1)^2, (q+1)^2 + q, 2q^2 + 3q + 1, 3q^2 + 3q + 1, (q+1)^3, \cdots),
\end{aligned}$$

where (5.1) was used to determine a_n recursively. We note that a_n has positive and negative integers as elements, as well as 0. For $m \geqq 3$, the system $S_{mn} = qS_n$ and $S^*_{mn} = q^* S^*_n$ is more complicated.

6. Solutions by sequences whose terms are fractions. The results below are stated for completeness.

THEOREM 4. *Let* k, m, *and* c *be positive integers with* $k \geqq 1$. *Then solutions of*

(6.1) $S_{2n} = 2^{-k} S_n$,

(6.2) $S_{mn} = (c/m)^k S_{cn}$, $\quad m > c \geqq 1$,

(6.3) $S_{m(cn+b)} = m^{-k} S_{cn+b}$, $c + b > 0$, $c \geqq 1$, $b = 0, \pm 1, \pm 2, \cdots$; $m \geqq 2$, *are given by the family*

(6.4) $S_n = n^{-k}$, *where* $a_1(k) = 1$, $a_n(k) = n^{-k} - (n-1)^{-k}$, $n = 2, 3, \cdots$.

Solutions of

(6.5) $S_{2n} = 2^{-k}(-1)^n S_n$, $\quad S_{mn} = m^{-k}(-1)^{(m-1)n} S_n$, $\quad m = 2, 3, \cdots$,

(6.6) $S_{mn} = (c/m)^k S_{cn}$, $m > c \geqq 1$, *with* m *and* c *of same parity,*

(6.7) $S_{m(cn+b)} = m^{-k} S_{cn+b}$, $c + b > 0$, $c \geqq 1$, $b = 0, \pm 1, \pm 2, \cdots$; m *odd,* $m \geqq 3$,
are given by the family

(6.8) $S_n = (-1)^n n^{-k}$, *where* $a_1(k) = -1$, $a_n(k) = (-1)^n(n^{-k} + (n-1)^{-k})$, $n = 2, 3, \cdots$.

REMARK. Unlike Theorems 1 and 2, S_n and $a_n(k)$ in Theorem 4 are not solutions of linear difference equations with constant coefficients.

7. Generating functions and Galileo sequences. Let $S(x) = \Sigma_{n=0}^{\infty} S_n x^n$, where $S_{2n} = qS_n$ (see (1.2)). Then we have

$$S(x) + S(-x) = 2q\,S(x^2). \tag{7.1}$$

If $S_n = n^k$ with $q = 2^k$ (see (2.1), (2.4)), then $S(x) = \Sigma_{n=0}^{\infty} n^k x^n$ is known (see [3], [4, p. 987, (4)]), namely,

$$(1-x)^{k+1} \sum_{n=0}^{\infty} n^k x^n = A_k(x), \tag{7.2}$$

$$(1+x)^{k+1} A_k(x) + (1-x)^{k+1} A_k(-x) = 2^{k+1} A_k(x^2), \tag{7.3}$$

where $A_k(x)$ are the well-known Eulerian polynomials, with $A_0(x) = 1$ and $A_1(x) = x$. Thus, $S(x) = (1-x)^{-k-1} A_k(x)$ is the solution of (7.1), where $q = 2^k$, which implies identity (7.3).

8. General remarks. In [1, p. 69, (9)], it is claimed that a strictly increasing GS for $S_{2n} = qS_n$, $q \geqq 5$, is given by $a_1 = 1$, (and with $[x]$ as the greatest integer function),

$$a_{2n} = [(qa_n)/2] + 1, \qquad a_{2n-1} = [(qa_n - 1)/2]. \tag{8.1}$$

Since $a_{2n-1} + a_{2n} = qa_n$, we must have $a_2 = (q-1)a_1 = q - 1$. For $q = 5$, (8.1) gives $a_2 = 3(\neq 4)$, an incorrect value. For $q = 2^k$, $k \geqq 3$, (8.1) gives $a_{2n} = 2^{k-1}a_n + 1$, or $a_2 = 2^{k-1} + 1\ (\neq q - 1)$, an incorrect value. It appears that (8.1) is correct only for $q = 4$. I thank the referee for pointing out the above situation, and for his many constructive suggestions.

References

1. K. O. May, Galileo sequences, a good dangling problem, this MONTHLY, 79 (1972) 67–69.
2. D. Zeitlin, On convoluted numbers and sums, this MONTHLY, 74 (1967) 235–246.
3. J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.
4. D. Zeitlin, Two methods for the evaluation of $\Sigma_{k=0}^{\infty} k^n x^k$, this MONTHLY, 68 (1961) 986–989.

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DETERMINATION OF POLYNOMIALS AND ENTIRE FUNCTIONS

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If $f(z)$ is a complex polynomial which has k distinct zeros, then to within a constant factor f is determined by the positions of the zeros and their multiplicities. In this note we show how polynomials and certain entire functions may be determined when the multiplicities of the zeros are not given, but enough coefficients of a Taylor expansion are.

First of all consider a complex polynomial which has exactly k distinct zeros at certain given positions. Then it may be shown that the polynomial is completely determined by the first $k + 1$ terms in its expansion about some point P which is not a zero. For suppose that P is the origin, and let

$$f(z) = \sum_{r=0}^{n} a_r z^r = a_0(1 - \lambda_1 z)^{c_1} \cdots (1 - \lambda_k z)^{c_k},$$

where $\lambda_1^{-1},\cdots,\lambda_k^{-1}$ are the zeros. When this equation is logarithmically differentiated we obtain, by expanding for sufficiently small $|z|$,

$$\frac{f'(z)}{f(z)} = -\sum_{r=1}^{k}\frac{c_r\lambda_r}{1-\lambda_r z} = -\sum_{m=1}^{\infty} s_m z^{m-1}$$

where $s_m = \sum_{r=1}^{k} c_r\lambda_r^m$. Now since $f(z)$ is given up to the term in z^k, f'/f is determined up to the term in z^{k-1}, and hence $s_1,\cdots,s_k$ are determined. The relationship between $c_1,\cdots,c_k$ and $s_1,\cdots,s_k$ is linear and invertible because the determinant of the coefficient-matrix $[\lambda_r^m]$ is $\Pi_{i>j}(\lambda_i-\lambda_j)\neq 0$. Thus if $a_0,\cdots,a_k$ are given, $c_1,\cdots,c_k$ and hence f are determined.

An interesting and immediate corollary of this is that if $f(z)=a_0+\sum_{r=k}^{n} a_r z^r$ $(a_0\neq 0)$, then either f has at least k distinct zeros, or else it is constant.

We have been assuming that the zeros are given, along with $a_0,\cdots,a_k$. If, however, only k, the number of zeros is given (without their positions), it may similarly be shown that f is determined by $a_0,a_1,\cdots,a_{2k}$, provided that $f(0)=a_0\neq 0$. For let two polynomials f and g satisfy the given conditions, where

$$f(z)=a_0(1-\lambda_1 z)^{c_1}\cdots(1-\lambda_k z)^{c_k}$$
$$g(z)=a_0(1-\mu_1 z)^{d_1}\cdots(1-\mu_k z)^{d_k},$$

and let $\nu_1,\cdots,\nu_q$ $(q\leqq 2k)$ be distinct and represent the union of $\{\lambda_1,\cdots,\lambda_k\}$ and $\{\mu_1,\cdots,\mu_k\}$. If $f/g=h$, then $h(z)=(1-\nu_1 z)^{p_1}\cdots(1-\nu_q z)^{p_q}$ for some integers $p_1,\cdots,p_q$, and from the fact that $h(z)=1+O(z^{2k+1})$ it follows, by the method used above, that each p_r is zero and hence $f=g$.

As one would expect, these results generalize to the case of polynomials defined over an arbitrary algebraically closed field. The complex case also has a generalization to entire analytic functions which will now be outlined.

$f(z)$ denotes an entire analytic function of finite order l. Suppose that f has k $(<\infty)$ zeros $\lambda_1^{-1},\cdots,\lambda_k^{-1}$, and that the expansion of f is given in powers of z up to the term in z^{l+k} (since f has only a finite number of zeros, l is necessarily an integer). Thus

$$f(z)=\sum_{r=0}^{\infty} a_r z^r = a_0 g(z)(1-\lambda_1 z)^{c_1}\cdots(1-\lambda_k z)^{c_k},$$

where $g(z)$ is an entire analytic function with no zeros which satisfies $g(0)=1$. Since g also has order l it follows (see, for example, Copson **[1]** pp. 166–7) that $g(z)$ can be written $e^{p(z)}$, where $p(z)$ is a polynomial of degree l. Thus when $|z|$ is sufficiently small,

$$\log f(z)=\log a_0-\sum_{r=1}^{\infty} s_r z^r/r+p(z).$$

$f(z)$ and therefore $\log f(z)$ are given up to the term in z^{l+k}, and since $p(z)$ is of degree l it follows that $s_{l+1},\cdots,s_{l+k}$ are determined. Exactly as before, then, $c_1,\cdots,c_k$ are determined. Thus $p(z)$ is determined and therefore so is $g(z)$. We deduce that the function $f(z)$ is determined uniquely by the positions of the zeros and the first $l+k+1$ terms of the power-series expansion.

Thus an entire analytic function of order $l<\infty$ which has an expansion $a_0+\sum_{r=l+k}^{\infty} a_r z^r$ $(a_0\neq 0)$ must have at least k distinct zeros, unless it is constant (and $l=0$). It may also be shown that an entire analytic function of order $l<\infty$ with k distinct zeros is uniquely determined by the first $l+2k+1$ terms in its expansion about some point which is not a zero.

Our final result concerns entire analytic functions of finite order which may have an infinite number of zeros. Let $f(z)$ be an entire function which is non-zero at the origin, with finite order less than the integer l, and let its zeros (including multiplicities) be $z_1,z_2,\cdots$ where $|z_1|\leqq|z_2|\leqq\cdots$.

The genus of $f(z)$ is an integer h satisfying $h \leqq \text{order}(f) \leqq h+1$, and Hadamard's factorization theorem (Copson [1], p. 174) states that $f(z)$ can be written $e^{p(z)}\Gamma(z)$, where $p(z)$ is a polynomial of maximum degree h, and $\Gamma(z)$ is the canonical product of the zeros, of genus h. Now for $|z| \leqq |z_1|$,

$$\log f(z) = \sum_{n=0}^{\infty} b_n z^n = p(z) + \sum_{k=1}^{\infty} [\log(1 - z/z_k) + z/z_k + (z/z_k)^2/2 + \cdots + (z/z_k)^h/h].$$

The second series on the right is uniformly convergent in a neighbourhood of the origin, and by considering the coefficient of z^n for $n > h$ we obtain the absolutely convergent sums

$$\sum_{k=1}^{\infty} z_k^{-n} = -nb_n \text{ for } n \geqq l.$$

Suppose that b_n is given for all $n \in N$ where N is an infinite set, arbitrary at this stage. It will be our purpose to find a restriction on N which causes $f(z)$ to be determined. Therefore let $g(z)$ be an entire function of order less than l which has zeros at $w_1, w_2, \cdots$ and let the expansion of $\log f(z)$ about the origin contain the same coefficients $\{b_n\}_{n \in N}$ as $\log f(z)$. Thus $\sum_{k=1}^{\infty} w_k^{-n} = -nb_n$ for $l \leqq n \in N$.

Consider the meromorphic function $f(z)/g(z)$, and on the assumption that this is not an entire function everywhere non-zero, let $\lambda_1^{-1}, \lambda_2^{-1}, \cdots$ be its *distinct* zeros and poles, where $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$. If the multiplicity of λ_j is c_j, counted positive for a zero and negative for a pole, it follows from the above that

$$0 = \sum_{k=1}^{\infty} c_k \lambda_k^n \text{ for } n \in M,$$

where $M = \{n \in N : n \geqq l\}$ and the sums converge absolutely.

After making a change of scale it may be assumed that $|\lambda_1| = 1$, and let $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_r| > |\lambda_{r+1}|$. Now

$$c_1\lambda_1^n + \cdots + c_r\lambda_r^n = -\sum_{k=r+1}^{\infty} c_k \lambda_k^n \text{ for } n \in M,$$

and if $\sum_{k=r+1}^{\infty} |c_k \lambda_k^l| = a$, $\sum_{k=r+1}^{\infty} |c_k \lambda_k^n| \leqq a\,|\lambda_{r+1}|^{n-l} \to 0$ as $n \to \infty$ in N. Hence $c_1\lambda_1^n + \cdots + c_r\lambda_r^n = \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$ in N.

At this point we introduce a restriction on N. Suppose that N has the property that it contains consecutive sequences of arbitrary length. Then there always exists an integer s, arbitrarily large, for which $s, s+1, \cdots, s+r-1$ all belong to N. Also,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{bmatrix} \begin{bmatrix} \lambda_1^s c_1 \\ \lambda_2^s c_2 \\ \cdots \\ \lambda_r^s c_r \end{bmatrix} = \begin{bmatrix} \varepsilon_s \\ \varepsilon_{s+1} \\ \cdots \\ \varepsilon_{s+r-1} \end{bmatrix}$$

and by inversion of these equations it can be seen that $c_1, c_2, \cdots, c_r$ tend to zero as $s \to \infty$ in N, and hence are zero. This contradiction establishes that f/g is an entire function, non-zero everywhere. $f(z)/g(z)$ can therefore be written $e^{q(z)}$ where $q(z)$ is a polynomial of maximum degree l, and f/g is thus identically 1 if b_k is given for every $k < l$ or, equivalently, if the expansion of $f(z)$ is given to within $O(z^l)$ as $|z| \to 0$. $f(z)$ is, in fact, uniquely determined by the first l terms in its expansion about *any* point which is not a zero, and this completes the proof of the following:

Let $f(z)$ be an entire analytic function which is of order (strictly) less than the integer l, and non-zero at the origin. Define, for sufficiently small $|z|$,

$$\log f(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then $f(z)$ is uniquely determined by the following information:

(i) *the first l terms in the power series expansion of $f(z)$ about any given point which is not a zero;*

and (ii) *the values of b_n for all $n \in N$, where the set N contains consecutive sequences of arbitrary length.*

There is a slight extension of this result, which we state without proof. Let $f^{(1)}(z)$ and $f^{(2)}(z)$ be entire functions with orders less than the integer l, and non-zero at the origin. If $\log f^{(j)}(z) = \sum_{n=0}^{\infty} b_n^{(j)} z^n$ for sufficiently small $|z|$, then $f^{(1)} = f^{(2)}$ identically provided that

(i) $f^{(1)}(z) - f^{(2)}(z) = O((z-a)^l)$ as $|z-a| \to 0$, where a is not a zero of the functions; and

(ii) $\sum_{n \in N} (b_n^{(1)} - b_n^{(2)}) z^n$ is an entire function, where N is a set containing consecutive sequences of arbitrary length.

Reference

1. E. T. Copson, Theory of Functions of a Complex Variable, Oxford, 1935.

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AN ALGEBRAIC CHARACTERIZATION OF LIMITS

J. D. GRAY

1. We intend showing that the following three properties characterize limits of functions at x:

(i) $\lim_x(\alpha f + \beta g) = \alpha \lim_x f + \beta \lim_x g$; $\alpha, \beta \in \mathbf{C}$;

(ii) $\lim_x f \cdot g = \lim_x f \cdot \lim_x g$;

(iii) $\lim_x h = 0$ whenever $h = 0$ on some punctured neighbourhood of x.

To be more specific, let X be a compact Hausdorff space, and denote by $\mathfrak{A}(X)$ the set of all complex-valued functions on X which have limits at every point. So $f \in \mathfrak{A}(X)$ if and only if for each non-isolated point[1] $x \in X$ there is an $l \in \mathbf{C}$ such that for each neighbourhood W of l there is a punctured neighbourhood V of x with $f(V) \subset W$. l is, of course, uniquely determined, and we write $l = \lim_x f$ — the limit of f at x. Thus, for each $x \in X$ we have a map $\lim_x : \mathfrak{A}(X) \to \mathbf{C}$. It is this map we intend characterizing.

Under pointwise operations $\mathfrak{A}(X)$ is a commutative complex algebra with 1, and $\lim_x$ is a complex homomorphism on $\mathfrak{A}(X)$. Further, as all functions in $\mathfrak{A}(X)$ are bounded we can norm $\mathfrak{A}(X)$ with the sup-norm, with respect to which it is complete. (Parenthetically we may observe that $\mathfrak{A}(X)$ comes equipped with a natural involution turning it into a C^*-algebra. As such, by the Gelfand theory ([1], p. 26) it is isometrically isomorphic to the algebra $C(M_X)$ of continuous functions on the maximal ideal space M_X *of* $\mathfrak{A}(X)$.) We shall eventually determine all the complex homomorphisms on the Banach algebra $\mathfrak{A}(X)$, but first some notation and a lemma.

For $f \in \mathfrak{A}(X)$ denote by $D(f)$ the set of points of discontinuity of f. Also, for $x \in X$ put $\omega(f;x) = f(x) - \lim_x f$, so that $x \in D(f)$ if and only if $|\omega(f;x)| > 0$.

LEMMA 1. *For each $f \in \mathfrak{A}(X)$, $D(f)$ is countable. Further, if $D(f) = \{x_n\}$ is infinite, $\omega(f;x_n) \to 0$ as $n \to \infty$.*

Proof. For each integer $n > 0$ put $A_n = \{x \in X : |\omega(f;x)| \geqq 1/n\}$. Then $D(f) = \cup_{n>0} A_n$ is countable as each A_n is finite; reason: if some A_n were infinite, it would, due to the compactness of X, have

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[1] If x is an isolated point of X we agree that for all functions $f: X \to \mathbf{C}$ the limit $\lim_x f$ exists and equals $f(x)$.

an accumulation point a. However, it is an easy exercise to show that f could not have a limit at a. The second part of the lemma follows similar lines and is also left to the reader.

Returning to the central discussion, we can immediately list two (classes of) non-zero complex homomorphisms on $\mathfrak{A}(X)$ viz, $\lim_x$ ('limit at x'), and δ_x ('evaluation at x') defined by $\delta_x(f)=f(x)$. In fact there are no others. Towards proving this, let h be any non-zero complex homomorphism on $\mathfrak{A}(X)$. For $f \in \mathfrak{A}(X)$ define $\hat{f}\colon X \to \mathbf{C}$ by $\hat{f}(x)=\lim xf$, i.e., remove the (possible) discontinuity at x. Then $\hat{f} \in C(X)$. By Lemma 1 we may enumerate $D(f)$ as $(x_n)_{n \geqq 0}$. Denote by χ_n the characteristic function of the singleton x_n. Via the second part of Lemma 1 it may be shown that the series

$$\sum_{n=0}^{\infty} \omega(f;x_n)\chi_n \tag{1}$$

converges uniformly on X, and that

$$f=\hat{f}+\sum_{n=0}^{\infty} \omega(f;x_n)\chi_n, \tag{2}$$

a result which, with a little extra effort, establishes the direct sum decomposition $\mathfrak{A}(X)= C(X) \oplus c_0(X')$. Here $c_0(X')$ is the Banach space of bounded functions ϕ on X' for which $\{x \in X' \mid |\phi(x)|>\varepsilon\}$ is finite for each $\varepsilon>0$, X' being the set of non-isolated points of X.

As every complex homomorphism on a Banach algebra is continuous, ([**1**], p. 22), it follows from (2) that

$$h(f)=h(\hat{f})+\sum_{n=0}^{\infty} \omega(f;x_n)h(\chi_n). \tag{3}$$

The determination of $h(\hat{f})$ offers no difficulty as ([**1**], p. 28) each non-zero complex homomorphism on $C(X)$ is δ_x for some $x \in X$. As for $h(\chi_n)$, note that χ_n being an idempotent of $\mathfrak{A}(X)$, $h(\chi_n)$ is either 0 or 1. A more detailed statement is included in:

LEMMA 2. *Let h be a complex homomorphism on $\mathfrak{A}(X)$. If the restriction $h|C(X)=\delta_x$ (respectively, 0) then $h(\chi_y)=0$ for $y \neq x$ (respectively, for all y).*

Proof. For each $g \in C(X)$ we have $g \cdot \chi_y = g(y)\chi_y$ so that $h(g)\cdot h(\chi_y)=g(y)h(\chi_y)$, and thus $g(x)h(\chi_y)=g(y)h(\chi_y)$ (respectively, $g(y)h(\chi_y)=0$). For $y \neq x$, if we now choose by Urysohn's lemma, g so that $g(x)=0$ and $g(y)=1$, it follows that in both cases, $h(\chi_y)=0$. Also, if $y=x$ it is clear that in the second case $h(\chi_y)=0$.

Reporting back to (3) we see that if $h|C(X)=0$ it would follow that $h=0$. Thus $h|C(X)=\delta_x$ for some x. If $x \notin D(f)$, $h(f)=h(\hat{f})=\hat{f}(x)=\lim_x f=f(x)=\delta_x(f)$; whereas if $x \in D(f)$.

$$h(f)=h(\hat{f})+(f(x)-\hat{f}(x))h(\chi_x)=\begin{cases}\delta_x(f) & \text{if } h(\chi_x)=1,\\ \lim_x f & \text{if } h(\chi_x)=0.\end{cases}$$

This completes the proof of

THEOREM 1. *Let X be a compact Hausdorff space and h a non-zero complex homomorphism on $\mathfrak{A}(X)$. Then $h|C(X)\neq 0$, and if $x \in X$ is that point for which $h|C(X)=\delta_x$, then, according as $h(\chi_x)=1$ or 0, either $h=\delta_x$ or $h=\lim_x$.*

The claim made at the beginning of the paper is contained in the next result, whose proof again relies on Urysohn's lemma.

COROLLARY. *Let X be as before and x a non-isolated point of X. Then $\lim_x$ is that non-zero complex homomorphism on $\mathfrak{A}(X)$ whose kernel contains the ideal of functions which vanish on punctured neighbourhoods of x.*

(With the aid of Theorem 1 it is possible to determine the structure of the topological space M_X. We leave details to the proverbial interested reader.)

2. Suppose now that X is only a locally compact Hausdorff space. By applying Theorem 1 to the one point compactification $X^+ = X \cup \{\infty\}$ of X it may be shown that the only non-zero homomorphisms on $\mathfrak{A}_\infty(X)$ — the algebra of functions on X with limits at every point of X^+ — are δ_x, $\lim_x$ and $\lim_\infty$. Hence $\lim_\infty$ is that non-zero complex homomorphism on $\mathfrak{A}_\infty(X)$ whose kernel contains the ideal of functions with compact support. For example, by taking $X = \mathbf{N}$ so that $\mathfrak{A}_\infty(\mathbf{N}) = c$ — the algebra of convergent sequences, we see that the following three properties characterize limits of sequences:

(i) $\lim\limits_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha \lim\limits_{n\to\infty} a_n + \beta \lim\limits_{n\to\infty} b_n;\qquad \alpha, \beta \in \mathbf{C};$

(ii) $\lim\limits_{n\to\infty} a_n \cdot b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n;$

(iii) $\lim\limits_{n\to\infty} \varepsilon_n^k = 0$ for each $k \in \mathbf{N}$. Here $\varepsilon_n^k = \begin{cases} 0 & \text{if } n \neq k, \\ 1 & \text{if } n = k. \end{cases}$

One consequence of this is that the only regular summability method **S** (i.e., **S** "sums" every convergent sequence to its limit; [**2**], p. 4) that sums the pointwise product of two sequences to the product of their **S**-limits, is ordinary convergence.

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References

1. G. M. Leibowitz, Lectures on Complex Function Algebras. Scott, Foresman, Glenview, Ill., 1970.
2. A. Wilansky, Functional Analysis, Blaisdell, Waltham, Mass., 1964.

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AN INTEGRAL TEST FOR SERIES AND GENERALIZED CONTRACTIONS

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1. Generalized geometric series. Let A be a class of real valued functions Q with the properties: (a) $0 < Q(s) < s$ for $0 < s \leqq s_1$ and (b) $g(s) = s/(s - Q(s))$ be nonincreasing. Put $s_{n+1} = Q(s_n)$, $n = 1, 2, \cdots$. Then the series $S = \sum_{n=1}^\infty s_n$ generated by Q is convergent if (c) $\int_0^{s_1} g(s)ds < \infty$.

Proof.

$$\sum_{i=n}^{m-1} s_i = \sum_{i=n}^{m-1} s_i(s_i - s_{i+1})/(s_i - Q(s_i)) \leqq \sum_{i=n}^{m-1} \int_{s_{i+1}}^{s_i} s[s - Q(s)]^{-1} ds = \int_{s_m}^{s_n} g(s)ds. \tag{1}$$

Hence, it follows that the series S is convergent and its remainder satisfies

$$\sum_{i=n}^{\infty} s_i \leqq \int_0^{s_n} g(s)ds,$$

since $s_n \to 0$ as $n \to \infty$.

Example 1.1. If $Q(s) = qs$ with $0 < q < 1$, then Q generates a geometric series and $g(s) = 1/(1-q)$ is constant. Thus, conditions (a), (b) and (c) are satisfied.

Example 1.2. If $g(s) \leqq M$, then $Q(s) \leqq qs$ with $q = (M-1)/M < 1$ and the case reduces to a geometric series.

Example 1.3. If g is not bounded, then (α) $g(s)\to\infty$ as $s\to 0$.

Let $Q(s)=s(1-s^{\alpha}/(1+\alpha))$ for $0<s\leqq s_1$ and $0<\alpha<1$. In this case $g(s)=(1+\alpha)/s^{\alpha}$ and conditions (a), (b), (c) and (α) are fulfilled.

2. Series $S=\Sigma_{n=1}^{\infty} f(n)$. Let f be a positive decreasing function such that $f(x)\to 0$ as $x\to\infty$ and put $Q(s)=f(f^{-1}(s)+1)$, $s=f(x)$, $s_n=f(n)$. Then $s_{n+1}=f(n+1)=f(f^{-1}(s_n)+1)=Q(s_n)$ and $g(s)=s/[s-f(f^{-1}(s)+1)]$. Thus, conditions (a), (b) and (c) can be written for Q in terms of f and f^{-1}.

If the function with values $f(x)[f(x)-f(x+1)]^{-1}$ is not decreasing, then g is not increasing and condition (b) holds true.

If f is differentiable, then condition (c) yields

$$\int_0^{s_1} g(s)ds=\int_0^{s_1} s[s-f(f^{-1}(s)+1)]^{-1}ds=\int_1^{\infty} f(x)\,|f'(x)|\,[f(x)-f(x+1)]^{-1}dx<\infty.$$

2.1 *Example.* Let $S=\Sigma_{n=1}^{\infty} 1/n^{\alpha}$, $1<\alpha$. Then $s=f(x)=1/x^{\alpha}$ and $Q(s)=(s^{-1/\alpha}+1)^{-\alpha}<s$. Since $Q(s)/s=1/(1+s^{1/\alpha})$, we have $g(s)=1/[1-Q(s)/s]=1/[1-1/(1+s^{1/\alpha})]$ and g is decreasing, that is, condition (b) is satisfied. The integral in (c) is

$$\int_0^{s_1} g(s)ds=\int_0^{s_1}[1-1/(1+s^{1/\alpha})]^{-1}ds=\alpha\int_1^{\infty}\{x^{1+\alpha}[1-(x/(x+1))^{\alpha}]\}^{-1}dx.$$

3. Necessity of (c). Suppose that in addition to (a) and (b) the slope of Q has the following property

(d) There exist positive numbers a and d such that $[Q(u)-Q(v)]/(u-v)\geqq d$ for all $0<v<u<a$.

Then condition (c) is necessary for the convergence of the series S generated by Q provided that the integrals $\int_{\alpha}^{s_1} g(s)ds$ exist for $0<\alpha<s_1$.

Proof. We have, by (b) and (d),

$$\int_{s_m}^{s_n} g(s)ds=\sum_{i=n}^{m}\int_{s_i}^{s_{i-1}} g(s)ds\leqq\sum_{i=n}^{m} s_i(s_{i-1}-s_i)/(s_i-Q(s_i))$$
$$=\sum_{i=n}^{m} s_i(s_{i-1}-s_i)/(Q(s_{i-1})-Q(s_i))\leqq d^{-1}\sum_{i=n}^{m} s_i.$$

Hence, it follows that the convergence of the series $S=\Sigma_{n=1}^{\infty} s_n$ implies the existence of the integral $\int_0^{s_1} g(s)ds$.

3.1 *Example.* The function $Q(s)=s[1-s^{\alpha}/(1+\alpha)]$ has property (d), since $Q'(s)=1-s^{\alpha}$.

The function $Q(s)=(s^{-1/\alpha}+1)^{-\alpha}$ with $1<\alpha$ which generates the series $S=\Sigma_{n=1}^{\infty} 1/n^{\alpha}$ has property (d), since $Q'(s)=(1+s^{1/\alpha})^{-(1+\alpha)}$. Since the series $S=\Sigma_{n=1}^{\infty} 1/n^{\alpha}$ with $\alpha>1$ is convergent, it follows that the integral

$$\int_0^{s_1} g(s)ds=\int_0^{s_1}[1-1/(1+s^{1/\alpha})]^{-1}ds=\alpha\int_1^{\infty}\{x^{1+\alpha}[1-(x/(x+1))^{\alpha}]\}^{-1}dx$$

exists.

Let the function $s=f(x)$, $x\geqq 1$, be positive, continuous and decreasing and let $f(x)\to 0$ as $x\to\infty$. Put $Q(s)=f(f^{-1}(s)+1)<s$ and assume that $f(x)[f(x)-f(x+1)]^{-1}$ is not decreasing. Then $g(s)$ is not increasing. Condition (d) for Q can be written as (d') $[f(x+1)-f(y+1)]/[f(x)-f(y)]\geqq d>0$ for sufficiently large $y>x$.

COROLLARY. *Suppose that f satisfies the hypotheses mentioned above. Then $\int_1^{\infty} f(x)dx<\infty$ if and only if $\int_0^{s_1} s[s-Q(s)]^{-1}ds<\infty$ or $\int_1^{\infty} f(x)\,|f'(x)|\,[f(x)-f(x+1)]^{-1}dx<\infty$, if f is continuously differentiable. Condition* (d') *can be replaced by $f'(x+1)/f'(x)\geqq d$ for sufficiently large x which is equivalent to $Q'(s)\geqq d$ for all positive s smaller than some a.*

REMARK. Under the hypotheses of the corollary except for (d) or (d′), $\int_0^{s_1} s[s-Q(s)]^{-1}ds<\infty$ or $\int_1^\infty f(x)\,|f'(x)|\,[f(x)-f(x+1)]^{-1}dx<\infty$ implies $\int_1^\infty f(x)dx<\infty$.

4. Generalized contractions. Let $F\colon X\to X$ be a mapping such that $d(Fx,Fy)\leqq Q(d(x,y))$ for all x,y (or with $y=Fx$) of the complete metric space X with distance d. Put $x_{n+1}=Fx_n$, $n=0,1,\cdots$ and $s_{n+1}=Q(s_n)$ with $s_1=d(x_1,x_0)$, where $Q\in A$ is a nondecreasing function satisfying conditions (a), (b) and (c). Then the sequence $\{x_n\}$ converges to a unique fixed point $x=Fx$.

Proof. We prove by induction that $d(x_n,x_{n+1})=d(Fx_{n-1},Fx_n)\leqq Q(d(x_{n-1},x_n))\leqq Q(s_n)=s_{n+1}$. Hence, we have

$$d(x_{n-1},x_{m-1})\leqq\sum_{i=n}^{m-1}d(x_i,x_{i+1})\leqq\sum_{i=n}^{m}s_{i+1}\leqq\int_{s_m}^{s_n}g(s)ds, \tag{2}$$

by (1), and $\{x_n\}$ converges to some $x\in X$. Since $d(Fy,Fz)\leqq Q(d(y,z))<d(y,z)$, the mapping F is continuous and $Fx=x$. If $x\neq y=Fy$, then $0<d(x,y)=d(Fx,Fy)\leqq Q(d(x,y))<d(x,y)$, yielding a contradiction. Letting $m\to\infty$ in (2) we obtain the error estimate $d(x_{n-1},x)\leqq\int_0^{s_n}g(s)ds$. Another application is given in [**1**], (see also [**2**]).

References

1. M. Altman, A class of majorant functions for contractors and equations, Bull. Austral. Math. Soc., 10 (1974) 51–58.

2. D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Am. Math. Soc., 20 (1969) 458–64.

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A VECTOR NORM INEQUALITY

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If z and w are complex numbers and $\lambda\geqq 2$, then van der Corput and Beth [**1**, p. 322] have shown that

$$|z+w|^\lambda+|z-w|^\lambda\geqq 2(|z|^\lambda+|w|^\lambda). \tag{1}$$

We extend this inequality by first showing that

$$S\equiv\sum|\pm\mathbf{V}_1\pm\mathbf{V}_2\pm\cdots\pm\mathbf{V}_n|^\lambda\geqq 2^n\left\{\sum_{i=1}^n\mathbf{V}_i^2\right\}^{\lambda/2} \tag{2}$$

where $\lambda>2$ or <0, $\mathbf{V}_i$ are vectors in E^m and the summation on the left hand side is taken over all 2^n permutations of the $\pm$ signs. The inequality is reversed for $0<\lambda<2$, while for $\lambda=0,2$ there is identity. A geometric interpretation, corresponding to the case $\lambda=1$, is that for all parallelotopes of given edge lengths, the rectangular one has the greatest sum of the lengths of the body diagonals.

Proof. $S=\Sigma\{\Sigma_{i=1}^n\mathbf{V}_i^2+\Sigma_{i,j}\pm2\mathbf{V}_i\cdot\mathbf{V}_j\}^{\lambda/2}$. Then, since $x^{\lambda/2}$ is convex for $\lambda>2$ or <0,

$$S\geqq 2^n\left\{\sum\left(\sum_{i=1}^n\mathbf{V}_i^2+\sum_{i,j}\pm2\mathbf{V}_i\cdot\mathbf{V}_j\right)\Big/2^n\right\}^{\lambda/2}$$

which reduces to (2).

By the inequality of power means [**2**],

$$\{\Sigma\,\mathbf{V}_i^2\}^{1/2}\geqq\{\Sigma\,|\mathbf{V}_i|^\lambda\}^{1/\lambda} \tag{3}$$

for $\lambda \geqq 2$. Combining (2) and (3), gives

$$S \geqq 2^n \Sigma |\mathbf{V}_i|^\lambda \tag{4}$$

which generalizes (1).

REMARK. If in (2), we let $\lambda = 1$ and $2\mathbf{V}_1 = \mathbf{A}+\mathbf{B}$, $2\mathbf{V}_2 = \mathbf{B}+\mathbf{C}$, $2\mathbf{V}_3 = \mathbf{C}+\mathbf{A}$, we obtain

$$|\mathbf{A}|+|\mathbf{B}|+|\mathbf{C}|+|\mathbf{A}+\mathbf{B}+\mathbf{C}| \leqq 2\{(\mathbf{A}+\mathbf{B})^2+(\mathbf{B}+\mathbf{C})^2+(\mathbf{C}+\mathbf{A})^2\}^{1/2}$$

which can be considered complementary to that of Hlawka's inequality [1, p. 171]. There is equality here *iff* $\mathbf{A}+\mathbf{B}$, $\mathbf{B}+\mathbf{C}$, $\mathbf{C}+\mathbf{A}$ are mutually orthogonal or, equivalently, if $|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = |\mathbf{A}+\mathbf{B}+\mathbf{C}|$.

References

1. D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970.
2. E. F. Beckenbach, R. Bellman, Inequalities, Springer-Verlag, Heidelberg, 1965, p. 18.

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A THEOREM ON RECIPROCAL POLYNOMIALS WITH APPLICATIONS TO PERMUTATIONS AND COMPOSITIONS

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Abstract: A polynomial $f(x) = a_n x^n + \cdots + a_0$ is called reciprocal if $f(x) = x^n f(1/x)$; it is called unimodal reciprocal if in addition $a_i - a_{i-1} \geqq 0$ for $1 \leqq i \leqq n/2$. It is here proved that the product of two reciprocal, unimodal polynomials with nonnegative coefficients is also a reciprocal, unimodal polynomial with nonnegative coefficients. This result is applied to problems in compositions and permutations.

1. Reciprocal polynomials. In this paper we shall prove a very elementary theorem on reciprocal polynomials (with real coefficients) and from it we shall deduce several combinatorial theorems that at first glance appear troublesome.

DEFINITION 1. A polynomial $f(x) = a_n x^n + \cdots + a_0$ is *reciprocal* if $f(x) = x^n f(1/x)$ (i.e., $a_r = a_{n-r}$).

DEFINITION 2. A polynomial $f(x) = a_n x^n + \cdots + a_0$ is called unimodal reciprocal if it is reciprocal and $a_i - a_{i-1} \geqq 0$ for $1 \leqq i \leqq n/2$.

THEOREM 1. *If $f(x) = \Sigma_{j=0}^n a_j x^j$ and $g(x) = \Sigma_{j=0}^m b_j x^j$ are unimodal reciprocal polynomials with nonnegative coefficients, then the same is true of their product.*

Proof. If we let $f(x)g(x) = h(x) = \Sigma_{j=0}^{n+m} c_j x^j$, then

$$h(x) = f(x)g(x) = x^n f(1/x) x^m g(1/x) = x^{n+m} h(1/x).$$

Also since each $a_i \geqq 0$ and $b_i \geqq 0$, we see that

$$c_j = \sum_{\substack{r+s=j \\ r\geqq 0,\, s\geqq 0}} a_r b_s \geqq 0.$$

Next we define $a_r = 0$ for $r < 0$ and $r > n$; also $b_s = 0$ for $s < 0$ and $s > m$. Note that now $a_r - a_{r-1} \geqq 0$ for $-\infty < r \leqq n/2$ and $b_s - b_{s-1} \geqq 0$ for $-\infty < s \leqq m/2$. Hence

$$2(c_j - c_{j-1}) = \sum_{r=-\infty}^{\infty} a_r b_{j-r} + \sum_{r=-\infty}^{\infty} a_{n-r+1} b_{j-n+r-1} - \sum_{r=-\infty}^{\infty} a_{r-1} b_{j-r} - \sum_{r=-\infty}^{\infty} a_{n-r} b_{j-n+r-1}$$

$$= \sum_{r=-\infty}^{\infty} (a_r - a_{r-1})b_{j-r} + \sum_{r=-\infty}^{\infty} b_{j-n+r-1}(a_{n-r+1} - a_{n-r})$$

$$= \sum_{r=-\infty}^{\infty} (a_r - a_{r-1})(b_{j-r} - b_{j-n+r-1}) \quad (\text{since } a_r = a_{n-r})$$

$$= \sum_{r=0}^{n+1} (a_r - a_{r-1})(b_{j-r} - b_{j-n+r-1})$$

$$= \sum_{r=0}^{(n+1)/2} (a_r - a_{r-1})(b_{j-r} - b_{j-n+r-1})$$

$$+ \sum_{r=0}^{(n+1)/2} (a_{n+1-r} - a_{n-r})(b_{j-n-1+r} - b_{j-r})$$

(note that if $(n+1)/2$ is an integer $a_{(n+1)/2} = a_{(n-1)/2}$)

$$= 2 \sum_{r=0}^{(n+1)/2} (a_r - a_{r-1})(b_{j-r} - b_{j-n-1+r})$$

$$= 2 \sum_{r=0}^{n/2} (a_r - a_{r-1})(b_{j-r} - b_{j-n-1+r}).$$

Therefore

$$c_j - c_{j-1} = \sum_{0 \leqq r \leqq n/2} (a_r - a_{r-1})(b_{j-r} - b_{j-n-1+r}). \tag{1}$$

Now since $0 \leqq r \leqq n/2$, we see that $a_r - a_{r-1} \geqq 0$ by hypothesis. If $j - r \leqq m/2$, then since $n + 1 \geqq 2r$, we see that $m/2 \geqq j - r \geqq j - n - 1 + r$ and so $b_{j-r} - b_{j-n-1+r} \geqq 0$. If $j - r > m/2$, then for $0 \leqq j \leqq (m+n)/2$ we see that $m + n + 1 \geqq 2j$ and so $m/2 > m - j + r \geqq j - n - 1 + r$; therefore $b_{j-r} - b_{j-n-1+r} = b_{m-j+r} - b_{j-n-1+r} \geqq 0$. Hence, in any event, both factors in the sum appearing in equation (1) are nonnegative provided $0 \leqq j \leqq (m+n)/2$, i.e., $c_j - c_{j-1} \geqq 0$ for $1 \leqq j \leqq (m+n)/2$. Thus $h(x) = f(x)g(x)$ is a unimodal reciprocal polynomial with nonnegative coefficients, and this proves Theorem 1.

2. Application to compositions. In [**8**], Z. Star develops an asymptotic series for $c(n,s,r)$ the number of compositions (i.e., partitions with order taken into account) of n into s parts with no part exceeding r. He begins by proving the following result:

THEOREM 2. *$c(n,s,r)$ is a nondecreasing function of n for $0 \leqq n \leqq \frac{1}{2}sr(r+1)$ and $c(n,s,r) = c(sr + s - n, s, r)$.*

Proof: Since [**6**, p. 151]

$$h(x) \equiv \sum_{n=0}^{sr} c(n,s,r)x^n = (x + x^2 + \cdots + x^r)^s = x^s(1 + x + \cdots + x^{r-1})^s,$$

we see that $x^{-s}h(x)$ is the product of s unimodal reciprocal polynomials with nonnegative coefficients. Therefore by Theorem 1, so is $x^{-s}h(x)$, and this implies Theorem 2.

3. Application to permutations. Let us consider permutations of a finite ordered multiset

$$M \equiv \{\underbrace{\xi_1, \xi_1, \cdots, \xi_1}_{a_1 \text{ times}}, \cdots, \underbrace{\xi_r, \cdots, \xi_r}_{a_r \text{ times}}\} \equiv \{\xi_1^{a_1} \cdots \xi_r^{a_r}\},$$

where $\xi_1 < \xi_2 < \cdots < \xi_r$. A multiset is like a set, except that it can have repetitions of identical elements [**5**; p. 22].

Suppose M' is a permutation of M, say

$$M' = \{x_1, x_2, \cdots, x_R\}, \quad \text{where} \quad R = a_1 + a_2 + \cdots + a_r.$$

The *inversion number* $S(M')$ is defined as the number of pairs (j, k) such that $j < k$ and $x_j > x_k$. The *index* $T(M')$ is defined as the sum of all integers j such that $1 \leqq j \leqq R - 1$ and $x_j > x_{j+1}$.

We let $\nu_M(n)$ denote the number of permutations of M with inversion number n, and we let $\xi_M(n)$ denote the number of permutations of M with index n. P.A. MacMahon [7] (see also [**1**; p. 52], [**3**]) proved that

$$\nu_M(n) = \xi_M(n)$$

for all M and all n.

THEOREM 3. *$\nu_M(n)$ is a nondecreasing function for*

$$0 \leqq n \leqq \tfrac{1}{2}\sigma_2(a_1, a_2, \cdots, a_r),$$

where $\sigma_2(a_1, \cdots, a_r)$ is the second elementary symmetric function of the a's.

Proof. MacMahon [**7**] showed that

$$\sum_{n \geqq 0} \nu_M(n) x^n = \frac{(x)_R}{(x)_{a_1}(x)_{a_2} \cdots (x)_{a_r}},$$

where $(x)_a = (1-x)(1-x^2)\cdots(1-x^a)$. Now

$$\frac{(x)_R}{(x)_{a_1}(x)_{a_2}\cdots(x)_{a_r}} = \frac{(x)_R}{(x)_{a_1}(x)_{R-a_1}} \cdot \frac{(x)_{R-a_1}}{(x)_{a_2}(x)_{R-a_1-a_2}} \cdots \frac{(x)_{R-a_1-a_2-\cdots-a_{r-1}}}{(x)_{a_r}(x)_{R-a_1-\cdots-a_r}}$$

$$= \begin{bmatrix} R \\ a_1 \end{bmatrix} \begin{bmatrix} R-a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} R-a_1-a_2 \\ a_3 \end{bmatrix} \cdots \begin{bmatrix} R-a_1-a_2-\cdots-a_{r-1} \\ a_r \end{bmatrix},$$

where $\begin{bmatrix} R \\ a \end{bmatrix}$ is the Gaussian polynomial defined to be $(x)_R (x)_a^{-1} (x)_{R-a}^{-1}$. E. B. Elliot [**2**; §§129, 130] (cf. [**4**] and [**9**; p. 85]) has shown that the Gaussian polynomials are unimodal reciprocal polynomials with nonnegative coefficients. Since $\Sigma_{n \geqq 0} \nu_M(n) x^n$ is a product of r Gaussian polynomials, Theorem 1 implies that $\Sigma_{n \geqq 0} \nu_M(n) x^n$ is also a unimodal reciprocal polynomial. Theorem 3 then follows directly from the fact that the degree of

$$\sum_{n \geqq 0} \nu_M(n) x^n \text{ is}$$

$$\binom{a_1 + a_2 + \cdots + a_r + 1}{2} - \binom{a_1 + 1}{2} - \binom{a_1 + 1}{2} - \cdots - \binom{a_r + 1}{2} = \sigma_2(a_1, a_2, \cdots, a_r).$$

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References

1. P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Math., No. 85, Springer, Berlin, 1969.

2. E. B. Elliot, An Introduction to the Algebra of Quantics, 2nd ed., Cambridge University Press, Cambridge, 1913 (reprinted: Chelsea, New York, 1964).

3. D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc., 19(1968) 236–240.

4. J. W. B. Hughes, Lie algebraic proofs of some theorems on partitions, (to appear).

5. D. E. Knuth, The Art of Computer Programming, Vol. 3, Sorting and Searching, Addison-Wesley, Reading, Mass., 1973.

6. P. A. MacMahon, Combinatory Analysis, Vol. 1, Cambridge University Press, Cambridge, 1915 (reprinted: Chelsea, New York, 1960).

7. ———, Two applications of a general theorem in combinatory analysis, Proc. London Math. Soc. (2), 15(1916) 314–321.

8. Z. Star, An asymptotic formula in the theory of compositions, (to appear).

9. K. B. Stolarsky, Higher partition functions and their relation to finitely generated nilpotent groups, Ph.D. thesis, University of Wisconsin, 1968.

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THE NIKODÝM AND VITALI-HAHN-SAKS THEOREMS FOR ALGEBRAS

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The theorems of the title are two fundamental theorems of measure theory. As discussed in [**2**], these theorems are stated for set functions defined on σ-algebras. There seem to be no examples given in the literature which show that the theorems are false for algebras, although this is part of the folklore of the subject. In this note we present a simple example showing that both theorems are false for algebras. For reference purposes we state the two theorems ([**2**], III.7.2, III.7.4).

Let Σ be a σ-algebra of subsets of a set S and let $\mathbf{R}$ denote the reals.

THEOREM 1. (Nikodým) *If $\mu_n : \Sigma \to \mathbf{R}$ is countably additive for each n and* $\lim \mu_n(E) = \mu(E)$ *exists for each $E \in \Sigma$, then*

(i) μ is countably additive, and

(ii) the family $\{\mu_n : n \geqq 1\}$ is uniformly countably additive.

THEOREM 2. (Vitali-Hahn-Saks) *If in addition to the hypothesis of Theorem 1, λ is a positive measure on Σ and each μ_n is absolutely continuous ($\varepsilon - \delta$ definition) with respect to λ then*

(I) *μ is absolutely continuous with respect to λ, and*

(II) *the family $\{\mu_n : n \geqq 1\}$ is uniformly absolutely continuous with respect to λ.*

For the example, let $\mathscr{A}$ be the algebra of subsets of $[0,1) = S$ generated by intervals of the form $[a,b)$, where $0 \leqq a < b \leqq 1$. Thus $\mathscr{A}$ consists of finite disjoint unions of intervals of the form $[a,b)$ plus the empty set. Let λ denote Lebesgue measure on S and for each $n \geqq 1$ set $\mu_n(A) = 2^n \lambda([1 - 1/2^n, 1) \cap A)$ for $A \in \mathscr{A}$. Then each μ_n is countably additive on $\mathscr{A}$ (and has a countably additive extension to the Borel sets of S). If $b < 1$, $\mu_n([a,b)) = 0$ for large n and $\mu_n([c,1)) = 1$ for large n so that $\lim_n \mu_n(A) = \mu(A)$ exists for each $A \in \mathscr{A}$. The limit function μ is a finitely additive set function on $\mathscr{A}$ which takes on only the values 0 and 1; μ is such that $\mu(A) = 1$ if $[1 - \delta, 1) \subseteq A$ for some $\delta > 0$ and $\mu(A) = 0$ otherwise (see [**3**], Theorem 2.2). In E. Hewitt's notation $\mu = \lambda_1$ and μ is a purely finitely additive set function ([**3**], Th. 2.2). Thus μ certainly does not satisfy condition (i) of Theorem 1; actually the example shows the limit function μ can be as "far" from being countably additive as possible.

To show the failure of condition (ii) of Theorem 1, let $E_k = [1 - 1/2^k, 1 - 1/2^{k+1})(k \geqq 1)$. Since $\bigcup_{k=j}^{\infty} E_k \in \mathscr{A}$ for any j and $\Sigma_{k=n}^{\infty} \mu_n(E_k) = 1$ for each n, the sequence $\{\mu_n : n \geqq 1\}$ is not uniformly countably additive on $\mathscr{A}$.

It is clear that each μ_n is absolutely continuous ($\varepsilon - \delta$ definition) with respect to λ, but certainly μ is not absolutely continuous with respect to λ. Hence condition (I) of Theorem 2 fails. Since $\mu_n([1 - 1/2^n, 1)) = 1$ for each n, the sequence $\{\mu_n : n \geqq 1\}$ is not uniformly absolutely continuous with respect to λ, and condition (II) of Theorem 2 does not hold.

There are also versions of the Nikodým and Vitali-Hahn-Saks Theorems for strongly bounded set functions defined on σ-algebras ([**1**], Cor. 1.2 and Theorem 2 and Theorem 3). In these versions the $\{\mu_n\}$ are assumed to be strongly bounded, the conclusions (I) and (II) remain unchanged, and conclusion (i) and (ii) become

(i)′ μ is strongly bounded,

(ii)′ $\lim_m \sum_{k=m}^{\infty} \mu_j(A_k) = 0$ uniformly for $j \geqq 1$ and $\{A_j\} \subseteq \Sigma$ disjoint. The example above shows that these results are false for strongly bounded set functions defined on algebras, although conclusion (i)′ does hold since μ is a bounded finitely additive set function.

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References

1. J. K. Brooks and R. S. Jewett, On finitely additive vector measures, Proc. Nat. Acad. Sci. USA, 67 (1970) 1294–1298.
2. N. Dunford and J. Schwartz, Linear Operators, Interscience, New York, 1958.
3. E. Hewitt, A problem concerning finitely additive measures, Mat. Tidsskrift B, (1951) 81–94.

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ESSENTIAL COMPONENTS IN DISCRETE GROUPS

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Let G be a discrete group, and let A and B be subsets of G. The *product* of A and B is the set $AB = \{ab \mid a \in A \text{ and } b \in B\}$. Let $|S|$ denote the number of elements in the set S. A finite subset B of G is an *essential component* if $|AB| > |A|$ for all finite nonempty subsets A of G with $A \neq G$. This is the analog for groups of the usual idea of essential component in additive number theory [1,2]. In this note we find the essential components of all groups.

Clearly, if B is an essential component and $g \in G$, then gB is also an essential component, since $|A(gB)| = |(Ag)B| > |Ag| = |A|$. It follows that we can suppose with no loss of generality that the identity e of G is in B.

Theorem. *Let B be a finite subset of the group G with $e \in B$, and let B^* be the set of products of positive powers of elements of B. Then B is an essential component if and only if either B^* is infinite or $B^* = G$.*

Proof. If B is not an essential component, then $|AB| = |A|$ for some finite nonempty subset A of G with $A \neq G$. Since $e \in B$, it follows that $A \subset AB$, and so $A = AB$. If $a \in A$ and $b_1, b_2, \cdots, b_n \in B$, then $ab_1 \in AB = A$. Therefore, $a(b_1b_2) = (ab_1)b_2 \in AB = A$. Continuing inductively, we obtain $ab_1b_2 \cdots b_n \in A$. Therefore, $A = AB^*$. Since $A \neq G$, then $B^* \neq G$, and since A is finite, then B^* is finite.

Conversely, suppose that B is an essential component. Since $B^*B = B^*$, it follows that either B^* is infinite or $B^* = G$.

Corollary. *Let G be an abelian group, and let B be a finite subset of G with $e \in B$. Then B is an essential component if and only if either B contains an element of infinite order or G is finite and B generates G.*

Remark. The theorem and proof above are still true if G is not a group but only a cancellation semigroup with identity.

References

1. H. Halberstam and K. F. Roth, Sequences, Oxford University Press, 1966.
2. H. B. Mann, Addition Theorems, Wiley, New York, 1965.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

GRAPH IMBEDDING PROBLEMS

SETH R. ALPERT AND JONATHAN L. GROSS

Drawing a graph on a surface means marking a separate dot for each vertex of the graph and sketching an arc between each pair of dots representing adjacent vertices, making sure that no arc crosses (or touches) itself or passes through any dot. A graph **imbedding** is a topological formalization of a graph drawing with the additional property that no arc crosses or touches another. This paper proposes four unsolved graph imbedding problems whose solutions might be obtainable by elementary methods.

The classification theorem for surfaces (see, for example, W. S. Massey [5, Chapter 1]) states that any closed orientable surface is homeomorphic to a sphere with g handles, for some unique nonnegative integer g, and that any closed nonorientable surface is homeomorphic to a sphere with q crosscaps, for some unique positive integer q. For a given surface, the number of handles or crosscaps is called its **genus** or **nonorientable genus**.

In what follows, a **graph** is connected and has no self-adjacencies or multiple edges. The **complete graph** K_p has p vertices and an edge between each pair of vertices.

The most common imbedding problem is to find a **minimal** imbedding surface for a given graph G, that is, an imbedding surface for G of either the minimum genus, $g(G)$, or the minimum nonorientable genus, $q(G)$. The parameters $g(G)$ and $q(G)$ are called the **genus**, and **nonorientable genus of the graph** G, respectively. If $g(G)=0$, then G is said to be **planar** and by convention we define $q(G)=0$, even though a surface with zero crosscaps is orientable.

An imbedding of a graph G in a surface M subdivides M into pieces called **faces**. Suppose there are V vertices, E edges, and F faces and that each face is simply connected. Then the number $V-E+F$ is called the **Euler characteristic** of the surface M. If the surface M is orientable of genus g, then

$$V-E+F=2-2g.$$

If M is nonorientable of genus q, then

$$V-E+F=2-q.$$

Both of these equations are called **Euler equations**.

Any minimal imbedding has only simply connected faces and so satisfies the Euler equations. From this and the fact that $2E \geqq 3F$ (because every edge is on the boundary of exactly two faces and every face is bounded by at least three edges) one may easily derive the following **Euler lower bounds**:

$$g(G) \geqq \lceil E/6 - V/2 + 1\rceil$$
$$q(G) \geqq \lceil E/3 - V + 2\rceil,$$

where $\lceil x\rceil$ denotes the **ceiling** of the real number x, that is, the least integer not less than x.

An important consequence of the Euler formula is that if E is divisible by 3, then there may be a

triangular imbedding, that is, one in which every face is bounded by exactly three edges. The Euler lower bounds imply that triangular imbeddings must be minimal.

The usual method of genus computation is to try to prove that the Euler lower bounds are exact by explicitly constructing minimal imbeddings. Undoubtedly the most famous result of this type is the computation by G. Ringel, J. W. T. Youngs, C. M. Terry, and L. R. Welch of the genus and, by Ringel of the nonorientable genus of complete graphs, thereby solving the Heawood map-coloring problem. The construction, now described fully in the new book by Ringel [**6**], uses the current graph method of W. Gustin as augmented by Youngs. A unification and extension of current graph theory is given by J. L. Gross and S. R. Alpert [**3**], but the method is somewhat too complicated for description here.

The interested reader may find an introduction to graph imbeddings in F. Harary [**4**, Chapter 11]. A more advanced treatment is provided by A. T. White [**7**].

The large number of classes of graphs for which the Euler lower bounds are exact is rather surprising. One wonders what properties of a graph are sufficient for this to be true. Our first two problems are closely related to this question.

PROBLEM 1. For each nonnegative integer n, let F_n denote the set of all planar graphs with exactly 12 vertices of degree five and n vertices of degree six. Decide for which numbers n the set F_n is empty.

Problem 1 was formulated by R. H. Fox, who told Gross he had proved that F_1 is empty. The nonemptiness of F_n, for small even n, is easily established with pencil and paper. For example, the imbedding of the icosahedral graph illustrated in Fig. 1 establishes that F_0 is nonempty.

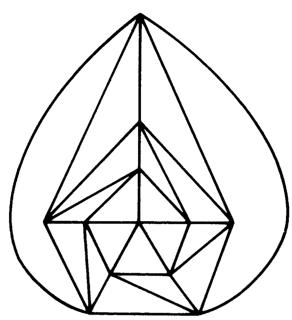

FIG. 1
The icosahedral graph.

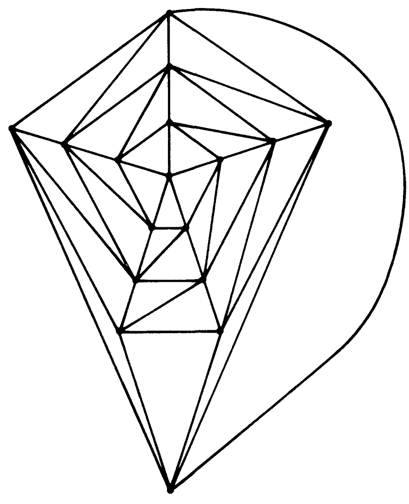

FIG. 2
A graph in F_5.

Ringel [**6**, p. 179] ascribes to Fox without specific publication reference the conjecture that whenever n is odd, the set F_n is empty. Fig. 2 proves that F_5 is nonempty.

A graph G is called **locally Hamiltonian** if for every vertex v there is a simple circuit (i.e., a circuit without self-crossings) passing through the set of all vertices adjacent to v, but through no other vertices of G. In order to have a triangular imbedding a graph must be locally Hamiltonian, for in such an imbedding each vertex would appear as the hub of a wheel whose rim is a simple circuit through all of its neighbors.

PROBLEM 2. Let c be an integer less than or equal to 2. Does a locally Hamiltonian graph with V vertices and E edges such that $V - E/3 = c$ necessarily have a triangular imbedding in some surface (possibly nonorientable) of Euler characteristic c? If not, what is the minimum number of vertices in a locally Hamiltonian graph for which $V - E/3 = c$ that cannot triangulate any surface of characteristic c?

It is possible to show with pencil and paper that each of the eleven graphs with 8 vertices and 24 edges has a triangular imbedding in either the orientable or nonorientable surface of characteristic 0. This shows that if G is a locally Hamiltonian graph with V vertices and $3V$ edges which cannot be imbedded in any surface of Euler characteristic 0, then the number V must be at least 9.

If a graph T is isomorphic to a subgraph of a graph G and also to a subgraph of a graph H, the graph $G \vee_T H$ obtained by identifying the copy of T in G with the copy of T in H is called an **amalgamation** of G and H along T. Alpert [1] has studied the genus of amalgamations of complete graphs along a complete graph, following the suggestion of Harary.

PROBLEM 3. Let m, n, p and q be positive integers such that $p \leqq q \leqq \text{minimum}(m, n)$. Is it necessarily true that $g(K_m \vee_{K_q} K_n) \leqq g(K_m \vee_{K_p} K_n)$?

The inequality in Problem 3 is satisfied in all cases where the genus of the amalgamations is known.

The fourth and last problem proposed here is probably the hardest. J. Edmonds [2] has obtained an elegant algorithm (detailed by Youngs [8] and by White [7]) for finding the genus of any graph, but the computation time is large, and no one knows whether it can be reduced sufficiently even to permit application to medium-sized graphs. For large graphs, perhaps the best general hope is a good approximation method. Since the genus of a V-vertex graph might have V^2 as its order of magnitude, knowing the genus within error tolerance V is probably an ambitious goal, suggesting the following problem.

PROBLEM 4. Find a class C of graphs of known genus and a constant k such that for any V-vertex graph G, there is a V-vertex graph in C differing from G by at most kV edge modifications (i.e., adding an edge or deleting an edge).

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References

1. S. R. Alpert, The genera of amalgamations of graphs, Trans. Amer. Math. Soc., 178 (1973) 1–39.
2. J. Edmonds, A combinatorial representation for polyhedral surfaces, Notices Amer. Math. Soc., 7 (1960) 646. Abstract #572–1.
3. J. L. Gross and S. R. Alpert, The topological theory of current graphs, J. Combinatorial Theory Ser. B., 17 (1974) 218–233.
4. F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.
5. W. S. Massey, Algebraic Topology: An Introduction, Harcourt, Brace, and World, New York, 1967.
6. G. Ringel, Map Color Theorem, Springer-Verlag, Berlin, 1974.
7. A. T. White, Graphs, Groups, and Surfaces, North-Holland, Amsterdam, 1973.
8. J. W. T. Youngs, Minimal imbeddings and the genus of a graph, J. Math. Mech., 12 (1963) 303–315. MR 26 #3043:

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CLASSROOM NOTES

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A CHARACTERIZATION OF THE DETERMINANT

DESMOND FEARNLEY-SANDER

The standard approach to the study of determinants is to prove the existence, for any n-dimensional vector space $\mathscr{V}$ over the field $\mathbf{K}$, of a non-zero alternating multilinear form d: $\mathscr{V}^n \to \mathbf{K}$, and then to define the determinant $\det T$ of a linear map T: $\mathscr{V} \to \mathscr{V}$ by the requirement that

$$\det T \cdot d(x_1, x_2, \cdots, x_n) = d(Tx_1, Tx_2, \cdots, Tx_n)$$

for all $(x_1, x_2, \cdots, x_n) \in \mathscr{V}^n$; these conditions determine d modulo a scalar multiple and hence completely determine the function det: $\mathscr{L}(\mathscr{V}) \to \mathbf{K}$, where $\mathscr{L}(\mathscr{V})$ is the algebra of all vector space endomorphisms of $\mathscr{V}$. At an elementary level one is chiefly interested in the function det defined on linear maps, rather than in the related function d defined on n-tuples. It is natural to ask for a characterization of det which is intrinsic in the sense that the characterizing properties refer to the action of det on linear maps. In this note we give an elementary proof of a result of this kind which does not seem to be widely known.

THEOREM. *Let $\mathscr{V}$ be an n-dimensional vector space over the field* $\mathbf{K}$.

(i) *If either* $\mathbf{K}$ *is the real field* $\mathbf{R}$ *and n is odd or* $\mathbf{K}$ *is the complex field* $\mathbf{C}$, *there exists a unique function* Δ: $\mathscr{L}(\mathscr{V}) \to \mathbf{K}$ *such that*

(1) *for T and S in $\mathscr{L}(\mathscr{V})$*

$$\Delta(TS) = \Delta(T)\,\Delta(S), \quad \textit{and}$$

(2) *for α in* $\mathbf{K}$, $\Delta(\alpha I) = \alpha^n$, *where I: $\mathscr{V} \to \mathscr{V}$ is the identity map.*

(ii) *If* $\mathbf{K} = \mathbf{R}$ *and n is even, there exists a unique function* Δ: $\mathscr{L}(\mathscr{V}) \to \mathbf{K}$ *satisfying* (1), (2) *and also*

(3) *for some reflection E in $\mathscr{L}(\mathscr{V})$, $\Delta(E) = -1$.*

We are not concerned here with existence—clearly the determinant function, defined as above, has these properties.

Observe that condition (1) above is not enough to characterize the determinant function; for example, the absolute value of det satisfies (1) (and (2) as well if $\mathscr{V}$ is real and of even dimension).

To prove the theorem, we begin by considering those linear maps E: $\mathscr{V} \to \mathscr{V}$ which leave fixed all the points of some $(n-1)$-dimensional subspace of $\mathscr{V}$—we shall call them *elementary maps.* A simple analysis shows that every elementary map is either a *shear* or an *axial stretch*; by a *shear* we mean a linear map which carries some ordered basis $(x_1, x_2, \cdots, x_n)$ for $\mathscr{V}$ onto the ordered basis $(x_1 + x_2, x_2, \cdots, x_n)$, and by an *axial stretch with stretch factor* $\alpha \in \mathbf{K}$ we mean a linear map which carries some ordered basis $(x_1, x_2, \cdots, x_n)$ for $\mathscr{V}$ onto the n-tuple $(\alpha x_1, x_2, \cdots, x_n)$. An axial stretch with stretch factor -1 is called a *reflection.* Note that every shear has determinant 1 and every axial stretch with stretch factor α has determinant α.

Every linear map T: $\mathscr{V} \to \mathscr{V}$ is a product of elementary maps. To see this, choose an ordered basis $Y = (y_1, y_2, \cdots, y_n)$ for $\mathscr{V}$ such that $(y_1, y_2, \cdots, y_p)$ is a basis for $T(\mathscr{V})$, and let $X = (x_1, x, \cdots, x_n)$

be an ordered basis for $\mathscr{V}$ which satisfies

$$Tx_i = \begin{cases} y_i & \text{for} \quad i = 1, 2, \cdots, \rho, \\ 0 & \text{for} \quad i = \rho + 1, \rho + 2, \cdots, n. \end{cases}$$

Using the well-known Steinitz exchange process (the key technique in the standard proof that all bases for a finite-dimensional vector space have the same number of elements) it is possible to go from X to Y by a finite sequence of steps (called *elementary operations*) each of which involves changing an ordered basis $Z = (z_1, z_2, \cdots, z_n)$ to a new ordered basis Z' obtained from Z by replacing one of the vectors z_i either by αz_i for some non-zero $\alpha \in \mathbf{K}$ or by $z_i + \lambda z_j$, for some j and some $\lambda \in \mathbf{K}$. Every such elementary operation determines a linear map $E\colon \mathscr{V} \to \mathscr{V}$ (namely the one which carries Z onto Z'), and E is easily shown to be elementary. Thus there is a finite sequence of elementary maps $E_1, E_2, \cdots, E_k$ whose product $E_k E_{k-1} \cdots E_1$ carries X onto Y. For $i = 1, 2, \cdots, \nu = n - \rho$ define $E'_i\colon \mathscr{V} \to \mathscr{V}$ to be the elementary map which carries Y onto the n-tuple $(y_1, y_2, \cdots, y_{\rho+i-1}, 0, y_{\rho+i+1}, \cdots, y_n)$. Clearly the product $E'_\nu E'_{\nu-1} \cdots E'_1$ carries Y onto the n-tuple $Y' = (y_1, y_2, \cdots, y_\rho, 0, 0, \cdots, 0)$. We conclude that T is identical with $E'_\nu E'_{\nu-1} \cdots E'_1 E_k E_{k-1} \cdots E_1$ since both these linear maps carry X onto Y'.

Since it satisfies (1), our function Δ is determined by what it does to elementary maps. If we can show that Δ agrees with det on elementary maps, we can conclude that the two functions agree everywhere and we shall be finished.

Let E be a shear. Then E^2 is also a shear; indeed if E takes the basis $(x_1, x_2, \cdots, x_n)$ onto $(x_1 + x_2, x_2, \cdots, x_n)$, then E^2 takes the basis $(x_1, 2x_2, x_3, \cdots, x_n)$ onto $(x_1 + 2x_2, 2x_2, x_3, \cdots, x_n)$. One easily sees that any two shears are similar to one another; in particular, E^2 is similar to E and so there exists an invertible $\mathbf{C} \in \mathscr{L}(\mathscr{V})$ such that

$$E = \mathbf{C}E\mathbf{C}^{-1}E^{-1}.$$

Using (1) we conclude that

$$\Delta(E) = \Delta(C)\Delta(E)\Delta(C^{-1})\Delta(E^{-1}) = (\Delta(I))^2 = 1.$$

Thus Δ agrees with det on shears.

Let E be the axial stretch which carries the basis $(x_1, x_2, \cdots, x_n)$ for $\mathscr{V}$ onto $(\alpha x_1, x_2, \cdots, x_n)$. For $i = 1, 2, \cdots, n$, let E_i be the axial stretch which carries $(x_1, x_2, \cdots, x_n)$ onto $(x_1, x_2, \cdots, x_{i-1}, \alpha x_i, x_{i+1}, \cdots, x_n)$; in particular, $E_1 = E$. It is easy to see that any two axial stretches with the same stretch factor are similar; hence each E_i is similar to E and so for every i

$$\Delta(E_i) = \Delta(E).$$

Obviously, $\alpha I = E_1 E_2 \cdots E_n$, and hence, using (2), we have

$$[\Delta(E)]^n = \Delta(E_1)\Delta(E_2) \cdots \Delta(E_n) = \Delta(\alpha I) = \alpha^n.$$

At this point, we must consider the different cases separately.

In the case where $\mathbf{K} = \mathbf{R}$ and n is odd, we can use the fact that any real number has a unique nth root (in $\mathbf{R}$) to conclude that

$$\det E = \alpha,$$

as required.

For each $\beta \in \mathbf{K}$, let E_β be the axial stretch which carries $(x_1, x_2, \cdots, x_n)$ onto $(\beta x_1, x_2, \cdots, x_n)$; in particular, $E_\alpha = E$.

In the case where $\mathbf{K} = \mathbf{C}$, let β be any nth root of α (in $\mathbf{C}$); then $E = E^n_\beta$ and so

$$\Delta(E) = [\Delta(E_\beta)]^n = \beta^n = \alpha. \qquad \text{as required.}$$

Finally, consider the case where $\mathbf{K}=\mathbf{R}$ and n is even. Since every positive real number has precisely two nth roots (in $\mathbf{R}$) we must have

$$\Delta(E)=\alpha \quad \text{or} \quad \Delta(E)=-\alpha.$$

If $\alpha \geqq 0$, then $\alpha=\gamma^2$ for some $\gamma \in \mathbf{R}$ and so $E=E_\gamma^2$ and $\Delta(E)=[\Delta(E_\gamma)]^2 \geqq 0$; hence $\Delta(E)=\alpha$ as required. If $\alpha<0$, then $E=E_{-\alpha}E_{-1}$; using the hypothesis (3), which says that

$$\Delta(E_{-1})=-1,$$

we conclude that

$$\Delta(E)=\Delta(E_{-\alpha})\Delta(E_{-1})=\alpha$$

as required. This completes the proof.

The determinant of an $n\times n$ matrix over $\mathbf{K}$ is, of course, defined to be the determinant of the corresponding linear map $T\colon \mathbf{K}^n \to \mathbf{K}^n$ (namely multiplication by that matrix); using this our theorem translates in the obvious way into a theorem about matrices.

While the theorem is interesting for purely aesthetic reasons it is also useful for proving certain properties of operators and matrices. As an example, consider the fact that the determinant of a matrix A equals the determinant of its transpose A^t; this is an immediate corollary of the theorem, since the function Δ defined on $n\times n$ matrices by

$$\Delta(A)=\det A^t$$

obviously has the properties (1), (2) and (3) and so coincides with det.

We conclude with remarks on the case of a general field $\mathbf{K}$ (which is elegantly treated by Artin [3]). A careful analysis of the Steinitz exchange process (or equivalently the matrix argument in [1], page 158) shows that the shears alone generate the special linear group $\mathcal{SL}(\mathcal{V})$; also, unless $n=2$ and $\mathbf{K}$ is the two element field, every shear is a commutator. Hence, with that single exception, $\mathcal{SL}(\mathcal{V})$ (being the kernel of det: $\mathcal{GL}(\mathcal{V})\to\mathbf{K}^*$) is the commutator subgroup of the general linear group $\mathcal{GL}(\mathcal{V})$. Any group homomorphism $\Delta\colon \mathcal{GL}(\mathcal{V})\to\mathbf{K}^*$ agrees with det on $\mathcal{SL}(\mathcal{V})$ and so $\Delta=f\circ\det$ where f is some endomorphism of $\mathbf{K}^*$, the multiplicative group of $\mathbf{K}$.
(In the more general case of a division ring $\mathbf{K}$, Dieudonné [2] actually defines a determinant function as a homomorphism from $\mathcal{GL}(\mathcal{V})$ to $\mathbf{K}^*$ having the commutator subgroup as its kernel.) If Δ also satisfies (2), we must have $f(\alpha^n)=\alpha^n$ for every $\alpha\in\mathbf{K}^*$ and so the map $\alpha\to f(\alpha)\alpha^{-1}$ is a homomorphism from $\mathbf{K}^*$ to the group of nth roots of 1 in $\mathbf{K}^*$ which sends all nth powers to 1. If every element of $\mathbf{K}$ is an nth power (for example, if $\mathbf{K}$ is algebraically closed) or if 1 itself is the only nth root of 1 in $\mathbf{K}$ (for example, if $\mathbf{K}=\mathbf{R}$ and n is odd), then $f(\alpha)=\alpha$ for all α and so Δ coincides with det on $\mathcal{GL}(\mathcal{V})$ and hence on $\mathcal{L}(\mathcal{V})$. In the case where $\mathbf{K}=\mathbf{R}$ and n is even, there are two possibilities for f and (3) serves to eliminate the non-trivial one.

The author wishes to thank a referee and P. Forrest for very helpful comments, and especially for the remarks in the previous paragraph.

References

1. J. Rotman, Theory of Groups, Allyn and Bacon, Boston, 1965.
2. J. Dieudonné, Les déterminants sur un corps non commutatif, Bull. Soc. Math. France, 71(1943) 27–45.
3. E. Artin, Geometric Algebra, Interscience, New York, 1957.

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THE KURATOWSKI CLOSURE-COMPLEMENT PROBLEM

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Let X be an arbitrary topological space. Denote the closure of a subset A by A^- and the complement by A'. An intriguing result in point-set topology is that at most 14 different sets can be derived from A by repeated applications of closure and complementation. This theorem was proved by Kuratowski [2], and popularized as an exercise in Kelley's *General Topology* [1, p. 57].

An *illustration* (X, A) of the closure-complement problem is a topological space X and a subset A which actually generates 14 different sets (including $A = A''$). Let R denote the reals with the usual topology and Q the set of rationals. It is known that R admits an illustration (R, A), e.g., $A = [0,1) \cup (1,2] \cup \{3\} \cup [4,5] \cup (Q \cap (6,7))$.

A natural question is whether an illustration (X, A) exists with X a finite topological space. If such an illustration exists, what is the minimal cardinality of X? This note answers these questions.

Figure 1 shows the 14 possible sets, plus X and the empty set $\varnothing$ [2, p. 186]. The edges denote set-theoretic inclusion. Note (X, A) is an illustration if and only if (X, A') is an illustration. If $A'^{-'} = \varnothing$, then $A'^{-'-} = A'^{-'}$ and (X, A) is not an illustration. Therefore $A'^{-'} \neq \varnothing$ and also $A'^{-} \neq X$. Similarly $A^{-'} \neq \varnothing$ and $A^{-} \neq X$. Therefore all of the edges in the figure represent proper inclusions. Hence $|X| \geqq 6$ where $|X|$ is the cardinality of X.

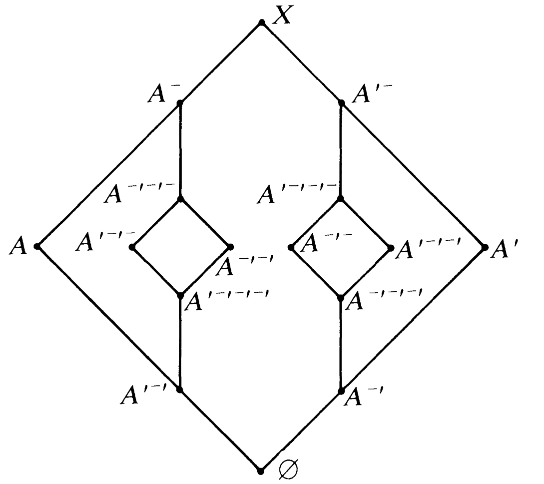

FIG. 1

Suppose $X = \{1,2,3,4,5,6\}$. If X admits an illustration (X, A), then it may be assumed that $A'^{-'} = \{1\}$, $A'^{-'-'-'} = \{1,2\}$, $A'^{-'-} = \{1,2,3\}$, $A^{-'-'-} = \{1,2,3,4\}$, and $A^- = \{1,2,3,4,5\}$. With the exception of A and A' this determines all of the sets in Figure 1. In particular $A'^- = \{2,3,4,5,6\}$. It is easy to check that $\{4\}$ is an open set. Now $4 \in A^-$, so $4 \in A$; similarly $4 \in A'$ so $4 \in A'^-$. This is impossible. So if (X, A) is an illustration, $|X| > 6$. This leads to the following result.

PROPOSITION: *The smallest cardinality of a topological space X admitting an illustration (X, A) of the closure-complement problem is seven.*

Sketch of proof. Let $X = \{1,2,3,4,5,6,7\}$. Form the base $\mathfrak{B} = \{\{1\}, \{1,2\}, \{4,5\}, \{6,7\}, \{7\}, X\}$. Then any of the following four subsets (or their complements) can serve as an illustration when X has topology generated by $\mathscr{B}$: $A_1 = \{1,3,4,6\}$, $A_2 = \{1,3,5,6\}$, $A_3 = \{1,4,6\}$, $A_4 = \{1,5,6\}$.

NOTE: If $X = \{1,2,3,4,5,6,7\}$ and (X, A) is an illustration, then (up to permutation of X) all of the derived sets in Figure 1 (except A and A') are unique. The base $\mathcal{B}$ generates the smallest topology on X containing these derived sets. Moreover, A_1, A_2, A_3, and A_4 are the only illustrations compatible with the figure and $\mathcal{B}$. In the $\mathcal{B}$-topology the group of homeomorphisms from X to X is $Z_2 \times Z_2$, generated by permutations (45) and (17) (26). The illustrations are related by homeomorphisms: (45) interchanges A_1 with A_2, and A_3 with A_4; (17) (26) interchanges A_1 with A'_4, and A_2 with A'_3.

The proposition has been independently obtained by H. H. Herda and R. Metzler (*Closure and interior in finite topological spaces*, Colloquium Math., 15 (1966) 211–216.)

References

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, N.J., 1955.
2. C. Kuratowski, Sur l'opération $\bar{A}$ de l'Analysis Situs, Fund. Math., 3 (1922) 182–199.

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ON THE EXPONENTIAL FUNCTION

Z. A. MELZAK

1. In this note we outline a development of the exponential function which uses only some simple mathematical facts (concerning the elements of limits and of continuity) and the arithmetic mean-geometric mean inequality. The latter is used in its simplest form: if $x_1, \cdots, x_N$ are positive, then

$$G(x_1, \cdots, x_N) \leqq A(x_1, \cdots, x_N). \tag{1}$$

Our plan is as follows. First, we define the harmonic mean $H(x_1, \cdots, x_N)$ and obtain very simply the full harmonic-geometric-arithmetic inequality. With this, we show that for $0 \leqq x \leqq 1$ the sequence of polynomials $\{(1 + x/n)^n\}$ is monotone nondecreasing while the sequence $\{(1 + x/n)^{n+1}\}$ is monotone nonincreasing. Also, the second sequence majorizes the first one term-by-term. Next, we conclude the existence of the limit

$$\phi(x) = \lim_{n \to \infty} (1 + x/n)^n \tag{2}$$

together with the crucial two-sided bracketing:

$$(1 + x/n)^n \leqq \phi(x) \leqq (1 + x/n)^{n+1}, \qquad 0 \leqq x \leqq 1. \tag{3}$$

Now, using (2) and (3) together, we develop the standard properties of $\phi(x)$ for $0 \leqq x \leqq 1$. Finally, relabeling $\phi(1)$ as e, it is shown that $\phi(x) = e^x$, and an extension is obtained from [0, 1] to all reals.

2. Since there are so many standard as well as less standard approaches to the exponential (see, e.g., [1], [2], [3], [4]) some motivation for a yet another way is in order. We observe that ours emphasizes an interesting application of one of the most important inequalities, that it avoids the use of series, and that it connects the development to one of the most basic natural laws of growth (cursed with the unfortunate name of 'compound interest law'). Above all, by emphasizing the two-sided bounding (3) our approach follows the constructionist thesis: an entity is given if it is computable to arbitrary accuracy. Moreover, this is built right into the definition, so to say. A somewhat unpleasant feature of our approach (one might object) is the need for 'smuggling in' the idea of uniformity. However, this introduces early and in the easiest possible fashion something which is perhaps delayed too much.

3. We define the harmonic mean $H(x_1,\cdots,x_N)$ as $1/A(1/x_1,\cdots,1/x_N)$. Since $G(x_1,\cdots,x_N)=1/G(1/x_1,\cdots,1/x_N)$ we conclude from (1) that

$$H(x_1,\cdots,x_N)\leqq G(x_1,\cdots,x_N)\leqq A(x_1,\cdots,x_N). \tag{3}$$

Let $0\leqq x\leqq 1$ and apply the G—A part of (3) to $N=n+1$ numbers: $1, 1+x/n,\cdots,1+x/n$, getting

$$(1+x/n)^n\leqq(1+x/(n+1))^{n+1}. \tag{4}$$

Next, with $0\leqq x\leqq 1$, apply the H—G part of (3) to $N=n+2$ numbers: 1, $1+x/n,\cdots,1+x/n$, getting

$$\left[(n+2)\Big/\left(1+\frac{(n+1)n}{n+k}\right)\right]^{n+2}\leqq(1+x/n)^{n+1}.$$

Since $0\leqq x\leqq 1$ it is a simple matter to show that

$$(n+2)\Big/\left(1+\frac{(n+1)n}{n+x}\right)\geqq 1+x/(n+1)$$

and hence

$$(1+x/(n+1))^{n+2}\leqq(1+x/n)^{n+1}. \tag{5}$$

By (4) and (5) the sequences whose nth terms are $(1+x/n)^n$ and $(1+x/n)^{n+1}$ are respectively nondecreasing and nonincreasing; the second sequence dominates the first one term-by-term; and the ratio of the two nth terms tends to 1 as n increases. Hence there exists the limit

$$\phi(x)=\lim_{n\to\infty}(1+x/n)^n,\qquad 0\leqq x\leqq 1,$$

and

$$(1+x/n)^n\leqq\phi(x)\leqq(1+x/n)^{n+1},\qquad 0\leqq x\leqq 1,\ n=1,2,\cdots. \tag{6}$$

From the above it follows that

$$\phi(x)\text{ is an increasing function} \tag{7}$$

since it is bracketed between two increasing functions whose difference can be made arbitrarily small:

$$(1+x/n)^{n+1}-(1+x/n)^n=x(1+x/n)^n/n\leqq\phi(1)/n. \tag{8}$$

Next,

$$\phi(x)\text{ is continuous in }x. \tag{9}$$

Here we use the idea, though not the name, of uniformity:

$$\begin{aligned}\phi(x+h)-\phi(x)=\phi(x+h)&-(1+(x+h)/n)^n\\&+(1+(x+h)/n)^n-(1+x/n)^n+(1+x/n)^n-\phi(x)\end{aligned}$$

and so by (6) and (8)

$$|\phi(x+h)-\phi(x)|\leqq 2\phi(1)/n+|(1+(x+h)/n)^n-(1+x/n)^n|.$$

The right-hand side can be made arbitrarily small: we fix $\varepsilon>0$ and choose n so that $2\phi(1)/n<\varepsilon/2$; next, n being fixed, we use the continuity of polynomials to make the second term on the right-hand side also $<\varepsilon/2$ by making h small enough. This proves (9). Now we prove the functional equation

$$\phi(x+y)=\phi(x)\phi(y),\qquad 0\leqq x,y,x+y\leqq 1. \tag{10}$$

In fact, by the definition of ϕ

$$\phi(x)\phi(y)=\lim_{n\to\infty}(1+(x+y+xy/n)/n)^n$$

and since xy/n can be made arbitrarily small by choosing n large, (10) follows from (7) and (9). To handle the derivative we have by (10)

$$[\phi(x+h)-\phi(x)]/h=\phi(x)[\phi(h)-1]/h. \tag{11}$$

Using (6) with $x=h$ and with $n=1$ on the left and $n=k$ on the right, we get

$$1+h\leqq\phi(h)\leqq(1+h/k)^{k+1}$$

so that, subtracting 1 and dividing by h, we have

$$1\leqq[\phi(h)-1]/h\leqq[(1+h/k)^{k+1}-1]/h.$$

We use the squeeze principle letting $h\to 0$ to get

$$1\leqq\lim_{h\to 0}[\phi(h)-1]/h\leqq 1+1/k$$

and since k is arbitrary the limit is 1. Hence from (11) $\phi'(x)=\phi(x)$.

Finally, we relabel $\phi(1)$ as e and we prove

$$\phi(x)=e^x,\qquad 0\leqq x\leqq 1. \tag{12}$$

First, we have by repeated use of (10)

$$[\phi(1/q)]^q=\phi(q/q)=e$$

for any positive integer q, so that

$$\phi(1/q)=e^{1/q}.$$

Next, we choose any integer p, $1\leqq p\leqq q$, and have again by (10)

$$\phi(2/q)=(e^{1/q})^2=e^{2/q},\qquad \phi(3/q)=e^{3/q},\cdots$$

and eventually

$$\phi(p/q)=e^{p/q}.$$

But this proves (12) for rational x, and the rest follows by continuity.

Having developed the principal properties of the exponential function on the interval $[0,1]$, we continue then to all real numbers by using the functional equation (10).

The referee and the editor have kindly suggested some corrections and improvement. In particular, they have brought to the author's attention the reference [5] outlining a program for e^x similar to ours. Also, the reader may wish to note the suggestions on page 37 of the well-known book [6] on inequalities.

References

1. K. Menger, Methods of presenting e and π, this MONTHLY, 52(1945) 28–37.
2. SMSG, Elementary Functions, Yale Press, 1959.
3. J. Tull, A discovery approach to e, this MONTHLY, 80(1973) 193–194.
4. H. Samelson, To e via convexity, this MONTHLY, 81(1974) 1012–13.
5. J. E. Wetzel, On the functional inequality, this MONTHLY, 74(1967) 1065.
6. G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, 1934.

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ON MARGINAL DENSITY FUNCTIONS OF CONTINUOUS DENSITIES

L. E. CLARKE

It is not infrequently implied, or even asserted, that if the density function $h(x, y)$ of a two-dimensional random variable (X, Y) is everywhere continuous, then so is the marginal density function

$$f(x) = \int_R h(x, y)\, dy \tag{1}$$

of X. That this is not so may be seen by taking

$$h(x, y) = \frac{|x|}{2\sqrt{(2\pi)}} \exp(-|x| - \tfrac{1}{2}x^2y^2),$$

when

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \tfrac{1}{2}\exp(-|x|) & \text{if } x \neq 0. \end{cases}$$

Then h is continuous and non-negative on R^2, and

$$\iint_{R^2} h(x, y)\, dx\, dy = \int_R f(x)\, dx = 1,$$

but f is not continuous at the origin.

Even worse may befall. Let $q_1, q_2, \cdots$ be the rational numbers in some order, and let

$$h_1(x, y) = \sum_1^\infty 2^{-n} h(x - q_n, y). \tag{2}$$

It is easily shown that h_1 is a density function which is everywhere continuous (note that h is bounded on R^2, and so the series on the right-hand side of (2) is uniformly convergent on R^2). The marginal density function $\int_R h_1(x, y)\, dy$ is

$$f_1(x) = \sum_1^\infty 2^{-n} f(x - q_n). \tag{3}$$

Since f is bounded on R, the series on the right-hand side of (3) is uniformly convergent on R. It follows readily that f_1 is not continuous at any rational point (though it is continuous at every irrational point).

But worse still. Suppose instead that

$$h(x, y) = \frac{\sqrt{|x|}}{2\pi\sqrt{2}} \exp(-|x| - \tfrac{1}{2}x^2y^2),$$

giving

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{2\sqrt{(\pi|x|)}}\exp(-|x|) & \text{if } x \neq 0, \end{cases}$$

and let h_1, f_1 be defined by (2), (3). Then h_1 is a continuous density function with marginal density f_1.

For each n

$$f_1(x) \geqq 2^{-n} f(x-q_n) \to \infty \quad \text{as} \quad x \to q_n.$$

Since the rationals are dense in the real line, it follows that f_1 takes arbitrarily large values in every open interval, and so

$$\limsup_{x \to c} f_1(x) = \infty \qquad \text{for all real } c.$$

Thus f_1 is nowhere continuous.

Two questions suggest themselves:

(a) Is there a continuous h for which the corresponding f is everywhere finite but nowhere continuous? Note that the second f_1 constructed above must be finite almost everywhere (since $\int_R f_1(x)\,dx < \infty$), but it is by no means clear that the q's can be ordered in such a way that f_1 is finite everywhere.

(b) More generally, given an arbitrary one-dimensional function f, i.e., a Borel measurable function $f: R \to [0,\infty]$ satisfying $\int_R f(x)\,dx = 1$, does there exist a continuous two-dimensional density function h for which (1) holds?

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

COMPUTER-ASSISTED TUTORIALS IN COLLEGE MATHEMATICS

J. L. CALDWELL AND DOUGLAS POLLEY

Many uses of computers in mathematics education are well known. We believe that our approach differs in important aspects from most standard uses. It is hoped that this note will encourage others to experiment with techniques similar to the one presented.

The programs which the authors have designed and written were specifically intended to analyze the students' answers to computational problems. We call this approach tutorial (although this terminology is nonstandard) because the programs mimic the role of a tutor in the traditional sense. No new material is presented via the computer. We assume that the student has received instruction in basic formulae before using the programs. If the student responds incorrectly to problems generated by the program, the program analyzes the student's answers in an attempt to identify common errors.

The first programs deal with quadratic equations and equations of straight lines. A program dealing with sums of rational functions is under construction.

The format of all the programs is the same. The student, seated at a teletype, receives a problem with randomly chosen parameters. The student then solves the problem and enters his solution. If his solution agrees with the computer's solution, then another problem is presented. If the student's solution is incorrect, the computer then works the problem incorrectly in several different ways, each time checking its answer against the student's. If a match is obtained, then the computer

suggests that the student has made a certain error and asks him to try again. If none of the incorrect answers match the student's answer, then the computer will check the student's work step by step. For example, in the case of the quadratic equation program, the student is first asked "What did you get for A,B,C?" and then later "What did you get for $B^2 - 4AC$?" Finally, if the student does not enter the correct answer, he receives a detailed solution. (For over 90% of the problems we have checked thus far, the student has eventually entered the correct answer.) If a student requires detailed solutions to several problems he is asked to contact the instructor.

The programs were written in FORTRAN and implemented on a CDC 6400 computer, but other choices would serve well also. In particular, the size of the CDC 6400 is unnecessary because such programs are easily divided into subprograms. FORTRAN could be replaced, and in fact the programs are currently being translated into BASIC to permit their use on other time sharing systems. The use of an instructional dialogue language (IDL) would greatly facilitate the writing; but unfortunately the authors know of no IDL which has the capability of generating problems at random. Hopefully this void will soon be filled.

The principal use of the programs has been by precalculus and linear algebra students at the University of Wisconsin-River Falls and the Minneapolis and Morris campuses of the University of Minnesota. In addition, several other members of the Minnesota Educational Regional Interactive Time-Sharing System (MERITSS) have been involved.

Those readers desiring additional information about the programs are invited to write UWRF.

It was our intention to identify the source of a student's error as soon as possible with minimal interrogation of the student. This presents some interesting theoretical, as well as practical, questions. On the theoretical side, we may ask just how much can we tell about the source of the error based on our knowledge of the incorrect response, the correct response, and a knowledge of the technique being used. On the practical side, we have considerations motivated by the following example of two incorrect solutions to the problem: "Find the equation of the line through the point $(2, -3)$ with slope 2."

I	II
$y - 3 = 2(x - 2)$	$y + 3 = 2(x - 2)$
$y - 3 = 2x - 4$	$y + 3 = 2x - 4$
$y = 2x - 1$	$y = 2x - 1$

In I the mistake occurs in the first line, while in II the error is in the subtraction of the last line. In both cases the answer given is the same. Although the existence of the error can be readily identified, and it may be reasonable to assume that any incorrect answer of the form $y = 2x - 1$ is due to one of these methods, we still cannot tell which is the case. In programming the response to this situation we may issue a message of the form "I think you computed $y - y_1$ incorrectly, or perhaps you combined terms incorrectly." Alternatively, we may inquire "What did you get for $y - y_1$?" While the latter choice is less ambiguous, it is more difficult to program because it involves additional decisions. In situations where the error is not among those errors deemed most likely to occur, it may be acceptable to respond in the former manner.

For those who want to develop a series of integrated tutorial programs there will be some additional system design considerations. One problem is how to allow a student to progress through problems of varying levels of difficulty. One possibility is to have a level indicator which determines the difficulty of the problem generated as well as the details of the interrogation of the student. In the case of a student who has made several mistakes we would assign a lower number and ask him to enter his work a step at a time. For a student who is not making many mistakes we would not ask for as many intermediate steps and give more opportunities to correct mistakes before we present the solution.

Another item of some import will be a series of software routines for such things as reading

student input (e.g., polynomials) and extracting relevant data. Other such routines might include a means for comparing algebraic expressions to determine if they are equivalent (see Uttral [6]). Of course, once such routines have been written they can serve a large number of tutorial programs.

We believe that this approach is a valid utilization of computer facilities because it offers immediate, individualized help to students in a manner that can be duplicated only by actual consultation with an instructor. The advantage to the student is a source of immediate feedback which might not be available if he had to compete with other students for help from the instructor. The advantage to the instructor is more time to deal with conceptual problems and other related matters because less time is spent in dealing with problems which are basically computational in nature.

We feel that this approach has advantages over the setting in which the entire course is presented via the computer. First there is a reduction in the teletype time required because the student first attends class or perhaps studies a programmed text, then comes to the computer to work problems. Even more teletype time can be saved if our system is modified by assigning to the students problems that are stored in the computer. Then the student would come to the computer only with problems he could not work. (Minimizing teletype time is important because it is often more scarce than computer time.) A second advantage is that tutorial programs are compatible with different presentations of the course material, thus leaving more control in the hands of the individual instructor.

Another benefit of a system of tutorial programs is the experience gained in developing them. The authors are currently having senior mathematics education students create additional programs because we believe that they will thus obtain a better understanding of the errors a student may make and also develop a facility in explaining the material. The student designer will, of course, learn much in the area of system programming.

Our experience with programs developed so far has indicated that there is considerable potential in this type of computer-assisted instruction, particularly at the college algebra level. The manipulative nature of many of the topics in college algebra (such as determinants, systems of equations, etc.) makes them suitable for the problem and answer analysis framework. Programs in this area would also provide review material for calculus students whose problem-solving techniques need improvement. In response to a questionnaire, students who have used existing programs indicated they considered the process most helpful and would recommend it to their fellow students. They also expressed a willingness to try new programs when they become available. While operating the programs, the students showed a most unusual enthusiasm for quadratic formula and line problems. It was most encouraging to see the students' satisfaction in obtaining the correct solutions with the assistance of the computer. For these reasons the authors hope that this approach will be expanded and improved.

References

1. E. B. Koffman, A generative CAI tutor for computer science concepts, AFIPS Conference Proceedings, 40, 1972. Spring Joint Computer Con., p. 379.

2. H. A. Lekan, Index to Computer Assisted Instruction, Harcourt, Brace, Jovanovich, New York, 1971.

3. A. W. Luehrmann, Should the computer teach the student, or vica versa?, AFIPS Conference Proceedings, 40, 1972. Spring Joint Computer Con., p. 407.

4. W. Stenberg, Computing in the high school — past present and future — and its unreasonable effectiveness in the teaching of mathematics, AFIPS Conference Proceedings, 40, 1972. Spring Joint Computer Con., p. 1051.

5. P. C. Suppes, M. Jerman, D. Brian et al., Computer Assisted Instruction: Stanford's 1965–1966 Arithmetic Program, Academic Press, New York, 1968.

6. W. R. Uttral, Reaction paper, Computer Assisted Instruction and the Teaching of Mathematics, National Council of Teachers of Mathematics, Inc., 1969.

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BEYOND BEHAVIORAL OBJECTIVES

ALVIN M. WHITE

In an earlier issue of the MONTHLY, Allendoerfer [1] discussed how behavioral objectives should be used when writing and testing elementary textbooks.

In response L. C. Jansson and R. T. Heimer [4] suggest that behavioral objectives should influence the actual instruction phases of education as well as the writing of textbooks.

The authors discuss the assessment of a student's performance based upon stated objectives. It is suggested that "behavioral objectives must be written for minimum course requirements and that instructors must then teach to, and test for those objectives. As mathematics teachers, however, we have various kinds of goals for our students. We expect, of course, that they will master certain basic facts..., but we also have higher level goals.... We want students to become creative and critical thinkers in mathematics."

I agree with Jansson and Heimer that the same kind of attention should be applied to the actual instruction phase as to the writing and selection of textbooks. I also agree with the objective that our students should become creative and critical thinkers in mathematics.

It may, however, be necessary to go beyond behavioral objectives as defined in [4] to achieve certain levels of creativity and critical thinking. There are significant aspects of mathematical education that are not often included in behavioral objectives. The National Academy of Sciences regrets that not even science teachers understand the scientific process: "The typical U.S. teacher... is not well equipped to guide his pupils in learning that science is more than a collection of facts to be memorized or techniques to be mastered..."

The Academy blames not the science teachers but the educational system in which they were trained [9].

Even some who have had extensive mathematical education seem to have a distorted image. The physicist Feynman in his *Messenger lectures* at Cornell in 1964 described a mathematician as "starting from a rigidly prescribed set of axioms (and presumably, rules of inference) and then slavishly grinding out all the consequences of these axioms."

The mistake is in confusing a formal description of mathematics as a deductive system with the living, creative activity of mathematicians [5].

Not every student in a mathematics class will make mathematics his life work. But every student in such a class should be imbued, to some extent, with an understanding of mathematics and an appreciation of the spirit which has engaged great minds over the centuries. The spirit of mathematics learned in the classroom is often quite different from the excitement of the creative process away from class. A student usually is unaware of the origin of the problem being studied, or why the solution proceeds in one way and not another.

In the long run it might be more important to try to develop a student's intuition than to be unduly concerned with the temporary mastery of basic theorems and techniques. Poincaré [7] describes intuition as the faculty which is "necessary to the explorer for choosing his route; it is not less so to the one following his trail who wants to know why he chose it."

Beauty is another quality of mathematics, the appreciation of which is not often a behavioral objective. Poincaré [7] asserts that the scientist does not study nature because it is useful; "he studies it because he delights in it, and he delights in it because it is beautiful."

Hermann Weyl [6] says it another way. "My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful."

Even if our objectives (behavioral and otherwise) include components which go beyond facts, formulas and techniques, these components may be lost in the classroom. In a traditional view, historical context, an appreciation of the difficulties associated with certain concepts, participation in discovery and creation, may be considered extra elements of the syllabus. These so-called extra elements may be the first casualty of the pressure of a crowded program.

R. C. Buck dramatically expresses his dismay during his encounter with misunderstood (?) objectives which emphasized facts and formulas. He describes his experience teaching an elementary course [**3**].

"The assigned text was personally embarrassing. Its preface was replete with reference to CUPM *Level I Guidelines*, some of which I had written myself during my term as Chairman of CUPM. Surely this dry and pedantic book, with tedious applications of the associative law, was not what we had envisioned as the outcome of those recommendations! Where was the sense of reality, the ties with the schoolroom, the special insights into the difficulty children have with understanding mathematical concepts? I reread the CUPM outlines, and found that these other aspects were mentioned, but smothered beneath the weight of algebraic structure."

Perhaps our guidelines and teaching objectives should not have as their major target or focus the mastery of facts and techniques. Rather, the facts and techniques should be the skeletal framework which supports our objective of imbuing our students with the spirit of mathematics and a sense of excitement about the historical development and the creative process. The concepts and relationships of mathematics should be presented as the building blocks of this magnificent edifice created by human imagination.

The student should not be denied the appreciation of the excitement of scientific discovery. M. Polanyi describes the passionate preoccupation with a problem which leads to discovery. As we pursue scientific discoveries through their consecutive publications on their way to the textbooks, we observe that the intellectual passions aroused by them appear gradually toned down to a faint echo of their discoverer's first excitement at the moment of illumination.

"A transition takes place here from the heuristic act to the routine teaching and learning of its results and eventually to the mere holding of these as known and true, in the course of which the personal participation of the knower is altogether transformed. The impulse which in the original heuristic act was a violent irreversible self-conversion of the investigator ... will assume finally a form in which all dynamic quality is lost." [**8**]

One of our objectives should be to return the dynamic quality which is present in the process of discovery and creation to the process involved in learning-teaching. What should be remembered after the details have been forgotten is that mathematics is connected with other human activity, historically and in its present creation.

I am not referring only to general education courses, but to mathematics at all levels. If mathematics is more than facts, formulas, and techniques, then that something more should be evident to all students and should be an intrinsic part of the syllabus.

My sophomore calculus students were responsive to some nontraditional elements. The first week of the semester was devoted to reading and discussing *Science and Human Values* by J. Bronowski [**2**]. Although our mathematics textbook did not even hint at the relationship between multiple integrals and human values, the ideas of that first week influenced some of the students in many ways.

Perhaps future texts will make more obvious those relationships. Then students may better understand the words of René Descartes:

"... But when I afterwards bethought myself how it could be that the earliest pioneers of Philosophy in bygone ages refused to admit to the study of wisdom anyone who was not versed in Mathematics ... I was confirmed in my suspicion that they had knowledge of a species of Mathematics very different from that which passes current in our time."

References

1. C. B. Allendoerfer, Evaluation of elementary textbooks, this MONTHLY, 78 (1971) 189–191.
2. J. Bronowski, Science and Human Values, Harper Torchbooks, New York, 1965.
3. R. C. Buck, 2 Experiments in Teaching, Houghton Mifflin, Mathematics News, March 1973, Boston.

4. L. C. Jansson, and R. T. Heimer, On behavioral objectives in mathematics education, this MONTHLY 80 (1973) 930–933.
5. A. Morley, Mathematics as process, Mathematics Teacher (January 1973) 39–45.
6. James Newman (ed.) The World of Mathematics, Simon and Schuster, New York, 1956, p. 1831.
7. H. Poincaré, The Value of Science, Dover, New York.
8. M. Polanyi, Personal Knowledge, Harper Torchbook, New York, 1964.
9. Science News, Editorial (8/26/72).

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before January 31, 1976.

E 2552. *Proposed by Philip Castevens, Winston-Salem, North Carolina*

Let A be an $n \times n$ real matrix with zeros on the main diagonal and ± 1 off the diagonal. Show that A is nonsingular if n is even, but that A may be singular if n is odd.

E 2553. *Proposed by V. B. Sarma, Mylapore, India*

Suppose that A, B, C, D are concyclic and that the Simson line of A with respect to triangle BCD is perpendicular to the Euler line of triangle BCD. Show that the Simson line of B will be perpendicular to the Euler line of triangle CDA. Is the above result true if we replace "perpendicular" by "parallel"?

E 2554. *Proposed by F. David Hammer, Stockton State College, New Jersey*

Can a polynomial function with integer coefficients be one-to-one when restricted to the rationals, but not one-to-one on the reals?

E 2555. *Proposed by T. W. Cusick, University of Illinois*

Let $A = (a_1 | a_2)$ be a nonsingular 2×2 matrix partitioned into columns. Show that

$$\min_A \max_x \frac{(a_1 \cdot x)(a_2 \cdot x)}{\det A} = \frac{1}{2},$$

where the max is over all x in the box $|x_i| \leqq 1$, and the min is over all such matrices A.

(*) Establish a corresponding result for higher dimensions.

E 2556. *Proposed by Leon Gerber, St. John's University*

Let $A = (a_1|\cdots|a_n)$ and $B = (b_1|\cdots|b_n)$ be two $2n \times n$ real matrices, partitioned into columns. Assume that $n \geqq 3$ and that the rank of A does not exceed $n-3$. Let $r_1, \cdots, r_n, s_1, \cdots, s_n$ be arbitrary positive numbers. For $i, j = 1, 2, \cdots, n$, define

$$t_{ij} = \frac{|a_i - b_j|^2 - r_i^2 - s_j^2}{2r_i s_j}.$$

Show that $\det(t_{ij}) = 0$.

E 2557. *Proposed by R. D. Nelson, Ampleforth College, York, England*

Find all cyclic quadrilaterals with integral sides, each of which has its perimeter numerically equal to its area. (The following references may be of interest: E 1168 [1955, 365; 1956, 43]; E 2420 [1973, 691; 1974, 662]; M. V. Subbarao, *Perfect triangles*, this MONTHLY 78 (1971), 384–385; R. W. Sielaff, *Perfect quadrilaterals*, this MONTHLY 80 (1973),414–415; M. J. Marsden, *Triangles with integer-valued sides*, this MONTHLY 81 (1974), 373–376, especially the last paragraph, p. 376. — Ed.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Squares Ending in Ones

E 2486 [1974, 776]. *Proposed by L. C. Washington, Princeton University*

Let p be an odd prime. Show that there exist perfect squares which when expressed in the base p terminate in arbitrarily many repetitions of the digit "1". (This is in contrast to the result for base 10; see the 1970–1971 Putman Examination, Problem A-3 [1971, 765–767].) Is there a similar result for higher powers?

Solution by the Temple University Problem Solving Group. Let $p > 1$ be any odd number. We proceed by induction on the number of 1's. Clearly 1^2 ends in a 1. Suppose x^2, in base p, ends in k 1's; that is

$$x^2 = 1 + p + \cdots + p^{k-1} + (ap^k + \text{higher terms}).$$

Clearly $(x, p) = 1$. Then

$$(x + cp^k)^2 = x^2 + 2cxp^k + c^2p^{2k} = 1 + p + \cdots + p^{k-1} + (a + 2cx)p^k + \text{higher terms}.$$

Since p is odd, $(2x, p) = 1$, and hence the congruence

$$2xc + a \equiv 1 \pmod p$$

is solvable for c. If c_0 is a solution, then $(x + c_0p^k)^2$ will end in $(k+1)$ 1's as required.

The same argument works for any base p and any power r which is relatively prime to p. For, if

$$x^r = 1 + p + \cdots + p^{k-1} + ap^{k+1} + \text{higher terms},$$

then, choosing c_0 to be a solution of

$$rx^{r-1}c + a \equiv 1 \pmod p, \tag{1}$$

$(x + c_0p^k)^r$ will end in $(k+1)$ 1's. (1) is always solvable for c since $(rx, p) = 1$.

Note: It is not necessary for p to be prime, only that it be relatively prime to r. This explains the difficulty referred to for base 10, as 2 and 10 are not relatively prime.

The general problem was solved also by David Allison (South Africa), Walter Bluger, Brother Alfred Brousseau, P. S. Bruckman, Cal Poly Solutions Group, P. J. Campbell, L. Carlitz, Neal Felsinger, T. H. Foregger, Peter Frankl (Hungary), J. G. Huard, O. P. Lossers (Netherlands), C. L. Mallows, D. C. B. Marsh, L. E. Mattics, Ram Murty & Kumar Murty, M. Perisastri (India), D. K. Pickard (Australia), Michael Shimsoni (Israel), Joseph Silverman, University of San Francisco Problem Group, I. K. Wolf, and the proposer.

The problem for the case of squares was solved by Ernst Adams (Germany), M. S. Demos, Michael Goldberg, W. H. Gustafson, Carl Hurd, I. G. Kastanas (Greece), Paul Kirchenmayer (Germany), Detlef Laugwitz (Germany), A. H. Stein, Marijke van Rossum, and Charles Wexler.

Allison and Lossers establish the following generalization: Let p be a prime, $(n,p)=1$. Let $a_0 \neq 0$ be an nth power residue mod p (i.e., there is an integer s $(1 \leqq s \leqq p-1)$, such that $s^n \equiv a_0 \pmod p$). For $i = 1,2,\cdots$, let a_i be an element of $\{0,1,\cdots,p-1\}$. Then, for $i \geqq 1$, there is an element x_i such that

$$x_i^n \equiv a_0 + a_1 p + a_2 p^2 + \cdots + a_i p^i \pmod{p^{i+1}}.$$

Arc Length and Functional Composition

E 2489 [1974, 776]. *Proposed by B. A. Reznick, Stanford University*

Suppose that f is a continuously differentiable function which maps $[0,1]$ onto itself and which satisfies $f(0)=0$ and $f(1)=1$. Can the arc length of f ever exceed the arc length of the composition $f \circ f$?

Solution by Cal Poly Solution Group, San Luis Obispo. The answer is yes. To demonstrate the existence of such a function, consider the piecewise linear function g which has corners at $(\frac{1}{4}, 0)$, $(\frac{1}{4}+\varepsilon, \frac{1}{4})$, $(\frac{1}{2}-\varepsilon, \frac{1}{4})$ and $(1-\varepsilon, \frac{1}{4}+\varepsilon)$, where $\varepsilon \leqq 1/100$. Denoting the arc length of f by $L(f)$, it is easy to see that

$$L(g\circ g) < (\tfrac{1}{2}-\varepsilon) + \frac{\sqrt{5}}{4} + \left(\frac{3}{4}+\varepsilon\right) = \left(\frac{5+\sqrt{5}}{4}\right) < 2 - 3\varepsilon$$
$$= (\tfrac{1}{4}) + (\tfrac{1}{4}) + (\tfrac{1}{4} - 2\varepsilon) + (\tfrac{1}{2}) + \left(\frac{3}{4} - \varepsilon\right) < L(g).$$

It remains to argue that "the corners can be rounded off" without changing the arc length of the function or of the composition significantly. Let $M = 1/\varepsilon$, which is larger than the slope of g at any point. Let F_n be a continuously differentiable function which maps $[0,1]$ onto itself, which satisfies $F_n(0)=0$ and $F_n(1)=1$, whose derivative is bounded by M, and which agrees with g except on a set of measure $1/n$. It follows that for each n,

$$|L(F_n) - L(g)| \leqq \sqrt{1+M^2}/n$$

and that

$$|L(F_n \circ F_n) - L(g\circ g)| \leqq \sqrt{1+M^4}/n.$$

Choosing n sufficiently large will guarantee that $L(F_n \circ F_n) < L(F_n)$.

Comment by J. E. leBel, University of Toronto, and the proposer (independently). It is possible to prove the following result: *Let us use the notation $f_1 = f$, $f_2 = f\circ f$, $f_3 = f\circ f\circ f, \cdots$. Then for any $\delta > 0$ and any N, there exists a continuously differentiable function f which maps* $[0,1]$ *onto itself, satisfies $f(0)=0$ and $f(1)=1$, and is such that $L(f_j) > 2-\delta$ for $j = 1, 2, \cdots, N-1$, while $L(f_N) < \sqrt{2}+\delta$.*

Space limitations prevent reproducing here the two lengthy and very similar proofs which have been submitted. The proofs depend upon the construction of a special piecewise linear, strictly increasing, function which has the above property, and then rounding off the corners.

Le Bel also provided a proof of the (geometrically clear) result that the arc length of $f\circ f$ cannot be shorter than that of f, provided that f is concave upwards (or downwards).

Also solved by J. E. leBel, O. P. Lossers (Netherlands), William Nuesslein, S. J. Sidney, J. S. H. Tung, M. M. Wells, and the proposer.

The Eigenvalues of a Matrix

E 2490 [1974, 776]. *Proposed by James Brink, Pacific Lutheran University*

Let $A = (a_{ij})$ be an $n \times n$ matrix with the property that there exist constants $c_1, \cdots, c_n$ such that

$$\sum_{i=1}^{k} a_{ij} = c_k$$

for $i \leqq j \leqq k \leqq n$. Find the eigenvalues of A.

I. *Solution by D. M. Bloom, Brooklyn College.* The operation, "Add ith row to jth row, then subtract jth column from ith column," which we denote by R_{ij}, replaces any matrix by a similar matrix and hence leaves the eigenvalues unchanged. If the operations $R_{12}, R_{23}, \cdots, R_{n-1\,n}$ are applied in succession to the given matrix the result is an upper triangular matrix whose diagonal entries (hence eigenvalues) are

$$c_1 - a_{12}, c_2 - (a_{13} + a_{23}), \cdots, c_{n-1} - \sum_{i=1}^{n-1} a_{in}, c_n.$$

II. *Solution by Ken Yocom, South Dakota State University.*
Since the case $n = 1$ is trivial, we assume that $n \geqq 2$. Setting $k = n$, we see that all column sums of A equal c_n. Since

$$c_k = \sum_{i=1}^{k} a_{ij} = \sum_{i=1}^{k-1} a_{ij} + a_{kj} = c_{k-1} + a_{kj}$$

for $j = 1, 2, \cdots, k-1$, we have, in row k, that $a_{k1} = a_{k2} = \cdots = a_{k,k-1} = c_k - c_{k-1}, 2 \leqq k \leqq n$. Now form $\det(a - \lambda I)$ and add the second through the nth rows to the first row and note that each element in the first row then equals $c_n - \lambda$. Now subtract column one from each of the second through the nth columns, expand the determinant about the first row and obtain

$$\det(A - \lambda I) = (c_n - \lambda) \prod_{k=2}^{n} (a_{kk} - a_{kl} - \lambda).$$

Thus the eigenvalues of A are c_n and $a_{kk} - a_{k1}$ for $k = 2, 3, \cdots, n$.

Also solved by R. L. Andrews, Anders Bager (Denmark), Gary Bates, Bennett College Team, Paul Chauveheid (Belgium), J. Delany, David Farnsworth & Rebecca Hill, T. H. Foregger, Peter Frankl (Hungary), E. W. Frees, Clark Givens, Emilie V. Haynsworth, E. T. Hoefer, Dennis Jesperson, I. N. Katz, Joel Levy, Graham Lord, O. P. Lossers (Netherlands), Carolyn MacDonald, C. L. Mallows, D. C. B. Marsh, L. E. Mattics, P. D. McCray, Ram Murty & Kumar Murty, F. D. Parker, Robert Patenaude, M. Perisastri (India), U. V. Mallikarjuna Rao (India), Frank Siwiec, Southern University Problem Solving Group, Allen Stenger, Temple University Problem Solving Group, H. S. Valk, D. A. Voss & J. M. Brown, Brian Wesselink, and the proposer. Six incorrect solutions were received.

Editor's Comment. The answer can be put in several different forms, e. g., c_n and $c_{k-1} - c_k + a_{kk}, k = 2, 3, \cdots, n$.

When is $[\sqrt{n}]$ a Divisor of n?

E 2491 [1974, 776]. *Proposed by S. W. Golomb, University of Southern California*

Find all natural numbers n with the property that $[\sqrt{n}]$ is a divisor of n.

I. *Solution By J. P. Comiskey, Monsignor Farrell High School, Staten Island, New York.* This is equivalent to asking for each m in N, what are the natural numbers n that are divisible by m with $m^2 \leqq n < (m+1)^2$? The only such numbers are clearly $m^2, m^2 + m$, and $m^2 + 2m$.

II. *Generalization by James Bartholomew, Ohio University.* For $k, n \in Z^+, k \geqq 2, [\sqrt[k]{n}]$ divides n if and only if n is of the form $a^k, a^k + a, a^k + 2a, \cdots, (a+1)^k - 1$. Notice that only for $k = 2$ is there a constant number of solutions between successive kth powers.

Also solved by the proposer and one hundred and ninety others.

Editor's Comment. Edward Dixon and Richard Mueller (independently) gave formulas for the rth natural number a_r which satisfies the conditions, essentially

$$a_r = \left(\left[\frac{r-1}{3}\right]+1\right)\left(r-2\left[\frac{r-1}{3}\right]\right),$$

so that $\{a_r\} = \{1, 2, 3, 4, 6, 8, 9, 12, 15, 16, 20, 24, \cdots\}$.

Some Sum

E 2492 [1974, 902]. *Proposed by Donald Knuth, Stanford University*

For natural numbers i and j, let "$i \bmod j$" denote the nonnegative remainder when i is divided by j; i.e., $i \bmod j = i - j[i/j]$. Evaluate the sum:

$$S_n = \sum_{k=1}^{2n^2} \binom{k \bmod n}{(2k+1) \bmod (2n+1)}.$$

Solution by M. Ram Murty and V. Kumar Murty, undergraduates, Carleton University. Given r and s, the system $k \equiv r \pmod n$, $2k+1 \equiv s \pmod{2n+1}$ has a unique solution for k modulo $n(2n+1)$ by the Chinese Remainder Theorem. Thus

$$\sum_{k=0}^{2n^2+n-1} \binom{k \bmod n}{(2k+1) \bmod (2n+1)} = \sum_{r=0}^{n-1} \sum_{s=0}^{2n} \binom{r}{s} = \sum_{r=0}^{n-1} 2^r = 2^n - 1.$$

But the sum on the left here equals S_n since $k = 0$ contributes nothing and

$$\sum_{k=2n^2+1}^{2n^2+n-1} \binom{k \bmod n}{(2k+1) \bmod (2n+1)} = \sum_{t=1}^{n-1} \binom{(2n^2+t) \bmod n}{(4n^2+2t+1) \bmod (2n+1)}$$

$$= \sum_{t=1}^{n-1} \binom{t}{2t+1} = 0 \text{ also.}$$

Thus $S_n = 2^n - 1$.

Also solved by 34 other solvers and the proposer.

D. P. Sumner remarks that for any integer $t \geqq 1$, we have

$$\sum_{k=1}^{tn^2} \binom{k \bmod n}{(tk+1) \bmod (tn+1)} = 2^n - 1.$$

$\sigma(n) = 2^n$

E 2493 [1974, 902]. *Proposed by C. D. H. Cooper, Macquarie University, Australia*

Prove that the sum of the (positive) divisors of the natural number n is a power of 2 if and only if n is a product of distinct Mersenne primes.

Solution by W. J. Dodge, El Paso, Texas. Let $\sigma(n)$ denote the sum of the positive divisors of n. Then $\sigma(rs) = \sigma(r)\cdot\sigma(s)$ for relatively prime r and s, and if $n = \Pi p^a$ denotes the prime decomposition of n, we have

$$\sigma(n) = \prod \sigma(p^a) = \prod (1 + p + p + p^2 + \cdots + p^a).$$

If each prime divisor of n is a Mersenne prime ($p = 2^b - 1$) and each index $a = 1$, we have

$$\sigma(n) = \prod (1+p) = \prod 2^b = 2^c, \text{ a power of 2.}$$

Conversely, suppose that $\sigma(n) = \Pi(1 + p + p^2 + \cdots + p^a) = 2^r$. Then each factor $(1 + p + p^2 + \cdots + p^a) = 2^s$ ($s \leqq r$). This requires a to be odd, say $2q+1$, lest $1 + p + \cdots + p^a$ be an odd number. Then

$$1 + p + \cdots + p^{2q+1} = (1+p)(1 + p^2 + p^4 + \cdots + p^{2q}) = 2^s,$$

implying (i) $1+p=2^t$, and (ii) $1+p^2+p^4+\cdots+p^{2q}=2^u$ for appropriate integers t and u. Condition (i) yields

$$p=2^t-1, \text{ a Mersenne prime.}$$

If the second factor materializes, that is, if $q>0$, then (ii) gives

$$1+(p^2)+(p^2)^2+\cdots+(p^2)^q=2^u,$$

implying (as for the value of a in the above argument) that q is odd, say $q=2v+1$. Factoring then yields

$$(1+p^2)(1+p^4+p^8+\cdots+p^{4v})=2^u,$$

and $1+p^2=2^w$ for some integer w. Recalling that $1+p=2^t$, we see that $(1+p)|(1+p^2)$ since $1+p<1+p^2$. But $1+p$ divides $1-p^2=(1+p)(1-p)$. Accordingly,

$$(1+p)\,\Big|\,[(1+p^2)+(1-p^2)]=2,$$

an impossibility since $1+p\geqq 1+2=3$. Thus q must be 0. This implies $a=1$, and $n=\Pi p^a$ is the product of distinct Mersenne primes.

Also solved completely by 70 others. There were also 2 incorrect submissions, 6 solutions which failed to establish that the prime divisors of n are distinct, and 3 solutions based on claims not given conclusive support. Andrzej MaKowski (Poland) observes that the problem is due to W. Sierpinski; see *Sur les nombres dont la somme de diviseurs est une puissance du nombre* 2, Calcutta Mathematical Society Golden Jubilee Commemorative Volume (1958–1959) Part 1, pp. 7–9.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before January 31, 1976.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6048*. *Proposed by H. M. Edgar, San Jose State University*

A positive integer n is said to be *harmonic* (See O. Ore, *On the averages of the divisors of a number*, this MONTHLY, December 1948, pp. 615–619) if the ratio $n\tau(n)/\sigma(n)$ is again integral. (Here $\tau(n)$ denotes the number of positive integral divisors of n and $\sigma(n)$ denotes their sum.)

(a) Are there any harmonic numbers other than the number one which are perfect squares?

(b) Do there exist infinitely many harmonic numbers?

6049. *Proposed by D. E. Knuth, Stanford University*

What group is generated by the two cyclic permutations $(1,2,\cdots,m)$ and $(1,2,\cdots,n)$, when $1<m<n$?

6050. *Proposed by D. E. Knuth, Stanford University*

Let $X_1, X_2, Y_1, \cdots, Y_{m+n}$ be independent random variables, where X_1 and X_2 have common distribution F and $Y_1,\cdots,Y_{m+n}$ have common distribution G. Prove that

$$\tfrac{1}{2}\leqq \Pr(X_1+\max(Y_1,\cdots,Y_m)\leqq X_2+\max(Y_{m+1},\cdots,Y_{m+n})\leqq\frac{n}{m+n}$$

when $m\leqq n$ and G is differentiable.

6051*. *Proposed by Jochem Zowe, University of Würzburg, Germany*

Let X be a real vector space, Y an ordered vector space and p a sublinear map of X into Y, i.e., $p(\lambda x)=\lambda p(x)$ and $p(x+x')\leqq p(x)+p(x')$ for all $x,x'\in X$ and all real non-negative λ. Does there always exist a linear map T of X into Y such that $Tx\leqq p(x)$ for all $x\in X$?

6052*. *Proposed by J. R. Gard, University of South Florida*

If G is a torsion group such that there exists an element $x\in G$ with the property that x and y generate G whenever $y\in G$ is not a power of x, is G finite? And what other properties does G have?

6053*. *Proposed by Raphael Finkelstein, Bowling Green State University*

Let $a+bi$ be a Gaussian integer with $(a,b)=1$ and let $A+Bi=(a+bi)^p$ where p is an odd prime. Let $C=\max(A,B)$ and $D=\min(A,B)$. Can C/D approach $\frac{1}{2}(1+\sqrt{5})$ arbitrarily closely?

SOLUTIONS OF ADVANCED PROBLEMS

Condition for a Quadratic

5880 [1972, 1043; 1974, 177]. *Proposed by Anon, Erewhon-upon-Yarkon*

Let $f(x)$ be a continuous function on $a<x<b$ such that $f'(x)$ exists at each point. Suppose for each x in this interval there exists a $\delta=\delta_x>0$ such that

$$\frac{f(x+h)-f(x-h)}{2h}=g(x)$$

for all h satisfying $0<h<\delta$. Prove that $f(x)$ is a quadratic polynomial. (This generalizes a problem in T. M. Flett, *Mathematical Analysis*, where f''' is assumed to exist.)

II. *Solution by David Borwein, University of Western Ontario.* Let n be a positive integer and let E_n be the set of $x\in(a,b)$ such that $f(x+h)-f(x-h)=2hg(x)$ whenever $0<h\leqq 1/n$. If $\{x_r\}$ is a sequence of points in E_n converging to $x\in(a,b)$, then for $0<h\leqq 1/n$,

$$\lim_{r\to\infty} g(x_r)=\frac{f(x+h)-f(x-h)}{2h}=g(x),$$

so that $x\in E_n$ and hence E_n is closed in (a,b). Since $(a,b)=\cup_{n=1}^{\infty}E_n$, we have, by a form of Baire's theorem, that there is an open interval $I\subset(a,b)$ and a positive integer m such that $I\subset E_m$. Hence, for $0<h\leqq 1/m$, $x\in I$,

$$f''(x)=g'(x)=\frac{f'(x+h)-f'(x-h)}{2h};$$

from which it follows that $f'''(x)$ exists for every $x\in I$.

Further, differentiation with respect to h yields that $f'''(x+h)+f'''(x-h)=0$ whenever $x,x-h,x+h\in I$ and $0\leqq h\leqq 1/m$. Consequently $f'''(x)=0$ throughout I, and hence $f(x)=Ax^2+Bx+C$ in I. Let $J=(\alpha,\beta)$ be the largest open interval such that $I\subset J\subset(a,b)$ and $f(x)=Ax^2+Bx+C$ in J.

If $\beta<b$, then there is a positive $\delta<\beta-\alpha$ such that, for $0\leqq h<\delta$,

$$\begin{aligned}f(\beta+h)&=2hf'(\beta)+f(\beta-h)\\&=2h(2A\beta+B)+A(\beta-h)^2+B(\beta-h)+C\\&=A(\beta+h)^2+B(\beta+h)+C\end{aligned}$$

which conflicts with the definition of J. Consequently $\beta=b$ and similarly $\alpha=a$; i.e., $J=(a,b)$ and this is the desired result.

III. *Solution by O. P. Lossers, Technological University, Eindhoven, the Netherlands.* Adding and subtracting $f(x)$ implies that $f'(x)=g(x)$. Let $c\in(a,b)$, and let S be the set of positive real

numbers δ with $a < c - \delta < c + \delta < b$ and such that

(1) $$f(c+h) - f(c-h) = 2hf'(c)$$

holds for $0 \leqq h \leqq \delta$, and let $\sigma \overset{\text{def}}{=} \sup S$. We show that $c - \sigma = a$ or $c + \sigma = b$. If $a < c - \sigma < c + \sigma < b$, then $\sigma \in S$, since f is continuous. Furthermore, for $0 < h < \sigma$ we have

(2) $$\rho(h) \overset{\text{def}}{=} f'(c+h) + f'(c-h) - 2f'(c) \equiv 0,$$

which formula is obtained by differentiating (1) with respect to h. Since ρ is the derivative of a function in a neighborhood of σ and $\lim_{h \uparrow \sigma} \rho(h) = 0$, we obtain $\rho(\sigma) = 0$, that is

(3) $$f'(c+\sigma) + f'(c-\sigma) - 2f'(c) = 0.$$

For small $h > 0$ we have:

(4) $$f(c+\sigma+h) - f(c+\sigma-h) = 2hf'(c+\sigma),$$

(5) $$f(c-\sigma+h) - f(c-\sigma-h) = 2hf'(c-\sigma).$$

Since $\sigma - h \in S$, we also have

(6) $$f(c+\sigma-h) - f(c-\sigma+h) = 2(\sigma-h)f'(c).$$

Adding (4), (5) and (6) and using (3) we obtain

$$f(c+\sigma+h) - f(c-\sigma-h) = 2(\sigma+h)f'(c)$$

for small $h > 0$. This contradicts the definition of σ.

Thus we have shown that relation (1) holds whenever $c \pm h \in (a,b)$. It follows that f' is differentiable and differentiating (2) with respect to h we see that $f''(c+h) = f''(c-h)$. Since any two points of (a,b) can be written $c+h$ and $c-h$, we see that f'' is constant. Hence f is a quadratic polynomial.

Editorial Note. The solution originally printed for this proposal is incorrect and is withdrawn.

Truncated Taylor Expansions and Function Values

5958 [1974, 292]. *Proposed by Alexander Abian, Iowa State University*

Let $r \neq 0$ be an interior point of the interval of convergence of the Taylor expansion $\Sigma_{n=0}^{\infty} a_n x^n$ of an analytic function f from reals to reals. Prove or disprove that for every interval $I(r)$ with center at r there exist a real number t and a natural number k such that $t \in I(r)$ and $f(r) = a_0 + a_1 t + \cdots + a_k t^k$.

I. *Solution by C. A. Kottman, Oregon State University.* Let $f(x) = e^x - \sqrt{e}x^2$ and $r = \frac{1}{2}$. Methods of elementary calculus show that the maximum of f in the interval $I_r = (\frac{1}{4}, \frac{3}{4})$ occurs at $x = r$. If $f_k = \Sigma_{n=0}^{k} a_n x^n$ denotes the kth partial sum of the Taylor expansion of f, then $f_0(x) = 1 < f(r)$ and $f_1(x) = 1 + x > f(r)$ for $x \in I_r$. And the sequence $f_2(x), f_3(x), \cdots$ is strictly monotone increasing to $f(x) \leqq f(r)$ for each $x \in I_r$. Thus there can exist no integer k and $t \in I_r$ with $f_k(t) = f(r)$.

II. *Solution by M. J. Hoffman, University of California at Berkeley.* If f and p_k are extended to complex functions of a complex variable z by putting $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and $p_k(z) = \Sigma_{n=0}^{k} a_n z^n$, then f is analytic on a disk $D = \{z : |z| < R\}$ about the origin, and the assertion holds in the following form:

THEOREM. *If $z_0 \in D$ and $\Delta = \{z : |z - z_0| < \delta\}$ is any closed disk around z_0 with $\Delta \subset D$, then there exist $k \geqq 0$ and $t \in \Delta$ with $p_k(t) = f(z_0)$.*

Proof. The sequence $p_k(z) - f(z_0)$ converges uniformly to $f(z) - f(z_0)$ on every closed disk in D. By Hurwitz' Theorem there is a sequence $z_k \to z_0$ such that $p_k(z_k) = f(z_0)$. For large enough k, these z_k lie in Δ.

Unfortunately, nothing says that these roots have to be real, even when z_0 and all the a_n are real,

as the above counterexample shows. However, the assertion does hold at points r where f does not have a local extremum, say if $f'(r) \neq 0$. Without loss of generality we may shrink $I(r)$ to an interval $[x_1, x_2]$ with r in its interior on which f is monotone. Since $p_k(x) \to f(x)$ uniformly on $[x_1, x_2]$, for large enough k we have

$$|p_k(x) - f(x)| < \tfrac{1}{4}\min(|f(x_i) - f(r)|), \quad (i = 1,2).$$

In particular, $f(r)$ lies between $p_k(x_1)$ and $p_k(x_2)$. By the intermediate value theorem, there is a $t \in [x_1, x_2]$ with $p_k(t) = f(r)$.

Also solved by S. P. Marin, Bryce Parry, M. K. Ritt, St. Olaf College Students, Oto Strauch (Czechoslovakia), and the proposer.

Locally Compact Topological Groups

5959 [1974, 292]. *Proposed by E. J. Howard, California State University, San Diego*

T. Husain in *Introduction to Topological Groups* (Saunders, Philadelphia, 1966), p. 69 proved that: If E is a locally compact topological group and F is any Hausdorff topological group and $f\colon E \to F$ is a continuous almost open homomorphism, then F is locally compact.

Establish the same conclusion for any topological group F. (Note: f is almost open at $x \in E$ if for each open neighborhood U of x, $\overline{f(U)}$, the closure of $f(U)$, is a neighborhood of $f(x)$.

Solution by P. R. Chernoff, University of California at Berkeley. Let $H \subset F$ be the closure of the identity e; H is a normal subgroup, and F/H is Hausdorff. Let $\pi\colon F \to F/H$ be the quotient map. Then $\pi \circ f$ is almost open because π is open. Hence, by the quoted result of Husain, F/H is locally compact. But from this it follows that F is locally compact, by the following argument.

First, every neighborhood U of e contains a "saturated" neighborhood V, i.e., $V = \pi^{-1}(\pi(V)) = V \cdot H$. Indeed, choose a neighborhood W with $W^2 \subset U$. Now $H \subset W$, so $V = W \cdot H \subset U$.

Next, if C is compact in F/H, then $\pi^{-1}(C)$ is compact in F. For, let $\{V_i\}_{i \in I}$ be an open cover of $\pi^{-1}(C)$. We may assume that each V_i is saturated. Finitely many of the open sets $\pi(V_i)$ cover C, and their inverse images cover $\pi^{-1}(C)$.

Finally, let U be any neighborhood of e. Let $V \subset U$ be a saturated neighborhood, and let $C \subset \pi(V)$ be a compact neighborhood of $\pi(e)$. Then $\pi^{-1}(C) \subseteq V \subseteq U$ is a compact neighborhood of e. Thus F is locally compact.

Also solved by Peter Garst, D. L. Grant, and the proposer.

Integrals of the Rademacher Function

5960 [1974, 292]. *Proposed by J. R. Higgins, The Cambridgeshire College of Arts and Technology, England*

Let $r_k(t)$ be the Rademacher "square sine wave" function defined by $r_k(t) = \operatorname{sgn}(\sin 2^k \pi t)$, $k = 1, 2, \cdots, t \in [0,1]$, and put

$$I(m, n, k) = \int_0^{m/2^n} r_k(t)\, dt,$$

where n is any positive integer, and m is an odd positive integer less than 2^n. Show that

$$\sum_{k=1}^{n} 2^{k-1}[I(m, n, k)]^2 = \frac{m}{2^n}\left(1 - \frac{m}{2^n}\right).$$

Solution by C. W. Onneweer, University of New Mexico. For given $n \geqq 1$ and m with $1 \leqq m < 2^n$

and m odd, we can express m as

$$m = a_1 2^{n-1} + a_2 2^{n-2} + \cdots + a_n 2^0,$$

with $a_i \in \{0,1\}$ for $1 \leqq i < n$ and $a_n = 1$. Also, for each k with $1 \leqq k \leqq n$ and each interval $I \subset [0,1]$ of length 2^{-k+1} we have $\int_I r_k(t)\,dt = 0$. Consequently,

$$I(m,n,k) = I(a_k 2^{n-k} + \cdots + a_n 2^0, n, k).$$

Next, since $r_k(t) = 1$ if $t \in (0, 2^{-k})$ and $r_k(t) = -1$ if $t \in (2^{-k}, 2^{-k+1})$ we have

$$I(m,n,k) = \begin{cases} a_{k+1} 2^{-(k+1)} + \cdots + a_n 2^{-n}, & \text{if} \quad a_k = 0, \\ a_k 2^{-k} - (a_{k+1} 2^{-(k+1)} + \cdots + a_n 2^{-n}), & \text{if} \quad a_k = 1. \end{cases}$$

Therefore,

$$\sum_{k=1}^{n} 2^{k-1}(I(m,n,k))^2 = \sum_{k=1}^{n} 2^{k-1}(a_k 2^{-k} + (-1)^{a_k}(a_{k+1} 2^{-(k+1)} + \cdots + a_n 2^{-n}))^2.$$

We now expand the quadratic expressions in the last sum and determine the coefficients of $a_i^2 = a_i$ for $1 \leqq i \leqq n$, and the coefficients of $a_i a_j$ if $i \neq j$ and $a_i = a_j = 1$. For fixed i the coefficient of a_i is equal to

$$\sum_{k=1}^{n} 2^{k-1}(2^{-i})^2 = (2^i - 1)2^{-2i}.$$

For fixed i, j with, say, $i < j$ and $a_i = a_j = 1$ the coefficient of $a_i a_j$ is equal to

$$\sum_{k=1}^{i-1} 2^{k-1}((-1)^{2a_k} 2^{-i} 2^{-j}) + 2^{i-1}(-1)^{a_i} 2^{-i} 2^{-j}$$
$$= (2^{i-1} - 1)2^{-i-j} - 2^{i-1}2^{-i-j} = -2^{-i-j}.$$

Hence

$$\sum_{k=1}^{n} 2^{k-1}(I(m,n,k))^2 = \sum_{k=1}^{n} (2^k - 1)2^{-2k} a_k - \sum_{1 \leqq i \neq j \leqq n} 2^{-i-j} a_i a_j$$
$$= \sum_{k=1}^{n} a_k 2^{-k} - \sum_{i=1}^{n} \sum_{j=1}^{n} 2^{-i-j} a_i a_j = \sum_{k=1}^{n} a_k 2^{-k} - \left(\sum_{k=1}^{n} a_k 2^{-k} \right)^2$$
$$= \frac{m}{2^n}\left(1 - \frac{m}{2^n}\right).$$

Also solved by Leonard Carlitz, L. E. Clarke (England), L. Kuipers, O. P. Lossers (Netherlands), Edward Lotkowski, L. E. Mattics, H. Reuvers (Netherlands), F. G. Schmitt, Jr., and the proposer.

Editor's Notes. Other solutions of this problem use induction on n. Kuipers observes that a similar proof yields the formula

$$\sum_{k=1}^{n} 2^{n-k}[I(m,n,k)]^2 = m(2^n - m).$$

Reuvers obtains $I(m,n,k) = \{\min(m, -m) \pmod{2^{n-k+1}}\}/2^n$ and then uses induction to establish

$$\sum_{k=1}^{n} 2^{k-1}\{\min(m, -m) \pmod{2^{n-k+1}}\}^2 = \{m \pmod{2^n}\}(2^n - \{m \pmod{2^n}\})$$

for all m.

The Twice Differentiation Operator

5961 [1974, 293]. *Proposed by S. Zaidman, University of Montreal*

Let X be the space of continuous functions on $[0,1]$ which are zero at 0 and 1; let $A = d^2/dt^2$ be the operator defined on the set $\mathcal{D}(A)$ of functions $\varphi(t) \in X$ which are in $C^2[0,1]$, such that also $d^2\varphi/dt^2 \in X$. Then prove:

(1) A is a linear closed operator with dense domain in X.

(2) For any $\varphi \in \mathcal{D}(A)$, there exists a linear continuous functional on X, F_φ, such that $F_\varphi(\varphi) = \|\varphi\|^2$ and $F_\varphi(A\varphi) \leqq 0$, where, as usual, $\|\varphi\| = \max_{0 \leqq t \leqq 1} |\varphi(t)|$.

Solution by S. J. Sidney, University of Connecticut. (1) If $T: X \to X$ is given by $(Tf)(x) = (Sf)(x) - x(Sf)(1)$ where

$$(Sf)(x) = \int_0^x \int_0^u g(t)dtdu = \int_0^x (x-t)g(t)dt,$$

then T is a continuous linear operator on X. Since $Tf = \phi$ is equivalent to $\phi \in \mathcal{D}(A)$ and $A\phi = f$, A is a closed linear operator on X. Its domain contains all polynomials in $x^m(x-1)^n$ for $m, n \geqq 3$; this family separates points on $[0,1]$ except for 0 and 1, so by the Stone-Weierstrass theorem $\mathcal{D}(A)$ is dense in X.

(2) We may assume $\phi \neq 0$, whence $A\phi \neq 0$.

If ϕ is an eigenvector of A, then $\phi(x) = a\exp(px) + b\exp(-px)$ for constants a and b, where p^2 is the corresponding eigenvalue. Since $\phi(0) = \phi(1) = 0$, $a + b = 0$ and $ae^p + be^{-p} = 0$, so $b = -a \neq 0$ and $e^p = e^{-p}$, whence $2p$ is an integral multiple of $2\pi i$ and $p^2 = -n^2\pi \leqq 0$ for some integer n. Thus if we select $x \in [0,1]$ at which $|\phi(x)|$ assumes its maximum, the functional $F_\phi(f) = \bar{\phi}(x)f(x)$ will do (and has minimum possible norm $\|\phi\|$).

If ϕ is not an eigenvector of A, then ϕ and $A\phi$ are linearly independent, so we can choose F_ϕ such that $F_\phi(\phi) = \|\phi\|^2$ and $F_\phi(A\phi) = 0$.

NOTE: If we insist further that F_ϕ have (minimum possible) norm equal to $\|\phi\|$, the problem has a solution for real scalars but not for complex scalars. Indeed, in the real case, let $F_\phi(f) = \phi(x)f(x)$, where $x \in (0,1)$ is chosen to maximize $|\phi(x)|$. Then $\|F_\phi\| = \|\phi\|$ and $F_\phi(\phi) = \|\phi\|^2$, but by the second derivative test, $A\phi(x) \leqq 0$ if $\phi(x) \geqq 0$ and $A\phi(x) \geqq 0$ if $\phi(x) \leqq 0$, so $F_\phi(A\phi) \leqq 0$. We infer that in the complex case the strengthened problem can be solved with $\mathrm{Re}(F_\phi(A\phi)) \leqq 0$ instead of $F_\phi(A\phi) \leqq 0$; however, an example can be given to show that it cannot be solved with $F_\phi(A\phi) \leqq 0$.

Also solved by D. A. Bondy, P. R. Chernoff, George Crofts, B. Fishel (England), Gary Gunderson, O. P. Lossers (Netherlands), George Luna, and W. F. Shreve.

Expectations in Decreasing Joint Densities

5963 [1974, 293]. *Proposed by A. de Falguerolles and G. Letac, Université de Clermont, France*

X and Y are two positive random variables such that the density $f(x, y)$ of their joint probability distribution is decreasing in each variable x and y. Prove or disprove: If $E(X)$ and $E(Y)$ are finite, then

$$E(X) + E(Y) \leqq 3E(|X - Y|);$$

and if X and Y are less than 1, then

$$4E(XY) \leqq 3E(|X - Y|).$$

Solution by S. M. Samuels, Purdue University. To obtain the first inequality, we have

$$3E(|X-Y|) - E(X) - E(Y) = 2E[(X - 2Y)I_{(X \geqq Y)} + (Y - 2X)I_{(Y \geqq X)}];$$

$$E[(X-2Y)I_{(X\geqq Y)}]=\int_0^\infty\int_0^x (x-2y)f(x,y)\,dy\,dx$$
$$\geqq\int_0^\infty\int_0^x (x-2y)f(x,x)\,dy\,dx=0.$$

Symmetrically, $E[(Y-2X)I_{Y\geqq X}]\geqq 0$.

For the second inequality, with $X,\ Y\leqq 1$, we see that

$$3E(|X-Y|)-4E(XY)=E[(3X-3Y-4XY)I_{(X\geqq Y)}+(3Y-3X-4XY)I_{(Y\geqq X)}];$$

$$E[(3X-3Y-4XY)I_{(X\geqq Y)}]=\int_0^1\int_0^x (3x-3y-4xy)f(x,y)\,dy\,dx$$
$$\geqq\int_0^1\int_0^x (3x-3y-4xy)f(x,x)\,dy\,dx$$
$$=\int_0^1 \tfrac{1}{2}(3x^2-4x^3)f(x,x)\,dx$$
$$\geqq\int_0^1 \tfrac{1}{2}(3x^2-4x^3)f(3/4,3/4)dx=0.$$

Symmetrically, $E[(3Y-3X-4XY)I_{(Y\geqq X)}]\geqq 0$.

Equality occurs whenever

$$f(x,y)=\begin{cases} pg(x)/x & \text{if } y<x \\ (1-p)h(y)/y & \text{if } x<y\end{cases}$$

with $0<p<1$, where $g(\cdot)$ and $h(\cdot)$ are any probability density functions on $(0,\infty)$ in the first case, or $(0,1)$ in the second case.

Also solved by L. E. Clarke (England), N. L. Johnson, and Harry Lass.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Differential Equations with Applications and Historical Notes. By George F. Simmons. McGraw-Hill, New York, 1972, xvii +465 pp. $11.95. (Telegraphic Review, June-July, 1972.)

With so many fine texts on differential equations available, a new one should either be very well done or have a fresh approach. George Simmons' new book satisfies both of these requirements. As the title indicates this book includes applications and historical notes. The author's treatment of applications is not the passive annexation of optional material often found in recent texts. Rather, it follows an "activist's approach" whose primary aim, according to Simmons, is "to develop methods for solving scientific problems in contrast to the contemplative school, which analyzes and organizes

the ideas and tools generated by the activists." So one is not surprised to see that the introductory chapter treats the separation of variables via the brachistochrone problem, the motion of a pendulum, first and second order chemical reactions and falling bodies. There is also an excellent account of the discovery and development of radiocarbon dating. Later chapters include the following (nonexhaustive) list of applications: Newton's law of gravitation, the prey-predator equations, nonlinear mechanics, mathematical functions of physics.

The historical notes are biographical pieces varying from a couple of sentences in a footnote to a chapter appendix of eight pages on Gauss. The historical material could be omitted and this would still be an excellent book. It does not interrupt the mathematical presentation and one does not have to teach around it. We used in total only two class hours on historical discussions, although the class was responsible for (and tested on) a fair amount of historical material.

We used this text in an introductory course for science and mathematics majors. We studied first order equations, second order linear equations, power series solutions and the Laplace transform (chapters 1, 2, 3, 5 and 10). The only prerequisite was a year of calculus. This is all the author assumes, too. The students reacted well to the activist's approach and were stimulated by the biographical material. They occasionally asked questions on unassigned sections which indicates how appealing the book is to students. There are six chapters which deal with oscillation theory, systems of first order equations, calculus of variations, existence and uniqueness of solutions in addition to topics already mentioned.

Simmons has a lucid, succinct style. He obviously has put considerable time and thought into this book. There are no distracting side issues. Unusual restraint and economy is apparent in the presentation of second order linear equations: linear algebra is not developed or assumed in spite of the clear temptation to do so. One does not need more than the linear dependence of two functions, a concept Simmons introduces without explicit mention of linear combinations. The point is that to develop even such an attractive topic as linear algebra would divert the activist from the immediate task, which is solving second order linear equations. We worked most of the exercises in the five chapters we studied without finding an error. No misprints were found. The sections are numbered one to fifty-seven without regard to chapter numbers. Exercises are listed at the end of each section. Their solutions (all of them) are listed, in large type, by section, at the end of the book. It is a convenient arrangement.

This is a fine book. It will become popular even against strong competition. I recommend it without reservation.

S. H. COX, JR., University of Puerto Rico

Abstract Algebra: A First Course. By Larry Joel Goldstein. Prentice-Hall, Englewood Cliffs, New Jersey, 1973. xii + 335 pp. $11.95. (Telegraphic Review, April 1974.)

I used this book in an honors algebra course for which Herstein's *Topics in Algebra* was the traditional choice. Goldstein's proofs are less elliptic than Herstein's and on the whole his exercises are easier, but the book is still written at a reasonably sophisticated level — too high, I believe, for an average modern algebra class. For my honors students, however, the level was quite suitable.

The book is very well written and it reads smoothly. When new concepts are introduced they are carefully explained and a generous supply of examples is provided (17 diverse examples of groups, 23 of rings, etc.). Additional examples are developed in the exercises, which on the whole are good. However, there is some unevenness here: many sections are followed by large sets of exercises of varied difficulty which reinforced and expanded upon the ideas of the section; in other sections there are few interesting problems, and it was necessary to supplement them with others.

Goldstein presents his book as "an historical approach to modern algebra." This manifests itself in two ways: (i) the solution of important historical problems using algebraic techniques, and (ii) short historical comments. The first is the major contribution of Goldstein's book to the literature and is generally exciting. In addition to solving the three classical construction problems and proving the insolubility of the quintic, as is done in many algebra texts, Goldstein offers us, among other things, a proof of Fermat's last theorem for $n = 3$ and a readable discussion of Kummer's work on the general case; solutions in radicals of equations of degree 2, 3, or 4; an essentially algebraic proof (due to Artin) of the Fundamental Theorem of Algebra; and a proof of a special case of Dirichlet's theorem on primes in arithmetic progressions. The historical comments, unfortunately, are much less successful. The page-long asides on the prime number theorem and on Gauss' law of quadratic reciprocity, although interesting to the more knowledgeable reader, seemed to distract the students, rather than enlighten them; and the calculation of Galois groups in the introductory chapter on "historical perspectives" overwhelmed them.

The only other complaint is that the proofreading got sloppy toward the end of the book. The first few chapters are almost error free, whereas chapter 9 has at least 25 "typos." A strange error, especially for a number theorist, is that primes of the form $2^{2^n} + 1$ are consistently referred to as Mersenne primes (rather than Fermat primes), and that 3 is omitted from the list of all such known primes.

Throughout the choice of topics and the way they are handled is excellent. A particularly nice touch is that in the chapter on Galois theory it is assumed that all fields are extensions of the rationals. By ignoring fields of characteristic p (a first year graduate course is time enough), he makes the students' first exposure to Galois theory much more comprehensible and enjoyable.

On balance I was very favorably impressed by this book. The "typos" and few small errors can be corrected in a second printing which, I believe, will be warranted. To anyone teaching a first algebra course to a class of good students I strongly recommend consideration of this text.

IRA K. WOLF, Brooklyn College

Elementary Linear Algebra. By Evar D. Nering. Saunders, Philadelphia, Pennsylvania. 1974 ix + 375 pp. $12.50. (Telegraphic Review, March 1974.)

Can linear algebra wander far from its geometric moorings and still be successful at the sophomore level? Those that are convinced that the answer must be "no" have heretofore been left with two choices—insubstantial books firmly rooted in geometry, and substantial books inadequately rooted in motivation. As an example of the first, *Introduction to Modern Algebra*, by John Kelly, D. Van Nostrand Co., 1960, has accompanying pictures for almost every topic, but he never gets past R^3. On the other hand, which is the easier way to remember Bessel's inequality, as

$$\sum_{k=1}^{m} \frac{|(\beta, \alpha_k)|^2}{\|\alpha_k\|^2} \leqq \|\beta\|^2$$

and equality holds if and only if

$$\beta = \sum_{k=1}^{m} \frac{(\beta, \alpha_k)}{\|\alpha_k\|^2}\alpha_k,$$

or as the geometric statement: "a vector is at least as long as its projection on any subspace, and equal to that projection in length if and only if it is in the subspace"? I claim that omitting the latter as do Hoffman and Kunze, *Linear Algebra*, Prentice Hall, Inc., 1961, p. 232 is inexcusable. A textbook should not be a list of facts.

Nering's book is a partially successful attempt to bridge that gap. For example, determinants are treated geometrically but axiomatically. Projections, rotations, reflections and shear transformations are illustrated geometrically in the text. Then an abstract definition of projection in the exercises (p. 183) paves the way for the spectral resolution theorem two chapters later (p. 268). Most books either play down numerical examples or present them mechanically. Nering does a good job of letting numerical examples expose the theory.

There is no discouraging introduction to set theory. (The discouragement of the first chapter is that matrix operations are unmotivated.) Instead, set theory is relegated to an appendix for reference. I introduced what notation I needed as I went along. Occasionally Nering, in his attempt to keep the notation uncluttered, uses a symbol both for a set of vectors and for the space they span (p. 102) or for a basis and the natural map from R^n to the space with that basis (consistently; see, e.g., p. 111). This confused the students, but helped them to develop a healthy criticism of material they read. A list of errors for the first five chapters (11 of them) is available from this reviewer.* Only one of them is conceptual: a coset is not a subspace (p. 203).

The students were pleased that current journal articles are accessible to them, as illustrated by the parallel between pp. 102–103 and W. S. Ericksen's article *The intersection of subspaces*, this MONTHLY, 81(1974) 159–60. Unfortunately that section is one of the most uneven expositions of the book. His comments range from fatherly advice to obscurities. The important concept of isomorphism receives only passing mention there, and one preparatory problem before that (12, p. 75) prior to its formal introduction on p. 177. The lack of problems on isomorphism is typical of the computational orientation of the problems in general. If you plan to use the book, plan to supplement it with problems of a theoretical nature. What little emphasis there is on proofs in the problems is for very special settings. Since one of the primary goals of linear algebra at the sophomore level is to increase the students' ability to do proofs, I feel that his failure here is Nering's major weakness.

The consistent use of the pivot operation unifies the treatment. A gentle introduction to mapping diagrams proves to be illuminating. The selection of topics is predictably standard—similar up to permutation to the rest of the deluge of linear algebra books. (Cf. reviews in this MONTHLY by D. E. Christie, June-July, 1973, and by David E. Kullman, March, 1974.) Student reaction uniformly applauded the pictures, criticized complicated notation, and wished that the text were more suitable for self-study. I too felt that lectures were necessary to fill in the gaps in the text.

Since the same students were taking advanced calculus from me, it was with some satisfaction that I was able to present the chain rule for derivatives from $R^m \to R^n$ in the "clean" matrix form because of their exposure to this concrete but modern introduction to linear algebra. I especially recommend the book to those instructors who like a text which is different from the way they will present the material in class. A stereoscopic view is more illuminating than a monocular view.

GENE B. CHASE, Messiah College

* Some but not all of these misprints have been corrected in a second printing, *Editor*.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook — P = professional reading
S = supplementary reading — L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks(?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(15-18), P*, L. *Mathematiker über die Mathematik*. Michael Otte. Springer-Verlag, 1974, 481 pp, $9.80 (P). A series of essays (all in German, sometimes by virtue of translation) by mathematicians about the nature of, research in, applications of, and teaching of mathematics. Most were written in the last five years, some specially written for this volume. The German isn't easy but sources for all essays are given and they do represent a significant body of statements about mathematics. JAS

GENERAL, S(13-16), L. *Die Kunst des Sehens in der Mathematik*. Bruno deFinetti. Birkhauser, 1974, 91 pp, $8. A book about imagination! Here is a presentation of nearly a score of examples of seeing mathematical patterns in practically anything. The mathematical concepts needed go no deeper than calculus. A nice collection of essays illustrating the role of "phantasy" in perceiving the mathematics in a problem. JAS

GENERAL, S*, L*. *6 Thèmes pour 6 Semaines*. André Myx. CEDIC, 1975, 341 pp, (P). Another good book from CEDIC. The six themes: logic, relations, order and equivalence, measure, enumeration, and geometry and topology. Intended for a "math-for-poets" course or for teachers. From the preface: "Ce n'est pas un livre, c'est un myxtification." PJM

GENERAL, T*(13-14: 1). *Mathematics, An Appreciation*. Michael Bernkopf. HM, 1975, xi + 276 pp, $10.95. Well-done, one-semester terminal course for liberal arts students. Euclid's fifth postulate, probability, mathematics of motion. LH

GENERAL, T(1). *Patterns in the Sand: An Exploration in Mathematics, Second Edition*. Maurice Bosstick, John L. Cable. Glencoe Pr, 1975, 384 pp, $10.95; *Instructor's Answer Booklet*, 42 pp, free (P). Additions to the first edition (TR, April 1972): an explicit treatment of sets; a brief discussion of FORTRAN IV; expanded discussion of the number system to include decimals, ratios and percents, with a section on money lenders; examples and exercises. MG

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1971, V. 24*. AMS, 1974, iii + 247 pp, $35.40.

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1973, V. 28*. AMS, 1975, 256 pp, $37.50.

GENERAL, S(16-18), P. *Les Mathématiques et la Réalité: Essai sur la Méthode Axiomatique*. Ferdinand Gonseth. Blanchard, 1974, 386 pp, (P). Reprint of the 1936 original. A non-technical philosophical investigation of mathematics, logic, and a little physics. JAS

GENERAL, S(17), P, L. *Mathematical Structure in Human Affairs*. R.H. Atkin. Crane, Russak, 1974, v + 212 pp, $18.75. What other book is illustrated with a picture of an English village street captioned "a visual 18-simplex"? Diverse aspects of human experience interpreted in the multidimensional realm of simplicial homology, including: chess (confusing misprints in diagrams, notation), a Mondrian painting (unconvincing analysis), physics ("the cocycle law"), urban structure, and faculty committee deliberations about moving the university bar (where the theory shows real possibilities). Text uses matrix algebra (connectivity of communication matrices), with hard stuff (deRham cohomology, Clifford algebras) relegated to appendices. PJC

GENERAL, L*. *Mathematik, 2. Auflage*. I. Szabó, K. Wellnitz, W. Zander. Springer-Verlag, 1974, xviii + 373 pp, $19.60. Something new in hand books! Tables, definitions and important theorems from a broad range of mathematical sciences including computer programming. An extensive bibliography and good indexing should make this a very handy volume. Basically includes all material found in undergraduate and beginning graduate mathematics with the exception of abstract algebra and topology. Includes: statistics, linear algebra, classical curve and surface theory, and even spline interpolation. JAS

BASIC, T**(13: 1, 2). *Contemporary Business Mathematics*. Ignacio Bello. Saunders, 1975, xv + 572 pp, $12.50; *Instructor's Manual*, iii + 158 pp, (P). Many practical examples. A clear, attractive, interesting presentation. No algebra background required. Instructor's manual with worked solutions. LH

BASIC, T(13: 1). *Applied Math for Technicians*. Claude S. Moore, Bennie L. Griffin, Edward C. Polhamus, Jr. P-H, 1975, xiv + 319 pp, $10.95. Arithmetic, simple algebra, metric system. Machine shop, construction job illustrations and problems. LH

BASIC, T(13: 1). *Mathematics of Business, Second Edition*. Jay Diamond, Gerald Pintel. P-H, 1975, x + 386 pp, $11.95. Reviews arithmetic, metric system. Applications to loans, discounts, taxes, bonds, annuities. LH

BASIC, T(13). *Technical Mathematics, Third Edition*. Harold S. Rice, Raymond M. Knight. McGraw, 1973, xvi + 697 pp, $10.95. Use of slide rule, algebra, and trigonometry. Lots of nice "word problems." PJM

BASIC, T(13-14: 1, 2). *Contemporary College Mathematics.* William F. Brett, Louis C. Contey, Michael Sentlowitz. West Pub, 1975, xi + 320 pp, $12.95. Intended for liberal arts students, business oriented students, and/or social science majors. Covers sets, logic, real numbers, arithmetic in various bases, probability, statistics, systems of equations, matrices and determinants, linear programming, simple and compound interest, annuities and perpetuities, life insurance. A clearly written book at a very elementary level. SG

PRECALCULUS, T*(13: 1, 2). *Introductory Mathematics: A Prelude to Calculus.* Robert C. Fisher, *et al.* Merrill, 1975, xii + 475 pp, $11.95. Many problems, clear examples. Nice section on A sin (ax+b). Functions, inverse functions, logarithms and exponentials, trigonometry, and complex numbers. LH

PRECALCULUS, T(13). *Precalculus Mathematics: A Functional Approach.* James F. Connelly, Robert A. Fratangelo. Macmillan, 1975, ix + 431 pp, $10.95; *Study Guide,* x + 322 pp, $4.95 (P). Algebra and trig, plus complex numbers, analytic geometry, and sequences. Study guide with more exercises and chapter summaries separate. Functions introduced before any algebra. PJM

PRECALCULUS, T*(13). *Algebra and Trigonometry.* Harley Flanders, Justin J. Price. Acad Pr, 1975, xiii + 425 pp, $11.95. Very well written. Attractive format. Basic algebra and trigonometry plus functions and inverse functions, vectors, complex numbers. 2300 exercises. LH

PRECALCULUS, T(13). *College Mathematics.* Eugene D. Nichols. Rinehart Pr, 1975, x + 382 pp, $9.95. Revision of 1970 text. Drill style. Algebra, geometry, matrices, graphs of functions, logic, probability and statistics. LH

PRECALCULUS, T(13: 1). *Elementary Functions.* Adil Yaqub. HM, 1975, 427 pp, $11.95. Many examples; problems. Very broad coverage of topics: complex numbers, trigonometric functions, analytic geometry, induction, progressions and binomial theorem. LH

PRECALCULUS, T(13). *College Algebra.* Louis Leithold. Macmillan, 1975, xi + 558 pp, $10.95. Real numbers, equations in one and two variables, functions, exp and log, systems of equations, inequalities, matrices, sequences and induction, permutations, complex numbers, polynomials ..."more topics than can be covered in...three semester hours." A good book, the author clearly intends it to be suitable to several different audiences. Neat chapter headings using Jasper Johns' lithographs. PJM

PRECALCULUS, T(13). *Algebra, Trigonometry, and Analytic Geometry, Second Edition.* Paul K. Rees, Fred W. Sparks, Charles Sparks Rees. McGraw, 1975, xi + 596 pp, $12.95. Interesting approach: trigonometry is done before algebra. Also includes sections on matrices, polar coordinates, vectors, surfaces. It would be hard to cover this whole book in one semester unless some of the material was review. Ample and good exercises. PJM

PRECALCULUS, T(13: 1), S*. *Geometry and Trigonometry for Calculus.* Peter H. Selby. Wiley, 1975, viii + 424 pp, $5.95 (P). A programmed text which covers elementary plane geometry, trigonometry, and analytic geometry, along with a brief introduction to limits. Includes precise objectives and frequent study questions. CEC

EDUCATION, P**, L. *The Origin of the Idea of Chance in Children.* Jean Piaget, Bärbel Inhelder. Trans: Lowell Leake, Jr., Paul Burrell, Harold D. Fishbein. Norton, 1975, xx + 251 pp, $12.50. Translation of the 1951 book by the noted Swiss developmental psychologist and his chief collaborator. Describes and illustrates by many experiments three stages in the development of ideas of chance, with the middle stage covering the period from approximately 7 or 8 to 11 or 12 years of age. Supplements the authors' previous publications on the development of thought in the child, but can be read independently. RSK

EDUCATION, T*(14-15: 1), L. *Mathematics, An Exploratory Approach.* Robert G. Stein. McGraw, 1975, xiii + 253 pp, $11.95. A good book, but very difficult to classify. A refreshing approach--experiment, and generalize--to elementary topics: negative and figurate numbers, graphs and rules, pythagorean theorem, polygons and polyhedra. Designed for elementary teachers. PJM

EDUCATION, T(14-15), L. *Modern Elementary Mathematics, Second Edition.* Malcolm Graham. HarBrace J, 1975, xi + 449 pp, $10.95. For prospective elementary teachers, sets, systems of numbers, geometry, and one chapter on probability and descriptive statistics. PJM

EDUCATION, S(15-16), P. *Beiträge zum Mathematikunterricht 1974.* Hermann Schroedel, 1974, 167 pp, 32,80 DM (P). Proceedings of a mathematics education conference at Berlin in March 1974. JAS

EDUCATION, T*(13-14: 2). *Fundamentals of Elementary Mathematics: Number Systems and Algebra.* Merlyn J. Behr, Dale G. Jungst. Acad Pr, 1971, xx + 419 pp, $4.20. Well-written, for students with little high school preparation. Elementary teachers should profit therefrom. Authors treat elementary logic, sets, functions and subsystems of reals slowly and carefully with motivations and abundant exercises. Some treatment of groups, fields, rings. AG

HISTORY, P, L***. *The Mathematical Papers of Isaac Newton, Volume VI, 1684-1691.* Ed: D.T. Whiteside. Cambridge U Pr, 1974, xxxv + 614 pp, $72.50. Extensively annotated treatment of Newton's 18 month response to Christopher Wren's challenge theoretically to define the paths of the solar planets. Exposes the evolution of Newton's *De Motu Corporum* from original to penultimate draft, from whence it was incorporated in *Principia.* A meticulously documented resource that will serve scholars for generations. LAS

HISTORY, L*. *The Collected Mathematical Papers of Leonard Eugene Dickson, V. I-V.* Ed: A. Adrian Albert. Chelsea, 1975. *V. I:* xvii + 680 pp; *V. II:* 766 pp; *V. III:* 580 pp; *V. IV:* 636 pp; *V. V:* 644 pp, $140 set. The major part of Dickson's published papers, keyed to a bibliography which concludes

Volume V. Remaining papers (those that proved difficult to locate) will be published later as Volume VI. Includes (in Vol. IV) a corrected edition of *Algebren und ihre Zahlentheorie*; other Dickson books are still in print. A major monument to one of America's premier mathematicians. LAS

FOUNDATIONS, P, *Modelltheorie II.* Wolfram Schwabhäuser. Bibliographisches Inst, 1972, 123 pp, 9,80 DM (P). Chapters on completeness, elementary extensions, ultraproducts, and classification by prefix type. JAS

FOUNDATIONS, S, L, *The Appeal of the Paradoxical: An Inaugural Lecture.* R.O. Davies. Leicester U Pr, 1974, 14 pp, £0.40 (P). A brief exposition of the role of paradoxes in the development of mathematics. JAS

COMBINATORICS, S(16-18), P*, L, *Algebraic Graph Theory.* Norman Biggs. Tracts in Math., No. 67. Cambridge U Pr, 1974, vii + 170 pp, $11.50. A marvelous book that brings together much of the known material on the algebraic approach to graph theory and much that is new. It is divided into three parts--the applications of linear algebra to graph theory, coloring problems, and symmetry and regularity of graphs. MG

NUMBER THEORY, S, L, *Fibonacci's Problem Book.* Ed: Marjorie Bicknell, Verner E. Hoggatt, Jr. Fibonacci Assoc, 1974, 66 pp, (P). A total of F_{12} choice problems, marked as elementary or advanced, together with solutions as they appeared in the *Fibonacci Quarterly*. Not just "further identities", but problems in algebra, geometry, and analysis too. PJC

NUMBER THEORY, T?(14-15: 1), S*, *Number Theory.* T.H. Jackson. Routledge & Kegan Paul, 1975, vii + 88 pp, $4.75 (P). A well written, but very brief introduction to elementary number theory. Primarily concerned with congruences and writing integers as sums of squares. Could use more exercises. CEC

LINEAR ALGEBRA, T(14-16: 1, 2), L, *Linear Algebra.* Ichiro Satake. Trans: Sebastian Koh, Tadatoshi Akiba, Shin-ichiro Ihara. Pure and Appl. Math., V. 29. Dekker, 1975, xi + 375 pp, $13.75. Linear Algebra is presented as algebra with interpretations in geometry and matrices. n-tuples are introduced first, axioms later. However, the material presented goes quite far and includes quadratic forms, Jordan form, and tensors. Relatively few exercises, good index. Quality mathematics appropriate for serious students. JAS

LINEAR ALGEBRA, T(14: 1, 2), *An Introduction to Linear Algebra.* Hans Samelson. Wiley, 1974, xi + 265 pp, $15.95. A relatively sophisticated introduction to linear algebra. Nicely written, it flows smoothly and exhibits considerable in-depth feel for the quintessence of the subject. Real, but definitely applicable, mathematics. JAS

ALGEBRA, T*(15-16: 2), L*, *Topics in Algebra, Second Edition.* I.N. Herstein. Xerox, 1975, xi + 388 pp, $14.95. Changes: direct products, more on Sylow theorems. A little more on Galois theory (including an explicit polynomial with group S_5). Still an excellent text. PJM

ALGEBRA, T*(14-17: 1, 2), S, P, L*, *Arrows, Structures, and Functors: The Categorical Imperative.* Michael A. Arbib, Ernest G. Manes. Acad Pr, 1975, xiii + 185 pp, $8.95. The title is to be taken seriously as an outline of the book. Designed not to depend on knowledge of algebra and topology at a high level but rather to introduce computer scientists and even mathematical biologists to material which is being used in the research literature. Generally very well written with an unusual chance of achieving its objective for a book that represents a rather new and radical turn in category theory texts. JAS

ALGEBRA, T(14-16: 1, 2), L, *An Introduction to Modern Algebra.* Burton W. Jones. Macmillan, 1975, xiii + 366 pp, $12.95. An abstract algebra text that starts right off with several examples of groups, then develops the theory. Also covers fields, a little linear algebra, ideals (with a view towards algebraic number theory), and Galois theory. Biographical notes are a pleasant addition. Plenty of exercises. PJM

ALGEBRA, S*(13-15), *Groups, A Concrete Introduction Using Cayley Cards.* John Mason. Transworld Pub, 1975, 125 pp, (P). A good introduction to group theory for the independent learner. One begins by manipulating Cayley Cards (included) which are in reality elements of A_4. Includes chapters on subgroups, homomorphisms, and permutations. Well written. CEC

ALGEBRA, P, *Semigroups.* E.S. Ljapin. Trans. Math. Mono., V. 3. AMS, 1974, xiii + 519 pp, $23.50. A translation of the revised (with supplemental bibliographies as of 1972) third Russian edition. The aim is to give a thorough up-to-date coverage of the general theory. JAS

ALGEBRA, T(18: 2), P, *Distributive Lattices.* Raymond Balbes, Philip Dwinger. U of Missouri Pr, 1974, xiii + 294 pp, $25. The foundations and techniques of the theory of distributive lattices using the methods of universal algebra and category theory. Includes numerous exercises and an extensive bibliography. An impressive book. CEC

ALGEBRA, S(18), P, *Lecture Notes in Mathematics-423: Geometric Theory of Algebraic Space Curves.* S.S. Abhyankar, A.M. Sathaye. Springer-Verlag, 1974, xiv + 302 pp, $11.50 (P). A complete proof (including lots of machinery) that all irreducible nonsingular space curves of degree at most five and genus at most one, over an algebraically closed ground field, are complete intersections. CEC

ALGEBRA, P, *Topics in the Homological Theory of Modules over Commutative Rings.* Melvin Hochster. CBMS Reg. Conf. in Math., No . 24. AMS, 1975, vii + 75 pp, $4.10 (P). Lectures on getting information about rings from Ext and Tor on modules. PJM

ALGEBRA, T?(15-17), L, *Einführung in die Algebra.* G. Fischer, R. Sacher. Teubner, Stuttgart, 1974, 238 pp, 15,80 DM (P). A short exposition of the basics of group theory (including solvability and Sylow theorems), ring theory, and Galois theory. A good source for some of the important things that lie just beyond the usual undergraduate courses. JAS

CALCULUS, T*(13-14: 2). *Calculus*. Geoffrey Matthews. Transatlantic Arts, 1964, x + 350 pp, $6.95. Ideas come quickly, informally, justification deferred. Min/max applications by page 34, definite integral by page 55 but gets to differential equations and partial derivatives. Excellent examples, problems, clear style. LH

CALCULUS, T(13-14: 1). *Practical Calculus for the Social and Managerial Sciences*. Laurence D. Hoffmann. McGraw, 1975, ix + 406 pp, $11.95. Highly intuitive (e.g., tangent not defined) but includes a few proofs. Gets to max/min for several variables. Adequate problem sets. LH

CALCULUS, T(13: 1, 2). *Concepts of Calculus with Applications to Business and Economics*. David G. Crowdis, Susanne M. Shelley, Brandon M. Wheeler. Glencoe Pr, 1975, vii + 340 pp, $11.95. An intuitive presentation with key concepts developed through examples and graded exercises instead of mathematical statements. Most applications are taken from business and economics. Contains optional exercises throughout that require a computer. BASIC is discussed explicitly in the last chapter. MG

CALCULUS, T*(13: 1, 2). *A Short Course in Calculus with Applications*. Hugh G. Campbell, Robert E. Spencer. Macmillan, 1975, viii + 328 pp, $11.95. A well-written book for social and life scientists. Ideas are strongly motivated by examples of considerable variety. In fact, there are so many that they may obscure the ideas. The intuition of definitions is stressed without obscuring the mathematics. Probability is well-integrated in the examples and applications--an admirable idea. Learning aids (introductions to chapters, vocabulary lists, suggested problem assignments, etc.) are good. MG

CALCULUS, T?(14-16: 1, 2), S, L. *Analysis I-III*. Christian Blatter. Springer-Verlag, 1974. *I:* xvi + 204 pp; *II:* xii + 180 pp; *III:* xii + 184 pp, $6.10 (P) each. Real numbers, complex numbers, sequences and series, continuity, differentiability, Riemann integral, power series, multivariable calculus through Stokes theorem: it's all here. Without exercises it's more a handbook than a text. JAS

CALCULUS, T(13-14: 2), S. *A Short Calculus for Management, Economics and the Life Sciences*. Kenneth Loewen. Prindle, 1975, xi + 318 pp, $11.95. Aimed at management, economics students with little math proficiency. Works directly towards optimization for several variables including constrained problems. Little discussion of integration. Clear style. LH

CALCULUS, T(13-14: 1). *Essential Calculus with Applications in Business, Biology, and Behavioral Sciences*. Margaret L. Lial, Charles D. Miller. Scott F, 1975, 346 pp, $11.95. Each chapter has diagnostic pretest of prerequisite skills plus a case study which applies new skills learned there. Moves from fundamentals of algebra to Lagrange multipliers in 300 pages. LH

CALCULUS, T(13-14: 3). *Calculus and Analytic Geometry*. Peter Frank, David A. Sprecher. Har-Row, 1975, xii + 734 pp, $15.95. A text for the standard three-semester course which contains applications from biology, social sciences, and physical sciences. Does not treat linear algebra or differential equations. New concepts are strongly motivated. The authors continually discuss the implications of what they are doing. The stress is on intuition, e.g., limits are defined by "the function f approaches a single number L as x approaches x_0." The formal definition is stated later. Difficult proofs and ideas are postponed. Illustrations are excellent and plentiful. There are many worked-out examples with steps clearly indicated. Most exercises are for the average student. MG

REAL ANALYSIS, P. *Lecture Notes in Mathematics-413: Variation Totale d'une Fonction*. Michel Bruneau. Springer-Verlag, 1974, xiv + 332 pp, $12.30 (P). Intended to give "an air of youth" to the study of real functions. Extensive indices and bibliographic material supplement exposition of functions of bounded p-variation and other classes of functions. JAS

COMPLEX ANALYSIS, T*(16-17: 1, 2). *Complex Variables*. Herb Silverman. HM, 1975, xi + 415 pp, $13.95. Unlike other first courses,whose goal appears to be making complex analysis "relevant" by getting to fluid flow in the shortest possible time. A solid one year introductory course with provocative questions to stimulate the student after each section. Can be used for a one semester course. The final 5 chapters offer introductions to more advanced topics. TAV

COMPLEX ANALYSIS, P*, L. *Le Théorème de Picard-Borel el la Théorie Des Fonctions Méromorphes*. Rolf Nevanlinna. Chelsea, 1974, x + 171 pp, $8.50. Of historical interest. An unabridged reprint of the 1939 classic. The results appear in English in Chapters VI to X of the author's *Analytic Functions*, Springer-Verlag, 1970. TAV

COMPLEX ANALYSIS, P. *The Neumann Problem for the Cauchy-Riemann Complex*. G.B. Folland, J.J. Kohn. Princeton U Pr, 1972, viii + 146 pp, $5.50 (P). A detailed solution to the Neumann problem--a boundary value problem for the complex Laplacian on subsets of C^n--followed by applications to complex function theory. LAS

DIFFERENTIAL EQUATIONS, T(14-16: 2). *A First Course in Differential Equations*. Frank G. Hagin. P-H, 1975, ix + 342 pp, $12.95. The author claims "the book was written with the typical sophomore firmly in mind." Does a good job in getting across some application of linear algebra without formally assuming it on the readers' part. Covers standard topics for an introductory text in ODE. Has nominal yet inspiring material on numerical methods: one sample program (written in both BASIC and FORTRAN), several flow charts, Runge-Kutta schemes and Predictor-Corrector methods. I-CH

DIFFERENTIAL EQUATIONS, P. *Notes on Time Decay and Scattering for Some Hyperbolic Problems*. Cathleen S. Morawetz. CBMS Reg. Conf. in Appl. Math., No. 19. SIAM, 1975, v + 81 pp, $7.65 (P).

NUMERICAL ANALYSIS, P. *Numerische Behandlung von Differentialgleichungen*. R. Ansorge, *et al.* Int. Ser. Num. Math., V. 27. Birkhauser, 1975, 355 pp, $18. Proceedings of the symposium in June 1974 at Oberwolfach. JAS

NUMERICAL ANALYSIS, T(15-16: 2), S. *Methods of Computation: The Linear Space Approach to Numerical Analysis.* Jens A. Jensen, John H. Rowland. Scott F, 1975, 303 pp, $13.95. Even with material carefully organized and well-referenced for matrix operations (esp. the sections on eigenvalues), iterations, interpolations and applications, and Chebyshev's best approximation, the theme does not quite support the book's subtitle. Occasionally some exercise problems involve flow-charting and call for canned sub-programs. The rest of the book can be taught without assuming programming language. I-CH

NUMERICAL ANALYSIS, T(14-15: 1). *Elementary Numerical Analysis.* J.T. Day. Vantage Pr, 1974, 117 pp, $13. Covers standard material on a very elementary level: iteration (no comparision of rate of convergence), Gaussian elimination (only for three by three systems), interpolation (no definition of interpolation polynomial given), numerical solutions of ODE's (no error analysis), etc. Contains FORTRAN IV Programs. Few exercises, none of them challenging. No index. I-CH

NUMERICAL ANALYSIS, T(18: 1, 2), P. *Analiza Numerica.* Gheorghe Marinescu. Editura Academiei Romania, 1974, 302 pp, Lei 22,50. Finite and infinite dimensional cases with extensions to general uniform spaces. JAS

NUMERICAL ANALYSIS, P. *Numerische Methoden der Approximationstheorie, Band 2.* L. Collatz, G. Meinardus. Int. Ser. Num. Math., V. 26. Birkhauser, 1975, 199 pp, $14. Lectures from the symposium in June 1973 at Oberwolfach. JAS

NUMERICAL ANALYSIS, T(16-17: 1, 2), L. *Numerical Computing and Mathematical Analysis.* Stephen M. Pizer. SRA, 1975, xxvii + 529 pp, $12.95. Oriented toward computer science students with an emphasis on the usefulness of analysis. Covers linear systems and eigenvalue problems; nonlinear equations; interpolation, integration and least squares; and the numerical solution of ODEs. Good sections discussing errors. Uses PL/1. RWN

FUNCTIONAL ANALYSIS, T*(17-18: 2), S. *Elements of Functional Analysis.* L.A. Lusternik, V.J. Sobolev. Halsted Pr, 1974, xvi + 360 pp, $21.50. An English translation of the second extensively enlarged and rewritten Russian edition. Some of the significant augumentations are: (S.L.) Sobolev spaces, Schauder principle and its applications, the spectral theory of non-bounded linear operators, etc. As in the earlier edition, the second edition avoids the coverage of some esoteric topics, such as normed rings, representation theory, semi-ordered spaces and generalized functions. I-CH

FUNCTIONAL ANALYSIS, T(18: 1), S, P. *Partially Ordered Topological Vector Spaces.* Yau-Chuen Wong, King-Fu Ng. Oxford U Pr, 1973, xi + 217 pp, $22.50. Theory for locally convex spaces, followed by applications for partially ordered Banach spaces and locally convex Riesz spaces. Concerned with 'influence' of order properties. LH

FUNCTIONAL ANALYSIS, T(17-18: 2), S. *An Introduction to Functional Analysis.* Mischa Cotlar, Roberto Cignoli. Trans: A. Torchinsky, A. González Villalobos. North-Holland, 1974, xiv + 585 pp, $23.10. A translated and revised version of the original Spanish edition. Covers the fundamental and traditional topics of linear analysis on normed and ordered vector spaces; also treats approximation theory, Banach algebras, spectral theorem and Fourier transform. Emphasizes didactical aspects in the motivation and promotion of concepts, and in the detailed explanation of the background material. I-CH

FUNCTIONAL ANALYSIS, P. *Topics in Operator Theory.* Ed: C. Pearcy. Math. Surveys, No. 13. AMS, 1974, ix + 235 pp, $23. The articles in this volume were originally commissioned for the MAA Studies series, but were transferred to AMS Mathematical Surveys by mutual agreement. Four main topics covered in these five expository articles are: 1) invariant subspaces, 2) weighted shift operators, 3) spectral multiplicity for normal operators on Hilbert space, and 4) canonical models for operators. I-CH

FUNCTIONAL ANALYSIS, S(18), P. *Funktionalanalysis.* Benno Fuchssteiner, Detlef Laugwitz. Bibliographisches Inst, 1974, 219 pp, (P). General material on normed spaces and operators introduces distribution theory (for physicists) in the latter half of the book. JAS

FUNCTIONAL ANALYSIS, P. *Hilberträume und elliptische Differentialoperatoren.* Alexander Voigt, Josef Wloka. Bibliographisches Inst, 1975, 260 pp, (P).

OPTIMIZATION, S*, P*, L*. *The Game of Business.* John McDonald. Doubleday, 1975, xxvii + 404 pp, $15. A practical view of real-life games of business and economic life, recreated and analyzed by an editor of *Fortune* who had first-hand access to sources and individuals involved. The game of Seagram vs. Schenley in 1948, the rise of GM at the expense of Ford in the twenties, J.P. Getty's choice of fortune over fame, W. Disney's game of art vs. business, H. Hughes's struggle over control of TWA, and others. Emphasizes the importance of coalitions, the players' values, and the concept of the core of the game. An exciting confirmation of the value of game theory. PJC

OPTIMIZATION, S(16-18), P, L. *Conjugate Duality and Optimization.* R. Tyrrell Rockafellar. CBMS Reg. Conf. in Appl. Math., No. 16. SIAM, 1974, vi + 74 pp, $5.60 (P). Brief introduction to application of conjugate convex functions to optimization. Examples include nonlinear (including nonconvex) programming, approximation, stochastic programming, calculus of variations, optimal control. LH

OPTIMIZATION, P. *Cooperative and Non-Cooperative Many Players Differential Games.* George Leitmann. Springer-Verlag, 1974, 77 pp, $5.70 (P).

ANALYSIS, P. *Éléments d' Analyse, Tome VI.* J. Dieudonné. Gauthier-Villars, 1975, xiv + 197 pp, 25F (P). Volume six and chapter 23 on harmonic analysis. Depends more on previous volumes than on Volume Five. PJM

ANALYSIS, P. *Éléments d'Analyse, Tome V.* J. Dieudonné. Gauthier-Villars, 1975, xv + 206 pp, 25F (P). Volume five (and chapter 21) of the series, covering compact and semi-simple Lie groups. Bibliography, index, problems, an annex on necessary algebraic background. An excellent book. Refers to earlier volumes, but can be read alone by anyone who has a good background in the subjects of those volumes. PJM

ANALYSIS, P. *Tables of Mellin Transforms.* Fritz Oberhettinger. Springer-Verlag, 1974, 275 pp, $14 (P). The Mellin transform, related to the Laplace and Fourier transform, is used primarily to solve integral equations and evaluate series. Tabulated here are the transforms for nearly every class of functions from algebraic to elliptic, with a somewhat shorter list of inverse transforms.TAV

ANALYSIS, T(17: 1), S. *Lecture Notes in Mathematics-408: Potential Theory.* John Wermer. Springer-Verlag, 1974, 146 pp, $7.40 (P). Presumes measure theory and familiarity with harmonic functions. A well organized, coherent presentation of the basics. Could serve nicely as a text, with supplementary exercises provided by instructor. TAV

ANALYSIS, S(14-15). *Calculus of Variations.* A.M. Arthurs. Routledge & Kegan Paul, 1975, vii + 80 pp, $4.75 (P). A brief, elementary introduction to variational problems; covers Euler-Lagrange, Hamilton-Jacobi, dynamic programming, direct methods. Few exercises. A rather high price to pay for a brief (74 page), small (4 3/4" x 7 1/4" pages) text. SG

ANALYSIS, T(15-16: 1, 2), S. *Differentialformen.* Erhard Heil. Bibliographisches Inst, 1974, v + 202 pp, (P). A rather classical approach closely allied to advanced multivariable calculus. Written with an eye to applications in differential equations for physics and engineering. The second part of the book introduces some of the general theory. A good source for a potential user or a student who wants insight into the sources of the subject. JAS

ANALYSIS, P. *Proceedings of the Royal Irish Academy, V. 74, Section A, Numbers 18-36: Spectral Theory Symposium.* Royal Irish Academy, 1974, 175 pp, £3.45 (P). A special issue containing papers from a March 1974 symposium held at Trinity College, Dublin. LAS

ANALYSIS, T(17-18: 1), S, P. *Nichtlineare Gleichungen und Abbildungsgrade.* Klaus Deimling. Springer-Verlag, 1974, viii + 131 pp, $6.90 (P). An introduction to the topological background, various mapping degrees, and fixed point theorems that are useful in the study of nonlinear equations. JAS

ANALYSIS, T(15-17: 1-3), L. *Analysis I.* Martin Barner, Friedrich Flohr. Walter de Gruyter, 1974, 489 pp, 48 DM. Beginnings of real analysis--real numbers, sequences and series, continuity, differentiation, integration (not Lebesgue) and Fourier Series. The second volume will contain multivariable analysis and Lebesgue integration. A thorough coverage of both volumes would require three terms. In German. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-429: Analytic Theory of the Harish-Chandra C-Function.* Leslie Cohn. Springer-Verlag, 1974, 154 pp, $7.40 (P).

ANALYSIS, P. *Lecture Notes in Mathematics-427: Infinite Dimensional Lie Transformation Groups.* Hideki Omori. Springer-Verlag, 1974, xi + 149 pp, $8.20 (P). An exploration of various new infinite dimensional analogues of finite dimensional Lie groups largely motivated by the fact that existing examples (e.g., Banach Lie groups) rarely act on finite dimensional manifolds. LAS

ANALYSIS, S(17-18), L. *Differentialrechnung.* Henri Cartan. Bibliographisches Inst, 1974, 236 pp, (P). A translation of the 1967 Hermann original *Calcul Différentiel.* JAS

ANALYSIS, T(17-18: 2), S, P. *Einführung in die Analysis.* Hans-Jörg Reiffen, Heinz Wilhelm Trapp. Hochschultaschenbucher, B. 786, 787. Bibliographisches Inst, 1973. *II:* 260 pp; *III:* 369 pp, 9,90 DM (P) each. Exposition of the theory of analytic and differentiable maps (with a chapter on differential equations), manifolds and differential forms, and Lebesgue integration with Stokes theorem. JAS

GEOMETRY, T*(14-15: 1, 2), S, P, L. *Geometry for Teachers.* Robert A. Nowlan, Robert M. Washburn. Har-Row, 1975, xvii + 391 pp, $11.95. An easily readable, intuitive presentation providing excellent geometrical background for elementary and junior high teachers. Includes experiments, pedagogical asides, a transformation approach to congruence and a non-frightening discussion of proof. JNC

GEOMETRY, T(16-18: 2), S. *Differentialgeometrie. Auflage 2.* Detlef Laugwitz. B.G. Teubner, 1968, 183 pp, 28 DM. Classical differential geometry with tensors and Riemannian geometry. A fairly sophisticated and condensed text with a brief historical appendix. English translation of the first edition appeared in 1965; the second edition is very similar to the first. JAS

GEOMETRY, T(15-16: 1), S. *Geometrie für Lehrer und Studenten, Band I, Kongruenzgeometrie.* Gerhard Holland. Hermann Schroedel, 1974, 264 pp, 21,80 DM. A moderately axiomatic approach leading to consideration of congruence transformations as a group. JAS

GEOMETRY, S(17-18), P. *Differentialgeometrie.* Ernst Peschl. Hochschultaschenbucher, B. 80. Bibliographisches Inst, 1973, 92 pp, 9,80 DM (P). Classical Euclidean and Riemannian differential geometry. Requires a good background in linear algebra and differential calculus. A substantial, moderately condensed introduction. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-425: Properties of Infinite Dimensional Hamiltonian Systems.* Paul R. Chernoff, Jerrold E. Marsden. Springer-Verlag, 1974, 160 pp, $8.20 (P). A Hamiltonian system is a (complex) manifold with some extra structure. These notes apply the theory of Hamiltonian systems to physics. PJM

TOPOLOGY, S**(13-16), L*. *From Geometry to Topology.* H. Graham Flegg. Crane, Russak, 1974, xii + 186 pp, $10.50. Not really a text but a coherent set of very readable "lectures" about the development of topology from geometry both conceptually and historically. (Geometry here means the study of congruence transformations and topology means point set topology.) Valuable for students and teachers alike on cultural grounds; not a substitute for the study of topology in any of its usual aspects. JAS

TOPOLOGY, T*(15-17: 1, 2), L. *A First Course in Topology, An Introduction to Mathematical Thinking.* Robert A. Conover. Williams & Wilkins, 1975, xix + 245 pp, $11.95. A substantial introduction to point set topology with a chapter on the fundamental groups at the end. Modified-Moore approach: most proofs are left to the student, but hard theorems are proved, and hints and references given. A nice chatty style in which the author takes time to stop and tell you what's really going on. JAS

TOPOLOGY, T(18: 2, 3), P. *Algebraic Topology--Homotopy and Homology.* Robert M. Switzer. Grund. math. Wissenschaften, B. 212. Springer-Verlag, 1975, xiii + 526 pp, $52.50. Meant to be a bridge between a year's graduate course in algebraic topology and the foothills of current research. Suitable as advanced reading or a major reference work. JAS

TOPOLOGY, T(18: 1, 2), P. *The Stone-Čech Compactification.* Russell C. Walker. Ergebnisse der Math., B. 83. Springer-Verlag, 1974, x + 332 pp, $30.40. A very nicely written reference work or text (each chapter ends with a number of problems) inspired by Gillman and Jerison's *Rings of Continuous Functions.* This book brings together more recent material. Good index and bibliography should make this as enjoyable to use as to read. JAS

TOPOLOGY, T(17-18: 1), S*, P. *Introduction to Topological Groups.* P.J. Higgins. London Math. Soc. Lect. Notes, No. 15. Cambridge U Pr, 1974, v + 109 pp, $5.95 (P). A brief introduction to topological groups and the Haar integral. Little or no background in topology, but a basic knowledge of group theory is assumed. Includes examples, exercises, and a brief bibliography. CEC

PROBABILITY, T(16-17), S. *Random Processes, Second Edition.* M. Rosenblatt. Grad. Texts in Math., V. 17. Springer-Verlag, 1974, x + 228 pp, $12.80. This new edition of this solid introduction has a new chapter on martingales. FLW

PROBABILITY, P. *Lecture Notes in Mathematics-379: Mésures Cylindriques, Espaces de Wiener et Fonctions Aléatoires Gaussiennes.* Albert Badrikian, Simone Chevet. Springer-Verlag, 1974, x + 383 pp, $13.20 (P). Nine articles on abstract probability theory; almost more of a treatise on measure theory. PJM

PROBABILITY, S?. *Stochastic Processes.* Rodney Coleman. Prob. Solvers, No. 14. Crane, Russak, 1974, 93 pp, $4.95 (P); £4.50. A smattering of definitions, techniques, theorems on a small sample of stochastic processes. Too incomplete for a text, too disorganized for supplementary reading. The paperback is overpriced; the hard cover price is incredible! TAV

PROBABILITY, T(13-14), S, L. *Probabilitiês Statistiques et Biologie.* J.L. Chassé, A. Pavé. CEDIC, 1975, 326 pp, (P). Biological examples, primarily probability but some sample theory and hypothesis testing. Calculus required. Very well written. PJM

STATISTICS, P**. *Index to Statistics and Probability: Locations and Authors.* Ian C. Ross, John W. Tukey. R & D Pr, 1973, xxiii + 1092 pp, $80. Volume 5 in The Information Access Series, of which Volumes 2-6 comprise the *Index to Statistics and Probability,* covering the literature through 1966 (Volume 2, TR June-July 1974). Contains two main sections. The first specifies full location, authorship and titles of all source items for the *Index,* listed by journal. The second is a listing by author of these items. May be used independently of the other volumes. RSK

STATISTICS, T(13: 1). *Introduction to Statistical Analysis: A Semi-programmed Approach.* Celeste McCollough. McGraw, 1974, xvi + 380 pp, $9.95. Combines standard textual material with programmed material. Includes one-way analysis of variance and some distribution-free tests in addition to the usual topics. RSK

STATISTICS, T?(13: 1). *Statistics, An Audio-Tutorial Approach.* Morton Goldberg, William G. Vick, Jr. Merrill, 1974, vi + 250 pp. $8.95 (P). Cursory treatment, designed to be used with a set of cassette tapes in a self-paced course. RSK

STATISTICS, T*(13-14: 1, 2). *Statistics for Management and Economics, Second Edition.* William Mendenhall, James E. Reinmuth. Duxbury Pr, 1974, xi + 596 pp, $13.95. Revision of the author's 1971 text (TR, October 1972). Many small changes, but the content remains basically the same. RSK

STATISTICS, T(17: 1), P*. *Methods for Statistical Analysis of Reliability and Life Data.* Nancy R. Mann, Ray E. Schafer, Nozer D. Singpurwalla. Wiley, 1974, ix + 564 pp, $24.95. In the Wiley Series in Probability and Mathematical Statistics. Self-contained introduction to reliability, the "science of predicting, estimating, or optimizing the probability of survival, the mean life or, more generally, the life distribution of components or systems." Contains new material on Bayesian methods and accelerated life testing. Excellent sets of references. RSK

STATISTICS, P. *Random Sets and Integral Geometry.* G. Matheron. Wiley, 1975, xxiii + 261 pp, $18.95. From the foreword: "The treatment is forbiddingly mathematical but the basic ideas are simple and elegant." The first half of the statement is more obvious than the second. The book attempts to break new ground in an area the author calls "mathematical morphology", the statistical analysis of the structure of geometrically involved sets, e.g., an analysis of the structure of porous "solids." Requires a high level of mathematical sophistication. TAV

STATISTICS, P. *Lecture Notes in Mathematics-424: Maximum Probability Estimators and Related Topics.* Lionel Weiss, Jacob Wolfowitz. Springer-Verlag, 1974, v + 106 pp, $7.40 (P).

APPLICATIONS (LINGUISTICS), T(15-17: 1), P, L. *Weighing Evidence in Language and Literature: A Statistical Approach.* Barron Brainerd. U of Toronto Pr, 1974, xi + 276 pp, $20. An introductory statistics text richly illustrated by examples drawn from the research literature of linguistics and literary analysis. Purports to require only high school algebra, but is written in a style that presumes considerable mathematical maturity. Designed to make "the innocent use of statistics a little less innocent." LAS

APPLICATIONS (MARKETING), P*. *An Investigation of Brand Choice Processes.* B. Wierenga. Rotterdam U Pr, 1974, xiii + 256 pp, $24. Primarily a study of several stochastic models for the processes by which consumers select products by brand name, showing the superiority of the linear learning model (from mathematical learning theory). Also investigates an alternative approach based on "poolsize," the number of different brands bought by a consumer during his last ten purchases, and studies the effects of other variables on brand choice. RSK

APPLICATIONS (MEDICINE), T(13-14). *Biomathematik für Mediziner.* E. Walter. Teubner, Stuttgart, 1975, 148 pp, (P). Mostly probability and statistics with a little computer programming (with generalities about computers) designed for pre-medical students. JAS

APPLICATIONS (METEOROLOGY), P. *Numerical Methods in Weather Prediction.* G.I. Marchuk. Acad Pr, 1974, x + 277 pp, $21.50. A book written by a Russian computer scientist, translated and edited by meteorologists. Covers topics in weather prediction, quasi-geostrophic approximation (the best studied model of atmospheric processes) etc. Contains author's formulation of general methods for solving problems of dynamic meteorology by splitting them into elementary algorithms that can be executed effectively on computers. I-CH

APPLICATIONS (PHYSICS), T(16-17: 1, 2). *The Theory of Relativity, Second Edition.* R.K. Pathria. Pergamon Pr, 1974, x + 316 pp, $24. An updating of the 1963 edition. In two parts (special and general theories) with a number of problems at the end of each chapter. A thorough standard treatment. JAS

APPLICATIONS (PHYSICS), S. *Fluid Mechanics.* J. Williams. Prob. Solvers, No. 15. Crane, Russak, 1974, 107 pp, $5.25 (P); £3.75. A potpourri of results and techniques to deal with problems in 2 and 3 dimensional flow. Includes complex potentials and mappings for steady flow, some results for axisymmetric flow. A source of supplementary problems, not a text. The hard cover price is ridiculous. TAV

APPLICATIONS (PHYSICS), S(18), P. *Homogeneous Relativistic Cosmologies.* Michael P. Ryan, Jr., Lawrence C. Shepley. Princeton U Pr, 1975, xv + 320 pp, $15; $7.50 (P). A highly mathematical account of relativistic cosomology theory. A brief historical sketch introduces a chapter presenting the basics of modern differential geometry. The remaining chapters develop differential geometry and physics side by side in the process of describing a number of homogeneous cosomologies. Asides on "what its all about" are interspersed. JAS

APPLICATIONS (PHYSICS), S(18), P, L. *Elements of Advanced Quantum Theory.* J.M. Ziman. Cambridge U Pr, 1969, xii + 269 pp, $6.95 (P). "An easy and often delightful path to the understanding of rather sophisticated concepts."--*Math. Rev.* LAS

APPLICATIONS (PHYSICS), P*. *Classical Dynamics: A Modern Perspective.* E.C.G. Sudarshan, N. Mukunda. Wiley, 1974, xi + 615 pp, $24.95. "We see classical dynamics not as part of physics, but as physics itself." A research monograph, twelve years in preparation, it progresses from a gentle survey of Newtonian mechanics to current research. LAS

APPLICATIONS (PHYSICS), P. *Methoden und Verfahren der mathematischen Physik.* Ed: Bruno Brosowski, Erich Martensen. Bibliographisches Inst, 1973. *Band 8*, 222 pp; *Band 9*, 201 pp; *Band 10*, 184 pp, (P). An irregular "journal" publishing assorted papers and reports of meetings. Mostly differential equations and applicable analysis. JAS

APPLICATIONS (PHYSICS), S(14). *Analytic Mechanics.* D.F. Lawden. Prob. Solvers, No. 4. Allen & Unwin (U.S. Distr: Crane, Russak), 1972, 78 pp, $2.75 (P). Essentially a collection of worked out problems with some text and exercises. Topics: mass distribution, motions in the plane and in space, Lagrange and Hamilton's equations, small oscillations. TAV

APPLICATIONS (SOCIAL SCIENCE), T(14-16: 1), S*, L*. *Game Theory and Politics.* Steven J. Brams. Macmillan, 1975, xix + 312 pp, $6.95 (P). "An explication of strategic features of actual political situations...emphatically not a work of mathematics." An effective synthesis of aspects of voting, employing and illustrating the concepts of game theory. Excellent resource for a seminar in mathematical aspects of political science. LAS

APPLICATIONS (TRAFFIC THEORY), P, L. *Transportation and Traffic Theory.* Ed: D.J. Buckley. Am Elsevier, 1974, 816 pp, $29.50. Proceedings of an international symposium at the University of New South Wales, Australia, August 1974. A thorough mix of theory and its application. LAS

Reviewers Whose Initials Appear Above

Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Marianne Gardner, Carleton; Arthur Gropen, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Ithaca College: Assistant Professors Diane Schmidt and Martin Sternstein have been promoted to Associate Professors.

Assistant Professor G. C. Branche, Worcester Polytechnic Institute, has been promoted to Associate Professor.

Associate Professor D. R. Byrkit, The University of West Florida, has been promoted to Professor.

Dr. P. C. Shields, Stanford University, has been appointed Associate Professor at the University of Toledo.

Dr. V. R. R. Uppuluri, Mathematics and Statistics Research Department, Computer Sciences Division, Oak Ridge, Tennessee, spent the 1975 Spring Quarter as a Visiting Professor in the Mathematics Department at Kent State University.

Professor Merritt S. Webster, Purdue University, has retired with the title of Professor Emeritus.

Reverend Donald T. Faught, University of Windsor, Ontario, died on February 19, 1975, at the age of 59. He was a member of the Association for sixteen years.

Mr. Alfred D. Sollins, Col Alpes, Mexico, died on March 16, 1975. He was a member of the Association for thirty-two years.

THE GREATER METROPOLITAN NEW YORK MATH FAIR

Fair Date: March 28, 1976

Eligibility: Students who have completed, or are currently taking, mathematics at the eleventh year level or higher in the public, private and parochial high schools in the New York City, Westchester, Putnam, Dutchess, or Rockland Counties are eligible to participate in the FAIR. Exceptional students who do not meet these requirements but wish to submit papers for consideration by the FAIR Committee are welcome to do so.

Purpose: The FAIR encourages a student to pursue, in depth, some phase of mathematics to which he is drawn.

Procedure: 1. Research a topic in mathematics. 2. Write a paper on this topic. 3. Give a talk (of no more than 15 minutes duration) on your paper to a group of judges.

Nature of Paper: The work need not be completely original, but the paper should reveal scholarship appropriate to the course level of the student. Teachers may suggest topics but the presentation should be voluntary and based entirely upon the student's own readings and investigations.

Further details and application forms may be obtained from: Dr. Theresa J. Barz, Secretary, MATH FAIR Committee, Department of Mathematics & Computer Science, St. John's University, Jamaica, New York 11439.

THIRD INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION — KARLSRUHE

The Third International Congress on Mathematical Education will be held in Karlsruhe, August 16–21, 1976. Following the examples set by the Congresses held in Lyons and Exeter, it is designed to offer participants the opportunity of briefing themselves on the latest developments in mathematical education at all levels. To this end the General Papers to be given in Plenary Sessions as well as lectures and discussions in various sections should be of assistance. Demonstrations, displays, workshops and practical involvement will enable participants to gain a thorough knowledge of important projects and teaching aids. It will also be one of the continuing aims of the Congress to facilitate and promote both scientific and personal contact amongst the participants.

Under the auspices of the International Commission on Mathematical Instruction (ICMI), the congress is being organized by the West German Sub-Committee of ICMI in collaboration with the Local Organizing Committee (Chairman: Professor Dr. H. Kunle, Karlsruhe; Secretary: E. F. an Huef, Karlsruhe).

The Official Congress Languages are English, French, Russian and German. During Plenary Sessions invited papers will be simultaneously interpreted into these languages.

For further information, please write to: Third International Congress on Mathematical Education, University, D 75 Karlsruhe, Federal Republic of Germany.

FOURTH INTER-AMERICAN CONFERENCE ON MATHEMATICAL EDUCATION

The fourth Inter-American Conference on Mathematical Education will be held in Caracas, Venezuela, December 1 to December 6, 1975. The first three conferences were held at Bogota, (1961), Peru, (1966), and Bahia Blanca (1972). The conference is sponsored by the Venezuelan Committee on Mathematical Education and supported by several other foundations. Invited speakers from Europe, Japan, Canada, USA and Latin America will speak on applications, ability grouping, extra-curricular activities, and teacher education. A Forum on the problem of curricular reform will bring both critics and supporters from Europe and the USA. There will be Plenary sessions, working groups, and short communication sessions.

Interested persons are urged to write for the second announcement to Professor Mauricio Orellano, Chacin, IV Conferencia Inter-Americana Sobre Educación Matemática, Instituto Pedagógico, Departamento de Matemáticas y Física, Avenida Paez, El Paraíso, Caracas 102, Venezuela.

WHAT HAS HAPPENED TO THE DOCTOR OF ARTS DEGREE?

The MAA's Committee on the Undergraduate Program in Mathematics took a leading role during the 60's in describing and publicizing the Doctor of Arts degree, an alternative to the Ph. D., based on critical or historical scholarship and oriented toward the training of teachers of undergraduates. A glimpse into what has happened to the D. A. degree is provided by a report, "Status of the Doctor of Arts Degree," by Robert H. Koenker, Dean of the Graduate School at Ball State University, Muncie, Indiana 47306. Koenker's report lists seven universities currently offering the D. A. in mathematics, two definitely planning to offer the degree in the near future, and three currently considering the possibility of offering the degree in the future. The report discusses various other information gleaned from a survey of the member institutions of the Council of Graduate Schools in the United States.

Interested persons may obtain copies of the report by writing to: Robert H. Koenker, Dean, Graduate School, Ball State University, Muncie, Indiana 47306.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

1975 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association expresses its deep appreciation to 164 of its members who have elected to be Contributing Members, Sponsors, or Patrons for 1975 by making contributions beyond the normal dues. There are 153 Contributing Members in 1975. In addition, the following members are 1975 Sponsors and Patrons:

Patrons	*Sponsors*
Nathaniel Chafee	G. H. Bridgman
Leonard Gillman	R. J. Engert
J. E. Kiefer	A. M. Gleason
H. J. Ryser	R. A. Good
One Anonymous Patron	W. E. Hartnett

The Association also acknowledges with deepest gratitude the following special gifts:

A gift of $1000 from H. M. Gehman.

An anonymous gift of $3,000 in partial support of the Visiting Lecturers and Consultants Program.

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifty-fifth regular meeting of the Southern California Section of the MAA was held on March 8, 1975, at Occidental College. The Chairman of the Section, Professor Donald Potts of Cal. State at Northridge, presided. The registered attendance was 97.

At the business meeting, the results of the election of section officers for 1975–76 were announced as follows:

Chairman, J. M. Hood, Occidental College; 1st Vice-Chairman, Betty Garrison, San Diego State Univ.; 2nd Vice-Chairman, Paul Yale, Pomona College; Program Chairman W. J. J. LeVeque, Claremont Graduate School. The luncheon speaker was Professor John Ernest, U. C. Santa Barbara, whose topic was *Mathematics and Sex.*

The following program was presented:

Sharp and dull points in a constructive continuum, by Joan Moschovakis, Occidental College.

Populations, hunting and fishing — with calculus and calculators, by David Sanchez, UCLA.

6 *Panel discussion: Mathematics preparation for college bound students*, by David Outcalt, UCSB, Chairman; Philip Curtis, UCLA; Jack Kifer, LAVC; Harry Pappus, Santa Monica H.S.

Summer research participation — one person's experience, (NSF Faculty Research Participation Program), by Mary Schrot, CSC Bakersfield.

Uniform distribution mod 1, with weights, by Jeff Valler, Cal. Tech.

The problem of characterizing derivatives, by Andrew Bruckner, UCSB.

Multiplicatively periodic rings, by Ted Chinburg, Harvey Mudd College.

JOHN GREEVER, *Secretary-Treasurer*

APRIL MEETING OF THE NORTH CENTRAL SECTION

The 1975 spring meeting of the North Central Section of the MAA was held at Hamline University, St. Paul, Minnesota, on April 25–26.

The invited speaker, Branko Grünbaum, University of Washington, Seattle, spoke Friday evening and Saturday morning. His titles were "Three Classical Theorems on Convex Polyhedra" and "Families of Boxes."

Other papers presented at the Saturday sessions included:

1. *Convolution arrays and the golden ratio*, by Gerald Bergum, South Dakota State University, Brookings, SD.
2. *African mathematics: a January '75 Interim Course*, by Paul Campbell, St. Olaf College, Northfield, MN.
3. *Communications with an angle trisector*, by David Uherka, University of North Dakota, Grand Forks, ND.
4. *What can we do with connectedness?*, by George Brauer, University of Minnesota, Minneapolis, MN.
5. *Citizenship and statistics: Interdisciplinary course*, by Robert Raymond and Donald Leavitt, University of Minnesota, Morris, MN.
6. *An old place for new examples of timeless concepts*, by Richard Jarvinen, Saint Mary's College, Winona, MN.
7. *A computer answers dumb calculus questions: ISCI*, by Leonard Shapiro, University of Minnesota, Minneapolis, MN.
8. *The Russian school of probability theory*, by Wei-Ching Chang, University of Minnesota, Morris, MN.
9. *A solution to E1075 (Monthly Problems, '53 & '74)*, by Terry Therneau (student), St. Olaf college, Northfield, MN.
10. *Some problems in generalized convex functions*, by Ronald Mathsen, North Dakota State University, Fargo, ND.
11. *Kepler and the wine merchants: An anecdote for freshman calculus*, by Pierre Malraison, Carleton College, Northfield, MN.
12. *Generalization of a Putnam Problem*, by Kenneth Yocom, South Dakota State University, Brookings, SD.
13. *Meromorphic vector fields and Gauss-Bonnet*, by Alan Mitchell, University of Minnesota, Minneapolis, MN.

Host presiders at the sessions were: Dale Varberg and Walter Fleming. Sectional chairman Sylvan Burgstahler presided at the Saturday afternoon session.

The following officers were elected to serve for the coming year: Chairman-elect — Gerald Bergum, South Dakota State University, Brookings, SD, Secretary/Treasurer — Louis Guillou, Saint Mary's College, Winona, MN, Member-at-large — V. C. Varadachari, Lakewood Community College, White Bear Lake, MN, Member-at-large — Lisl Gaal, University of Minnesota, Minneapolis, MN.

The newly elected officers will serve on the North Central Section's Executive Committee for the 1975–76

academic year along with: Chairman — Kent Carlson, St. Cloud State College, St. Cloud, MN, Governor — Murray Braden, Macalester College, St. Paul, MN, Past-chairman — Sylvan Burgstahler, University of Minnesota, Duluth, MN.

LOUIS GUILLOU, *Secretary-Treasurer*

APRIL MEETING OF THE SEAWAY SECTION

The Spring meeting of the Seaway Section of the MAA was held at York University, Toronto, Ontario, on April 26, 1975, with a registered attendance of 85 people, including 79 members of the Association. Professor D. O. McKay of the University of Western Ontario, Chairman of the Section, presided.

At the morning session the members of the Seaway Section were welcomed by President H. I. MacDonald of York University.

T. M. O'Loughlin, State University College at Cortland, gave a talk, "The Effect of an Electronic Programmable Calculator on a First Course in Calculus."

P. A. Lindstrom of Genesee Community College, Second Vice-Chairman of the Section, talked on "Some Recent Activities from the Two-Year College."

Erik Hemmingsen, Syracuse University, presented the Harry M. Gehman Lecture, the title being "Topological Dynamics on Surfaces."

At the business meeting the following officers were elected: Chairman, Mabel D. Montgomery, State University College at Buffalo; First Vice-Chairman, F. D. Parker, St. Lawrence University; Second Vice-Chairman, L. A. Trivieri, Mohawk Valley Community College; Secretary-Treasurer, E. C. Stopher, State University College at Oswego. It was announced that Frank Hacker, Mohawk Valley Community College, had been appointed as Committee Chairman for the year 1975–76 with the responsibility for the Annual High School Mathematics Examination in the Upper New York State Region.

Grant Roberts of the University of Waterloo and Peter Saxe of Union College were announced as sharing the distinction of having the best score of anyone in the Section in the William Lowell Putnam Competition. They will each receive congratulatory checks for $10 from the Section.

During the afternoon the following contributed papers were presented:

Can college students reason? by Larry Copes, Ithaca College.

The differential and the second derivative test, by D. S. Martin, State University College at Brockport.

Games of chance and probability: A historical anecdote, by Peter Tan, Carleton University.

The trace, inverse and characteristic polynomial — a complete set, by Donald Fama, Auburn Community College.

Restricting and inducing on inner products of representations of a finite group, by G. de B. Robinson, University of Toronto.

Distribution of initial digits in tables of physical data, by N. M. Rice, Queen's University.

From Burali Forti to Gödel: Early efforts to axiomatize set theory, by G. H. Moore, University of Toronto.

EMMET STOPHER, *Secretary-Treasurer*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the MAA was held at Angelo State University, San Angelo, Texas, on April 11 and 12, 1975. There were 215 registered persons attending.

Professor James Hodge, Angelo State University, Vice-Chairman of the Section, became Chairman. Second Vice-Chairman Professor G. R. Blakley, Texas A & M University, became First Vice-President. The following additional officers were elected: Professor Howard Rolf, Baylor University, was elected Second Vice-President. Professor R. G. Dean, Stephen F. Austin State University, was elected for a one-year term as Director-at-Large. Professor Archie Brock, East Texas State University, was elected to a two-year term as Level II Director, and Professor J. C. Bradford was elected for a three-year term as Secretary-Treasurer. Level I Director Dr. Amogene Devaney, Amarillo College, will serve for one more year. Professor J. R. Boone, Texas A & M University, was elected to a three-year term as MAA High-School Contest Director and by vote of the Section was added to the Executive Committee of the Texas Section.

Professor Jurgen Schmidt of the University of Houston presented an invited address on "Partial Fractions and Invertible Prime Ideals." Dr. H. O. Pollak of Bell Laboratories and President of the MAA spoke on "On the Nature of Applied Mathematics."

The following papers were presented:

1. *Completions of commutative rings*, by J. L. Hein, University of Texas of the Permian Basin.
2. *Rings generated by their units*, by J. W. Fisher, University of Texas at Austin.
3. *Graph theoretic decomposition of a poset*, by E. L. Perry, Baylor University.
4. *One algebraic system*, by Ernest Ratliff, Jr., Southwest Texas State University.
5. *Groups of infinite exponent whose proper quotient groups are of finite exponent*, by Shirley Tucker, Texas Christian University.
6. *Linearization of expansive homeomorphisms*, by R. K. Williams, Southern Methodist University.
7. *Some branch point covering theorems for n-pseudo-confluent mappings*, by D. R. Read, Lamar University.
8. *A classification of continua*, by M. R. Hagan, North Texas State University.
9. *The Wallman compactification and the product topology*, by S. E. Green, East Texas State University.
10. *Convergence of the sequence of successive approximations for nonexpansive mappings through abstract cones*, by J. C. Bolen, A. A. Gillespie, B. B. Williams, University of Texas at Arlington.
11. *Bounded slope variation and generalized convexity*, by F. N. Huggins, University of Texas at Arlington.
12. *On the structure of Zermelo and Hamiltonian operators via multitensor transform analysis*, by D. M. Kulvicki, University of Texas at Austin.
13. *A three-dimensional surface equation for 'smooth' solids via a two-parameter family of polynomials*, by D. V. Goulet, University of Houston; J. R. Cuzzi, Texas Institute for Rehabilitation and Research.
14. *Report on an alternate structuring of high school geometry*, by Don Edmondson, University of Texas at Austin.
15. *Personalized instruction in mathematics at UTA*, by G. B. Turney, University of Texas at Arlington.
16. *Predicting success in college algebra at Tarleton State University*, by Linda M. Neal, Tarleton State University.
17. *Sound page as a tutorial aid*, by L. F. Heath, University of Texas at Arlington.
18. *Teaching the unwilling the unwanted (or Life in the pits of finite mathematics)*, by J. W. Strain, Midwestern University.
19. *A panel on uses of hand calculators*, by Martha Daley, Texas Southern University; Cris Boltz, Eastfield College; Omer Jenkins, Texas A & M University; Tom Bohanon, Tarleton State University, chairman.
20. *Techniques of linear algebra in geometry*, by A. R. Amir-Moéz, Texas Tech University.
21. *Some problems Kepler did not solve*, by J. M. Stark, Lamar University.
22. *Applying mathematics to the computer*, by K. L. Park, East Texas State University.
23. *Constrained optimization in finite dimensional Euclidean spaces*, by D. K. Hughes, Abilene Christian College.
24. *Transposing the impossible dream*, by Bonnie Kelterborn, East Texas State University.
25. *On mathematical aesthetics*, by A. A. Mullin, Killeen, Texas.
26. *Factoring with Cuisenaire rods*, by J. A. Bell, Texas A & I University at Laredo.
27. *An experiment in secondary teacher preparation*, by Pat Hickey, Baylor University.
28. *Some recent projects of CUPM*, by Paul Knopp, University of Houston.
29. *A new Association and a New Journal: The Graduate Mathematical Association and the Mathematics Microjournal*, by Dennis Kulvicki, University of Texas at Austin.
30. *What an engineer wants to know about linear algebra*, by Betty Barr, University of Houston.
31. *Using Boolean logic in developing probability functions*, by J. A. Nickel, University of Texas of the Permian Basin.
32. *Function collection characterizations and Hellinger type integrals for bounded finitely additive set functions*, by W. C. Bell, North Texas State University.
33. *Orthogonality and derivatives of the norm in spaces of measures*, by Russell Bilyeu, North Texas State University.
34. *A generalization of the Hermite polynomials*, by Russell Cowan, Lamar University.
35. *Solutions of differential systems in K-Banach spaces with applications to non-linear operators*, by R. L. Tennison and A. R. Mitchell, University of Texas at Arlington.
36. *Some Tauberian theorems for a class of regular summability methods*, by T. A. Keagy, North Texas State University.
37. *Matrix identities and substitutions in summability*, by D. F. Dawson, North Texas State University.
38. *A look at mathematics for business administration majors*, by R. J. Martin, Amarillo College.
39. *Developmental mathematics — What are the results?*, by Jim Bennett, Eastfield College.

J. C. BRADFORD, *Secretary-Treasurer*

MAY MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the MAA was held at the General Motors Institute, Flint, on May 1–3, 1975. Approximately 160 persons attended the meeting. The program included three invited addresses, an open forum, seven applied mathematics seminars conducted by representatives from industry, four contributed papers and three student papers.

The open forum on May 1 was led off by four invited speakers (Merle DeMoss, General Motors Institute; Allen Butterworth, Research Laboratories, General Motors Corporation; Harvey Arnold, Oakland University; Henry Brysk, KMS Fusion) and the mathematics departmental chairpersons of several colleges.

At the business meeting, Professor Alavi, chairperson of the Michigan Section, reported on items and news from the MAA office in Washington. The Section presented a resolution honoring Professor Lyle Mehlenbacher upon his retirement as chairman of the Committee on Sections. This was followed by reports by the Secretary-Treasurer of the Section, the Director of the Michigan Mathematics Prize Competition and the chairperson of the High School Visiting Lecturer Program Committee.

The most important item of business decided upon was the approval of the revisions of the Section Bylaws which proposed the creation of another vice-chairpersonship in addition to the present one. Copies of the revised Bylaws as approved by the members of the Michigan Section will be sent to the Committee on Sections for approval.

Upon the suggestion of Dr. Henry Pollak, President of MAA, Professor Yousef Alavi, as Chairperson of the Michigan Section (1974–75), will submit a supplemental report on this year's meeting for presentation at the AMS–MAA summer meeting in Kalamazoo in August.

The newly elected officers of the Michigan Section are: C. B. Stortz, Northern Michigan University, Chairperson; Elliot Tanis, Hope College, Vice-chairperson. Each will serve for one year (1975–76).

The following program was presented:

Invited Addresses:

1. *On the relationship between applications of mathematics and teaching of mathematics,* by H. O. Pollak, Bell Laboratories and President MAA.
2. *Random observations on a number of things,* by F. B. Quackenboss, Director of Market Studies, General Motors Corporation.
3. *Directions for education in applicable mathematics,* by W. F. Lucas, Director, Applied Mathematics Center, Cornell University.

Applied Mathematics Seminars:

1. *You never step into the same river twice — applying mathematics in industry,* by A. V. Butterworth, General Motors Corporation.
2. *Selected examples of mathematical applications in the chemical industry,* by R. R. Klimple, Dow Chemical Company.
3. *An approach to file contention resolution,* by Ron Hall, Management Scientist, Burroughs Corporation.
4. *Ore blending control in the mining industry: a case history in the analysis and implementation of an industrial control strategy,* by K. W. Luke, Cleveland-Cliffs Iron Company.
5. *The actuary — his education and his work,* by Ray Nacin, Maccabees Life Insurance Company.
6. *The realities of business math, or 'You mean there really is a use for the logarithm?',* by F. S. Patterson, Dow Corning Corporation.
7. *Mathematical techniques in computer aided design and numerical control,* by M. W. Sterling, System Associates, Inc.

Contributed Papers:

1. *Quadratic programming: an application to Markov transition probability estimation,* by V. R. Hoffner, Jr., Eastern Michigan University.
2. *Determination of a certain parameter in Sobolev equations,* by Richard Ewing, Oakland University.
3. *Covers of Abelian groups,* by J. D. O'Neill, University of Detroit.
4. *Equations for the determination of prime twins,* by G. P. Loweke, Wayne State University.

Student Papers:

1. *Mathematical model for dynamics of railway vehicles on curved track,* by Donald Bolt, General Motors Institute.
2. *Snakes on an n-dimensional cube,* by Robert Myers, Hope College.
3. *Computations in quantum mechanics,* by James Hanson, Kalamazoo College.

DELIA KOO, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A & M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Valparaiso University, Valparaiso, November 15, 1975.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., November 22, 1975.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY

NORTH CENTRAL, Southwest Minnesota State College, Marshall, October 25, 1975.

NORTHEASTERN, Simmons College, Boston, Mass., November 29, 1975.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

OHIO

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Ark., March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Franklin and Marshall College, Lancaster, November 22, 1975.

ROCKY MOUNTAIN, Fort Lewis College, Durango, Colorado, May 1–2, 1976.

SEAWAY, State University College, Cortland, N. Y., October 31–November 1, 1975.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A & M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday) and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 20–22, 1975.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 1976.

FIBONACCI ASSOCIATION, California State University, San Francisco, October 18, 1975.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, MGM Grand Hotel, Las Vegas, November 17–19, 1975.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Sheraton-Palace Hotel, San Francisco, December 3–5, 1975 (SIAM-SIGNUM 1975 Fall Meeting).

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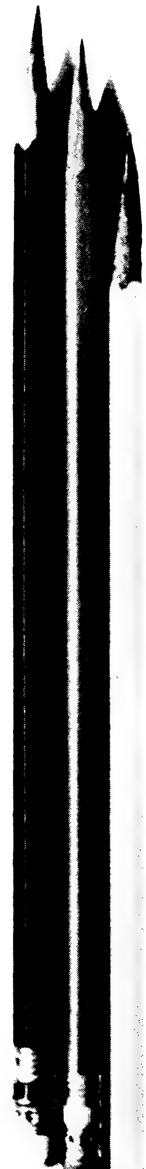

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NOVEMBER 1975

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SOME MATHEMATICAL PROBLEMS FROM NEUROBIOLOGY

S. P. HASTINGS

1. Introduction. Mathematical models of physiological processes are not usually received with enthusiasm by biologists. Although many of us might like to ascribe this to a blinkered attitude on the part of our scientific cousins, it must be acknowledged that such models are often deficient in several respects. They may be derived from an incomplete biological theory, which leads to the over-zealous introduction of "black boxes," and a consequent failure of the model to explain anything. Further, we often lack sufficient experimental data to determine all of the parameters in the model. The unknown parameters must then be adjusted to reproduce observed experimental behavior. As a result, the model has little or no capacity for prediction, and without such a capacity, the physiological assumptions on which it is based may be viewed with skepticism.

In this paper we discuss one of the few mathematical models which do give quantitatively accurate predictions of an important physiological process, the conduction of electrical impulses in a nerve axon. This model, which is due to the British physiologists A. E. Hodgkin and A. F. Huxley [**26**], does involve a black box, but because its detailed predictions provide strong evidence for the validity of the underlying theory, it is undoubtedly the most important mathematical model in neurobiology.

Although the Hodgkin-Huxley (HH) theory first appeared in 1952, it seems that until recently few mathematicians were aware of its existence, or of the interesting problems in ordinary and partial differential equations which it raises. After introducing the model, we shall consider some of these problems, including several which seem far from a satisfactory resolution. We shall also mention an instance where mathematical analysis suggests a direction for further experimentation.

2. Physiological background. We begin by describing, briefly, the physical setting of the model, and a little of the experimental evidence, in order to motivate the mathematical questions to be asked later. For a fuller explanation of the basic neurophysiology, see [**29**].

The function of the nervous system is, of course, to transmit information, and for this purpose it uses electrical impulses initiated and transmitted by individual nerve cells, or neurons. Hodgkin and Huxley developed a set of differential equations to describe the ionic and electrical events occurring during the transmission of an impulse along an axon, which is usually the filament carrying signals from the nerve cell body to other parts of the organism. These equations are based on the assumption that an axon behaves like a cylindrical electrical cable, with a conducting core and partially insulating sheath, or membrane. Both the core and the surrounding medium are fluids containing significant concentrations of charged metallic ions, mostly sodium and potassium. The currents which flow are due either to the capacitance of the membrane or to the movement of ions either through the membrane or longitudinally along the axon. Perhaps the most important feature of the theory is its reliance on changes in the permeability of the membrane to certain of these ions during the course of an impulse. (The molecular structure of the membrane which allows it to play an active role in signal transmission is unknown. The membrane is the black box referred to earlier.) The resulting nonlinear equations can be written in the form

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+I(u,w), \tag{1}$$

$$\frac{\partial w}{\partial t}=P(u)w+q(u). \tag{2}$$

Here $u=u(x,t)$ is the electrical potential across the axonal membrane as a function of time t and distance x along the axon. Also, $w=w(x,t)$ is a three-dimensional vector function of x and t which determines the permeability of the membrane to the principal ionic components of the transmem-

brane current. The 3×3 matrix function P and the vector function $\boldsymbol{q}$ depend nonlinearly on u, and $I(u, w)$ is linear in u but nonlinear in w. (In the appendix we write the equations out in more detail.) The right-hand side of (1) gives the net current across the membrane, as the sum of a capacitive term $\partial u/\partial t$ and an ionic term $I(u, w)$. Assuming the axon to be of infinite length, we study the system (1), (2) either on the half plane $-\infty < x < \infty$, $t \geqq 0$, or on the quarter plane $x \geqq 0$, $t \geqq 0$. The appropriate initial, or initial-boundary, value problems are discussed in §4, 5.

The solutions of (1), (2) which are observed in the laboratory are primarily of two types, those which are independent of the space variable x and those which depend on the single quantity $s = x + ct$, where $c \neq 0$ is constant. In each of these categories there are found solutions representing a single nerve impulse, and also solutions representing a train of impulses following each other in rapid succession.

A solution u, w which depends only on $x + ct$ is often called a travelling wave. Observe that if $x_1 \neq x_2$, then the two functions $u_i(t) = u(x_i + ct)$, $i = 1, 2$, are simply translates of each other, with no distortion or change of amplitude. Physically this means that a signal appears to move along the axon, with no loss of strength, and in a direction determined by the sign of c. The rate of propagation, or wave speed, is $|c|$, and in cases where the direction is important, c can be called the wave velocity.

The appearance of travelling wave solutions of (1), (2) is interesting mathematically, since this kind of behavior is not seen when studying linear parabolic partial differential equations, such as the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial t$. However, equations of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + f(u),$$

where f is nonlinear, have long been known to admit travelling waves. Such equations appear in genetics, flame propagation theory, and other areas, and their mathematical history goes back at least to an important paper of A. Kolmogoroff, I. Petrovsky, and N. Piscounov in 1937 [**30**]. The particular interest of (1), (2) is in the presence of the additional variable w, since this gives rise to significant new features in the analysis of wave solutions. It should be noted, as well, that there is no "diffusion" term $\partial^2 w/\partial x^2$ in the second equation. Thus, the nerve equations display different phenomena from those exhibited by equations describing chemical reactions, as in [**5**] and [**36**].

Spatially homogeneous solutions, which do not depend on x, are found experimentally either by subjecting the axon to an initial stimulus which is uniform along its length, or by inserting a conducting wire along the core, thereby eliminating longitudinal potential gradients and currents. The latter procedure is called a space clamp, and reveals the crucial role played by the membrane in nerve conduction. Observe that if $\partial^2 u/\partial x^2 = 0$, then (1), (2) reduces to a fourth order system of ordinary differential equations.

When a space clamped axon is subjected to a short burst of stimulating current, the potential u is raised to a value $u(0)$ above the resting level (which is taken here to be $u = 0$). When $u(0) > 0$ is small, u returns to the resting level almost immediately, but if $u(0)$ exceeds a critical value u_T, called the threshold, a rapid and large increase in u is observed, followed by a slightly slower fall to below the resting level and then a gradual return to rest. This is the famous "action potential," known to physiologists since the nineteenth century (Figure 1).

A similar threshold effect is found in an unclamped axon stimulated at one end, except that now u varies with x and t. When the stimulus is strong enough, $u(x, t)$ appears to approach a function of the form $u(x + ct)$, as t increases. (This is made precise in §4.) The shape of the wave resembles the action potential in Figure 1, with the addition of a tail tending to zero at $-\infty$.

Consider once again the space-clamped axon, and suppose that a steady stimulating current I_0 is applied instead of a short burst. This has the effect of replacing equation (1) with

$$I_0 = \frac{du}{dt} + I(u, w). \tag{3}$$

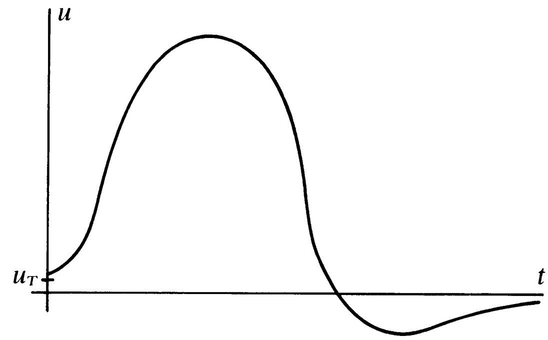

FIG. 1

If I_0 is sufficiently large, then several impulses appear in succession. Numerical integration of the ordinary differential equations (2), (3) indicates the existence of periodic solutions, representing an infinite train of impulses, but these are not found experimentally in the particular nerve preparation (the so-called "giant" axon of a squid) on which the HH equations are based. Thus, as has been remarked by R. FitzHugh [**15**], it appears that the model could be improved by the addition of a variable to control the long term accomodation of the axon to a prolonged stimulus.

Finally, it is found that the unclamped axon also produces repeated pulses in response to a maintained stimulating current. In some axons the frequency of these pulses varies widely with signal strength, and this is thought to be an important component of the information transmitted by the system. However, the squid axon does not display this frequency variation, perhaps because in the living animal this axon is only required to transmit an all-or-none type of signal.

3. A simpler model. Mathematical analysis of the Hodgkin-Huxley equations is technically very difficult, because of the complicated nonlinear functions I, P, and q which appear on the right-hand sides. Fortunately, a simpler formulation has been discovered which seems to reproduce most of the qualitative features of the original system, and yet is more amenable to analytical manipulation. Henceforth in this article we shall discuss only the simplified model, for which several positive results have been obtained. Every problem to be mentioned can also be raised in connection with HH, but progress on the original system has been much slower.

The initial form of the equations in question is due to FitzHugh [**14**], who discussed and simplified the space-independent system of ordinary differential equations (2), (3). His model is considered in §7. In this section we are concerned with the propagated action potential, as this is the more important case physically and presents the most challenging mathematical problems. The relevant equations involve partial derivatives, and were first given by J. Nagumo, S. Arimoto, and S. Yoshizawa [**33**], who relied heavily on FitzHugh's work. These equations are

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} - u(1-u)(u-a) + w \\ \frac{\partial w}{\partial t} &= bu, \end{aligned} \tag{4}$$

where a and b are positive constants with $a < 1$.The variable u, as before, represents the potential across the membrane, and one additional variable, the scalar w, is needed. The parameter b is taken to be "small", because w represents a pair of variables in the original HH system which change slowly relative to u. (See the appendix.)

As mathematicians, our first goal is to show that the system (4) has solutions which behave

qualitatively like those described in §2, and to look for other solutions of possible physical interest. We then want to investigate whether the solutions obtained are likely to be observable experimentally. This second question is one of stability, in a sense to be defined later, and is much more difficult than existence.

In the previous section it was stated that a single nerve impulse appears to tend, as t increases, to a travelling wave. Hence one is led to seek solutions u, w of (4), not identically zero, of the form

$$u = u(x + ct)$$

$$w = w(x + ct)$$

for some $c \neq 0$. Substitution into (4) gives a set of ordinary differential equations which, upon introduction of the variables $v = u'$, $s = x + ct$, takes the form

$$\begin{aligned} u' &= v \\ v' &= cv - f(u) + w \\ w' &= \frac{b}{c}u, \end{aligned} \tag{5}$$

where $f(u) = u(1-u)\,(u-a)$ and $'$ denotes differentiation with respect to s. The problem is to choose $c \neq 0$ so that (5) has a non-constant solution (u, v, w) with $u(\pm\infty) = 0$. The latter condition implies that v and w also tend to 0 at $\pm\infty$, and reflects the resting state of a nerve before and after an impulse.

The system (5) is readily seen to have $u = v = w = 0$ as its only equilibrium point (where $u' = v' = w' = 0$). To emphasize the nonlinear character of the problem just posed, "linearize" around this point. That is, expand $f(u)$ in a Taylor series around $u = 0$:

$$f(u) = -au + (1+a)u^2 - u^3,$$

and drop all but the first term. This gives a linear set of differential equations which can be written in vector form as

$$\boldsymbol{x}' = A\boldsymbol{x} \tag{6}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ a & c & 1 \\ b/c & 0 & 0 \end{pmatrix}.$$

It is easily shown that A has one real positive eigenvalue λ_1 and two eigenvalues λ_2 and λ_3 with negative real parts. For small $b > 0$, λ_2 and λ_3 are real, and every real solution of (6) is then of the form

$$\boldsymbol{x}(s) = \alpha_1 e^{\lambda_1 s}\boldsymbol{p}_1 + \alpha_2 e^{\lambda_2 s}\boldsymbol{p}_2 + \alpha_3 e^{\lambda_3 s}\boldsymbol{p}_3, \tag{7}$$

where p_i is the eigenvector of A corresponding to λ_i and the α_i are arbitrary. Therefore (6) has a two-dimensional family of solutions ($\alpha_1 = 0$, α_2 and α_3 arbitrary) which tend to $\mathbf{0} = (0,0,0)$ as $s \to +\infty$, and a one-dimensional family of solutions which tend to $\mathbf{0}$ at $-\infty$. Furthermore, there is no non-zero solution which approaches the origin in both directions, so to obtain the desired trajectory it is necessary to investigate the full nonlinear system (5).

For this purpose it is convenient to discuss the three-dimensional phase space of (5), by which we mean the curves $s \to (u(s), v(s), w(s))$ in R^3 defined by solutions (u, v, w). Observe that the phase space of (6) is easy to describe from (7). If $\boldsymbol{x}$ is a solution such that $\boldsymbol{x}(0)$ lies in the plane determined by $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$, then $\boldsymbol{x}(s)$ remains in this plane and tends to $\mathbf{0}$ at $+\infty$, while if $\boldsymbol{x}(0)$ lies in the line determined by $\boldsymbol{p}_1$, then $\boldsymbol{x}(-\infty) = \mathbf{0}$ (Figure 2). Other solutions are unbounded as $s \to \pm\infty$.

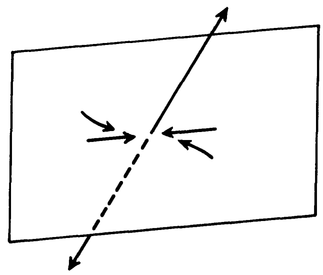

FIG. 2
Phase space of (6). Arrows indicate direction of increasing s.

One hopes that the linearized system (6) gives some information about (5), at least in a neighborhood of the origin where $f(u)$ is well approximated by $f'(0)u$. In fact, a fundamental result in ordinary differential equations, sometimes called the stable manifold theorem [4], [21], tells us that there is a two-dimensional manifold $\mathcal{S} \subset R^3$ containing $\mathbf{0}$ in its interior, such that if $y = (u, v, w)$ is a solution of (5) with $y(s_0) \in \mathcal{S}$ for some s_0, then $y(s) \in \mathcal{S}$ for all $s \geqq s_0$, and $y(+\infty) = \mathbf{0}$. There is also a one-dimensional manifold (curve) $\mathcal{U}$, again containing $\mathbf{0}$ in its interior, such that solutions starting on $\mathcal{U}$ tend to $\mathbf{0}$ at $-\infty$. $\mathcal{U}$ intersects $\mathcal{S}$ non-tangentially at $\mathbf{0}$, and by considering these manifolds only in some small neighborhood of $\mathbf{0}$, we can assume that they intersect nowhere else. Therefore $\mathcal{S}$ divides the set $\mathcal{U} - \{\mathbf{0}\}$ into two disjoint branches, $\mathcal{U}^+$ and $\mathcal{U}^-$ (Figure 3.a).

Consider a fixed $b > 0$. Since the system (5) depends on c, so also do the manifolds $\mathcal{S} = \mathcal{S}_c$ and $\mathcal{U} = \mathcal{U}_c$. If $y(0) \in \mathcal{U}_c$, then $y(s)$ exists on some interval $(-\infty, \sigma)$ containing $s = 0$, and tends to the origin at $-\infty$. Our problem reduces to choosing c so that if $y(0)$ lies in the correct branch of $\mathcal{U}_c$, then $y(s)$ exists for $-\infty < s < +\infty$ and intersects $\mathcal{S}_c$ for large s. For such a solution $y(+\infty) = \mathbf{0}$ as well, giving the desired trajectory (Figure 3.b). An orbit of this sort (beginning and ending at the same equilibrium point) is sometimes called "homoclinic." In §8 it is shown that such a solution cannot exist unless $0 < a < \frac{1}{2}$. With this restriction, the existence of homoclinic orbits for sufficiently small $b > 0$ has recently been proved in at least two different ways [2], [24]. A weaker result is sketched in §8, to illustrate one approach to this kind of problem.

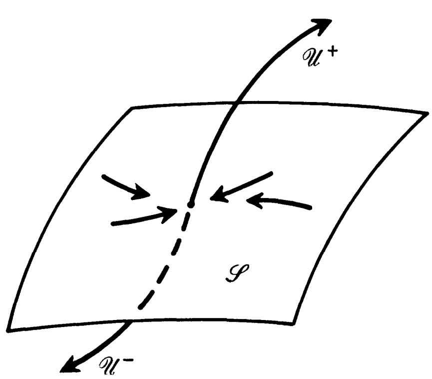

FIG. 3.a

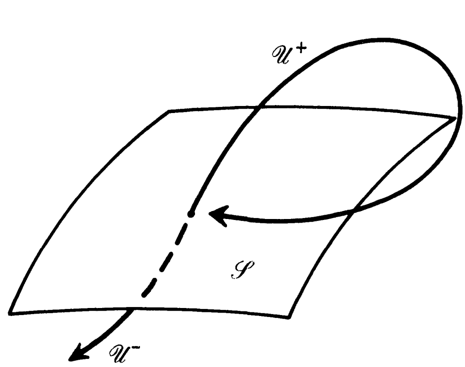

FIG. 3.b

It is possible to discuss the existence of homoclinic orbits for the full Hodgkin-Huxley equations as well, though the results obtained so far are less satisfactory. The system of ordinary differential equations for the travelling wave (corresponding to (5)) is five-dimensional, and the stable and unstable manifolds are of dimensions four and one, respectively. The author knows of progress on

this problem only under certain assumptions about the right-hand sides of the equation which may not be justified physically [2], [22].

4. Stability of travelling waves. The experimental evidence described in §2 suggests that there is only one positive value of c such that (5) has a homoclinic orbit. (It is no loss of generality to take $c > 0$, as is seen by considering the transformation $s \to -s$, $c \to -c$, $v \to -v$ in (5).) However, the situation is not quite so simple. One can prove [24] that there are at least two such values of c, and numerical evidence points to there being exactly two possible wave speeds, say $0 < c_* < c^*$ [5], [15], [33]. Why, then, do we not see waves moving at two different speeds in the laboratory?

First, it must be remembered that in the living axon, one only observes solutions of the partial differential equations (4), and these solutions appear to approach a travelling wave rapidly. Hence, the observability of a particular homoclinic orbit of (5) is a question of the stability of the corresponding wave solution of (4). Do all solutions of (4) in some class, such as those which, initially, are "close" to the travelling wave, approach the travelling wave solution with increasing t?

Numerical computations indicate that the faster of the two waves is the stable one, but the problem seems difficult analytically. (See [10], [11], and [12] for some preliminary results.) It might be useful, here, at least to formulate the question in mathematical terms and point out where the difficulties seem to lie. To do this, it is necessary to take more care about the region in which we look for solutions of (4). The simplest approach is to consider an initial value problem, in which data is specified along the line $t = 0$:

$$\left.\begin{aligned} u(x,0) &= u_0(x) \\ w(x,0) &= w_0(x) \end{aligned}\right\} \qquad -\infty < x < \infty \tag{8}$$

and solutions are sought in the half plane $t > 0$. Suppose now that $(\tilde{u}(s), \tilde{w}(s))$ is a given travelling wave solution of (4) which tends to the origin at $s = \pm\infty$. (Again, $s = x + ct$.) One hopes that if (u_0, w_0) is close enough to $(\tilde{u}, \tilde{w})$, for instance in the sense that

$$|u_0(x) - \tilde{u}(x)| + |w_0(x) - \tilde{w}(x)|$$

is small uniformly in x, then $(u(x,t), w(x,t))$ tends to a travelling wave uniformly in x as $t \to \infty$. It is not reasonable, however, to expect (u, w) to be asymptotic to the particular wave $(\tilde{u}, \tilde{w})$. To see why, observe that any translation $\tilde{u}_h(s) = \tilde{u}(s+h)$, $\tilde{w}_h(s) = \tilde{w}(s+h)$ is also a travelling wave of the desired kind. If h is small, then $(\tilde{u}_h, \tilde{w}_h)$ is close to $(\tilde{u}, \tilde{w})$, but the difference between these functions of $x + ct$ does not tend to zero uniformly in x as t increases. This complicates the stability problem, but one possible formulation is the following.

Suppose that $\sup_{-\infty<x<\infty}\{|u_0(x) - \tilde{u}(x)| + |w_0(x) - \tilde{w}(x)|\}$ *is small. Is it then the case that for some constant* h,

$$\lim_{t\to\infty} \sup_{-\infty<x<\infty} \{|u(x,t) - \tilde{u}(x+ct+h)| + |w(x,t) - \tilde{w}(x+ct+h)|\} = 0?$$

We assume that the reader can be even more precise if he wishes. One can question the use of the sup norm, for perhaps some other measure of closeness is better. For instance, in [40] D. Sattinger uses a more restrictive norm to study some other stability problems for travelling waves. In particular, he treats equation (4) with $b = 0$. In this case (5) is essentially two-dimensional, and the theory is well developed. A recent paper by D. Aronson and H. Weinberger [1] gives some elegant stability results of a different kind, but again, these are for $b = 0$; and as with Sattinger's methods, the extension to $b > 0$ seems difficult. Both of these papers make it clear that the homoclinic waves are hard to study because they are not monotonic in any reasonable sense.

5. Thresholds. A basic property of nerve axons, and many other biological or bio-chemical media, is the existence of an excitability threshold. A small stimulus will not produce an impulse; only when the stimulus strength rises above a certain level will an action potential occur. To study this phenomenon mathematically, we must consider a semi-infinite axon, rather than one which extends along the whole x-axis. This is, in any case, more realistic, since usually a signal is initiated at one end of the axon and travels towards the other end. Accordingly, we look for solutions of (4) in the quarter plane $t > 0$, $x > 0$, with initial and boundary conditions

$$\begin{aligned} u(x,0) &= w(x,0) = 0, \qquad && x \geqq 0 \\ u(0,t) &= U_0(t), && t > 0. \end{aligned}$$

It appears, numerically and physically, that if the initial function U_0 is sufficiently large, then the solution is asymptotic to a travelling wave. However, it is not at all clear what measure of the size of U_0 is most appropriate. It is thought that if U_0 has sufficiently small compact support, that is, if U_0 vanishes outside some interval $[0, T]$ where T is small, then the size of $\int_0^T U_0(t)\,dt$ is crucial. Physically this integral represents the total stimulating charge. However, a constant initial function $U_0(t) \equiv \varepsilon > 0$ of sufficiently small amplitude will not produce an impulse, so some other factor is important as well.

6. Periodic travelling waves. It has long been recognized that an important feature of nervous system behavior is the existence of trains of impulses in response to a prolonged stimulus. In Hodgkin-Huxley type systems this is reflected in the existence of periodic solutions, that is, solutions y such that for some $P > 0$, $y(s) = y(s + P)$ for all s. In this section we describe some results and problems concerning periodic travelling waves, while in §7 we deal with space independent periodic solutions.

It turns out that the study of periodic traveling waves is to some extent complementary to the study of homoclinic orbits of (5). As remarked above, it has been shown that for small $b > 0$, there are at least two wave speeds, $0 < c_* < c^*$ which allow homoclinic solutions. Further, c_* and c^* can be chosen so that in the interval $c_* < c < c^*$, no homoclinic orbit is possible. It is exactly in this interval, however, that the existence of periodic solutions of (5) can be demonstrated, by a slight extension of methods in [**23**]. (This is done in [**24**].) These correspond to periodic travelling waves of (4). As we shall see, such solutions may be present for other values of c as well. Other proofs that (5) has periodic solutions are given in [**8**] and in [**2**].

Further insight into what to expect can be obtained from numerical computation, and also by considering even simpler models which still preserve the features of interest. In one of the first mathematical papers on the FitzHugh-Nagumo model, H. P. Mckean proposed that the cubic polynomial f in (5) be replaced by a piecewise linear function, in such a way that one could find exact solutions of the resulting system [**31**]. This can be done in several ways, but the particular suggestion

$$f_1(u) = \begin{cases} -u, & u \leqq a \\ a - u, & u > a \end{cases}$$

has received the most attention, and leads to some interesting conjectures about (5). Since f_1 is discontinuous, the "solution" cannot be too smooth. One looks for u and w which are continuous, together with their first derivatives, and which satisfy the condition that u'' jumps up (down) by 1 each time u crosses down (up) across the line $u = a$.

This model has been thoroughly studied by J. Rinzel [**37**] and by Rinzel and J. Keller [**38**]. As expected, homoclinic orbits were obtained at two different wave speeds, with periodic solutions in between. Furthermore, and rather curiously, periodic solutions were found for values of c at, and in an interval below, the slower speed c_*, for some values of a and b. It was also found that the period

of these solutions varied continuously with c, becoming infinite as the periodic solution tended to either of the homoclinic orbits. This is illustrated in Figure 4, where the period P is plotted against c for two parameter pairs (a, b). Note the non-uniqueness of periodic solutions for some values of a, b, and c.

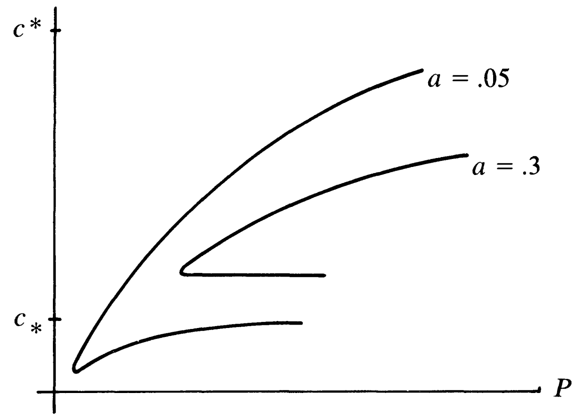

FIG. 4
Taken from [29] or [38], but stylized to emphasize qualitative features described in text. The critical wave speeds c^* and c_* are indicated for the case $a = .05$. In both cases, $b = .05$.

It is natural to ask if these phenomena carry over to (5) (and to the Hodgkin-Huxley equations). R. Casten, H. Cohen, and P. Lagerstrom [3] have used methods of singular perturbation theory to study (5), and obtained some of the structure outlined above, albeit non-rigorously. Results are given in [2] for both the FitzHugh-Nagumo and the Hodgkin-Huxley models when c is close to c^*. Beyond this, little is known.

Before leaving the work of Rinzel and of Rinzel and Keller, we should mention that they have, in addition, studied the stability properties of periodic and homoclinic travelling waves of (4), with f_1 instead of f. Their analysis indicates that the slower waves are unstable and the faster waves stable.

7. Spatially homogeneous solutions. R. FitzHugh, in developing the first simplified nerve model based on Hodgkin-Huxley, sought a pair of coupled ordinary differential equations which mimic the HH equations for a space clamped axon, in which u and w are independent of x. Starting from the classical van der Pol equation, he arrived at essentially the following system, in which α, β, and γ are parameters, with γ assumed to be "large."

$$0 = u' + w + \frac{u^3}{3} - u \tag{9}$$

$$w' = (\alpha + u - \beta w)/\gamma. \tag{10}$$

Modern conventions about the sign of the potential u differ from those in the original HH papers, and in FitzHugh's earlier work. Our equations therefore differ from FitzHugh's in sign. The FitzHugh-Nagumo system (4) is derived from (9), (10) by adding a diffusion term $\partial^2 u/\partial x^2$ to (9), setting $\beta = 0$ in (10) to obtain a simpler system, and changing variables appropriately. It has been suggested [18], [19] that the "damping" term $-\beta w/\gamma$ should, in fact, be retained, but this proposal has yet to be thoroughly investigated. It appears numerically that solutions behave satisfactorily without this term, but any analytical advantage arising from the simpler looking dependence of w' on u may be illusory.

The constants α and β in (10) are limited by the constraints

$$0<\beta<1, \qquad 1-\tfrac{2}{3}\beta<\alpha<1, \tag{11}$$

to insure that (9), (10) has a unique, asymptotically stable, equilibrium point*. Equation (9) is written so that the right-hand side represents the net transmembrane current I_m. If the axon is subjected to a prolonged constant stimulus, then this current is a non-zero constant I, and (9) is replaced by

$$I = u' + w + \frac{u^3}{3} - u. \tag{12}$$

We wish to discuss the phase space of the two-dimensional system (10), (12) for various values of I. First, linearize around the equilibrium point (u_0, w_0), which depends on I, by letting $x_1 \sim u - u_0$, $x_2 \sim w - w_0$. This gives the system

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = B(I)x, \qquad B(I) = \begin{pmatrix} 1-u_0^2 & -1 \\ \gamma^{-1} & -\beta\gamma^{-1} \end{pmatrix}.$$

Considering first $I = 0$, a little algebra, and use of (11), shows that the eigenvalues of $B(0)$ have negative real parts, in accordance with our assertion that when $I = 0$, (u_0, w_0) is asymptotically stable.

Still taking $I = 0$, we consider to what extent the phase space of (9), (10) reflects the experimentally observed behavior of a space-clamped axon. Referring to §2, it is not hard to see that the basic experiment of stimulating a resting axon with a brief current pulse corresponds to an initial value problem for (9), (10) with $w(0) = w_0$, $u(0) > u_0$. Thus, in the phase plane, relevant trajectories start on the line $w = w_0$ and to the right of $u = u_0$. In Figure 5 we have plotted the curves $u' = 0$ and $w' = 0$, and the line $w = w_0$, which intersects $u' = 0$ at three points, $u_0 < u_1 < u_2$. Note that if $u(0) < u_1$, then $u'(0) < 0$, while if $u(0) > u_1$, then $u'(0) > 0$. Further, $dw/du = w'/u'$ is small unless $|u'|$ is small, since γ is large.

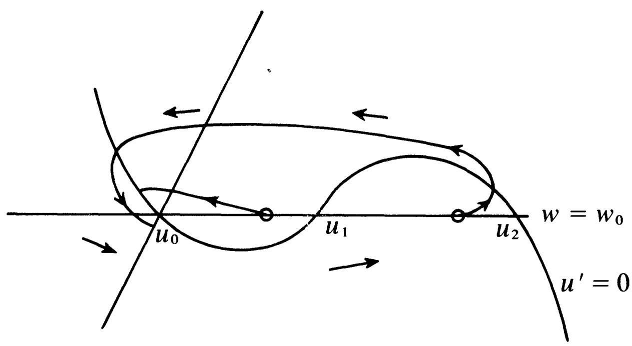

FIG. 5
○'s indicate starting points $(u(0), w_0)$ of two trajectories

Also in Figure 5 we have sketched the tangent vector (u', w') at several points in the plane, as well as a couple of solution trajectories. Observe that if $u(0) < u_1$, then u decreases to below u_0, and never crosses the line $u = u_1$, while if $u(0) > u_1$, then u rises almost to u_2. This is the mathematical version of the threshold effect for a space clamped axon as described in §2, with $u_T = u_1$. Of course, a precise statement requires a bit more care, which we trust the reader can take if desired.

* See any of the standard references on ordinary differential equations for the definition of asymptotic stability. Roughly speaking it means that if $|u - u_0|$ and $|w - w_0|$ are small initially, then they remain small and $u \to u_0$, $w \to w_0$ as $t \to \infty$.

Now suppose that $I>0$, corresponding to a steady stimulating current. This has recently been investigated by W. Troy, and the following observations are due to him **[41]**. Over some interval $0<I<I_0$, the equilibrium point remains asymptotically stable, and the phase picture is roughly as for $I=0$. However, at a critical value $I=I_0$ the matrix $B(I)$ has purely imaginary eigenvalues, and in an interval $I_0<I<I_1$ the eigenvalues of $B(I)$ have positive real parts, so that (u_0, w_0) is an unstable equilibrium point. Such a change in stability is usually associated with the appearance of qualitatively different solutions of the system under discussion. In particular, one suspects from other examples that in the interval $I_0<I<I_1$ there are periodic solutions. In phase space such solutions have trajectories which are closed curves surrounding the equilibrium point. As $I\to I_0$ or $I\to I_1$, the closed curve shrinks down to the equilibrium point. This phenomenon is called "bifurcation" of a family of periodic solutions from equilibrium, and was first investigated carefully by E. Hopf **[27]**. An accessible account of the two-dimensional case is found in **[17]**. Troy has shown that bifurcation does occur in (9), (12), and gone on to investigate the continued existence and amplitude of periodic solutions as I increases. He has also found bifurcation of periodic solutions for some fourth order systems resembling the HH space-clamp equations. This work is of interest biologically because the periodic solutions of small amplitude which are found when I is near I_0 or I_1 have not been reported before in the experimental literature. There is hope that these solutions can be detected experimentally, since, in the two-dimensional version, they have been shown to be stable for some values of the parameters. As far as we know, however, they have not yet been sought in the laboratory.

Added in proof: Small amplitude oscillations have been seen before in numerical simulations **[13]**. Computations in **[39]** indicate that for the four dimensional HH system these oscillations are unstable near the lower bifurcation point I_0, but stable near the upper bifurcation point I_1.

8. A little mathematics. We have referred briefly to some results which have been obtained for the nerve equations, with no indication of the proofs. In this section proofs of some simpler propositions are sketched, partly to attain a bit of mathematical respectability **[20]**, and partly to impart some of the flavor of one kind of analysis which has been used in this area.

A basic question is the existence of homoclinic orbits for the system of ordinary differential equations (5); that is, does there exist a $c>0$ such that (5) has a non-constant solution tending to $\mathbf{0}$ at $\pm\infty$? The following lemma shows the answer to be "no", unless $0<a<\frac{1}{2}$.

LEMMA 1. *If* $\frac{1}{2}\leqq a<1$, *then all non-constant solutions which intersect the unstable manifold* $\mathcal{U}$ *are unbounded.*

Proof. Recall from §3 that if $y(0)\in\mathcal{U}$, then $y(-\infty)=\mathbf{0}$. The stable manifold theorem also implies that solutions which never intersect $\mathcal{U}$ cannot approach $\mathbf{0}$ at $-\infty$. $\mathcal{U}$ is a smooth curve which is tangent to the eigenvector p_1 of A at $\mathbf{0}$, again according to the stable manifold theorem, and by an easy computation, all three components of p_1 have the same sign. Hence, a solution $y(s)$ tending to $\mathbf{0}$ at $-\infty$ must lie, for large negative s, either in the positive octant $(u, v, w>0)$ or in the negative octant.

Suppose that for some s_0, $u(s_0)$, $v(s_0)$, and $w(s_0)$ are all negative. Since $f(u)>0$ if $u<0$, (5) implies that u', v', and w' are all negative at s_0. Therefore (proof by contraction), u, v, and w remain negative and decreasing for $s\geqq s_0$. Hence, $u(s)\not\to 0$, and in fact, $y(s)$ tends to infinity in the set Q^-: $u<0$, $v<0$. (It is convenient in what follows to ignore w in describing how a solution becomes unbounded.)

As a consequence, the only possible candidates for homoclinic solutions are those which, for large negative s, lie on $\mathcal{U}$ and in the positive octant.

Notation: For any $c>0$, let $y_c=(u_c, v_c, w_c)$ denote any solution y of (5) such that for large negative s, $y(s)\in\mathcal{U}$ and all three components of $y(s)$ are positive.

We show that if $\frac{1}{2} \leqq a < 1$, then $y_c(s)$ remains in the set Q^+: $u > 0$, $v > 0$, as long as the solution y_c is defined. To do this, we use the standard technique of defining an "energy" function

$$\psi(s) = \tfrac{1}{2} v_c(s)^2 + F(u_c(s)),$$

where

$$F(u) = \int_0^u f(x)\, dx.$$

Differentiating ψ and using (5) gives

$$\psi' = cv_c^2 + v_c w_c,$$

which is positive as long as v_c and w_c are positive. From the first and third equations of (5), u_c and w_c remain positive as long as $v_c > 0$.

Since $y_c(-\infty) = 0$, $\psi(-\infty) = 0$ and therefore $\psi > 0$ and $\psi' > 0$ as long as $v_c > 0$. But if $\psi > 0$, then $\frac{1}{2} v_c^2 > -F(u_c)$. This is where the condition $a \geqq \frac{1}{2}$ is used, for it insures that $F(u) \leqq 0$ for all u. The point is that in the graph of f in Figure 6, the area A_1 is no smaller than the area A_2.

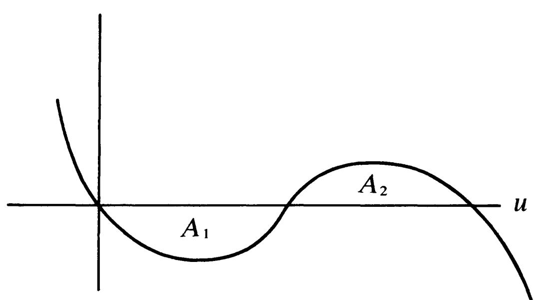

FIG. 6

Another proof by contradiction now shows that $v_c(s) > 0$ as long as $y_c(s)$ is defined, and hence u_c and w_c continue to increase, completing the proof of Lemma 1.

Thus, in order for homoclinic orbits to exist, a must be less than $\frac{1}{2}$, or, equivalently, $\int_0^1 f(x)\, dx > 0$. This condition has a physiological interpretation (which we shall not go into here) and has been known to biologists for some time [**35**]. There is a direct analogue in the case of the full Hodgkin-Huxley equations [**22**].

The principal result in this section is

THEOREM 1. *Let y_c denote a solution of* (5) *which intersects $\mathcal{U}_c$ in the positive octant. For fixed positive $a < \frac{1}{2}$ and sufficiently small $b > 0$, there is a $c^* > 0$ such that y_{c^*} exists and is bounded on* $(-\infty, \infty)$.

Proof. We use a topological method of considerable importance in the geometric theory of ordinary differential equations. This technique, sometimes called the "shooting method", is especially suited to boundary value problems on infinite or semi-infinite intervals, as, for example, in [**32**]. Its exploitation here is particularly simple, because the only topological concept required is connectedness on the real line.

LEMMA 2. *Under the hypotheses of the theorem there is, for sufficiently small $b > 0$, a $c_1 > 0$ such that $y_{c_1}(s)$ tends to infinity in the set Q^- as s increases. There is also a $c_2 > c_1$ such that $y_{c_2}(s)$ tends to infinity in the set Q^+ as s increases.*

Proof. First consider (5) when $b = 0$. It is important to realize that the unstable manifold is still well defined [**21**]. Then (5) is essentially two-dimensional, with $w = 0$ on $\mathcal{U}$. Suppose in addition that $c = 0$. If $\psi = \frac{1}{2}v_c^2 + F(u_c)$, then $\psi' = 0$ and ψ is constant. But $\psi(-\infty) = 0$, so ψ is identically zero. Hence the solution trajectory lies in the plane $w = 0$ along the curve δ defined by $\frac{1}{2}v^2 + F(u) = 0$. Since $0 < a < \frac{1}{2}$, δ is a closed curve (Figure 7.a).

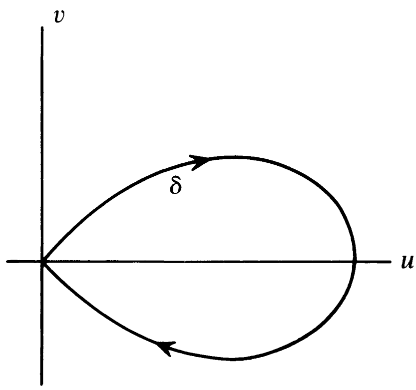

FIG. 7.a

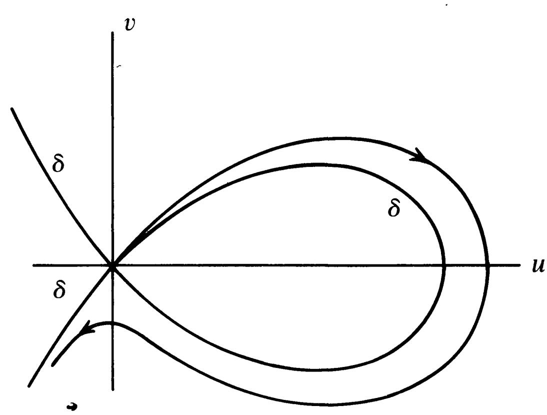

FIG. 7.b

Now suppose that $b = 0$ and c is positive but small. A sufficiently general statement of the stable manifold theorem, as in [**21**], implies that $\mathcal{U}_c$ varies continuously with c, for $c \geqq 0$, so y_c still remains close to the curve δ. (A uniformity argument is needed to make this precise.) Now, however, $\psi' = cv^2 > 0$ (when $v \neq 0$), $v^2 > -2F(u)$, and the point (u, v) lies outside δ. After extending δ to the left-half plane, it can be shown that for some sufficiently small $c_1 > 0$, v_{c_1} tends to $-\infty$ (Figure 7.b). Note also that u_{c_1} is eventually negative, and hence there is an s_0 such that

$$u_{c_1}(s) < 0,\ v_{c_1}(s) < 0, \quad \text{and} \quad c_1 v_{c_1}(s) < -\int_{-\infty}^{s} u_{c_1}(\omega)\, d\omega \tag{13}$$

for $s \geqq s_0$.

Next fix $c = c_1$, but let b be positive. The manifold $\mathcal{U}_{c_1}$ depends continuously on b, so for small b the inequalities (13) are still valid at $s = s_0$. From this we conclude that $c_1 v_{c_1}(s_0) < -w_{c_1}(s_0)$ if b is small, and therefore u'_{c_1}, v'_{c_1}, and w'_{c_1} are negative at s_0. A differentiation of (5) then leads to the inequality $v''_{c_1}(s_0) < 0$, and the first assertion of Lemma 2 follows in a couple of steps.

The proof of the second part is easier, and can proceed from the relation $\psi' = cv^2 + vw$. For large c it is shown that u_c, v_c, and w_c are positive as long as the solution y_c exists. We omit the details.

In order to prove the theorem, consider a fixed $b > 0$, as in Lemma 2, and define three subsets of the half-line $0 < c < \infty$.

$$A = \{c \mid y_c \text{ is unbounded}\}$$
$$B = \{c \mid y_c(s) \to \infty \text{ in } Q^+\}$$
$$C = \{c \mid y_c(s) \to \infty \text{ in } Q^-\}.$$

Then use the next lemma.

LEMMA 3. *The sets B and C are open in* $(0, \infty)$, *and* $A = B \cup C$.

Idea of proof. The hard part is to show that $A = B \cup C$. This is done by elementary, but slightly tedious, computations which eliminate the possibility of unbounded solutions oscillating between

Q^- and Q^+. These computations work because, for large u, $u'' = cu' - f(u) + w \sim u^3$ if u' and w' are small. See [**23**] and [**24**] for details.

Now observe that, by Lemma 2, B and C are non-empty. They are clearly disjoint as well, so A is disconnected. Therefore $A \neq (0, \infty)$, and we can choose $c^* > 0$ to be in the complement of A. This proves Theorem 1.

With more careful analysis one can show that the set B contains an interval (c_*, c^*), $c_* > 0$, the end points of which correspond to homoclinic orbits, and a further refinement of the shooting technique reveals periodic orbits if $c_* < c < c^*$.

Appendix, the Hodgkin-Huxley equations. A major challenge to those working in this area remains the set of equations developed by Hodgkin and Huxley to model impulse transmission in the squid axon. In presenting these equations we follow the original notation [**26**] by letting V denote the potential across the membrane and m, n, and h the components of $\mathbf{w}$ in (1), (2). Physically, m and h jointly determine the permeability of the membrane to sodium, while n plays this role for potassium. A thorough treatment of the theory, describing the beautifully conceived experiments used to determine the various parameters, is contained in [**25**], and other useful surveys, covering subsequent developments and extensions of the theory, are in [**15**] and [**34**].

The HH equations can be written as follows:

$$\partial^2 V/\partial x^2 = R[(\partial V/\partial t) + 36n^4(V - V_K) + 120m^3h(V - V_{Na}) + .3(V - V_L)], \tag{a.1}$$

$$\begin{cases} \partial n/\partial t = (n_\infty(V) - n)/\tau_n(V) \\ \partial m/\partial t = (m_\infty(V) - m)/\tau_m(V) \\ \partial h/\partial t = (h_\infty(V) - h)/\tau_h(V). \end{cases} \tag{a.2}$$

Here R, V_K, V_{Na}, and V_L are constants, while τ_n, τ_m, τ_h, n_∞, m_∞, and h_∞ are known functions. A scale change in V will put these equations in the form (1), (2). We do not write out the precise expressions for the constants and given functions in this paper (see [**26**]), but merely note some of their properties.

$$R > 0, \qquad V_K < V_L < V_{Na}. \tag{a.3}$$

(a.4) All given functions are real analytic and positive on $-\infty < V < \infty$.

$$n'_\infty,\ m'_\infty\ - h'_\infty > 0;\ n_\infty(-\infty) = m_\infty(\infty) = h_\infty(-\infty) = 1; \tag{a.5}$$
$$n_\infty(-\infty) = m_\infty(-\infty) = h_\infty(\infty) = 0.$$

In addition, τ_n and τ_h are considerably larger than τ_m, with the result that n and h are slow variables, relative to m and V.

Important digital computer studies of the equations are in [**7**], [**9**], and [**16**], and references to further numerical work can be found in [**13**] and [**15**].

Finally, it should be noted that while the theory of Hodgkin and Huxley is widely accepted in broad outline, certain details have been, and continue to be, criticized. Suggestions have been made on how the equations might be modified to achieve even better quantitative agreement with experiment [**28**], and one or two of these might affect the mathematical analysis of the system. Nevertheless, it seems unlikely that the qualitative features we have described will undergo significant revision in the foreseeable future.

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References

1. D. Aronson and H. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve propagation, to appear in Proceedings of the Tulane Program in Partial Differential Equations, Lecture Notes in Mathematics, Springer Verlag, 1975.

2. G. Carpenter, Travelling wave solutions of nerve impulse equations, Ph. D. Thesis, Univ. of Wisconsin, 1974.

3. R. Casten, H. Cohen, and P. Lagerstrom, Perturbation analysis of an approximation to Hodgkin-Huxley theory, Quart. Appl. Math., 1975.

4. E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955, p. 330.

5. D. Cohen, Multiple solutions of nonlinear boundary value problems arising in chemical reactor theory, SIAM J. Applied Math., 20 (1971) 1–13.

6. H. Cohen, Nonlinear diffusion problems, in Studies in Applied Mathematics, A. H. Taub, editor, Prentice-Hall, Englewood Cliffs, N. J., 1971.

7. K. S. Cole, H. A. Antosiewicz, and P. Rabinowitz, Automatic computation of nerve excitation, J. SIAM, 3 (1955) 153–172.

8. C. C. Conley, Travelling wave solutions of nonlinear diffusion equations, to appear in Proceedings, Tulane Conference in Partial Differential Equations, Springer Verlag Lecture Notes, 1975.

9. J. W. Cooley and F. A. Dodge, Digital computer solutions for excitation and propagation of the nerve impulse, Biophysical Journal, 6 (1966) 583–599.

10. J. W. Evans, Nerve axon equations, I: linear approximations, Indiana Univ. Math. J., 21 (1972) 877–885.

11. ———, Nerve axon equations, II: stability at rest, Indiana Univ. Math. J., 22 (1972) 75–90.

12. ———, Nerve axon equations, III: stability of the nerve impulse, Indiana Univ. Math. J., 22 (1972) 577–594.

13. R. FitzHugh, Thresholds and plateaus in the Hodgkin-Huxley nerve equations, J. Gen. Physiol., 43 (1960) 867–896.

14. ———, Impulses and physiological states in theoretical models of nerve membrane, Biophysical J., 1 (1961) 445–466.

15. ———, Mathematical models of excitation and propagation in nerve, in Biological Engineering, H. P. Schwan, editor, McGraw-Hill, New York, 1969.

16. R. FitzHugh and H. Antosiewicz, Automatic computation of nerve excitation — corrections and additions, J. SIAM, 7 (1959) 447–458.

17. K. O. Friedrichs, Advanced Ordinary Differential Equations, Springer Verlag, 1971.

18. J. M. Greenberg, A note on the Nagumo equation, Quart. J. Math. Oxford, 2 (1973) 307–314.

19. M. W. Green and B. D. Sleeman, On FitzHugh's nerve axon equations, J. Math. Biology, 1 (1974) 153–163.

20. P. Halmos, How to talk mathematics, Notices AMS, 21 (1974) 155–158.

21. P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964, p. 242.

22. S. P. Hastings, On travelling wave solutions of the Hodgkin-Huxley equations, to appear in Arch. Rational Mech. Anal.

23. ———, The existence of periodic solutions to Nagumo's equation, Quart. J. Math. Oxford, 25 99 (1974) 369–378.

24. ———, The existence of homoclinic and periodic orbits for Nagumo's equation, to appear in Quart. J. Math., 1976.

25. A. L. Hodgkin, The conduction of the nervous impulse, Charles Thomas Company, Springfield, Ill., 1964.

26. A. L. Hodgkin and A. F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J. Physiol., 117 (1952) 500–544.

27. E. Hopf, Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems, S.-B. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig, 94 (1942) 1–22.

28. E. Jakobsson, The physical interpretation of mathematical models for sodium permeability changes in excitable membranes, Biophysical J., 13 (1973) 1200–1211.

29. B. Katz, Nerve, Muscle and Synapse, McGraw-Hill, New York, 1966.

30. A. Kolmogorov, I. Petrovsky, and N. Piskunov, Étude de l'équation de la chaleur avec croissance ... et son application à un problème biologique, Bull. U. État Moscou, 6 (1937) 1–25.

31. H. P. Mckean, Nagumo's equation, Adv. in Math., 4 (1970) 209–223.

32. J. B. McLeod and J. Serrin, The existence of similar solutions for some boundary layer problems, Arch. Rational Mech. Anal., 31 (1968) 288–303.

33. J. Nagumo, S. Arimoto, and S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE, 50 (1962) 2061–2070.

34. D. Noble, Applications of Hodgkin-Huxley equations to excitable tissues, Physiol. Rev., 46 (1966) 1–50.

35. D. Noble and A. E. Hall, The conditions for initiating all-or-nothing repolarization in cardiac muscle, Biophysical J., 3 (1963) 261–274.

36. A. B. Poore, A model equation arising in chemical reactor theory, Arch. Rational Mech. Anal., 52 (1973) 358–388.

37. J. Rinzel, Travelling wave solutions of a nerve conduction equation, Doctoral thesis, New York University, 1973.

38. J. Rinzel and J. Keller, Travelling wave solutions of a nerve conduction equation, Biophysical J., 13 (1973) 1313-1337.

39. N. H. Sabah and R. A. Spangler, Repetitive response of the Hodgkin-Huxley model for the Squid Giant Axon, J. Theor. Biol., 29 (1970) 155–171.

40. D. Sattinger, On the stability of waves of nonlinear parabolic systems, to appear.

41. W. Troy, Doctoral thesis, SUNY, Buffalo, 1974.

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MATHEMATICIANS IN OPERATIONS RESEARCH CONSULTING

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The purpose of this article is to describe the author's perception of the operations research (OR) consulting environment as it relates to mathematicians. It is presumed that the employment market and related issues in graduate education will be foremost in minds of most readers, and I attempt to respond to these interests. Much has been written and said about these matters in the last few years (not peculiar to OR), and some of the assertions below will be at variance* with typical positions taken elsewhere. The main points are the following:

(1) It is quite feasible for a Ph.D. in basic mathematics, as well as one in applicable mathematics, who obtains a position in OR to have a fruitful career compatible with his education. Some important qualifications on this contention are given in section 5.

(2) A desire on the part of a graduate mathematics department to include in its objectives that of preparing students for an OR career, and indeed other fields of mathematical applications, need not have drastic effects on existing curricula (section 6).

The term "basic" mathematics is used here in place of "pure" mathematics. Mathematics which is researched or taught with a view to possible applications, but outside the context of specific applications to non-abstract problems, is termed "applicable." This reserves the term "applied" mathematics to refer to such actual applications. These usages follow some recent trends.

The first section describes the background on which this article is based. The succeeding two sections discuss OR in general and OR consulting in particular. The fourth section presents some examples from mathematically oriented OR. This is followed by a section on employment potential, and the final section comments on implications for graduate education.

I am indebted to my colleagues, a reviewer, and other mathematicians for much valuable criticism. Most of the points raised here are concurred in by some colleagues and disagreed with by others. I appreciate the editor's initial stimulus to prepare this article, his subsequent encouragement, and his kind permission to publish abbreviated versions as [31] and [32].

1. Basis for this exposition. It is necessary to say something about the experience background on which this article is based, and this requires describing the operations research consulting firm that I presently head.

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* See [8], which reports a survey of mathematics department heads in industry. I apologize for not responding to this survey, largely by oversight. The opinions quoted in [8] appear to be much more representative of attitudes in industry than are those stated in this article. While this article disagrees with much of [8], I agree with some of the quotes there, particularly the first (by respondent [19] on page 700).

The firm in question was formed in 1963 and currently has a full-time consulting staff of 14 Ph.D.'s in the mathematical sciences, supported by two part-time graduate students and four programmers. The doctoral theses were, approximately chronologically, in group theory, probability theory (four), ergodic theory, differential geometry, functional analysis (three), statistics, number theory, control theory and complex analysis. Since 1968 the staff has submitted 34 papers which were accepted by research journals, averaging a little over one paper per two man-years. Most of these are in applicable mathematics; about half are referenced in section 4. That our consulting staff is homogeneously from the mathematical sciences is by intent and is a fairly unique feature. We have no OR degrees, but that is not by intent.

The majority of our work is for the U. S. Navy in the areas of anti-submarine warfare, search for lost objects, and logistics. Most of our colleagues have undertaken, enjoyed, and benefitted professionally from field assignments with the fleet, either on search operations for a few months or with tactical development activities for one to three years. The balance of our work is largely for non-defense government agencies such as the Department of the Interior, the Coast Guard, a transportation authority, and the National Institutes of Health. We have also served a large New York bank extensively in the municipal bond market.

2. General remarks about operations research. One of the extant definitions of OR is that it is the application of scientific methods to provide a basis for executive decision and action. I would accept that definition but would not regard it as exclusive. The adjective "scientific" excludes most (but not all) forms of accounting and management consulting. It also excludes Hitler's astrologer, who has been offered as a counterexample to some definitions. Those who perform OR are referred to here as operations analysts.

"Operations" to which OR applies generally involve not only "things," but also people, which tends to be a point distinguishing OR from engineering. While OR is sometimes applied to problems which are almost entirely concerned with people, e.g., racial tensions, I believe such applications are less successful than when "things" are also involved with people in important ways. Another point which distinguishes OR from other sciences and from engineering is that OR deals with parameters ordinarily not measured in a laboratory, e.g., costs, profits, probabilities of various kinds of success or failure, etc.

As an identified science, OR had military origins in 1937–40 with a group of British scientists investigating ways to obtain the best operational effectiveness of radar, then a new device. This work was of vital importance in the Battle of Britain. Nobel laureate physicist P. M. S. Blackett was a prominent early leader. OR spread to this country in 1942 during the anti-submarine Battle of the Atlantic with MIT physicist Philip Morse organizing the initial group. Military use of OR became more widespread as World War II proceeded, and after the war it became a permanent part of the establishments of each of the armed forces.

Industrial use of OR evolved after World War II, initially through some of the large management consulting firms. Today most large corporations have their own in-house OR groups. Numerous service organizations specialize in OR consulting, while others, such as large accounting firms and management consulting firms, consider OR to be a natural adjunct to the primary services they offer their clients. Non-defense governmental use of OR has increased markedly: transportation, crime control, public health, and pollution control have been important areas in recent years, and the energy shortage problem will make new demands on OR.

Over the years numerous significant improvements in operational effectiveness have been attributed to OR in a wide variety of endeavors. One notes, however, that its acceptance in various quarters ebbs and flows. My opinion is that acceptance of OR in the business community is well below its potential, and the same is to be said of, for example, some important governmental areas of transportation.

One reason why OR encounters difficulties in acceptance at times is a feeling on the part of the

prospective user executive that if he resorts to OR, he is admitting inadequacy as a manager. This is related to a deceptively erroneous appearance that OR conclusions are often obtained so easily and frequently seem obvious after the fact. I am of the firm opinion that the most effective users of OR are generally very capable executives (self-serving bias admitted here).

Empiricism plays an important role in OR. Data describing the operation under study are desirable to enhance realism, to reveal which parameters are crucial, to suggest tradeoffs, etc. Many striking payoffs achieved by OR have resulted primarily from the search for good data to describe what is going on with a realism not otherwise available to the user executive; **[14]**, a classic text on OR, has several examples of this. However, when, for example, one is analyzing the operational requirements for a new system to be developed, it is entirely possible that there are simply no reasonable operational data available to describe the future environment in which the system will function.

What is the role of mathematicians in OR? A long-standing precept of many in OR is that an OR group should bring together talents from a variety of scientific disciplines. This concept certainly has merit, but it is by no means a dominating requisite, as may be inferred from the continued existence of my firm. I prefer a different emphasis as expressed in **[14]**, where the accent is not so much on the multi-disciplinary nature of the group as on the principle that operations analysts can be drawn from any of a variety of scientific disciplines, providing they have strength in research ability within their own disciplines, among other qualifications. (However, **[14]** also expresses mildly negative feelings about mathematicians in OR.) At any rate mathematics has been one of the several disciplines thus brought together, and it was recognized from early beginnings that elementary methods in probability and in optimization would be of fundamental importance to the conduct of OR.

Among the U. S. mathematicians who were pioneers in OR, probably the most prominent to remain active in the OR profession subsequent to World War II is B. O. Koopman (see section 4). He retained an academic base for many years. Many academic mathematicians who participated in wartime OR left the field thereafter, e.g., Norman Steenrod.

Two important centers of mathematicians among military OR organizations should receive special mention: the RAND Corporation mathematics group during the 1950's, serving the Air Force, and the mathematical programming group of the Research Analysis Corporation during the 1960's, serving the Army. The main strength of both of these groups has since been dispersed elsewhere.

The broad areas of mathematics which have the most general usefulness in OR are optimization methods, probability theory, and statistics. Optimization questions arise quite naturally from the desire to choose from among the available alternatives (i.e., subject to stipulated constraints) that course of action which maximizes payoff. Since most OR deals with operational problems in which considerable uncertainty is present, the need for probability and statistics is evident. Typically one uses probability theory for modeling uncertainty and as a calculus to deduce consequences of uncertainty. Statistics is used to interpret data, to design experiments (operational trials), and to draw inferences from their results.

One wishes to give a quantitative description of the operation in question in a way which is reasonably faithful to reality and which is simple enough to be tractable for analysis; such a description is called a model, or rather, a good model. The ability to devise such models is one of the prime ingredients for success as an operations analyst, and, I presume, as an applied scientist in other fields.

Numerical analysis and computer methods are natural adjuncts to mathematically oriented OR work, as they are to most work in applied mathematics.

OR has provided a tremendous stimulus for the development of new methods of constrained optimization during the past 30 years. Let us note the best known areas and some mathematicians who achieved initial breakthroughs. Game theory (J. von Neumann) and linear programming (G. B. Dantzig) were the earliest branches of new development to be identified in this connection. The

latter term has given rise to the term mathematical programming, synonymously non-linear programming (H. W. Kuhn and A. W. Tucker), which may be defined as choosing a point in a given subset S of Euclidean space (or of a function space) to maximize a given real-valued function on S subject to inequality constraints on a given finite family of real-valued functions on S. If S is countable, this is called integer programming. Dynamic programming (Richard Bellman) is a recursive method of constrained optimization. There is much current interest in combinatoric optimization questions.

Some pioneers have been mentioned. It is hopeless to try to enumerate current researchers in mathematical programming. Most such research is in academia or research institutes rather than in OR consulting. Few of the mathematicians involved hold appointments in mathematics departments and much of the significant progress is due to non-mathematicians.

3. Operations research consulting. We now discuss the conduct of OR consulting.

The genesis of an OR consulting case usually begins with discussions among the prospective OR user, here called the client, and the prospective workers on the case, in a preliminary effort to define the problem. It is imperative that the operations analyst be able to conduct his own communications with the client. The problem often arises as merely something vaguely bothersome to the client in his concern about his operation. Perhaps he perceives a conflict between two influences, both of which make cost and effectiveness contributions to payoff. At this very early stage, OR competence should assert itself in a most useful way by efforts to obtain a *quantitative* description of the situation and by asking probing questions which endeavor to disclose the parameters that bear significantly on the issues at hand. In the process of these problem formulation discussions, the operations analyst must quickly become familiar with some of the technology and jargon peculiar to the client's operation.

When the OR consultant believes he has acquired a reasonable preliminary understanding of the problem (he probably will not really understand the problem until he gets well into the research), he ordinarily conveys this understanding in an outline of anticipated approaches toward a solution in the form of a "technical proposal" (accompanied by a cost proposal). Depending on whether he has had a past consulting relationship with this prospective client, he might demonstrate competence through performance of some preliminary analysis, e.g., postulating simplified models and performing computations to illustrate the kinds of tradeoffs which might be present. If the prospective client is satisfied with the proposal, an agreement is negotiated and the work proceeds. If the client is a government agency, considerable red tape can intervene.

The effort assigned to an OR case is usually between a few man-weeks and a few man-years. For my firm, two analysts working half-time for six months each is fairly typical. (We also endeavor to assign at least two cases to each analyst.) If the deadline for finishing the analysis is short, e.g., less than a month away, it is usually not practical to assign a large group; then the total effort is small and shortcuts must be taken. Nevertheless, much very useful OR work is done in just this "quick and dirty" way.

Whether the group assigned is large or small, one finds that, as in other research endeavors, the important contributions are made by individuals at times when they are working by themselves. Group discussions to stimulate and group review to check the soundness of conclusions are very important, but individual creativity is paramount.

The work on the case usually proceeds with efforts to further define the problem, in quantitative terms, of course. One must identify a suitable measure of effectiveness, i.e., a quantity which directly reflects success from the client's viewpoint and which can be related to parameters that are at least partly under the client's control. When this much is accomplished, one has probably already made an important contribution to serving the client's needs. In fact, the remaining efforts to achieve the solution are in some cases quite easy.

Modeling the operation in question, in a faithful but tractable way, is an important early effort and might be part of the problem definition. "Analytic" models are preferred when feasible; this

term means, loosely, those models which avoid monte carlo simulation. When monte carlo simulation is used, the model would almost surely be programmed for use on a computer; this, of course, is also often done with analytic models.

In general, a monte carlo simulation is easy to formulate and program, and often permits one to employ more realistic models by reducing the requirements for artificial simplifying mathematical assumptions. Some problems defy analysis without use of monte carlo, possibly on a large scale, either because of complexity or massiveness. Monte carlo simulation also has disadvantages which should be considered in advance of its use; these include

(1) hiding the intermediate stages of the operation between input and output,

(2) forcing optimization to be carried out by trial and error, and

(3) sometimes incurring unacceptable computer costs.

After the problem is defined and the operation is modeled, it remains to perform whatever additional analysis is needed to solve the problem. At this point, the work might become one of abstract mathematics, but intuition which is inspired by one's appreciation of the physical operation can still play an important role. The analysis might be one of finding constrained extrema or of performing descriptive calculations. In any event, through analysis of the model one determines indicated courses of action which give rise to the principal conclusions or recommendations of the study. Often the indicated precise course of action is too complicated to implement or too unwieldy to be computed so that simplifying approximations are needed.

With the analysis satisfactorily achieved, one usually reports to the client both verbally and in writing (a) the conclusions and recommendations, (b) the assumptions on which these are based, and (c) the analysis which led to the results. Even though the client may understand little of the technicalities of the analysis, both (b) and (c) are necessary to expose the work to external critique if the client should so desire, and also to enable those who performed the analysis to satisfy themselves, by review of their own written work, that it makes sense.

4. Examples. Three examples† from mathematically oriented OR are presented in general terms in this section. Again, I draw upon matters most familiar to me, primarily work of my colleagues. The first is a specific OR example: an important contribution to a problem in distance estimation. The second is an outline history of extensive work on choosing an optimal plan to search for an object one desires to detect; this was an evolution of OR applications and theoretical investigations in applicable mathematics that were thereby motivated. The third is a body of work which is partly of interest as basic mathematics and is motivated by OR needs: some contributions to stochastic process first passage problems.

The first example is the problem of finding the distance from a moving observation platform to an observed object moving with constant unknown velocity, using only directional observations. For motion in the plane, four observations are required to solve for the four parameters of the object motion (two position and two velocity components). If direction to the object is observed twice before and twice after a change in the observer's velocity (the change in velocity is required to obtain an independent set of equations), solution for the object's motion is straightforward. However, the distance estimate is generally sensitive to directional errors, whose distribution is assumed known.

This problem had been very much studied in certain naval circles, but while on field assignment in 1968, my colleague D. C. Bossard [3] made a penetrating observation which yielded new breakthroughs in the subject. He noted that if the distance estimate were made at a well-chosen and calculable time, the principal contribution to error in the estimate could be largely eliminated. Moreover, by suitable choice of change in the observer's velocity (i.e., maneuver), the time for

† *Added in proof.* In a related paper [33], Raisbeck gives a more extensive, and very sound, discussion of the anatomy of OR cases.

favorable estimation of distance may be made to occur at a moment close to the observer's preference (before, during, or after the direction observations used to make the estimate).

Considerable practical, useful applications ensued from these findings, which evolved from a succession of theoretical inspirations and experimental findings from operational trials. In its fundamentals, this work did not require tools more advanced than differential calculus and very elementary statistics. It was nevertheless first class OR and required strong mathematical talent and training in research. In a more complete treatment, one considers a heavy redundancy of observations, and then the analysis uses more advanced filtering theory. If the observed object is permitted to change its velocity, we have a vastly more difficult problem; significant results in this area have been obtained, but important unsolved problems remain.

The search history comprising our second example begins with World War II work by Koopman and his colleagues in naval operations research. As part of a general treatise on search and detection in naval problems [**9**] (see also [**10**], [**11**] and [**12**]), Koopman solved a problem in allocating a given resource, as an amount of search effort, of known effectiveness, to maximize the probability of finding an object whose position has a known probability distribution. This was a striking early example of solving an extremal problem with inequality constraints, not amenable to classical calculus of variations. Koopman used variational methods; modern ways of using Lagrange multipliers are easier. The text [**9**] and this example in particular are classics in search theory. Bibliographies of search theory up to 1966 and 1968 are given in [**6**] and [**7**] respectively.

The remaining discussion of search outlines subsequent contributions by my colleagues, principally H. R. Richardson and L. D. Stone, beginning with on-scene analysis by the former in the 1966 Mediterranean H-bomb search and by both in the 1968 search for the remains of the lost submarine Scorpion [**17**]. In both of these major search operations a Bayesian approach was undertaken. Plausible scenarios were postulated to explain what could have happened to the missing object. A probability distribution of the object's position was derived from each scenario, and a composite distribution was computed as a weighted average using as weights subjective probabilities that the respective scenarios occurred. This exploited an idea of the physicist J. P. Craven. The prior distributions thus obtained (with much effort, including monte-carlo simulation) were valuable planning aids in estimating the duration of the search and in indicating the best places to search at a given stage. For the latter purpose the distributions were updated in Bayesian fashion to account for non-success of effort already applied. Along with these fundamentals, analyses were done on a variety of tasks pertaining to the diversity of search sensors employed, navigational accuracies, amount of investigation needed to identify a non-visual contact as valid or false, etc. The experience thus gained pointed up the need for better theoretical tools to deal with the false contact problem just mentioned, and the problem of uncertainty in the search sensor capabilities (e.g., not knowing whether the object was buried or resting proud of the ocean bottom), among other problems.

To fill this need the Office of Naval Research subsequently supported a significant amount of theoretical research on search problems. Among the results, which would be considered applicable mathematics, were [**15**], [**23**], [**24**], [**28**], [**29**] and [**30**]. Reference [**28**] provides a sufficient condition for optimality of a plan to search for a moving object, in an important class of cases elegantly characterized by factorability of the Jacobian of the object's motion transformation in certain ways. Subsequent work by the Finnish mathematician U. Pursiheimo and by Stone [**26**] has applied functional analysis techniques to finding necessary conditions for a more general class of object motions than that considered in [**28**]. These techniques are related to methods of A. Dubovitsky and A. Milyutin which have been useful in control theory. There remain a variety of important unsolved search problems complicated by object motion, false contacts, or both.

An example of a moving object of a search is a drifting life raft. This was highlighted when we subsequently developed, under Richardson's leadership, computer software for use by the Coast Guard to render real-time computer assistance to search and rescue operations.

Our recent on-scene analysis [**16**] in the search for unexploded explosive ordnance in the Suez

canal revealed new search problems arising from the multiplicity of search objects and from the confining nature of the sloping banks of the canal. On-scene testing of search equipment and procedures played a big role in this operation.

Compendia of search work subsequent to [9], including numerous references to work of other organizations, are [18] on the pragmatic side and [25] and [26] on the theoretical side. F. A. Andrews, coauthor of [18], is a Professor at Catholic University and as a Navy Captain commanded the 1963 search for the lost submarine Thresher.

For the third example, let us consider how stochastic process first passage problems arise in OR. Suppose that the output of a system (e.g., an aid to aircraft collision avoidance or a detection system) is represented by a real stochastic process ξ on $[0, T]$ in such a way that the system succeeds (e.g., avoids collision or detects) at time $t \in [0, T]$ if $\xi(t) \geqq K$ for some fixed threshold K.

It is typical for engineers designing such a system to focus on the instantaneous behavior of each $\xi(t)$. However, it is also typical that the operational user of the system is more interested in the probability that success will be obtained *throughout* the interval of exposure. $[0, T]$ (e.g., in collision avoidance), or that success will be obtained *at least once* in $[0, T]$ (e.g., in detection), as opposed to the probability of success at a given instant. That is, one is interested in (in the former case)

$$p = \Pr\{\xi(t) \geqq K \text{ for all } t \in [0, T]\}.$$

The problem of finding p (equivalently, the distribution of $\sup_{0 \leq t \leq T} \xi(t)$) is called a first passage problem and such problems arise frequently in OR by the means just described. A related problem is that of finding the distribution of time above a threshold, which arises when success in the foregoing example requires that for some $\delta > 0$, $\{t\colon \xi(t) \geqq K \text{ and } 0 \leqq t \leqq T\}$ has Lebesgue measure at least δ. Additional variations on this theme are also useful. Assuming that ξ is a correlated process, these problems are generally difficult and are, in fact, of substantial interest as abstract basic mathematics outside the context of the applications just suggested. Among recent contributions to these problems are [1], [2], [13], [19], [20], [21], [22], and [27], by my colleagues largely under support by the Office of Naval Research.

5. The employment problem. As noted earlier, I think there is an excellent potential for future employment of Ph.D. mathematicians in OR and in OR consulting in particular. Certainly OR itself will expand. The world grows increasingly complex, and new theoretical tools and computer capabilities bring more and more of these complexities within reach of OR and other forms of analysis. The question is one of how much mathematicians will participate in this growth as applied practitioners. The full potential will be difficult to realize.

The best support I offer for the contention that a Ph.D. in basic mathematics can flourish in OR consulting is that I have seen it happen repeatedly in my own firm. The key requisites for success in such a transition are, in my opinion: (1) strength in research ability, preferably evidenced by a good thesis; (2) a sincere interest in applications (not just a flight from the academic job market); (3) the ability to devise good models of operational problems (hard to acquire or discover in the classroom); (4) research versatility, which enables one to shift with facility from one type of application to another; (5) written and verbal expository ability; (6) personal attributes which enable one to work with and communicate with scientists of other disciplines and non-scientist users; and (7) a temperament which accepts inexactitudes.

An element of (2) is that one must be able to derive satisfaction from the importance of one's applications. Applied work is not for one whose satisfaction derives solely from the beauty of his methods in abstraction. One should also be disinclined to chide onself that the value of all the effort put into studying advanced algebra, etc., was wasted, once a career in OR was undertaken. The value of such academic experiences is noted in the next section.

Although there are many times that a mathematician's accomplishments in OR make demands on advanced levels of his knowledge of, for example, analysis or probability theory, most of his OR

makes little *ostensible* demands on the *technology* of his graduate education. Nevertheless, if he is successful he will realize that many of his OR results which appear to be within the reach of scientists other than mathematicians, for some reason are *not* done by such others, or at least not as well. Ergo his graduate work and talent in mathematics made a difference.

The biggest obstacle to realizing the potential referred to above is that among employers the views expressed here towards basic mathematics are very much in a minority. Overwhelmingly, industrial employers of Ph.D. mathematicians, including employers who are themselves mathematicians, prefer those whose education has a strong applied label. In fact they usually have more specialized preferences such as education in computers, statistics, control theory, etc. Employers can generally find applicants with good credentials in their desired specialties, and consequently, even a very capable Ph.D. in basic mathematics will find vigorous competition for obtaining (not so much for retaining) employment in industry. Our own firm absorbs only a very small part of the supply, and we do focus some of our recruiting effort on specialties, to acquire needed augmentations in the variety of our capabilities.

As to measures that the profession can take to overcome this obstacle, I see none with promise of much short term progress. What would help in the long term, would be for more mathematicians to be placed in positions of employment decision-making in research and development activities in industry and government.

Of various comments that have been made about the employment problem, I align myself most closely with some things William Browder has said **[4]**, **[5]**. He has noted that the present problem has arisen from turning out too many doctorates in mathematics in the 1960's, some under weakened standards in response to pressures of the era, and that a logical remedy is to reverse that trend by making the new doctorates fewer and subject to higher standards. I gather that many graduate schools are doing so. Unfortunately this will decrease the amount of graduate instruction in mathematics, but that is a bullet that the profession must bite.

I strongly disagree, as do most of my colleagues, with the suggestion of granting non-research doctorates, certainly in respect to preparation for industrial employment. It has amazed me to have been asked at times whether industry could absorb much of such output. There has been a common misconception among mathematics professors that industry is a logical home for their less talented but relatively personable students. (I *frequently* hear comments which reflect this attitude during reference checks. Does this also carry the curious implication that personality characteristics are unimportant in classroom instruction?) Now there are in industry many uses of mathematics in non-demanding ways. However, these are not the roles for which Ph.D.'s should be produced. A Ph.D. is, by definition, supposed to be capable of independent research and this ability is the dominating value in research and development activities in industry as well as in academia.

It is true that most of the very great minds in mathematics, who make broad and penetrating advances in the frontiers of mathematical knowledge, are found in academia. Recall, however, that the very greatest in past generations moved freely between pure and applied research, e.g., Gauss, Hilbert, Poincaré, and von Neumann, in contrast to the disdain for applied work or even applicable mathematics that has been typical among many academic mathematicians of the past 25 years. Disdainers in the opposite direction should also consider whether Courant and Wiener, for example, could have risen to their heights as applied mathematicians without first magnitude strength in basic mathematics.

To a relatively new Ph.D. in basic or applicable mathematics who is seeking employment in industry, I would give general advice as follows: "First of all, if your heart is really in academia and your primary interest in an industrial position is bread and butter, I doubt that you will succeed. If you have a sincere interest in applications, perhaps merely a healthy curiosity, and if you have done excellent work in your academic research, then you should not lack confidence in your ability to do a good job merely by virtue of lack of applied training. On the other hand, you must realize that most industry employers are not mathematicians and even those who are will lack confidence in you

unless you present yourself with some applied trappings. So at least for that reason you should obtain some training in applied areas, e.g., elementary use of computers. You will moreover really need a variety of tools which are usually not taught in a basic mathematics program, many of them fairly easy to acquire, and you will be more credible to employers if you broaden yourself in these ways before confronting them . You are also best advised to seek out employers or managers who are themselves mathematicians. If you succeed in obtaining a position in an organization which is active in identifying and solving complex quantitative scientific problems, your graduate education, particularly your thesis research, should hold you in good stead."

6. Implications for graduate education. To conclude, I offer some opinions on implications of the foregoing for graduate education in mathematical sciences. They are intended to pertain to choices of curricula taken by students and offered by departments.

In the case of a department in basic mathematics, a premise is that a larger fraction of students than before will pursue careers as applied practitioners, possibly in OR. Some of these students will have such an objective in mind from the outset of their graduate work, while others might thus redirect themselves at an advanced point in their studies.

It seems obvious to many that a graduate student in mathematics should be learning things he will use to earn his living. To me it is equally obvious that his overriding objective is to learn mathematics.

An initial objective is to acquire a good general knowledge of what mathematics is all about, including set-theoretic foundations. This objective is usually met by first year studies in the areas of analysis, algebra, and topology. Fulfillment of this objective of general comprehension is highly desirable to prepare for applied as well as basic research.

Beyond this one pursues increasingly advanced work, to prepare for original research on his dissertation, on the role of which I again align myself with William Browder [5]. This requires some degree of specialization, but the accent is on originality. In the dissertation research, probably for the first time the student solves difficult problems without knowing in advance whether they have reasonably nice solutions. An applied practitioner certainly needs this training.

What about the choice of specialty? It is of crucial importance that the thesis be well done. Since the thesis and the preparatory advanced courses are to be demanding and should stretch the student's intellectual capacity, the speciality should by all means be an area of the student's liking. If he happens to warm up to an area of applicable mathematics, and therefore does his thesis in such an area, he will have an easier time finding employment; however, excellence in this climax of his formal education should be the overriding consideration. Does it matter that much if one changes fields in one's post-doctoral work? Indeed among academic mathematicians I venture that numerous examples can be cited of men whose main marks were made in fields quite different from those of their theses.

There is nothing new in the description I have just given of a desired graduate education program. Accordingly, I am not recommending drastic curricular changes.

The student in basic mathematics who contemplates an applied career will need to acquire at some point what I refer to loosely in the preceding section as "tools" useful in applied work. This takes time, for which provisions should be made. Perhaps it would be in the form of two to four one-semester specialized courses, which in turn could be part of the basis of his preliminary examination. I should think a second or third year graduate student could alternatively do very well with a reading course program under one or two professors in applicable mathematics, with a goal of covering fundamentals of a wide variety of applicable tools.

A school which teaches basic and applicable mathematics in the same department will offer its students more flexibility in pursuing the above objectives than if these are separate departments; under the present employment stresses such flexibility becomes more important. Let us consider the case, however, of a graduate department identified as applied (i.e., applicable) mathematics. I would

respectfully suggest to such departments, and this really is raising a question rather than making an assertion, that they ask themselves whether their curricula have too much emphasis on teaching students to solve problems which have already been solved, rather than equipping them to devise their own tools to solve the unknown new problems that they will confront in applied work. If there exists an imbalance not favorable to the latter, an indicated remedy should probably be to increase the amount of basic mathematics being taught. Within a practitioner's career technical tools will change drastically (particularly in a profession as young as OR), while the basics change more slowly. In any event an applied mathematician should be educated in basic mathematics well beyond the level at which he makes his applications.

If I were to pick the single most important graduate course in mathematics to prepare for a career in OR, I would definitely choose real analysis. Among other reasons, real analysis is necessary for a good comprehension of probability, statistics, optimization methods, and numerical analysis. Twenty years ago complex analysis and differential equations typically received the most emphasis in applied mathematics departments, with applications in engineering and physics primarily in view. I would think that today, real analysis is at least as important even for such traditional fields of application. I also regard convex analysis as an attractive specialty, for OR potentiality among other reasons.

There are several well-established departments in OR across the country, generally relatively young as formal programs of instruction. My direct acquaintance with them is rather limited. My suggestions for applied mathematics departments I would generally offer to OR departments also. However, I would be inclined to put somewhat stronger emphasis on the need for basic mathematics beyond the minimum levels needed by the students to understand the various tools and techniques that are useful in OR. I would also emphasize the difficulty of learning OR in the classroom, although I speak as one who never had such instruction and has learned his OR through practice. As OR departments probably realize, modeling ability is crucial, and somehow this must be learned through confrontation with live problems.

Such suggestions that I would have for statistics departments would not be dissimilar to what I have suggested for departments in applied mathematics and in OR.

Certainly my purpose in these suggestions is not to promote ascendance of one department of a school over another. Graduate education in mathematics and related fields is best served by emphasis on standards of excellence, which should advance commensurately with advances in knowledge in the respective fields. It is unfortunate that there has been considerable aloofness in both directions between basic and applied mathematicians in recent decades. What is greatly to be desired is interdisciplinary mutual respect for the presence of and cultivation of ability to do original research.

References

1. B. Belkin, First passage to a general threshold for a process corresponding to sampling at Poisson times, J. Appl. Prob., 8 (1971) 573–588.

2. ———, L. D. Stone, and M. A. Snyder, Crossing problems for non-constant thresholds and certain non-Markov processes, J. Appl. Prob., 9 (1972) 769–788.

3. D. C. Bossard, Passive localization in noise, in preparation.

4. W. Browder, Comments on panel discussion: Graduate education in mathematics in the coming decade, Notices, Amer. Math. Soc., 145 (1973) 130.

5. ———, The role of the dissertation in the Ph.D. program, Notices, Amer. Math. Soc., 154 (1974) 180.

6. J. M. Dobbie, A survey of search theory, Operations Res., 16 (1968) 525–537.

7. P. H. Enslow, A bibliography of search theory and reconnaissance theory literature, Naval Res. Logist. Quart., 13 (1966) 177–202.

8. R. E. Gaskell and M. S. Klamkin, The industrial mathematician views his profession: a report of the Committee on Corporate Members, this MONTHLY, (1974) 699–716.

9. B. O. Koopman, Search and Screening, Operations Evaluation Group Report No. 56, Office of the Chief of Naval Operations, 1946.

10. B. O. Koopman, The theory of search, part I. Kinematic bases, Operations Res., 4 (1956) 324–346.

11. ———, The theory of search, part II. Target detection, Operations Res., 4 (1956) 503–531.

12. ———, The theory of search, part III. The optimum distribution of searching effort, Operations Res., 5 (1957) 613–626.

13. B. J. McCabe, Comparison of stochastic processes used in sonar detection models, Naval Res. Logist. Quart, 21 (1974) 673–682.

14. P. M. Morse and G. E. Kimball, Methods of Operations Research, The M.I.T. Press, 1951.

15. H. R. Richardson and B. Belkin, Optimal search with uncertain sweep width, Operations Res., 20 (1972) 764–784.

16. ———, T. L. Corwin, L. D. Stone, F. P. Engel, B. E. Scranton and W. J. Browning, Operation analysis during the 1974 search for explosive ordnance in the Suez Canal, in preparation.

17. ——— and L. D. Stone, Operations analysis during the underwater search for Scorpion, Naval Res. Logist. Quart., (1971) 159–167.

18. ———, L. D. Stone, and F. A. Andrews, Manual for the Operations Analysis of Deep Ocean Search, Supervisor of Salvage, Naval Ship Systems Command (NAVSHIPS 0994–010–7010), March 25, 1971.

19. M. A. Snyder and B. Belkin, Expansion of the Weber function D_υ for small order, with an application, J. Math. Anal. Appl., 38 (1972) 320–327.

20. L. D. Stone, On the distribution of the maximum of a semi-Markov process, Ann. Math. Statist., 39 (1968) 947–956.

21. ———, On the distribution of the supremum functional for semi-Markov processes with continuous state spaces, Ann. Math. Statist., 40 (1969) 844-853.

22. ———, Distribution of time above a threshold for semi-Markov jump processes, J. Math. Anal. Appl., 30 (1970) 576–591.

23. ———, Incremental approximation of optimal allocations, Naval Res. Logist. Quart., 19 (1972) 111–122.

24. ———, Total optimality of incrementally optimal allocations, Naval Res. Logist. Quart., 20 (1973) 419–430.

25. ———, Notes on search theory, U. S. Naval Postgraduate School, Monterey, California (1973).

26. ———, Theory of Optimal Search, Academic Press (to appear 1976).

27. ———, B. Belkin, and M. A. Snyder, Distribution of time above a threshold for Markov processes, J. Math. Anal. Appl., 30 (1970) 448–470.

28. ——— and H. R. Richardson, Search for targets with conditionally deterministic motion, SIAM J. Appl. Math., 27 (1974) 239–255.

29. ——— and J. A. Stanshine, Optimal search using uninterrupted contact investigation, SIAM J. Appl. Math., 20 (1971) 241–263.

30. ———, J. A. Stanshine, and C. A. Persinger, Optimal search in the presence of Poisson-distributed false targets, SIAM J. Appl. Math., 23 (1972) 6–27.

31. D. H. Wagner, Mathematicians can also practice operations research, SIAM News, 7 (October, 1974).

32. ———, Letter to the Editor, Notices, Amer. Math. Soc., 157 (1974) 359–360.

33. G. Raisbeck, Mathematicians in the practice of operations research, to appear in this MONTHLY.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A. P. HILLMAN

The following results of the thirty-fifth William Lowell Putnam Mathematical Competition, held on December 7, 1974, have been determined in accordance with the regulations governing the competition. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, was awarded to the faculty of Mathematics of the **University of Waterloo,** Waterloo, Ontario. The members of the team were Richard P. Anstee, Stephen C. Locke, and Edward A. Severn; a prize of one hundred dollars was awarded to each of these students.

The second prize, four hundred dollars, was awarded to the Department of Mathematics of the **University of Chicago,** Chicago, Illinois. The members of the team were Franklin T. Adams, Thomas P. Branson, and Andrew M. McLennan; each was awarded a prize of seventy-five dollars.

The third prize, three hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology,** Pasadena, California. The members of the team were Christopher L. Henley, Frank M. Liang, and James B. Shearer; each was awarded a prize of fifty dollars.

The fourth prize, two hundred dollars, was awarded to the Department of Mathematics of the **Massachusetts Institute of Technology,** Cambridge, Massachusetts. The members of the team were David J. Anick, Sheldon H. Katz, and Steven M. Pincus; each was awarded a prize of fifty dollars.

The fifth prize, one hundred dollars, was awarded to the Department of Mathematics of the **University of British Columbia,** Vancouver, British Columbia. The members of the team were D. Henry King, J. Bruce Neilson, and John L. Spouge; each was awarded a prize of fifty dollars.

The five highest ranking individual contestants, in alphabetical order, were **Thomas G. Goodwillie,** Harvard University; **Grant M. Roberts,** University of Waterloo; **Karl C. Rubin,** Princeton University; **James B. Saxe,** Union College; and **Philip N. Strenski,** Armstrong State College. Each of these students has been designated as a Putnam Fellow by the Mathematical Association of America and was awarded a prize of two hundred and fifty dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *Frederic G. Commoner,* Harvard University; *Christopher L. Henley,* California Institute of Technology; *Stephen C. Locke,* University of Waterloo; *Mark W. Saaltink,* University of Victoria; and *James B. Shearer,* California Institute of Technology. Each of these students was awarded a prize of one hundred dollars.

The following teams, named in alphabetical order, won honorable mention: *Harvard University,* with team members Anders E. Carlsson, Frederic G. Commoner, and Lloyd N. Trefethen; *University of Illinois,* with team members Marlies Gerber, Bruce E. Hajek, and Daniel D. Sleator; *Michigan State University,* with team members Russel E. Caflisch, Charles E. Meeker, and Mark P. Merriman; *Oberlin College,* with team members Kendall H. Barker, James A. Paget, and Spencer W. Thomas; and *Princeton University,* with team members James M. Lyon, Karl C. Rubin, and Roger S. Schlafly.

Honorable mention was given to the following twenty-nine individuals, named in alphabetical order: *Franklin T. Adams,* University of Chicago; *Mark D. Anderson,* Lehigh University; *Robert L. Anderson,* Princeton University; *David J. Anick,* Massachusetts Institute of Technology; *Richard P. Anstee,* University of Waterloo; *Jonathan D. Arnon,* Princeton University; *Thomas P. Branson,* University of Chicago; *Peter G. deBuda,* University of Toronto; *David R. Cok,* Calvin College; *David S. Dummit,* California Institute of Technology; *Sholom Feldblum,* Harvard University; *David C. Garlock,* Harvard University; *Alan S. Grenadir,* Harvard University; *Joseph Y. Halpern,* University of Toronto; *Barry W. Hill-Tout,* University of British Columbia; *Sheldon H. Katz,* Massachusetts Institute of Technology; *Donald T. Kersey,* McMaster University; *Eric S. Lander,* Princeton University; *Frank J. Lhota,* Wayne State University; *James M. Lyon,* Princeton University; *William J. Magoon,* Amherst College; *Andrew M. McLennan,* University of Chicago; *Mark P. Merriman,* Michigan State University; *Philip E. Moore,* Harvard University; *Wayne J. Noss,* Worcester Polytechnic Institute; *James A. Paget,* Oberlin College; *Karl W. Pettis,* Michigan State University; *Edward A. Severn,* University of Waterloo; *John L. Spouge,* University of British Columbia.

The other individuals who achieved ranks among the top one hundred, in alphabetical order of their schools, were: University of Alberta, *Arthur T. Whitney*; Augustana College (Rock Island),

Örjan Smedry; University of British Columbia, *D. Henry King, J. Bruce Neilson*; Brown University, *Douglas N. Arnold, Anselm C. Blumer, David E. Wallace*; California Institute of Technology, *Frank M. Liang*; California Polytechnic State University, *Richard H. Watrous*; Carleton College, *Albert T. Borchers*; Carleton University, *Kumar P. Murty*; University of Chicago, *Peter L. Dordal, Leslie F. Reid*; Claremont Men's College, *Kenneth R. Drew*; Colgate University, *Thomas E. Dinger*; Columbia University, *John P. Matelski*; Concordia College, *Karl W. Heuer*; Grinnell College, *Dale R. Worley*; Harvard University, *Steven A. Ballmer, Jeffrey M. Dielle, Charles A. Hornig, Nathaniel S. Kuhn, David R. Richman, David Y. Sze, S. Tucker Taft*; Harvey Mudd College, *Theodore C. Chinburg, Daniel A. Rawsthorne*; Haverford College, *Leonard A. Hausner*; University of Illinois (Urbana), *Marlies Gerber, Daniel D. K. Sleator*; University of Kansas, *Stephen M. Paneitz*; Lehigh University, *Douglas A. Kurtze*; Macalester College, *Daniel P. Johnson*; University of Maryland (College Park), *Stephen C. Spriggs*; Massachusetts Institute of Technology, *David B. Feinberg, Steven M. Pincus, Richard E. Stone*; McGill University, *Norman J. Goldstein*; Michigan State University, *Jesse O. Hobbs*; University of Michigan (Ann Arbor), *Barrett P. Eynon*; University of Minnesota (Minneapolis), *Thomas A. Doan*; University of New Mexico, *Erik J. Gilbert, H. Turner Laquer, Michael L. Wilson*; University of North Carolina, *Wesley H. Presler*; Northwestern University, *Henry C. Cejtin*; Oberlin College, *Spencer W. Thomas*; University of Pennsylvania, *Seth T. McCormick*; Princeton University, *David R. Morrison, Adam N. Rosenberg*; University of San Francisco, *Christopher F. Freiling*; University of Saskatchewan, *Alan D. Listoe*; University of Texas (Austin), *Robert L. Toellner, Jr.*, University of Waterloo, *Michael J. Boyle, Gregory J. Fee, Ian P. Goulden, Robin D. Jackson, Peter F. Schneider, Douglas R. Stinson*; University of Wisconsin (Madison), *David C. Ullrich*; Yale University, *Philip M. Rennert, Robert C. Weissler.*

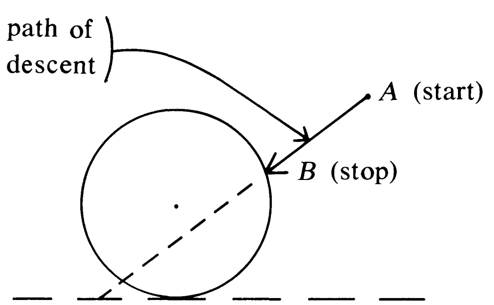
There were 2159 individual contestants from 374 colleges and universities in Canada and the United States in the competition of December 7, 1974. Teams were entered by 306 institutions.

The Questions Committee, consisting of G. D. Chakerian, D. J. Newman (Chairman), and J. I. Richards, prepared the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS, PART A

A–1. Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. (A set of integers is "pairwise relatively prime" if no pair of them has a common divisor greater than 1.) What is the largest number of elements in any "conspiratorial" subset of the integers 1 through 16?

A–2. A circle stands in a plane perpendicular to the ground and a point A lies in this plane exterior to the circle and higher than its bottom. A particle starting from rest at A slides without friction down an inclined straight line until it reaches the circle. Which straight line allows descent in the shortest time? [Assume that the force of gravity is constant over the region involved, there are no relativistic effects, etc.]



The starting point A and the circle are fixed; the stopping point B is allowed to vary over the circle.

Note. The answer may be given in any form which specifies the line of descent in an unambiguous manner; it is not required to find the coordinates of the point B.

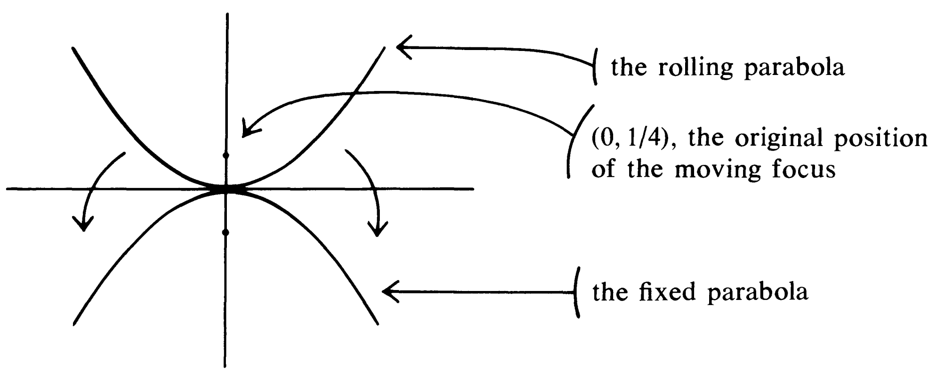
A–3. A well-known theorem asserts that a prime $p > 2$ can be written as the sum of two perfect squares ($p = m^2 + n^2$, with m and n integers) if and only if $p \equiv 1 \pmod 4$. *Assuming* this result, find which primes $p > 2$ can be written in each of the following forms, using (not necessarily positive) integers x and y:
(a) $x^2 + 16y^2$;
(b) $4x^2 + 4xy + 5y^2$.

A–4. An unbiased coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails? In other words, evaluate in *closed form*:

$$\frac{1}{2^{n-1}} \sum_{k<n/2} (n-2k)\binom{n}{k}.$$

(In this problem, "closed form" means a form not involving a series. The given series can be reduced to a single term involving only binomial coefficients, rational functions of n and 2^n, and the greatest integer function $[x]$.)

A–5. Consider the two mutually tangent parabolas $y = x^2$ and $y = -x^2$. [These have foci at (0, 1/4) and (0, −1/4), and directrices $y = -1/4$ and $y = 1/4$, respectively.] The upper parabola rolls without slipping around the fixed lower parabola. Find the locus of the focus of the moving parabola.



A–6. It is well known that the value of the polynomial $(x+1)(x+2)\cdots(x+n)$ is exactly divisible by n for every integer x. Given n, let $k = k(n)$ be the *minimal degree* of any monic integral polynomial

$$f(x) = x^k + a_1x^{k-1} + \cdots + a_k$$

(with integer coefficients and leading coefficient 1) such that the value of $f(x)$ is exactly divisible by n for every integer x.

Find the relationship between n and $k = k(n)$. In particular, find the value of k corresponding to $n = 1\,000\,000$.

PROBLEMS, PART B

B–1. Which configurations of five (not necessarily distinct) points $p_1, \cdots, p_5$ on the circle $x^2 + y^2 = 1$ maximize the sum of the ten distances

$$\sum_{i<j} d(p_i, p_j)?$$

[Here $d(p, q)$ denotes the straight line distance between p and q.]

B–2. Let $y(x)$ be a continuously differentiable real-valued function of a real variable x. Show that if $(y')^2 + y^3 \to 0$ as $x \to +\infty$, then $y(x)$ and $y'(x) \to 0$ as $x \to +\infty$.

B–3. Prove that if α is a real number such that

$$\cos \pi\alpha = 1/3,$$

then α is irrational. (The angle $\pi\alpha$ is in radians.)

B–4. In the standard definition, a real -valued function of two real variables $g\colon R^2 \to R^1$ is *continuous* if, for every point $(x_0, y_0) \in R^2$ and every $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that $[(x - x_0)^2 + (y - y_0)^2]^{1/2} < \delta$ implies $|g(x, y) - g(x_0, y_0)| < \varepsilon$.

By contrast, $f\colon R^2 \to R^1$ is said to be *continuous in each variable separately* if, for each fixed value y_0 of y, the function $f(x, y_0)$ is continuous in the usual sense as a function of x, and similarly $f(x_0, y)$ is continuous as a function of y for each fixed x_0.

Let $f\colon R^2 \to R^1$ be continuous in each variable separately. Show that there exists a sequence of continuous functions $g_n\colon R^2 \to R^1$ such that

$$f(x, y) = \lim_{n\to\infty} g_n(x, y) \text{ for all } (x, y) \in R^2.$$

B–5. Show that $1 + (n/1!) + (n^2/2!) + \cdots + (n^n/n!) > e^n/2$ for every integer $n \geqq 0$.

REMARKS. You may assume as known Taylor's remainder formula:

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{1}{n!} \int_0^x (x - t)^n e^t \, dt,$$

as well as the fact that

$$n! = \int_0^\infty t^n e^{-t} \, dt.$$

B–6. For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 \pmod 3$, $\equiv 1 \pmod 3$, $\equiv 2 \pmod 3$? In other words, calculate

$$s_{i,n} = \sum_{k \equiv i \pmod 3} \binom{n}{k} \qquad \text{for } i = 0, 1, 2.$$

Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n. In particular, show the relationships among these three sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \cdots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqq i \geqq 0$ is the number of students among the top 205 contestants, who received i points for the problem and n_{-1} is the number of students not submitting solutions.

A–1. (89, 14, 12, 14, 16, 3, 29, 1, 9, 0, 13, 5)

A conspiratorial subset (CS) of $\{1, 2, \cdots, 16\}$ has at most two numbers from the pairwise relatively prime set $\{1, 2, 3, 5, 7, 11, 13\}$ and so has at most $16 - (7 - 2) = 11$ numbers. But

$$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$$

is a CS with 11 elements; hence the answer is 11.

A–2. (18, 5, 3, 1, 1, 4, 0, 0, 8, 62, 25, 78)

Let C be the other point of intersection of line AB with the circle and let θ be the inclination of AB. Let $\overline{AB} = b$ and $\overline{AC} = c$. The square of the time of descent is proportional to $b/\sin\theta$ and hence to $1/(c \sin\theta)$, since it is well known that bc is constant with respect to θ. The time is minimized by maximizing $c \sin\theta$; this is done by choosing C as the bottom of the circle.

A–3. (63, 20, 10, 7, 2, 18, 8, 3, 11, 2, 18, 43)

If $p \equiv 1 \pmod 4$, either (A): $p \equiv 1 \pmod 8$ or (B): $p \equiv 5 \pmod 8$. We show that (A) and (B) are necessary and sufficient for (a) and (b), respectively. If $p = m^2 + n^2$ and p is odd, one can let m be

odd and n be even. Then $p = m^2 + 4v^2$ with $m^2 \equiv 1 \pmod 8$. With (A), v is even and $p = m^2 + 16w^2$. Conversely, $p = m^2 + 16w^2$ implies $p \equiv m^2 \equiv 1 \pmod 8$. With (B), v is odd, $m = 2u + v$ for some integer u, and $p = (2u+v)^2 + 4v^2 = 4u^2 + 4uv + 5v^2$. Conversely, $p = 4u^2 + 4uv + 5v^2$ with p odd implies $p = (2u+v)^2 + 4v^2$ with v odd and hence $p \equiv 5 \pmod 8$.

A–4. (31, 7, 3, 3, 2, 3, 5, 10, 11, 7, 14, 109)

The answer is

$$\frac{n}{2^{n-1}}\binom{n-1}{[(n-1)/2]}$$

since

$$\begin{aligned}\sum_{k<n/2}(n-2k)\binom{n}{k} &= \sum_{k<n/2}\left\{(n-k)\binom{n}{k} - k\binom{n}{k}\right\} \\ &= \sum_{k<n/2}\left\{n\binom{n-1}{k} - n\binom{n-1}{k-1}\right\} = n\sum_{k<n/2}\left\{\binom{n-1}{k} - \binom{n-1}{k-1}\right\} \\ &= n\binom{n-1}{[(n-1)/2]}.\end{aligned}$$

A–5. (70, 5, 5, 4, 1, 3, 2, 0, 29, 4, 16, 66)

Let F be the fixed focus, M be the moving focus, and T be the (varying) point of mutual tangency. The reflecting property of parabolas tells us that the tangent line at T makes equal angles with FT and with a vertical line. This and congruence of the two parabolas imply that MT is vertical and that the segments $\overline{FT}$ and $\overline{MT}$ are equal. Now M must be on the horizontal fixed directrix $y = 1/4$ by the focus-directrix definition of a parabola.

A–6. (0, 0, 0, 0, 9, 14, 4, 6, 16, 20, 48, 88)

Let $p(k,x)$ be the monic polynomial $(x+1)(x+2)\cdots(x+k)$ and let m be an integer. Then $p(k,m)$ is exactly divisible by $k!$ since the absolute value of the quotient is a binomial coefficient (even when m is negative). Hence, if $n \mid k!$ there is a monic integral polynomial $f(x)$ of degree k with $n \mid f(m)$ for all integers m. Conversely, the condition $n \mid k!$ is necessary since the k-th difference $k!$ of a monic integral polynomial of degree k is divisible by any common divisor of all the values $f(m)$.

In particular, $k(10^6) = k(5^6 2^6) = 25$ since the smallest s with $5^6 \mid s!$ is $s = 25$.

B–1. (0, 1, 4, 3, 2, 2, 1, 9, 6, 14, 79, 84)

Since the p_i need not be distinct, the sum is a continuous function on the compact set $C \times C \times C \times C \times C$, where C is the circle. Hence maxima exist. One proves that the maximum occurs when the p_i are the vertices of a regular pentagon by showing that this configuration simultaneously maximizes both of the sums:

$$S = d(p_1,p_2) + d(p_2,p_3) + d(p_3,p_4) + d(p_4,p_5) + d(p_5,p_1),$$

$$T = d(p_1,p_3) + d(p_2,p_4) + d(p_3,p_5) + d(p_4,p_1) + d(p_5,p_2).$$

For S or T, one can fix four of the points; then the varying part of the sum is of the form.

$$D = d(p,a) + d(p,b), \text{ with } a \text{ and } b \text{ fixed.}$$

Using the Law of Sines, one shows that D is a constant times $\sin\alpha + \sin\beta$ where $\alpha = \measuredangle pab$, $\beta = \measuredangle pba$, and $\alpha + \beta$ is constant. Then it is easy to show that D is not a maximum unless p is symmetrically situated with respect to a and b.

B–2. (1, 0, 0, 0, 1, 1, 3, 6, 11, 24, 57, 101)

If $y'(x_n) = 0$ for a sequence $\{x_n\}$ approaching $+\infty$, the hypothesis insures that $y(x_n) \to 0$. Since these x_n may include any relative maxima and minima, this case must have $y(x) \to 0$ as $x \to +\infty$. Then one also has $y'(x) \to 0$ as $x \to +\infty$.

In the remaining case, there is an x_0 such that for $x > x_0$ one has $y' \neq 0$ and so $(y')^2 > 0$. We restrict ourselves to the x's with $x > x_0$ and consider two subcases:

(a) $y' > 0$. If y is unbounded above, so are y^3 and $(y')^2 + y^3$. This contradicts the hypothesis $(y')^2 + y^3 \to 0$ as $x \to +\infty$. If y is bounded above, it approaches a finite limit. Then y^3, $(y')^2$, and y' approach limits. Since y is bounded, the limit for y' must be 0. Then y also has 0 as its limit.

(b) $y' < 0$. There is no problem unless y is unbounded below. Then we may assume that $y < 0$ and compare y to a solution of the differential equation

$$y' = -(1/2)\,|\,y\,|^{3/2}, \qquad y < 0.$$

Every solution diverges to $-\infty$ in a finite interval, hence so does $y(x)$; this contradicts the hypothesis that y is defined and smooth for all large x.

B–3. (24, 4, 3, 2, 2, 3, 15, 19, 7, 16, 21, 89)

If $\alpha = r/s$ with r and s integers and $s > 0$, then $\cos(n\pi\alpha)$ takes on at most $2s$ distinct values for integral choices of n. When $\cos \pi\alpha = 1/3$, the formula $\cos 2\theta = 2\cos^2\theta - 1$ and mathematical induction can be used to show that

$$\cos(2^m \pi\alpha) = t/3^{2^m - 1} \qquad [m = 1, 2, 3, \cdots],$$

with t an integer not divisible by 3, and hence that these cosines form an infinite set of distinct values. Thus α is irrational.

B–4. (0, 2, 4, 10, 3, 2, 3, 0, 0, 2, 47, 132)

For each n, we construct the function $g_n(x, y)$ as follows: First divide the xy-plane into vertical strips of width $1/n$ separated by the lines $\{x = m/n\}$, m an integer. Now set $g_n(x, y) = f(x, y)$ along each vertical line $x = m/n$, and interpolate linearly (holding y fixed and letting x vary) in between. Then $g_n(x, y)$ is continuous because $f(x_0, y)$ is continuous in y; $g_n(x, y) \to f(x, y)$ because $f(x, y_0)$ is continuous in x.

REMARKS. This result has two interesting consequences for functions which are continuous in each variable separately:

(i) Such functions are Borel measurable.

(ii) They are continuous (in the usual sense) except on a set of points of the first Baire category. (In particular, there is no function which is continuous in each variable separately and yet discontinuous at every point.)

B–5. (0, 1, 0, 0, 0, 0, 5, 4, 34, 18, 34, 109)

We want to show that

$$\sum_{k=0}^{n} \frac{n^k}{k!} = e^n - \frac{1}{n!}\int_0^n (n-t)^n e^t\, dt > \frac{e^n}{2}$$

or, equivalently, that

$$n! > 2e^{-n}\int_0^n (n-t)^n e^t\, dt,$$

$$\int_0^\infty t^n e^{-t}\, dt > 2e^{-n}\int_0^n (n-t)^n e^t\, dt.$$

Letting $u = n - t$, this can be transformed into

$$\int_0^\infty t^n e^{-t}\,dt > 2\int_0^n u^n e^{-u}\,du,$$

which is equivalent to

$$\int_n^\infty u^n e^{-u}\,du > \int_0^n u^n e^{-u}\,du.$$

Let $f(u) = u^n e^{-u}$. Then it suffices to show that

$$f(n+h) \geqq f(n-h) \quad \text{for} \quad 0 \leqq h \leqq n.$$

This is equivalent to

$$(n+h)^n e^{-h} \geqq (n-h)^n e^h,$$

$$n\ln(n+h) - h \geqq n\ln(n-h) + h.$$

Let $g(h) = n\ln(n+h) - n\ln(n-h) - 2h$. Then $g(0) = 0$ and

$$\frac{dg}{dh} = \frac{n}{n+h} + \frac{n}{n-h} - 2 = \frac{2n^2}{n^2 - h^2} - 2 > 0$$

for $0 < h < n$. Hence $g(h) > 0$ for $0 < h < n$. The desired result follows.

B–6. (26, 8, 15, 10, 6, 9, 25, 14, 13, 19, 13, 47)

Let $n \equiv r \pmod 6$ with r in $\{0,1,2,3,4,5\}$. Then the pattern is

r	0	1	2	3	4	5
$s_{0,n}$	$a+1$	b	c	$d-1$	e	f
$s_{1,n}$	a	b	$c+1$	d	e	$f-1$
$s_{2,n}$	a	$b-1$	c	d	$e+1$	f

This is easily proved by mathematical induction using the formulas

$$s_{i,n} = s_{i-1,n-1} + s_{i,n-1}. \qquad [\text{Here } 0-1 = 2 \pmod 3).]$$

These formulas follow immediately from the rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The sums may be computed readily using the above patterns and

$$s_{0,n} + s_{1,n} + s_{2,n} = 2^n.$$

For $n = 1000$, $r = 4$ and

$$s_{0,1000} = s_{1,1000} = s_{2,1000} - 1 = (2^{1000} - 1)/3.$$

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QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

Reply to Query 23. In this Query references before 1900 to the "utility problem" were sought. R. K. Guy points out that this problem occurs as problem 251 in H. E. Dudeney's "Amusements in Mathematics," Nelson, 1917. Dudeney calls the problem "ancient" and says that it is "older than electric lighting, or even gas." Dudeney may have borrowed this, as he did much other material, from Sam Loyd or earlier writers, but this Department has received no more information on this point.

26. J. A. Murtha. Can anyone supply information about efforts to teach college students how to read mathematics more effectively? We are interested in supporting traditional courses as well as in developing non-traditional delivery systems.

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

ON THE DETERMINATION OF THE BIVARIATE NORMAL DISTRIBUTION FROM DISTRIBUTIONS OF LINEAR COMBINATIONS OF THE VARIABLES

G. G. HAMEDANI AND M. N. TATA

J. Behboodian [1], has given a rather general example of n normal random variables whose joint distribution is not normal, but some of whose linear combinations are normal. Unfortunately when $n = 2$, his example does not work. We give below a simple example for this case. While working on this example, we succeeded in constructing simple examples of two normal variables X and Y such that given N linear combinations of them, the linear combinations are normal, but X and Y are not jointly normal. This constitutes part of the statement of a theorem of T. Ferguson [3]. Ferguson's theorem also asserts that this statement is false for an infinite number of linear combinations. We give a proof of this, any proof of which, as far as the authors have gathered, remains unpublished.

THEOREM 1. *There exists a (non-normal) bivariate density $f_{X,Y}(x, y)$ such that the marginal densities are normal, and also $X+Y$ is a normal variable.*

Proof. Let

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} [1 + xy(x^2 - y^2) e^{-\frac{1}{2}(x^2+y^2+2k)}], \qquad (x, y) \in R^2,$$

where k is such that $|xy(x^2-y^2)e^{-\frac{1}{2}(x^2+y^2+2k)}|\leqq 1$. Then

$$\phi(t_1,t_2)=\iint_{R^2}\exp[it_1x+it_2y]f_{X,Y}(x,y)\,dxdy$$

$$=\exp[-\tfrac{1}{2}(t_1^2+t_2^2)]+\frac{1}{32}t_1t_2(t_1^2-t_2^2)\exp[-k-(t_1^2+t_2^2)/4].$$

Now it remains to note that

(a) $\phi(0,0)=1$; i.e., $f_{X,Y}(x,y)$ is a density,
(b) $\phi(t,0)=\phi(0,t)=e^{-\frac{1}{2}t^2}$; i.e., X and Y are each standard normal,
(c) $\phi(t,t)=e^{-\frac{1}{2}(2t^2)}$; i.e., $X+Y$ is normal with mean zero and variance 2.

Also, clearly (X, Y) is not bivariate normal.

Note. We have a bonus in this example that $X-Y$ is also normal, and X and Y are uncorrelated.

The characteristic function in Theorem 1 has a natural generalisation, which is given in the following:

THEOREM 2*. *Given* $\{(a_k,b_k)\in R^2\colon k=1,2,\cdots,N\}$. *Consider*

$$\phi_{X,Y}(t_1,t_2)=\exp[-\tfrac{1}{2}(t_1^2+t_2^2)]+\exp[-k-c(t_1^2+t_2^2)/2]\left[\prod_{k=1}^{N}(b_k^2t_1^2-a_k^2t_2^2)\right].$$

Then $\phi_{X,Y}(t_1,t_2)$ *is the characteristic function of a bivariate distribution function; the characteristic function of* a_kX+b_kY *is*

$$\phi_{X,Y}(a_kt,b_kt)=\exp[-\tfrac{1}{2}(a_k^2+b_k^2)t^2],\qquad k=1,2,\cdots,N,$$

which is normal, but $\phi_{X,Y}(t_1,t_2)$ *is not the characteristic function of a bivariate normal distribution.*

Proof. The inverse Fourier transform of $\phi_{X,Y}(t_1,t_2)$ is

$$f_{X,Y}(x,y)=\frac{1}{(2\pi)^2}\iint_{R^2}\exp[-it_1x-it_2y]\phi_{X,Y}(t_1,t_2)\,dt_1dt_2$$

$$=\frac{1}{2\pi}\exp[-\tfrac{1}{2}(x^2+y^2)]+\frac{e^{-k}}{(2\pi)^2}\iint_{R^2}\exp[-it_1x-it_2y]\exp[-c(t_1^2+t_2^2)/2]$$

$$\times\prod_{k=1}^{N}(b_k^2t_1^2-a_k^2t_2^2)\,dt_1dt_2.$$

Consider

$$\left|\frac{e^{-k}}{(2\pi)^2}\iint_{R^2}\exp[-it_1x-it_2y]\exp[-c(t_1^2+t_2^2)/2]\left[\prod_{k=1}^{N}(b_k^2t_1^2-a_k^2t_2^2)\right]dt_1dt_2\right|$$

$$=\left|\frac{e^{-k}}{(2\pi)^2}\exp[-(x^2+y^2)/2c]\iint_{R^2}\exp[-c/2[(t_1-ix/c)^2+(t_2-iy/c)^2]]\right.$$

$$\left.\times\left[\prod_{k=1}^{N}(b_k^2t_1^2-a_k^2t_2^2)\right]dt_1dt_2\right|$$

(*) The referee has suggested the following generalisation for which we are grateful.

THEOREM. *Let* $f(t)=\exp(-t^2/2)$ *and let* $P(t,u)$ *be a polynomial whose terms are of positive even degree. Then for all sufficiently small* k, $f(t)f(u)+k\,P(t,u)f(t/2)f(u/2)$ *is a (non-Gaussian) characteristic function.*

$$= \left| \frac{1}{(2\pi)^2} \exp[-k-(x^2+y^2)/2c]\, P_N(x,y) \right|,$$

where $P_N(x, y)$ is a polynomial in (x, y) of degree at most $2N$. We can choose constants k and c such that

$$\left| \frac{1}{(2\pi)^2} \exp[-k-(x^2+y^2)/2c]\, P_N(x,y) \right| \leqq \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}, \text{ for all } (x,y) \in \mathbf{R}^2,$$

and for such a choice of k and c, the function $\phi_{X,Y}(t_1, t_2)$ is the characteristic function of a distribution. Obviously $\phi_{X,Y}(t_1, t_2)$ is not a normal characteristic function.

THEOREM 3. *Given $\{(a_k, b_k);\ k = 1,2,\cdots\}$, a countable "distinct" sequence in $\mathbf{R}^2$ such that for each k, $a_kX + b_kY$ is a normal random variable, then (X, Y) is a bivariate normal variable.*

Note. By a "distinct" sequence $\{(a_k, b_k):\ k = 1,2,\cdots\}$ we mean a sequence $\{(a_k, b_k);\ k = 1,2,\cdots\}$ such that the parametric equation $(t_1 = a_kt, t_2 = b_kt)$ represents an infinite number of lines in $\mathbf{R}^2$.

LEMMA. *Given the functions g_1, g_2 in $C^\infty(R)$ and a sequence (x_n) such that $x_n \to x_0$ as $n \to \infty$. If*

(1) $$g_1(x_n) = g_2(x_n) \qquad \textit{for all } n = 1,2,\cdots,$$

(2) $$\begin{cases} g_1(x) = \displaystyle\sum_{k=0}^{\infty} \frac{g_1^{(k)}(x_0)}{k!}(x-x_0)^k, & -\infty < x < \infty \\ g_2(x) = \displaystyle\sum_{k=0}^{\infty} \frac{g_2^{(k)}(x_0)}{k!}(x-x_0)^k, & -\infty < x < \infty, \end{cases}$$

then $g_1 = g_2$ on R.

Proof. See [2], page 41.

Proof of Theorem 3. Without loss of generality we may assume that X and Y are standard normal random variables with correlation coefficient ρ.

Let (a_k, b_k) be as given, and let $\phi_{X,Y}(t_1, t_2)$ be the characteristic function of (X, Y). Without loss of generality we may assume all a_k's are nonzero, and $m_k = b_k/a_k$, $k = 1,2,\cdots$, is such that there exists a subsequence with limit point m_0 finite. (If not, we shall rotate our coordinate axes.)

We know that

(3) $$\phi_{X,Y}(t, m_kt) = \exp[-\tfrac{1}{2}(t^2 + m_k^2t^2 + 2\rho m_kt^2)], \qquad k = 1,2,\cdots.$$

Let t_0 be nonzero arbitrary and fixed. Then $\{(t_0, m_kt_0);\ k = 1,2,\cdots\}$ is a set of points on the line $\{(t_0, mt_0):\ -\infty < m < \infty\}$, with a cluster point, namely, (t_0, m_0t_0). Since all the moments of $a_kX + b_kY$, $k = 1,2,\cdots$ exist (they are normal random variables), the moments of (X, Y) also exist, and hence $\phi_{X,Y}(t_1, t_2)$ is infinitely differentiable everywhere and is equal to the sum of its Taylor series. The lemma now applies and the theorem is proved.

References

1. J. Behboodian, A simple example on some properties of normal random variables, this MONTHLY, 79 (1972) 632–4.

2. H. Cartan, Elementary theory of analytic functions of one or several complex variables, Addison-Wesley, Reading, Mass., 1963, p. 41.

3. T. Ferguson, On the determination of joint distributions from the marginal distributions of linear combinations (Abstract), Ann. Math. Stat., 30 (1959).

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A PROOF OF THE NORMAL BASIS THEOREM

T. R. BERGER AND I. REINER

Let $\mathbf{L}$ be a finite Galois extension of the field $\mathbf{K}$, with the Galois group G. We say that $\mathbf{L}$ has a *normal* $\mathbf{K}$-basis if there exists an element $x \in \mathbf{L}$ whose conjugates $\{\sigma(x) \mid \sigma \in G\}$ form a $\mathbf{K}$-basis for $\mathbf{L}$. The following result is important in many investigations.

NORMAL BASIS THEOREM: *If* $\mathbf{L}$ *is a finite Galois extension of* $\mathbf{K}$, *then* $\mathbf{L}$ *has a normal* $\mathbf{K}$*-basis.*

This well-known result is proved in many books on algebraic number theory by calculations with determinants. Here we give a new proof based upon the Krull-Schmidt theorem and a special case of the Jacobson-Bourbaki theorem. We shall include a proof of this special case below.

To begin with we introduce a $\mathbf{K}$-algebra which is the *crossed product algebra* of $\mathbf{L}$ by G, defined as follows: set

$$\mathbf{A} \simeq \sum_{\sigma \in G}^{\oplus} \mathbf{L} u_\sigma, \quad u_1 = 1,$$

where the symbols $\{u_\sigma\}$ are an $\mathbf{L}$-basis for $\mathbf{A}$, and where multiplication in $\mathbf{A}$ is defined by the formula

$$(xu_\sigma)(yu_\tau) = x\sigma(y)u_{\sigma\tau}, \quad x, y \in \mathbf{L}; \quad \sigma, \tau \in G.$$

We first prove the following special case of the Jacobson-Bourbaki theorem (see [**1**, Th. 2, p. 22]).

THEOREM. *Keeping the above notation, let* $\mathbf{L}^+$ *be the additive group of* $\mathbf{L}$, *viewed as a* $\mathbf{K}$*-vector space. Then there is an isomorphism of* $\mathbf{K}$*-algebras*

$$\mathbf{A} \simeq \mathrm{Hom}_{\mathbf{K}}(\mathbf{L}^+, \mathbf{L}^+). \tag{1}$$

Proof. We show first that $\mathbf{A}$ is a simple ring. Suppose that $\mathbf{X}$ is a nonzero two-sided ideal of $\mathbf{A}$, and let

$$x = a_1 u_{\sigma_1} + \cdots + a_r u_{\sigma_r} \in \mathbf{X}, \quad \text{all} \quad a_i \in \mathbf{L},$$

be a nonzero element of $\mathbf{X}$ of minimal length r. If $r > 1$, we choose $b \in \mathbf{L}$ such that $\sigma_1(b) \neq \sigma_2(b)$. Direct computation shows that

$$x - \sigma_1(b)^{-1} \cdot x \cdot b$$

is a nonzero element of $\mathbf{X}$ of shorter length. Hence r is necessarily equal to 1, so $\mathbf{X}$ contains a nonzero element au_σ, $a \in \mathbf{L} - \{0\}$. This element is a unit of $\mathbf{A}$, and therefore $\mathbf{X} = \mathbf{A}$. This proves that $\mathbf{A}$ is a simple ring.

For $x \in \mathbf{L}$, let x' denote left multiplication by x upon $\mathbf{L}^+$. We shall establish (1) by exhibiting an isomorphism

$$\varphi : \mathbf{A} \to \mathrm{Hom}_{\mathbf{K}}(\mathbf{L}^+, \mathbf{L}^+).$$

We set

$$\varphi(xu_\sigma) = x' \circ \sigma, \quad x \in \mathbf{L}; \quad \sigma \in G.$$

It is easy to check that φ is a nonzero $\mathbf{K}$-algebra homomorphism. Since Ker φ is a two-sided ideal of $\mathbf{A}$ and $\mathbf{A}$ is simple, it follows that φ is monic. Finally, if $[\mathbf{L}:\mathbf{K}] = n$, then

$$\dim_{\mathbf{K}} \mathbf{A} = n^2 = \dim_{\mathbf{K}} \mathrm{Hom}_{\mathbf{K}}(\mathbf{L}^+, \mathbf{L}^+).$$

This proves that φ is an isomorphism, and establishes (1).

By (1) we may view $\mathbf{L}^+$ as a left $\mathbf{A}$-module, with

$$(xu_\sigma)a = x' \circ \sigma(a), \quad x \in L, \sigma \in G, a \in L^+. \tag{1*}$$

The structure of the full ring of linear transformations $\mathrm{Hom}_{\mathbf{K}}(\mathbf{L}^+, \mathbf{L}^+)$ of the vector space $\mathbf{L}^+$ is well known. Viewed as a left module over itself, the ring is isomorphic to the direct sum of n copies of $\mathbf{L}^+$. Let $\mathbf{A}^+$ denote the ring $\mathbf{A}$, viewed as a left $\mathbf{A}$-module. The isomorphism (1), the action (1*), and our preceding remarks show that

$$\mathbf{A}^+ \simeq \mathbf{L}^{+(n)} \qquad \text{as left } \mathbf{A}\text{-modules.}$$

Now we set

$$\mathbf{B} = \mathbf{K}[G] = \sum\nolimits_{\sigma \in G}^{\oplus} \mathbf{K}u_\sigma,$$

the *group algebra* of G over $\mathbf{K}$. Then $\mathbf{B}$ is a subring of $\mathbf{A}$, and so we now have

(2) $$\mathbf{A}^+ \simeq \mathbf{L}^{+(n)} \text{ as left } \mathbf{B}\text{-modules.}$$

Choose $x_1, \cdots, x_n \in \mathbf{L}$ as a $\mathbf{K}$-basis for $\mathbf{L}^+$. Then

$$\mathbf{A} = \sum\nolimits_{\sigma}^{\oplus} \mathbf{L}u_\sigma = \sum\nolimits_{\sigma}^{\oplus} u_\sigma \mathbf{L} = \sum\nolimits_{\sigma, i}^{\oplus} \mathbf{K}u_\sigma x_i = \sum_{i=1}^{n}{}^{\oplus} \mathbf{B}x_i.$$

Hence $\mathbf{A}^+ \simeq \mathbf{B}^{+(n)}$ left $\mathbf{B}$-modules, whence

(3) $$\mathbf{B}^{+(n)} \simeq \mathbf{L}^{+(n)} \quad \text{as left } \mathbf{B}\text{-modules.}$$

At this point we may invoke the Krull-Schmidt theorem to prove

LEMMA. *Let* $\mathbf{B}$ *any finite dimensional* $\mathbf{K}$*-algebra, and let* X, Y *be a pair of finitely generated* $\mathbf{B}$*-modules. If* $X^{(n)} \simeq Y^{(n)}$ *as* $\mathbf{B}$*-modules, then* $X \simeq Y$.

Proof. Since $\mathbf{A}$ is finite dimensional and X, Y are finitely generated, both X and Y have $\mathbf{A}$-composition series. Therefore, both $X^{(n)}$ and $Y^{(n)}$ have $\mathbf{A}$-composition series. Write $X \simeq P_1^{(n_1)} \oplus \cdots \oplus P_t^{(n_t)}$ and $Y \simeq Q_1^{(m_1)} \oplus \cdots \oplus Q_s^{(m_s)}$, where the P_i's and Q_j's are indecomposable $\mathbf{A}$-modules, and for different μ, ν, $P_\mu \neq P_\nu$ and $Q_\mu \neq Q_\nu$. By hypothesis

$$X^{(n)} \simeq P_1^{(nn_1)} \oplus \cdots \oplus P_t^{(nn_t)} \simeq Y^{(n)} \simeq Q_1^{(nm_1)} \oplus \cdots \oplus Q_s^{(nm_s)}.$$

By the Krull-Schmidt theorem $s = t$, and we may renumber the Q_j's so that $Q_i \simeq P_i$ and $nn_i = nm_i$. In particular,

$$X \simeq P_1^{(n_1)} \oplus \cdots \oplus P_t^{(n_t)} \simeq Q_1^{(m_1)} \oplus \cdots \oplus Q_s^{(m_s)} \simeq Y.$$

This completes the proof of the lemma.

Applying this lemma to (3), we obtain

(4) $$\mathbf{B}^+ \simeq \mathbf{L}^+ \quad \text{as left } \mathbf{B}\text{-modules.}$$

Now $\mathbf{B}^+ = \mathbf{B} \cdot u_1$, where 1 is the identity element of G. If the isomorphism (4) maps u_1 onto $x \in \mathbf{L}$, we have (by (1*))

$$\mathbf{L}^+ = \mathbf{K}[G]x = \sum\nolimits_{\sigma} (\mathbf{K}u_\sigma)x = \sum\nolimits_{\sigma} \mathbf{K}\sigma(x).$$

The sum $\Sigma_\sigma \mathbf{K}\sigma(x)$ must be direct, since its $\mathbf{K}$-dimension is n. This completes the proof of the Normal Basis Theorem.

Usually one proves the Theorem of the Primitive Element separately for finite and infinite fields. Using the Normal Basis Theorem, we can give a uniform proof for any extension of $\mathbf{K}$ contained in a finite Galois extension.

THEOREM OF THE PRIMITIVE ELEMENT. *Let* $\mathbf{L}$ *be a finite Galois extension of* $\mathbf{K}$. *If* $\mathbf{F}$ *is a subfield of* $\mathbf{L}$ *containing* $\mathbf{K}$ *then there is a primitive element* θ *for* $\mathbf{F}$ *over* $\mathbf{K}$, *that is,* $\mathbf{F} = \mathbf{K}(\theta)$ *for some* $\theta \in \mathbf{F}$.

Proof. Let G be the Galois group of $\mathbf{L}$ over $\mathbf{K}$, and let H be the subgroup of G fixing all elements of $\mathbf{F}$. Set $m = [G: H] = [\mathrm{F}: \mathrm{K}]$, and let $\tau_1, \cdots, \tau_m$ be a complete set of left coset representatives for H in G. Choose $x \in \mathbf{L}$ generating a normal basis for $\mathbf{L}$ over $\mathbf{K}$, and set

$$\theta = \sum\nolimits_{\sigma \in H} \sigma(x).$$

Then $\theta \in \mathbf{F}$, since θ is fixed by each element of H. Furthermore, the elements $\{\tau_i\sigma(x): \sigma \in H, 1 \leqq i \leqq m\}$ are a $\mathbf{K}$-basis for $\mathbf{L}$, so the elements $\{\tau_1\theta, \cdots, \tau_m\theta\}$ are linearly independent over $\mathbf{K}$, and hence are distinct. This shows that θ has precisely m distinct conjugates over $\mathbf{K}$, and so the minimal polynomial of θ over $\mathbf{K}$ must be of degree m. Therefore $[\mathbf{K}(\theta): \mathbf{K}] = m = [\mathbf{F}: \mathbf{K}]$, which proves that $\mathbf{K}(\theta) = \mathbf{F}$, since $\mathbf{K}(\theta) \subset \mathbf{F}$. This completes the proof.

Both authors were partially supported by National Science Foundation grants.

Reference

1. N. Jacobson, Lectures in Abstract Algebra, Vol. I, Van Nostrand, Princeton, N.J., 1964.

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A NOTE ON THE FUNDAMENTAL THEOREM FOR RIEMANN-INTEGRALS

J. van de Lune

Usually the fundamental theorem for R-integrals

$$\int_a^b f'(x)dx = f(b) - f(a)$$

is proved under the assumption that f' is R-integrable over $[a, b]$. Compare, for example, [**2**, p. 115].

It appears that there is no answer in the literature to the question under what conditions the derivative f' of a differentiable function f is R-integrable (without making use of specific measure theoretic notions). Compare, for example, [**1**, p. 47].

In this note we prove the following

Theorem. *If $f: [a, b] \to \mathbf{R}$ is differentiable then the derivative f' is R-integrable over $[a, b]$ if and only if there exists an R-integrable function $\phi: [a, b] \to \mathbf{R}$ such that*

$$f(x) = f(a) + \int_a^x \phi(t)dt.$$

In order to prove this theorem we use two lemmas.

Lemma 1. *Let $\phi: [a, b] \to \mathbf{R}$ be R-integrable and assume that $m \leqq \phi(x) \leqq M, \forall\, x \in [a, b]$. Define*

$$\Phi(x) = \int_a^x \phi(t)dt, \qquad \forall\, x \in [a, b],$$

and assume that Φ is differentiable on $[a, b]$. Then we have

$$m \leqq \Phi'(x) \leqq M, \qquad \forall\, x \in [a, b].$$

Proof. Define $K(x) = \Phi(x) - m(x - a)$. Then K is differentiable and $K'(x) = \Phi'(x) - m$. We also have

$$K(x) = \int_a^x \phi(t)dt - m(x - a) = \int_a^x (\phi(t) - m)dt.$$

Since $\phi(t) \geqq m$ we see that K is monotonically nondecreasing. Hence $K'(x) \geqq 0$, or

$$\Phi'(x) \geqq m, \qquad \forall\, x \in [a, b].$$

In a similar fashion one proves

$$\Phi'(x) \leqq M, \qquad \forall\, x \in [a, b].$$

LEMMA 2. *Under the same conditions in Lemma* 1 *we have that* Φ' *is R-integrable over* $[a, b]$.

Proof. Lemma 1 is applicable on every closed subinterval of $[a, b]$. From this it follows that the fluctuation of Φ' is not larger than the fluctuation of ϕ.

According to a well-known criterion for R-integrability [**2**, p. 107] we may conclude that Φ' is R-integrable over $[a, b]$.

Proof of the theorem. It is clear that the given condition is necessary; take $\phi = f'$.

To prove the sufficiency we write

$$f(x) - f(a) = \int_a^x \phi(t)dt.$$

Then it is clear that f' is the derivative of a function of the form $\int_a^x \phi(t)dt$. According to Lemma 2 we obtain that f' is R-integrable over $[a, b]$, completing the proof.

REMARK. Clearly we have

$$\int_a^x f'(t)dt = \int_a^x \phi(t)dt, \qquad \forall\, x \in [a, b].$$

However, it is easy to construct examples in which we do not have $f' = \phi$.

In general ϕ may have simple discontinuities, whereas, according to a theorem of Darboux, a derivative can only have discontinuities of the second kind. Compare [**2**, p. 94].

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References

1. H. Kestelman, Modern theories of integration, Dover Publications, New York, 1960.
2. W. Rudin, Principles of mathematical analysis, McGraw-Hill, New York, 1964.

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A SIMPLE PROOF FOR PARTIAL PIVOTING

DONALD J. ROSE

Let A be a (real), $n \times n$, nonsingular matrix and suppose we wish to solve numerically the linear system of equations

$$Ax = b. \tag{1}$$

This task is straightforward if we can factor A as $A = LU$ where $L = (l_{ij})$ is lower triangular ($l_{ij} = 0$ for $i < j$) and $U = (u_{ij})$ is upper triangular ($u_{ij} = 0$ for $i > j$) since then we solve

$$Ly = b \tag{2}$$

and

$$Ux = y, \tag{3}$$

obtaining first $y_1, y_2, y_3, \cdots, y_n$ from (2) and finally $x_n, x_{n-1}, x_{n-2}, \cdots, x_1$ from (3). Unfortunately, it is not always possible to factor $A = LU$ as above; take, for example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The purpose of this note is to offer a simple inductive (and constructive) proof of the following fundamental theorem of numerical linear algebra.

THEOREM. *For any nonsingular $n \times n$ matrix, A, there exists an $n \times n$ permutation matrix P such that*

$$PA = LU,$$

where L is lower triangular with ones on the diagonal and U is upper triangular with nonzero diagonal elements.

Proofs of the "$PA = LU$" theorem are offered by relatively few authors of numerical analysis texts. Wendroff [**3**, pp. 127–129] presents a complete proof which is similar in spirit to Wilkinson's sketch in [**4**, pp. 206–207]. Stewart [**2**, Chapt. 3, §2] gives a proof as part of his general discussion of Gaussian elimination while Forsythe and Moler [**1**, Chapt. 16, pp. 63–64] imbed a proof in their algorithm and subsequent discussion.

Suppose A is written as

$$A = \begin{bmatrix} a & r \\ c & B \end{bmatrix} \tag{4}$$

where $a \neq 0$ is scalar, c is $(n-1)\times 1$, r is $1\times(n-1)$ and B is $(n-1)\times(n-1)$. Most authors view Gaussian elimination as effecting elementary row operations on A and its transforms until one obtains an upper triangular matrix. For example, the first step takes $A \equiv A^{(1)}$ into $A^{(2)}$ by

$$L_1 A^{(1)} = \begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix} = A^{(2)},$$

where

$$L_1 = \begin{bmatrix} 1 & 0 \\ -c/a & I \end{bmatrix}, \qquad B^{(2)} = B - cr/a.$$

One then continues (formally) finding elementary lower triangular matrices ([**2**], p. 115), L_i such that

$$L_i L_{i-1} \cdots L_1 A = A^{(i+1)} \equiv (a_{ij}^{(i+1)}) \tag{5}$$

has zero below the diagonal in its first i columns. $A^{(n)}$ is then upper triangular, $L = L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}$ is lower triangular so $A = LU$. The hitch in this formal discussion is that possibly some $a_{ii}^{(i)} = 0$ so, in general, (5) must be replaced by an expression with permutation matrices sandwiched in between the L_i (see [**3**, p. 128] or [**2**, p. 124]); this causes some untidyness. An alternative formal derivation of $A = LU$ proceeds from (4) to

$$A = \begin{bmatrix} 1 & 0 \\ c/a & I \end{bmatrix} \begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix} \tag{6}$$

and supposing $B^{(2)} = L_2 U_2$ to

$$A = \begin{bmatrix} 1 & 0 \\ c/a & L_2 \end{bmatrix} \begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix} = LU.$$

There is nothing "existential" about this argument; one would begin to factor $B^{(2)}$ exactly as one began to factor A itself.

Proof of Theorem. The proof of the theorem is shorter than its motivation so let us begin. The nonsingular matrix A must have a nonzero in its first column; hence there exists a permutation matrix P_1 such that

$$P_1A = \begin{bmatrix} a & r \\ c & B \end{bmatrix}, \qquad a \neq 0,$$

as in (4). Then as in (6)

$$P_1A = \begin{bmatrix} 1 & 0 \\ c/a & I \end{bmatrix} \begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix},$$

$B^{(2)} = B - cr/a$. Now $B^{(2)}$ is nonsingular so by induction (the 1×1 case is clear) there exists an $(n-1) \times (n-1)$ permutation matrix $\bar{P}_2$ such that $\bar{P}_2 B^{(2)} = L_2 U_2$. Thus

$$P_1A = \begin{bmatrix} 1 & 0 \\ c/a & \bar{P}_2^T \end{bmatrix} \begin{bmatrix} a & r \\ 0 & L_2U_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & \bar{P}_2^T L_2 \end{bmatrix} \begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix}$$

and

$$P_2P_1A = \begin{bmatrix} 1 & 0 \\ \bar{P}_2 c/a & L_2 \end{bmatrix} \begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix}, \quad \text{where} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}. \tag{7}$$

Hence $PA = LU$ as in the theorem.

If algorithms can contain proofs, then perhaps proofs can also contain algorithms. Notice that in our algorithm the $\bar{P}_2 c/a$ expression in (7) is important. Its presence says that as we overwrite A by successive columns of L and rows of U, any row interchanges on the subsequent submatrices must in fact be done on the whole evolving matrix. (One can also avoid any physical interchanging as discussed in [**1**, Chapt. 16].) Finally, when executing our proof on any modern day computer with inexact arithmetic (and therefore round-off error), choose a (and subsequent "pivots") to be the element of largest magnitude in the column (on or below the diagonal). The proof is then called "Gaussian elimination with partial pivoting." We refer to [**1**] or [**2**] for more extensive discussions and to [**4**] for the classic on the subject.

References

1. G. Forsythe and C. B. Moler, Computer Solution of Linear Algebraic Systems, Prentice-Hall, Englewood Cliffs, N. J., 1967.

2. G. W. Stewart, Introduction to Matrix Computations, Academic Press, New York, 1973.

3. B. Wendroff, Theoretical Numerical Analysis, Academic Press, New York, 1966.

4. J. H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.

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EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

MAC MAHON'S PRIME NUMBERS OF MEASUREMENT

GEORGE E. ANDREWS

Let us define a sequence of integers m_n by $m_1 = 1$, and for $n > 1$ let m_n be the least positive integer that is not the sum of consecutive terms of $m_1, m_2, \cdots, m_{n-1}$. Thus $m_2 = 2$; since $3 = m_2 + m_1$, $m_3 = 4$; $m_4 = 5$; since $6 = m_3 + m_2$, $7 = m_3 + m_2 + m_1$, $m_5 = 8$, etc.

P. A. MacMahon [1] called these numbers "segmented numbers" and also defined $M_n = m_1 + m_2 + \cdots + m_n$ as the "prime numbers of measurement." MacMahon posed the problem of placing division marks on an infinite ruler so that every integral distance can be measured between some two division points on the ruler; if in addition the marks are put on the scale sequentially so that successively larger distances may be measured, then the marks must be placed at distances M_n from the end.

The first few values of m_n and M_n are given below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m_n	1	2	4	5	8	10	14	15	16	21	22	25	26	28
M_n	1	3	7	12	20	30	44	59	75	96	118	143	169	197

N. J. A. Sloane in [2] lists these sequences as # 363, and # 1044 respectively.

Apparently no results of any real depth are known concerning m_n. MacMahon asserts that: "It would be interesting to determine how either set of numbers behaves at great distances from the origin."

Conjecture: $\lim\limits_{n\to\infty} \dfrac{n \log n}{m_n \operatorname{loglog} n} = 1.$

The following table indicates the behavior of the above ratio:

n	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
m_n	1477	3165	4963	6853	8802	10725	12704	14624	16521	18524
$\dfrac{n \log}{m_n \operatorname{loglog} n}$	1.1516	1.1293	1.1109	1.0937	1.0802	1.0766	1.0709	1.0724	1.0759	1.0732

While this is certainly not conclusive evidence for the conjecture, it is not atypical behavior. For example, we know that $p_n/n \log n$ tends to 1 as $n \to \infty$ (p_n is the nth prime); however, $p_{5000}/5000 \log 5000 \approx 1.14$. The truth of our conjecture would directly imply $M_n \sim \frac{1}{2} n^2 \log n / \log\log n$.

Problems of less difficulty than the conjecture also suggest themselves:

PROBLEM 1. Prove: $\lim\limits_{n\to\infty} n^{-\Delta} m_n = 0$ for some $\Delta < 2$.

PROBLEM 2. Prove: $\lim\limits_{n\to\infty} \dfrac{m_n}{n} = +\infty$.

PROBLEM 3. Prove: $m_n < p_n$ for every n, where p_n is the nth prime.

It is not difficult to show that for $n \geqq 1$, $n \leqq m_n \leqq \binom{n}{2} + 1$. The left hand inequality is obvious. To see the right hand inequality, we note that since there are only

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = \binom{n+1}{2}$$

possible sums of consecutive members of $m_1, m_2, \cdots, m_n$, we must have m_{n+1} among the first $\binom{n+1}{2} + 1$ integers.

Added in proof. N.J.A. Sloane and Victor Meally have pointed out to me that the sequence $M_n + 1$ appears as sequence No. 416 in [2]. This observation supplies two further relevant references: Problem E1910 by R.L. Graham, this MONTHLY, 75 (1968) 80; *Elementary Theory of Numbers* by W. Sierpiński, Warszawa, 1964, pp. 411–412.

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References

1. P. A. MacMahon, The prime numbers of measurement on a scale, Proc. Cambridge Phil. Soc., 21 (1923) 651–654.

2. N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, New York, 1973.

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ANOTHER TREE LABELLING PROBLEM

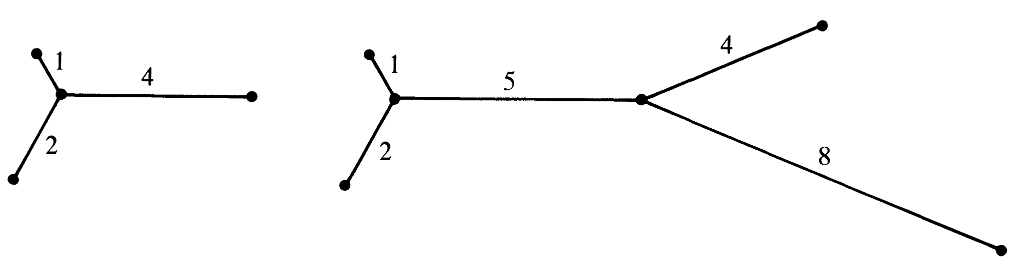
JOHN LEECH

Each edge of a finite tree is labelled with an integer, which we call its **length.** The **distance** between two nodes is the sum of the lengths on the (unique) path between the nodes.

PROBLEM. *For what integers n is it possible to find a labelled tree with n nodes in which the distances between the* $\frac{1}{2}n(n-1)$ *pairs of nodes take the consecutive values* $1, 2, 3, \cdots, \frac{1}{2}n(n-1)$?

For unbranched trees there are solutions for $n \leqq 4$ only, namely

Among branched trees I find the solutions



for $n = 4$ and 6, and there are no others for $n \leqq 6$. In particular there is no solution for $n = 5$; for $n > 6$ I can offer no information.

We may consider this as a case of the more general problem:

PROBLEM. *For each integer n what is the greatest integer N such that there exists a labelled tree with n nodes in which the distances between pairs of nodes include the consecutive values* $1, 2, \cdots, N$? *What can we say about N as* $n \to \infty$?

As far as I can trace, this problem has been considered previously only for unbranched trees; extended accounts are given by Leech [2] and Miller [3], who give results of searches by hand and computer respectively, and who give further references. For these unbranched trees the problem has two main divisions: the distance between the extreme nodes may be **restricted** not to exceed the upper limit N of the consecutive distances occurring, or it may be **unrestricted** and allowed to exceed N. (Branched trees might be classified similarly; I have not considered this.)

The table below gives values of N for $n \leqq 12$ for restricted unbranched (RU), unrestricted unbranched (UU) and branched (B) trees; these are the best possible or (bracketed) best known in each case. The results for unbranched trees are taken from [2, 3]. Note the gap at $n = 9$ for unrestricted unbranched trees — it is an unsolved problem whether $N = 30$ is possible, and this is the smallest value of n for which it is not known whether N for unrestricted unbranched trees can exceed the maximum for restricted unbranched trees. The branched trees are given in an obvious notation in which the numbers in a bracket are the lengths of concurrent edges and the number between them is the length of the edge joining the points of concurrence. For $n > 6$ the trees given belong to a family whose members have the form that the first bracket contains the numbers $1, 2, \cdots, k-1$, with 3 omitted, the second bracket contains the first three or more multiples of k, and the join is $2k-3$. (This construction fails for $k = 6$, the distance 8 being absent.) These are the best I have constructed for $n \leqq 12$, though some alternatives are possible. For $n > 12$ trees of this family are inferior to the best known unbranched trees as given in [3].

The following are the known results for unbranched trees as $n \to \infty$. Let N_r, N_u be the greatest values of N for each n for restricted and unrestricted unbranched trees respectively. Then the limits

$$l_r = \lim_{n \to \infty} n^2/N_r, \qquad l_u = \lim_{n \to \infty} n^2/N_u$$

exist;

$$2.4344 \cdots \leqq l_r \leqq 3, \qquad 2.4344 \cdots \leqq l_u \leqq 2.6571 \cdots$$

(see [2] for existence and lower estimates, [4] for the upper estimate for l_r, and [1] for the upper estimate for l_u). These results are not applicable to branched trees, and we have only the crude estimates $n^2/N_b > 2$, from $N_b \leqq \frac{1}{2}n(n-1)$, and $n^2/N_b \leqq n^2/N_u$. The values of n^2/N_b may well not tend to a limit.

Should anyone think that these problems are abstrusely impractical, I remark that they seem to have originated in the context of electrical networks in which the nodes are the terminals and the lengths and distances are the electrical resistances between pairs of nodes.

TABLE

n	RU	UU	B	tree
2	1			
3	3			
4	6		6	(1, 2, 4)
5	9	9	9	(1, 2, 4, 7)
6	13	13	15	(1, 2) 5 (4, 8)
7	17	18	(20)	(1, 2) 5 (4, 8, 12)
8	23	24	(26)	(1, 2, 4) 7 (5, 10, 15)
9	29		(31)	(1, 2, 4) 7 (5, 10, 15, 20)
10	36	(37)	(38)	(1, 2, 4, 5, 6) 11 (7, 14, 21)
11	43	(45)	(45)	(1, 2, 4, 5, 6) 11 (7, 14, 21, 28)
12	50	(51)	(52)	(1, 2, 4, 5, 6, 7) 13 (8, 16, 24, 32)

References

1. M. J. E. Golay, Notes on the representation of $1, 2, \cdots, n$ by differences, J. London Math. Soc., (2) 4 (1972) 729–734.
2. J. Leech, On the representation of $1, 2, \cdots, n$ by differences, J. London Math. Soc., 31 (1956) 160–169.
3. J. C. P. Miller, Difference bases: three problems in the additive theory of numbers, pp 299–322 of A. O. L. Atkin and B. J. Birch, editors, Computers in Number Theory, Academic Press, London and New York, 1971.
4. B. Wichmann, A note on restricted difference bases, J. London Math. Soc., 38 (1962) 465–466.

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CLASSROOM NOTES

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A GENERALIZED MAGIC SQUARE

DAVID LEGG AND JOE WARD

The purpose of this note is to provide a generalization of "magic squares."

THEOREM. *Let $0 < r < 1$. Assuming the Continuum Hypothesis, there is a bijection f from the unit square I^2 onto the unit interval I such that*

$$\int_0^1 f(x, y)dy = \int_0^1 f(x, y)dx = r \quad \textit{for all } x \textit{ and } y \textit{ in } I.$$

Proof. Let $K \subset I$ be the Cantor ternary set, and express $I - K$ as $\bigcup_{n=1}^{\infty}(a_n, b_n)$.

Call any subset of I whose complement is homeomorphic to $\bigcup_{n=1}^{\infty}[a_n, b_n)$ a modified *Cantor set.* For each $n \geqq 1$, let $C_n \subset (a_n, b_n)$ be a non-empty, nowhere dense modified Cantor set. Extract from $I - \bigcup_{n=1}^{\infty} C_n$ a non-denumerable collection $\{K_\lambda\}$ of sets satisfying (a), (b), and (c) below:

(a) Each K_λ is a non-empty, nowhere dense modified Cantor set.

(b) $K_\lambda \cap K_\mu$ is empty for $\lambda \neq \mu$.

(c) Each K_λ contains points greater than r and less than r.

Let g_λ be a monotonically increasing function from I onto K_λ. For each λ, there is a $p > 0$ such that

$$\int_0^1 g_\lambda(t^p)dt = r.$$

To see this, let λ be fixed, and suppress it temporarily. By condition (c) on the sets $\{K_\lambda\}$, we can find positive numbers ε and δ such that

$$g(t) < r - \varepsilon \quad \text{for} \quad t < \delta, \text{ and}$$

$$g(t) > r + \varepsilon \quad \text{for} \quad t > 1 - \delta.$$

Then

$$\int_0^1 g(t^p)dt = \frac{1}{p}\int_0^1 g(u)u^{-1+1/p}du \geqq \frac{1}{p}\int_{1-\delta}^1 g(u)u^{-1+1/p}du \geqq (r + \varepsilon)[1 - (1-\delta)^{1/p}].$$

Hence $\int_0^1 g(t^p)dt > r$ for sufficiently small p. On the other hand, let $p > 1$, and let q be defined by $1/p + 1/q = 1$. Then Hölder's inequality gives

$$\begin{aligned}\int_0^1 g(t^p)dt &= \frac{1}{p}\int_0^1 g(u)u^{-1+1/p}du \\ &= \frac{1}{p}\int_0^\delta g(u)u^{-1+1/p}du + \frac{1}{p}\int_\delta^1 g(u)u^{-1+1/p}du \\ &\leqq (r-\varepsilon)\delta^{1/p} + 1/p(1-\delta)^{1/p}(\log 1/\delta)^{1/q}.\end{aligned}$$

Hence $\int_0^1 g(t^p)dt < r$ for sufficiently large p. It is easy to see by the Lebesgue Dominated Convergence Theorem that $\int_0^1 g(t^p)dt$ is a continuous function of p. It follows that for some $p > 0$, $\int_0^1 g(t^p)dt = r$.

Let $f_\lambda(t) = g_\lambda(t^p)$, where p is chosen so that

$$\int_0^1 f_\lambda(t)dt = r.$$

Note that each f_λ is a bijection from I onto K_λ.

Divide the collection $\{K_\lambda\}$ into two collections, $\{K_\alpha\}_{\alpha<\Omega}$ and $\{K^\alpha\}_{\alpha<\Omega}$, where Ω is the first uncountable ordinal. The functions f_λ will now be denoted by either f_α or f^α, according to whether their range is K_α or K^α.

We now define f on I^2. As a matter of notation, set $I(s) = I \times \{s\}$ and $J(s) = \{s\} \times I$ for $s \in I$. Let $\{s_\alpha\}_{\alpha<\Omega}$ be an ordering of $[0,1]$. If $(x, y) \in I(s_1)$, define $f(x, y) = f_1(x)$, and if $(x, y) \in J(s_1)$, define $f(x, y) = f^1(y)$. Obviously $f(s_1, s_1)$ is not well defined. Take $f(s_1, s_1) = f_1(s)$, and call $f^1(s_1)$ a deleted value.

Let $\beta < \Omega$, and suppose we have defined f on $\bigcup_{\alpha<\beta} I(s_\alpha) \cup \bigcup_{\alpha<\beta} J(s_\alpha)$. Let $(x, y) \in I(s_\beta)$ and define $f(x, y) = f_\beta(x)$ unless a value has previously been assigned. If $(x, y) \in J(s_\beta)$, define $f(x, y) = f^\beta(y)$ unless a value has previously been assigned. All values of f_β and f^β at points where f had previously been defined will be called deleted values also. Note $f(x, y) = f_\beta(x)$ for all but countably many $(x, y) \in I(s_\beta)$ and $f(x, y) = f^\beta(y)$ for all but countably many $(x, y) \in J(s_\beta)$. Hence

$$\int_0^1 f(x, s_\beta)dx = \int_0^1 f_\beta(x)dx = r, \quad \text{and} \quad \int_0^1 f(s_\beta, y)dy = \int_0^1 f^\beta(y)dy = r.$$

By transfinite induction we have defined f on the entire square I^2. However, this f is not onto I. We shall change f on the slice $I(0)$. Let

$$S = I - \left(\bigcup_{\alpha<\Omega} K_\alpha \cup \bigcup_{\alpha<\Omega} K^\alpha \cup \bigcup_{n=1}^\infty C_n\right),$$

let D be the set of all deleted values obtained in the construction, and let E be the set of all values taken on by f on $I(0)$. Referring to the first paragraph of the proof, let ϕ_n be a monotonically increasing function from $[a_n, b_n]$ onto C_n, and let ϕ be an arbitrary bijection from $I - \bigcup_{n=1}^\infty [a_n, b_n]$ onto $D \cup E \cup S$. Define a function g on I by $g(t) = \phi_n(t)$ if $t \in [a_n, b_n]$, and $g(t) = \phi(t)$ if $t \in I - \bigcup_{n=1}^\infty [a_n, b_n]$. As before, we can pick $p > 0$ so that $\int_0^1 g(t^p)dt = r$. Then by letting $f(x,0) = g(x^p)$, we have evidently defined an f satisfying the requirements of the theorem.

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A NOTE ON LANDAU'S PROBLEM FOR BOUNDED INTERVALS

CHARLES K. CHUI AND PHILIP W. SMITH

1. Introduction. In this note we shall consider the following problem: Suppose that a particle moves in one dimension so that its motion as a function of time can be described by a real-valued function f. It is assumed that the motion f is restricted to a bounded interval, and, due to the energy shortage, the acceleration f'' is bounded. The question which we pose and answer in this note is: Under the above restrictions, what is the largest magnitude of the velocity f' that the particle can attain in a given time interval?

In order to formulate the above problem in a precise way, we will assume that f and f' are continuous and for all a, b in the domain of definition of f we have

$$\int_a^b f''(t)dt = f'(b) - f'(a).$$

We remark that this last requirement on f does not mean that f'' must exist for all t.

When the time interval is unbounded, Landau [1, 2] solved this problem completely. We use the notation $\|f\|_J = \sup_{t\in J}|f(t)|$, $R = (-\infty, \infty)$, and $R^+ = [0, \infty)$.

THEOREM A. (Landau) *If* $\|f\|_R \leqq 1$ *and* $\|f''\|_R \leqq 8$, *then* $\|f'\|_R \leqq 4$.

THEOREM B. (Landau) *If* $\|f\|_{R^+} \leqq 1$ *and* $\|f''\|_{R^+} \leqq 8$, *then* $\|f'\|_{R^+} \leqq 4\sqrt{2}$.

For the proofs of these theorems and for more information on problems of this type we refer the reader to Schoenberg [3]. We remark that both of the above theorems are sharp and that the extremizing functions are essentially unique (cf. [3], p. 135 and p. 148).

Now we consider a bounded time interval. For convenience, let $I = [0, 1]$. We have

THEOREM 1. *Let* $\|f\|_I \leqq 1$ *and* $\|f''\|_I \leqq A$. *Then*

$$\|f'\|_I \leqq \begin{cases} (4+A)/2 & \text{if} \quad 0 < A \leqq 4 \\ 2\sqrt{A} & \text{if} \quad A > 4. \end{cases}$$

Furthermore, these inequalities are sharp.

As an immediate consequence the following corollary may be obtained by a change of scale.

COROLLARY. *If* $\|f\|_{[0,a]} \leqq 1$ *and* $\|f''\|_{[0,a]} \leqq 8$, *then*

$$\|f'\|_{[0,a]} \leqq \begin{cases} (2+4a^2)/a & \text{if} \quad 0 < a \leqq 1/\sqrt{2} \\ 4\sqrt{2} & \text{if} \quad a \geqq 1/\sqrt{2}, \end{cases}$$

and again these inequalities are sharp.

We remark that if $a \geqq 1/\sqrt{2}$ then the upper bound $4\sqrt{2}$ is independent of a and it also agrees with Landau's result for the half line. This means, in particular, that the particle cannot build up any more speed even if it is given an infinite amount of time to do so.

2. Proof of Theorem 1. We divide the proof into three cases: (i) $0 < A \leqq 4$, (ii) $A \geqq 16$, and (iii) $4 < A < 16$. Our proofs for cases (i) and (ii) are straightforward, while case (iii) requires a little more computation.

Case (i). Suppose now $0 < A \leqq 4$. Our proof for this case is similar to the proof of Theorem A as given in [3]. For each $x \in I = [0, 1]$, we take the difference of the following two Taylor formulas

$$f(1) - f(x) = (1-x)f'(x) + \int_x^1 (1-t)f''(t)dt$$

and

$$f(0) - f(x) = -xf'(x) + \int_0^x tf''(t)dt$$

to obtain

(1) $$f'(x) = f(1) - f(0) + \int_0^1 K(x,t) f''(t)\,dt,$$

where

$$K(x,t) = \begin{cases} t & \text{if} \quad 0 \leqq t < x \\ t-1 & \text{if} \quad x \leqq t < 1. \end{cases}$$

Since

$$\int_0^1 |K(x,t)|\,dt = x^2 - x + \tfrac{1}{2} \leqq \tfrac{1}{2}$$

for all $x \in I$, we can conclude that $\|f'\|_I \leqq 2 + A/2$.

Case (*ii*). Now let $A \geqq 16$. For each $x \in [0, 1 - 2/\sqrt{A}]$, we have

$$f\left(x + \frac{2}{\sqrt{A}}\right) - f(x) = \frac{2}{\sqrt{A}} f'(x) + \int_x^{x+2/\sqrt{A}} \left(x + \frac{2}{\sqrt{A}} - t\right) f''(t)\,dt$$

so that

(2) $$|f'(x)| \leqq \frac{\sqrt{A}}{2}\left\{2 + \frac{A}{2}(2/\sqrt{A})^2\right\} = 2\sqrt{A}.$$

By considering $g(x) = f(1-x)$, we also obtain (2) for $x \in [2/\sqrt{A}, 1]$. Since $[0, 1-2/\sqrt{A}] \cup [2/\sqrt{A}, 1] = I$ for $A \geqq 16$, we have $\|f'\|_I \leqq 2\sqrt{A}$ as required.

Case (*iii*). We now consider $4 < A < 16$. Let $S_A = [0, 1-2/\sqrt{A}] \cup [2/\sqrt{A}, 1]$ and set $M = \|f'\|_I = |f'(x_0)|$. Without loss of generality, we can assume that $f'(x_0) = M$. Now if we can find $x_0 \in S_A$, we can use the proof of case (ii) to conclude that $\|f'\|_I = f'(x_0) \leqq 2\sqrt{A}$. Hence, we assume that $1 - 2\sqrt{A} < x_0 < 2/\sqrt{A}$. Consider the function $F(x) = -A|x - x_0| + M$. Since $\|f''\|_I \leqq A$, we have, for $x \geqq x_0$,

$$\begin{aligned} f'(x) &= f'(x_0) + \int_{x_0}^x f''(t)\,dt \\ &= M + \int_{x_0}^x f''(t)\,dt \geqq M - A(x - x_0) = F(x); \end{aligned}$$

and similarly for $0 \leqq x \leqq x_0$,

$$f'(x) = f'(x_0) - \int_x^{x_0} f''(t)\,dt \geqq M - A(x_0 - x) = F(x).$$

That is, we have $f'(x) \geqq F(x)$ for all $x \in I$. This gives

(3) $$\int_0^1 F(x)\,dx \leqq \int_0^1 f'(x)\,dx = f(1) - f(0) \leqq 2.$$

However, it is clear that

(4) $$\int_0^1 F(x)\,dx = M - A[x_0^2 - x_0 + \tfrac{1}{2}].$$

Since $\frac{1}{2}$ is the mid-point of $(1 - 2/\sqrt{A},\ 2/\sqrt{A})$, and $y = x^2 - x + \frac{1}{2}$ is a parabola with minimum at 1/2, it follows that

(5) $$x_0^2 - x_0 + \tfrac{1}{2} < \frac{4}{A} - \frac{2}{\sqrt{A}} + \tfrac{1}{2},$$

recalling that $x_0 \in (1-2/\sqrt{A}, 2/\sqrt{A})$. Hence, combining (3), (4), and (5), we have

$$M < 2 + A\left(\frac{4}{A} - \frac{2}{\sqrt{A}} + \frac{1}{2}\right) = 6 - 2\sqrt{A} + \frac{A}{2}.$$

We must show that $M \leqq 2\sqrt{A}$. Suppose not, then

$$2\sqrt{A} < M < 6 - 2\sqrt{A} + \frac{A}{2},$$

or $A - 8\sqrt{A} + 12 > 0$.

Completing the square, we obtain $(\sqrt{A}-4)^2 > 4$. That is, either $\sqrt{A}-4>2$ or $\sqrt{A}-4<-2$. Hence, either $A > 36$ or $A < 4$. This is a contradiction. Hence, $\|f'\|_I = M \leqq 2\sqrt{A}$.

To see that the inequalities are sharp, we consider the functions

$$f(x) = \frac{-A}{2}x^2 + \left(2+\frac{A}{2}\right)x - 1 \quad \text{if} \quad 0 \leqq A \leqq 4,$$

and

$$f(x) = \begin{cases} -\dfrac{A}{2}(x-2/\sqrt{A})^2 + 1 & \text{for} \quad 0 \leqq x \leqq \sqrt{2/A} \\ 1 & \text{for} \quad x > \sqrt{2/A} \end{cases}$$

if $A > 4$.

3. Final remarks. In this problem piecewise polynomials occur quite naturally as the solution to an extremal problem. One could then use this problem as an introduction to more general piecewise polynomials and hence the study of splines.

It should be pointed out that these Landau type problems have the following equivalent formulation. Suppose $\|f\|_J \leqq 1$ and it is required to find a function f so that $\|f'\|_J = B$ which minimizes $\|f''\|_J$. That is, find the motion of a particle which is bounded between -1 and 1 and achieves a given velocity B while minimizing the maximum of the magnitude of the acceleration.

References

1. E. Landau, Einige Ungleichungen für zweimal differentzierbare Funktionen, Proc. London Math. Soc., 13 (1913) 43–49.

2. E. Landau, Die Ungleichungen für zweimal differentzierbare Funktionen, Danske Vid. Selsk, Math. Fys. Medd., 6 (1925) No. 10, 49 pages.

3. I. J. Schoenberg, The elementary cases of Landau's problem of inequalities between derivatives, this MONTHLY, 80 (1973) 121–158.

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ALTERNATIVES TO TAYLOR'S THEOREM IN PROVING ANALYTICITY

J. A. EIDSWICK

Any power series is infinitely differentiable in its interval of convergence. On the other hand, there are C^∞ functions which do not have power series expansions. This is illustrated by the function $f(x) = \exp(-x^{-2})$ for $x \neq 0$, $f(0) = 0$, whose Maclaurin series converges but not to f (see, e.g., [**3**, p. 96]). In addition, by a theorem of Borel (see [**2**, p. 50]), there are C^∞ functions whose Maclaurin series have zero radius of convergence. Despite this, essentially all of the familiar C^∞ functions can be represented, at least locally, by power series. How can such expansions be justified? A popular method is to apply Taylor's theorem and to reduce the problem to showing that sequences of the

While Theorem 3 has great practical value, its theoretical value is limited. The following result, on the other hand, has theoretical value (e.g., it can be used to show that the product of two power series is a power series which converges whenever both factors converge).

THEOREM 4. *A necessary and sufficient condition that a C^∞ function f have a power series expansion about x_0 valid in $(x_0-\rho, x_0+\rho)$ is that for each positive number $r<\rho$ there exist a constant $A = A_r$ such that*

$$|f^{(n)}(x)| \leqq A(r-|x-x_0|)^{-n} n!$$

whenever n is a nonnegative integer and $|x-x_0|<r$.

Proof. The condition of the theorem implies that the Taylor series corresponding to f has radius of convergence at least ρ, and, by Theorem 3, this series represents f in $(x_0-\rho, x_0+\rho)$.

If f has a power series expansion in $(x_0-\rho, x_0+\rho)$, then so does $g=f'$ (see, e.g., [**1**, p. 24]). Hence, by the proof of Theorem 1,

$$|f^{(n)}(x)| = |g^{(n-1)}(x)| \leqq r(r-|x|)^{-n}(n-1)!$$

for all sufficiently large n and $|x-x_0|<r$, from which the necessity follows.

References

1. H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, Addison-Wesley, Reading, Mass., 1963.

2. W. F. Donoghue, Jr., Distributions and Fourier Transforms, Academic Press, New York, 1969.

3. C. Goffman, Introduction to Real Analysis, Harper and Row, New York, 1966.

4. A. E. Taylor and W. R. Mann, Advanced Calculus, 2nd ed., Xerox, Lexington, 1972.

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AN INTERESTING SUBSERIES OF THE HARMONIC SERIES

A. D. WADHWA

The series $\Sigma\, 1/n$ diverges, but it is known that if we omit the terms corresponding to the integers whose decimal representations contain a specified digit at least once (the digit 0, for example), the resulting series converges ([**1**], [**2**], [**3**], [**4**], [**5**]).

This seems surprising at first sight; but for very large n the integers that do not have a particular digit in their decimal representations are quite scarce. Indeed, there are $9\cdot 10^{n-1}$ integers with n-digit decimal representations, but only 9^n of them omit 0, and $9^n/9\cdot 10^{n-1}\to 0$ as $n\to\infty$. Since few people seem to be aware of the convergence of the harmonic series thinned out in this way, we give a brief proof, and show that the sum of the series is between 20.2 and 28.3. The series with some other digit omitted can be treated similarly. We also show that the series is considerably more than merely convergent: in fact $\Sigma\, n^{-\alpha}$ converges, for the integers n in question, if $\alpha > \log_{10} 9 = 0.95+$. There are corresponding results if integers are represented in other bases. In particular, if we use base 100 we see that, if we delete from the harmonic series only the reciprocals of integers whose decimal representations contain at least two zeros, the resulting series also converges (similarly for any fixed number of zeros).

The n-digit integers which do not have any digit 0 can be written $a_1\cdots a_n$ (understood to mean, as usual, $10^{n-1}a_1+10^{n-2}a_2+\cdots+a_n$), where $0<a_k\leqq 9$. Since all the terms are positive, the convergence of $\Sigma\, 1/k$ over all k of this form is equivalent to the convergence of

$$S=\sum_{n=1}^{\infty}\sum\frac{1}{a_1\cdots a_n}.$$

For a given n there are 9^n terms in the inner sum, and each exceeds 10^{n-1}, so

$$S \leqq \sum_{n=1}^{\infty} 9^n/10^{n-1} = 90.$$

To go further, we notice that of the n-digit integers, one-ninth have leading digit 1, one-ninth have leading digit 2, and so on; so of the 9^n n-digit integers, 9^{n-1} are between 10^{n-1} and $2\cdot 10^{n-1}$, and so on. These integers then contribute at most

$$10^{-n+1}\cdot 9^{n-1}(1+\tfrac{1}{2}+\cdots+\tfrac{1}{9}) = (0.9)^{n-1}(1+\tfrac{1}{2}+\cdots+\tfrac{1}{9})$$

to S, and consequently

$$S < (1+\tfrac{1}{2}+\cdots+\tfrac{1}{9})\sum_{n=1}^{\infty}(0.9)^{n-1} = 10(1+\tfrac{1}{2}+\cdots+\tfrac{1}{9}) < 28.3.$$

On the other hand, the integers between 10^{n-1} and $2\cdot 10^{n-1}$ are less than $2\cdot 10^{n-1}$, and so on, so that for $n \geqq 2$ the n-digit integers contribute at least

$$(0.9)^{n-1}(\tfrac{1}{2}+\cdots+\tfrac{1}{10})$$

to S. Hence

$$S > 1+\tfrac{1}{2}+\cdots+\tfrac{1}{9}+\sum_{n=2}^{\infty}(0.9)^{n-1}(\tfrac{1}{2}+\cdots+\tfrac{1}{10}) > 20.189.$$

Closer bounds are given in **[2]**.

The remainder after the reciprocals of the k-digit integers have been added is

$$R_n = \sum_{n=k+1}^{\infty}\sum \frac{1}{a_1\cdots a_n},$$

(where again the denominator means $10^{n-1}a_1 + 10^{n-2}a_2 + \cdots + a_n$, $0 < a_j \leqq 9$). Since each denominator is less than 10^n and the inner sum contains 9^n terms,

$$R_n > \sum_{n=k+1}^{\infty}(0.9)^n = 9\cdot(0.9)^k.$$

Thus $R_n \to 0$ quite slowly (for $k = 20$ it is still greater than 1), and so (even though $R_n \to 0$) there is no possibility of calculating S by adding up the successive terms. However, S can be calculated by first estimating the remainder more carefully; a machine calculation gave $S = 23.10$, correct to two decimal places.

If we write integers in base k instead of base 10, we see that the corresponding thinned-out harmonic series has a sum between

$$k\left\{1+\tfrac{1}{2}+\cdots+\frac{1}{k-1}\right\} - (k-1)^2/k \quad \text{and} \quad k\left\{1+\tfrac{1}{2}+\cdots+\frac{1}{k-1}\right\}.$$

Similarly, it follows that $\Sigma\, 1/n^{\alpha}$ over the integers whose base k representations contain no 0 is convergent if $\alpha > [\log(k-1)]/\log k$, and its sum is between

$$\frac{k^{\alpha}}{k^{\alpha}-k+1}\{2^{-\alpha}+\cdots+k^{-\alpha}\} \quad \text{and} \quad \frac{k^{\alpha}}{k^{\alpha}-k+1}\{1+2^{-\alpha}+\cdots+(k-1)^{-\alpha}\}.$$

I wish to thank Professor S. D. Chopra for bringing the problem to my attention and for his interest in the solution. I am indebted to R. A. Honsberger for references and to the referee for some numerical results.

References

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1960, p. 384, Ex. 12–16.
2. F. Irwin, A curious convergent series, this MONTHLY, 23 (1916) 149–152.
3. A. J. Kempner, A curious convergent series, this MONTHLY, 21 (1914) 48–50.
4. E. J. Moulton, Solution of Problem 453, this MONTHLY, 23 (1916) 302–303.
5. G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis, Springer, 1925, Problem I 124.

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MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

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"I DO AND I UNDERSTAND": A PROJECT IN PROBABILITY

MAITA LEVINE AND RAYMOND H. ROLWING

Many articles have been written recently encouraging instructors of undergraduate mathematics courses to stress applications of mathematics and to involve the students in the learning process beyond the traditional lecture-discussion setting. Although these goals are worthwhile, they are not always easy to achieve. The content of elementary probability and statistics, however, lends itself particularly well to the use of experiments and projects to stimulate creativity, provide for the pursuit of individual interests, and demonstrate a wide variety of illustrations and applications.

In this paper we would like to share some experiences which we have had with probability and statistics classes. The students responded enthusiastically to the challenge of "doing something on their own."

The Department of Mathematics at the University of Cincinnati offers an elementary probability and statistics course, four hours per week for two quarters. Class size ranges from 15 to 45 students, mostly from the social sciences, the College of Engineering, and the College of Education. During the 1972–73 and 1973–74 academic years the textbook was *Probability and Statistics for Engineers and Scientists* by Walpole and Myers. After a one week introduction to the concepts of probability, a set of seventeen problems was distributed. Some of the examples were chosen merely because they were interesting, nonintuitive, or entertaining illustrations of probability theory. Others were selected as examples of applications of probability to the natural, social, or behavioral sciences. Further applications involving queuing theory, information theory, programming, or game theory might be added to the list. Included in the category of illustrations were the following three problems:

1. Put two black marbles in an urn labeled X and two white marbles in an urn labeled Y. Draw a marble from each urn. Put the one from X into Y and put the one from Y into X. Repeat, each time recording the number of black marbles in urn X. Continue until you have a sequence of 30 entries. What is the probability that the 30th entry will be 1? What is the probability that the 1000th entry will be 1?

2. What is the probability that of n people in a room, at least two of them of them have the same birthday (month and day)?

3. Suppose that n men check their hats in a restaurant. The hatcheck attendant fails to place the numbered checks on the hats. Hence, as the men leave the restaurant, the hats are distributed to them at random. What is the probability that at least one man receives his own hat?

Among the "real life" applications were the following examples:

1. An individual suspected of high alcohol content in his blood may be given a "rapid test" which has a probability of .75 of being correct. If the test gives a positive result, the individual is subjected to a further test that detects alcohol content with .90 validity. The second test will always diagnose a low alcohol content correctly. Answer the following questions, assuming that the results of the two tests are independent of each other and an individual is classified as suspect when the probability of having a high alcoholic blood content is .2. What proportion of suspects will pass the second test? What is the probability that a suspect with high alcohol content will pass the second test? What proportion of suspects are not given the second test? What is the probability that a suspect had a high alcohol content if he is not given the second test?

2. Parents have an average of nearly 14 years of "post-parental life" (the time between the marriage of the youngest child and the death of one or both partners). In an article entitled "The Quality of Postparental Life: Definitions of the Situation" (*Journal of Marriage and the Family*, 26 (No. 1): 53, 1964), I. Deutscher asked a group of postparental couples to rate the postparental against the preparental and parental phases. The results for two socioeconomic groups are as follows:

Evaluation	*Lower-middle*	*Upper-middle*
Favorable	68%	86%
Neutral or indeterminate	21%	5%
Unfavorable	11%	9%

If we were to pick a postparent off a street having 2/3 lower-middle and 1/3 upper-middle pedestrians, what is the probability, according to the data, that he will rate these postparental years favorably? Unfavorably? If he rates them unfavorably, what is the probability that he is of the lower-middle class?

3. Albinism is a trait characterized by milky skin color, very light hair color, and pinkish eye color. The relative frequency of albinos in the United States is 1 person in 20,000. Whether a person is an albino or a non-albino depends on a single pair of genes which we will denote by A and a. Persons who are non-albino can have genotype AA or Aa. Albinos always have genotype aa. Using the dominant-recessive terminology, A is a dominant gene and a is a recessive gene. For convenience, we shall refer to A as the non-albino gene and to a as the albino gene. Can albino children be born to non-albino parents? In view of the dominance of the non-albino gene, A, will the proportion of albinos in the population become smaller as time goes on? If 1 person out of 20,000 is an albino (aa), what is the relative frequency of carriers (Aa)?

After a discussion of the techniques of simulating an experiment, three hours of class were devoted to gathering experimental data for each of the 17 problems. Suggested simulation techniques involved the use of cards, dice, numbered slips of paper drawn at random from a box, and a table of random numbers. Students worked in pairs, and pooled their data at the end of the three hours. In the discussion that followed, students were encouraged to make conjectures concerning the answers to the problems and possible mathematical techniques for solving them.

At the close of the discussion, the following assignment was given to the students: Choose 6 problems and present a complete or partial solution for each problem. You may consult textbooks, solve the problem by your own mathematical reasoning, or make a conjecture based on further simulation of the experiment. You may work with another student and you may use the computer to simulate an experiment. Each of your solutions must explain how you arrived at your conclusion, including any references used; what your conclusion is; and what you learned about probability as you attempted to solve this problem.

Only basic mathematical concepts had been discussed in detail. Included in the discussion were compound events, conditional probability formulas, and tree diagrams. Students were told, however, that they might gain insight into some of the problems by investigating such topics as the binomial distribution, the geometric distribution, the hypergeometric distribution, Bayes' theorem,

and Markov chains. Furthermore, some of these concepts would be discussed in class, at a later date. The students were also told that there was considerable variance in the difficulty of the problems and in the frequency with which solutions or hints appeared in standard probability textbooks.

The same assignment was given to a total of three classes during the academic years 1972–73 and 1973–74. The largest class consisted almost exclusively of fourth year College of Engineering students. (Most engineering students at the University of Cincinnati are enrolled in a five-year coop workstudy program.) The majority of the students in the remaining two sections were majoring in the social or behavioral sciences. Although the students seemed unanimous in their enjoyment of the project, the engineering students exhibited the greatest enthusiasm, the most cooperation and efficiency in working in small groups, and the highest degree of ingenuity in simulating experiments and solving the problems.

The varied ability and mathematical background of the students were evident in the solutions which they submitted. Those who had had linear algebra employed matrix techniques. Several students carried out elaborately programmed simulations on the computer. Many students found hints or solutions in textbooks or recreational mathematics books.

Although there was no formal, objective evaluation, the success of the assignment was reflected in the students' informal evaluation of the project. Their written comments at the end of their sets of solutions and their comments in class indicated that they were overwhelmingly enthusiastic at the opportunity to choose their methods, to experiment, and to work together. The content of an elementary probability course provided an ideal setting for this type of learning experience.

The success of the project suggested the use of a similar type of assignment in the second quarter of the course. The students were asked to carry out a modest statistical experiment to test a hypothesis and to analyze the results. Again, they responded enthusiastically to the opportunity to work independently on a topic of their choice.

References

1. H. D. Brunk, An Introduction to Mathematical Statistics, Ginn/Blaisdell, Waltham, Mass., 2nd ed., 1965.

2. W. Feller, An Introduction to Probability Theory and Its Applications, Volume I, Wiley, New York, 3rd ed., 1967.

3. B. Gelbaum and J. March, Mathematics for the Social and Behavioral Sciences: Probability, Calculus, and Statistics, Saunders, Philadelphia, 1969.

4. J. G. Kemeny, J. L. Snell, and G. L. Thompson, Introduction to Finite Mathematics, Prentice-Hall, Englewood Cliffs, N. J., 2nd ed., 1966.

5. M. F. Neuts, Probability, Allyn and Bacon, Boston, 1973.

6. A. E. Scheerer, Probability on Discrete Sample Spaces with Applications, International Textbook Company, Scranton, Pennsylvania, 1969.

7. School Mathematics Study Group, Introduction to Probability, Parts I and II, A. C. Vroman, Inc., Pasadena, Calif., 1966.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before February 29, 1976.

E 2558. *Proposed by A. Torchinsky, Cornell University*

Suppose that Σa_n is a divergent series of positive terms, and let $s_n = a_1 + \cdots + a_n$ for $n = 1, 2, \cdots$. For which values of p does the series $\Sigma a_n / s_n^p$ converge? (*Remark*: The special cases $p = 1$ and $p = 2$ are Problem 11, p. 70, of Walter Rudin, *Principles of Mathematical Analysis* (Second Edition), McGraw-Hill, New York, 1964.)

E 2559. *Proposed by Hugh L. Montgomery, University of Michigan*

Determine whether the following matrix is singular or nonsingular:

$$\begin{pmatrix} 51237 & 79922 & 55538 & 39177 \\ 46152 & 16596 & 37189 & 82561 \\ 71489 & 23165 & 26563 & 61372 \\ 44350 & 42391 & 91185 & 64809 \end{pmatrix}.$$

E 2560. *Proposed by Richard Madsen, University of Missouri, Columbia*

Let $n_1, \cdots, n_k$ be natural numbers. Define $d_1 = 1$ and $d_i = \mathrm{GCD}(n_1, \cdots, n_{i-1})/\mathrm{GCD}(n_1, \cdots, n_i)$ for $i \geqq 2$. Show that the $d_1 \cdots d_k$ possible sums

$$\sum_{i=1}^{k} a_i n_i \qquad a_i \in \{1, 2, \cdots, d_i\}$$

are all distinct mod n_1.

E 2561. *Proposed by J. M. Simon, Philadelphia, Pennsylvania*

Let (p_1, p_2, p_3) be a prime triplet spaced by the common interval d. Show that if d is not a multiple of 6, then $p_1 = 3$ and necessarily the triplet is unique. Discuss the situation if d is a multiple of 6.

E 2562. *Proposed by N. C. K. Phillips, University of the Witwatersrand, South Africa*

Each of the $\binom{m}{2}$ edges of the complete graph on m vertices is assigned a direction and one of n colors in such a way that there is no monochromatic directed path $\overrightarrow{AB}$, $\overrightarrow{BC}$ of length 2. How large can m be in terms of n?

E 2563. *Proposed by J. Th. Korowine, Athens, Greece*

Let f_1 and f_2 be nonnegative periodic functions of period 2π and let $h > 0$. Let $P_1(\theta)$ and $P_2(\theta)$ be the points whose cylindrical coordinates are $(f_1(\theta), \theta, 0)$ and $(f_2(\theta), \theta, h)$ respectively. Find integrals for the volume and surface area of the solid bounded by the planes $z = 0$, $z = h$ and the lines $P_1(\theta)P_2(\theta)$.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Long Lost Problem

E 2344 [1972, 303; 1974, 530]. *Proposed by Jordi Dou, Barcelona, Spain*

Consider a square array of red dots and blue dots with 50 rows and 50 columns. Whenever two dots of the same color are adjacent in the same row or column, connect them with a segment of that color; if they are adjacent of different color, connect them with a black segment. There are 1269 red dots, among them 99 on the border, none of them at the corners. There are 1035 black segments. Find the number of red segments and the number of blue segments.

Solution by Anita Grossman. There are 4900 segments and, since there are 1035 black segments, the red segments and the blue segments total 3865. There are 1269 red dots, 99 on the border; hence there are 1170 interior red dots.

The number of segments radiating from an interior red dot is 4. Since none of the border red dots are at the corners, the number of segments radiating from each border red dot is 3. So we have the total number of segments radiating from the red dots is

$$3(99) + 4(1170) = 4977.$$

Each red segment is counted twice while each black segment is counted once in this sum. So $4977 = 2$ (the number of red segments) + (the number of black segments), where the number of black segments is given to be 1035. Hence the number of red segments is $\frac{1}{2}(4977\text{–}1035) = 1971$; the number of blue segments is $4900 - 1971 - 1035 = 1894$.

The result could have been obtained as well by counting segments radiating from the blue dots.

Also solved by Jean H. Anderson, D. M. Bloom, R. J. Douglas, R. A. Gibbs, Steven Russ, A. A. Sardinas, A. H. Sherman, D. B. Weinberger, and the proposer.

Editor's comment. Douglas suggests that the proposer chose the number of red segments (1971) to be the year he submitted this problem.

Weinberger mentions that the grid could be considered to be a supply-demand network with the red dots, say, as supply nodes and the blue dots as demand nodes. If each node has supply or demand equal to the number of segments radiating from it and the arc capacities are defined to be 1 for black segments and 0 for red and blue segments, then the total amount sent through the network will be equal to the number of black segments, and each red (blue) segment will correspond to 2 units of unused supply (unsatisfied demand).

This is the Limit

E 2495 [1974, 902]. *Proposed by M. S. Klamkin, University of Waterloo, Ontario*

Let n be a natural number. Evaluate the following limit:

$$I_n = \lim_{x\to\infty}\left\{\frac{(\log x)^{2n}}{2n} - \int_0^x \frac{(\log t)^{2n-1}}{1+t}\,dt\right\}.$$

Solution by Watson Fulks, University of Colorado. Consider the slightly modified problem where $2n$ is replaced by an arbitrary positive integer k. Then the substitutions $y = \log x$ and $u = \log t$ reduce the problem to the determination of $f(\infty) = \lim_{y\to\infty} f(y)$ where f is given by

$$f(y) = \frac{y^k}{k} - \int_{-\infty}^{y} \frac{u^{k-1}}{1+e^{-u}}\,du.$$

We note that

$$f(0) = -\int_{-\infty}^{0} \frac{u^{k-1}}{1+e^{-u}}\,du = (-1)^k \int_0^\infty \frac{u^{k-1}}{1+e^{u}}\,du$$

so that

$$f(y) = f(0) + \int_0^y f'(t)\,dt = f(0) + \int_0^y \frac{u^{k-1}}{1+e^u}\,du$$

from which

$$f(\infty) = [1+(-1)^k] \int_0^\infty \frac{u^{k-1}}{1+e^u}\,du.$$

By formula (6), p. 312 of *Tables of Integral Transforms*, Vol. 1, Erdélyi et al., (McGraw-Hill, 1954), or by expanding $(1+e^u)^{-1}$ in powers of e^{-u} and integrating termwise, this is

$$f(\infty) = [1+(-1)^k]\,\zeta(k)\,\Gamma(k)\,[1-2^{1-k}].$$

Further, f can be written in the form

$$f(y) = f(\infty) - \int_y^\infty \frac{u^{k-1}}{1+e^u}\,du$$

from which the asymptotic behavior of f as $y\to\infty$ is easily deduced, again by expanding $(1+e^u)^{-1}$. In particular

$$\begin{aligned} f(y) &= f(\infty) - e^{-y}\sum_{j=0}^{k-1}\binom{j}{k-1} y^j\,\Gamma(k-j) + O(e^{-2y}y^{k-1}) \\ &= f(\infty) - e^{-y}y^{k-1}[1+O(1/y)]. \end{aligned}$$

Also solved by 41* other solvers and the proposer.

Editor's comment. As noted by a number of solvers, the solution may be expressed in terms of Bernoulli numbers using the formula $(2n)!\,\zeta(2n) = 2^{2n-1}\pi^{2n}|B_{2n}|$.

* From now on, if more than 40 correct solutions are submitted, it will no longer be possible to print the names of the solvers.

A Nonsingular Matrix

E 2496 [1974, 902]. *Proposed by R. D. Whittekin, Metropolitan State College*

Show that the square matrix $M = (m_{ij})$ is nonsingular if it satisfies the following conditions:

(i) $m_{ii} \neq 0$ for all i;

(ii) If $i \neq j$ and $m_{ij} \neq 0$, then $m_{ji} = 0$;

(iii) If $m_{ij} \neq 0$ and $m_{jk} \neq 0$ then $m_{ik} \neq 0$.

I. *Solution by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.* We prove the stronger conclusion that $\det M = \Pi_i m_{ii}$. Let $\sigma \in S_n$, where S_n denotes the symmetric group of degree n. Express σ as the product of disjoint cycles. If $\sigma \neq 1$, then at least one of these cycles, $(i_1, i_2, \cdots, i_k)$, has length $k \geqq 2$. Suppose $\Pi_i m_{i\sigma(i)} \neq 0$. Then, in particular, $m_{i_1 i_2} \neq 0$, $m_{i_2 i_3} \neq 0$, $\cdots m_{i_{k-1} i_k} \neq 0$, $m_{i_k i_1} \neq 0$. Repeated application of (iii) yields $m_{i_j i_k} \neq 0$. Since $m_{i_k i_1} \neq 0$, this is a contradiction.

II. *Solution by Richard Poppen, Stanford University.* Let M be an $n \times n$ matrix. We can define a relation R on the numbers $1, 2, \cdots, n$ by $i\mathrm{R}j$ if and only if $m_{ij} \neq 0$. The conditions (i), (ii), (iii) state respectively that R is reflexive, antisymmetric, and transitive, i.e., is a partial ordering. So we can find a permutation σ of $1, 2, \cdots, n$ such that $i\mathrm{R}j$ implies $\sigma(i) \leqq \sigma(j)$. (For example, find an R-minimal number m_1 and set $\sigma(m_1) = 1$. Then find an R-minimal element m_2 of $\{1, 2, \cdots, n\} \setminus \{m_1\}$, and set $\sigma(m_2) = 2$, and so on.)

Then the matrix $M_\sigma = (m_{\sigma(i)\sigma(j)})$ is an upper triangular matrix with nonzero diagonal elements, hence is nonsingular. But M_σ is obtained from M by only row and column interchanges; so M is nonsingular as well.

Also solved by 50 other solvers and the proposer.

Editor's comment. Solution II above also gives the value of $\det M$: since the same permutation σ is applied to both the rows and the columns of M we have $\det M = \det M_\sigma = \Pi_i m_{ii}$.

A Fibonacci-type Congruence

E 2497 [1974, 902]. *Proposed by Jim King and Phil Hosford, New Mexico State University*

Let a_1 and a_2 be arbitrary integers and define the doubly infinite sequence $\cdots, a_{-1}, a_0, a_1, a_2, \cdots$ by $a_{n+1} = a_n + a_{n-1}$. Show that $(a_{k+2j} + a_{k-2j})$ is divisible by a_k for all integers j, k.

I. *Solution by E. S. Lander, Princeton University.* Choose a k and set

$$m_j^{(k)} = a_{k+2j} + a_{k-2j}, \qquad n_j^{(k)} = a_{k+2j+1} - a_{k-2j-1}.$$

Then $n_j^{(k)} + m_j^{(k)} = m_{j+1}^{(k)}$ and $n_j^{(k)} + m_{j+1}^{(k)} = n_{j+1}^{(k)}$. Clearly a_k divides $m_0^{(k)} = 2a_k$ and $n_0^{(k)} = a_k$. Moreover, if a_k divides $m_j^{(k)}$ and $n_j^{(k)}$ then, from the above relations, a_k divides $m_{j+1}^{(k)}$ and $n_{j+1}^{(k)}$. Thus, by induction, a_k divides $m_j^{(k)}$ and $n_j^{(k)}$ for all j.

II. *Comment by Michael Goldberg, Washington, D. C.* This problem is solved by I. D. Ruggles in *Some Fibonacci Results using Fibonacci-type sequences*, the Fibonacci Quarterly, vol. 1, no. 2 (1963), 75–80.

III. *Comment by D. M. Bloom, Brooklyn College.* The stronger result

$$a_{k+i} + (-1)^i a_{k-i} = a_k L_i,$$

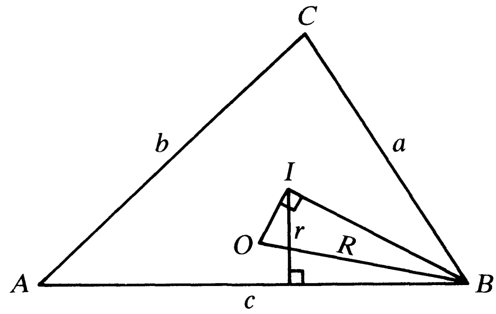
where (L_n) is the Lucas sequence $(L_0 = 2, L_1 = 1, L_n + L_{n+1} = L_{n+2})$ appears in my paper, *On periodicity in generalized Fibonacci sequences*, this MONTHLY, 1965, p. 857.

One hundred and seven additional solutions were received.

A Right Triangle in a Right Triangle

E 2501 [1974, 1026]. *Proposed by Steven Conrad, Benjamin R. Cardozo High School, Bayside, New York*

Let ABC be a right triangle with $\measuredangle C > \measuredangle B \geqq \measuredangle A$ and let O and I be the circumcenter and incenter respectively. Show that the triangle BIO is a right triangle if and only if $BC : CA : AB = 3:4:5$.



Solution with generalization and correction by Anders Bager, Hjørring, Denmark.

THEOREM. *In a general triangle, $\overrightarrow{IO} \perp \overrightarrow{BI}$ is valid if and only if $b = \frac{1}{2}(a + c)$ (the zero vector being perpendicular to all vectors).*

Proof. Using, among others, the relations $BO = R$,

$$BI = r/\sin\tfrac{1}{2}B, \qquad r = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C, \quad \text{and} \quad IO^2 = R^2 - 2Rr,$$

we have the following chain of biimplications:

$$\begin{aligned}\overrightarrow{IO} \perp \overrightarrow{BI} &\Leftrightarrow BI^2 + IO^2 = BO^2 \Leftrightarrow r = 2R\sin^2\tfrac{1}{2}B \\ &\Leftrightarrow 2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}C = \sin\tfrac{1}{2}B \\ &\Leftrightarrow \cos\tfrac{1}{2}(A - C) - \cos\tfrac{1}{2}(A + C) = \sin\tfrac{1}{2}B \\ &\Leftrightarrow \cos\tfrac{1}{2}(A - C) = 2\sin\tfrac{1}{2}B \\ &\Leftrightarrow 2\sin\tfrac{1}{2}(A + C)\cos\tfrac{1}{2}(A - C) = 4\cos\tfrac{1}{2}B\sin\tfrac{1}{2}B \\ &\Leftrightarrow \sin A + \sin C = 2\sin B \Leftrightarrow a + c = 2b.\end{aligned}$$

We now consider the original problem (with $C = \frac{1}{2}\pi$). In the triangle BIO, we never have $\measuredangle IBO = \frac{1}{2}\pi$. Clearly (but overlooked by the proposer) $\measuredangle IOB = \frac{1}{2}\pi$ if and only if triangle ABC is right-isosceles. Finally, it follows from the theorem above that $\measuredangle BIO = \frac{1}{2}\pi$ if and only if $b = \frac{1}{2}(a + c)$. Putting $c = b + d$, $a = b - d$, we get $(a, b, c) = (3d, 4d, 5d)$ by "Pythagoras."

Also solved by David Beran, Walter Bluger (Canada), Robert Breusch (New Zealand), P. S. Bruckman, Leon Gerber, G. J. Griffith, J. R. Herring & J. G. Huard, G. A. Heuer, M. Ram Murty & V. Kumar Murty (Canada), David Penner, Ben Sapolsky, Southern University Primer for Research Group, and Aleksandras Zujus. Partial solutions by Leon Bankoff, Brother Alfred Brousseau, Jordi Dou (Spain), Ragnar Dybvik (Norway), Frank Eccles, Jane Evans, J. Garfunkel, Michael Goldberg, M. G. Greening (Australia), Rebecca Hill, J. D. E. Konhauser, L. Kuipers, M. A. Laframboise, Tak-Shing Leung (Hong Kong), H. S. Lieberman, Graham Lord, Carolyn T. MacDonald, Henrik Meyer (Denmark), D. W. Oman, M. R. Railkar (India), H. Reuvers (Netherlands), St. Olaf College Students, L. R. Tanner, C. S. Venkataraman (India), Julius Vogel, William Wernick, Charles Wexler, and the proposer. Kay King noted the mistake but did not submit a solution.

Editor's comment. The mistake in the original statement of the problem went unnoticed by more than half of those who submitted solutions.

A Sum of Sines

E 2502 [1974, 1026]. *Proposed by Jean-Marie DeKoninck, Université Laval, Quebec*

For each natural number n, let

$$f(n) = \sin\pi\left(\frac{(n-1)!+1}{n}\right).$$

Prove that

$$\sum_{n=1}^{N} f(n) = \pi \log\left(\frac{N}{\log N}\right) + O(1).$$

Solution by L. Carlitz, Duke University. For $n = ab$, $a > b > 1$,

$$(ab-1)! = 1\cdot 2\cdots b\cdots a\cdots(ab-1) \equiv 0 \pmod{2ab}.$$

For $n = p^2$, p prime > 2, $(p^2-1)! = 1\cdot 2\cdots p\cdots 2p\cdots(p^2-1) \equiv 0 \pmod{2p^2}$. Hence

$$\sin \pi\left(\frac{(n-1)!+1}{n}\right) = \sin\frac{\pi}{n} \qquad (n \neq p,\ n \neq 4).$$

By Wilson's theorem we have

$$\sin \pi\left(\frac{(p-1)!+1}{p}\right) = 0.$$

Thus (for $N \geqq 4$)

$$\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} \sin\frac{\pi}{n} - \sum_{p\leqq N} \sin\frac{\pi}{p} - 2\sin\frac{\pi}{4}\cdot$$

Since

$$\sum_{n=1}^{N} \sin\frac{\pi}{n} = \sum_{n=1}^{N}\left(\frac{\pi}{n} + O\left(\frac{1}{n^3}\right)\right) = \pi\log N + O(1)$$

and

$$\sum_{p\leqq N} \sin\frac{\pi}{p} = \sum_{p\leqq N}\left(\frac{\pi}{p} + O\left(\frac{1}{p^3}\right)\right) = \pi\log\log N + O(1),$$

it follows that

$$\sum_{n=1}^{N} f(n) = \pi\log\frac{N}{\log N} + O(1).$$

Also solved by Anders Bager (Denmark), Bethany College Problems Group, D. R. Beuerman, Robert Breusch (New Zealand), D. M. Bressoud, P. R. Chernoff, M. S. Demos, Leon Gerber, S. W. Golomb, Aleksandar Ivić (Yugoslavia), L. Kuipers, O. P. Lossers (Netherlands), L. E. Mattics, Henrik Meyer (Denmark), M. R. Modak & M. R. Railkar (India), M. Ram Murty & V. Kumar Murty (Canada), Allen Stenger, Temple University Problem Solving Group, Charles Wexler, and the proposer. Partial solutions were submitted by J. L. Davison (Canada), Emil Grosswald, J. R. Kuttler, Jeff Lagarias, J. B. van Rongen (Netherlands), O. G. Ruehr, and R. Shantaram.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before February 29, 1976.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6054. *Proposed by Lung Ock Chung, University of California at Los Angeles*

Let ϕ: $\{0,1,\cdots,N-1\}\to\{0,1,\cdots,N-1\}$ be a permutation for $N \geqq 2$. Then ϕ induces a function ϕ^*: $(0,1)\to(0,1)$ from the open unit interval to itself as

$$\phi^*\left(\sum_{i=1}^{\infty}\frac{m_i}{N^i}\right) = \sum\frac{\phi(m_i)}{N^i},$$

where $m_i = 0, 1, \cdots$, or $N-1$; $m_i \not\to 0$; and $N^i = N\cdot N\cdots N$ (i times).

Find the subgroup H of the permutation group such that ϕ^* is continuous if $\phi \in H$. Further, show that ϕ^* is differentiable for such ϕ.

6055. *Proposed by S. Zaidman, University of Montreal*

Let $u_\alpha(x,t)$ be the complex-valued function defined for $x \in R^n$, $t \geqq 0$, through the formula

$$u_\alpha(x,t) = (2\pi)^{-n/2} \int_{s_1^2+\cdots+s_n^2 \leqq 1} \cdots \int \exp(-i(x_1 s_1 + \cdots + x_n s_n))\, g_\alpha(s_1,\cdots,s_n,t)\, ds_1 \cdots ds_n$$

where

$$g_\alpha(s_1,\cdots,s_n,t) = |s|^{-\alpha-2}(1-e^{-|s|^2 t}), \qquad |s| \leqq 1,\ t \geqq 0,$$

$|s| = (s_1^2 + \cdots + s_n^2)^{1/2}$, α is a real number.

Find a number α such that

$$\lim_{t\to\infty} \int_{R^n} |u_\alpha(x,t)|^2\, dx_1 \cdots dx_n = +\infty.$$

6056*. *Proposed by Simeon Reich, Tel-Aviv University, Israel*

Let $\{a_n\}$, an increasing sequence of real numbers, tend to infinity and set $p_n(t) = \Sigma_{k=0}^n a_{n-k} t^k/k!$. Is it true that $\lim_{n\to\infty} e^{-a_n} p_n(a_n)/a_n = 0$?

(REMARK: It can be shown, for example, that if $a_n = n$ for all n, then the answer is in the affirmative.)

6057. *Proposed by Anon, Erewhon-upon-Yarkon*

Let A, B, C, D be $n \times n$ matrices such that $CD' = DC'$, where the prime denotes transpose. Prove that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD' - BC'|.$$

6058. *Proposed by Larry Taylor, New York, N.Y.*

(A) If $p \equiv 31$ or $39 \pmod{40}$ is prime and if $a \equiv (\sqrt{5}+2)/3$ and $b \equiv (\sqrt{5}-2)/3$ are of even order $(\bmod\, p)$, prove that either $a-1$, a and $a+1$ or $b-1$, b and $b+1$ are quadratic nonresidues of p.

(B) If $p \equiv 19 \pmod{24}$ is prime and $a \equiv \sqrt{-1/3}$ is of even order $(\bmod\, p)$, prove that $a-1$, a and $a+1$ are quadratic nonresidues of p.

(For example, (A) 21, 22 and 23 are quadratic nonresidues of 31; (B) 32, 33 and 34 are quadratic nonresidues of 43.)

6059. *Proposed by S. Baskaran, University of Madras, India*

In his book, *The Theory of Groups*, M. Hall calls a group G metacyclic if the derived group G' and the factor group G/G' are both cyclic. Prove that if G is a finite metacyclic group and p is the smallest prime dividing the order of G then a Sylow p-subgroup of G is cyclic.

SOLUTIONS OF ADVANCED PROBLEMS

Complete Linear Spaces

5773 [1971, 84]. *Proposed by L. S. Hahn, University of Washington*

Give an example of a linear space complete under each of a pair of "inconsistent" norms in the sense of Katznelson (*Introduction to the Harmonic Analysis*, Wiley, New York, p. 94).

Solution by J. P. Williams, Indiana University. We first prove a lemma.

LEMMA. *If* $|\ \ |_0$ *and* $|\ \ |_1$ *are consistent norms on a linear space B and if the spaces* $(B, |\ \ |_0)$ *and* $(B, |\ \ |_1)$ *are both complete, then these norms are equivalent, that is,* $a|\ \ |_0 \leqq |\ \ |_1 \leqq b|\ \ |_0$ *for some positive constants a and b.*

Proof. Fix $\alpha \in (0, 1)$ and consider the interpolation norm $|f|_\alpha$ on B defined on p. 94 of Katznelson. Then $|f|_\alpha \leqq |f|_0^{1-\alpha}|f|_1^\alpha$ for all $f \in B$ (see p. 95).

Let $|f| = |f|_0 + |f|_1$ for $f \in B$. Then $|\ \ |$ is a norm on B and the space $(B, |\ \ |)$ is complete. In fact if $\{f_n\}$ is a Cauchy sequence in $(B, |\ \ |)$ then $\{f_n\}$ is also Cauchy in both $(B, |\ \ |_0)$ and $(B, |\ \ |_1)$. Hence there are vectors f, g in B such that $|f_n - f|_0 \to 0$ and $|f_n - g|_1 \to 0$ as $n \to \infty$. Since $|f_n - g|_0 \leqq |f_n - f|_0 + |f - g|_0$ is bounded, it follows that $|f_n - g|_\alpha \leqq |f_n - g_0|^{1-\alpha}|f_n - g_1|^\alpha$ tends to 0 as $n \to \infty$. Similarly $|f_n - f|_\alpha$ is bounded so that $|f_n - f|_\alpha \to 0$. Therefore $f = g$ because $|\ \ |_\alpha$ is a norm on B. Consequently $|f_n - f| = |f_n - f|_0 + |f_n - f|_1 = |f_n - f|_0 + |f_n - g|_1 \to 0$ so that $\{f_n\}$ has limit f in $(B, |\ \ |)$.

Now since $|\ \ |_0 \leqq |\ \ |$, and since B is complete with respect to each of these norms, the Closed Graph Theorem implies that $|\ \ | \leqq a|\ \ |_0$ for some constant a, and $|\ \ | \leqq b|\ \ |_1$ for some constant b. Therefore $|\ \ |_0 \leqq b|\ \ |_1 \leqq b|\ \ | \leqq ab|\ \ |_0$.

Returning to the original problem, it now suffices to show that if $\mathscr{H}$ is an infinite dimensional Hilbert space then there exists an (inner product) norm on $\mathscr{H}$, not equivalent to the given norm, with respect to which $\mathscr{H}$ is complete. Such a norm can be obtained by putting $|f| = \|Af\|$ for $f \in \mathscr{H}$ where A is a one-one linear transformation from $\mathscr{H}$ onto itself that is not continuous. To obtain such an operator, let $\{e_n\}$ be an orthonormal basis of $\mathscr{H}$, let $\mathscr{B}$ be a Hamel basis of $\mathscr{H}$ that contains each e_n, and define A by linearity and the conditions $Ae_n = ne_n$, $Ae = e$ for $e \neq e_n$, $n = 1, 2, 3, \cdots$.

A Separable Hausdorff Space not σ-compact

5962 [1974, 293]. *Proposed by Lee Erlebach, University of Arizona*

Find an elementary solution to the "unsolved problem" of finding a separable locally compact Hausdorff space which is not σ-compact, proposed on page v of Steen and Seebach, *Counterexamples in Topology.*

Solution by Roy Olson, University of Hawaii. Let βN denote the Stone-Čech compactification of the positive integers N, and let X be any open subset that is not Lindelőf (for example, the complement of a non-isolated point.) Then X is separable, X is locally compact Hausdorff because X is open in βN (a compact Hausdorff space), and X is not σ-compact because X is not Lindelőf.

Also solved by Bo Berndtsson & Thomas Gunnarsson & Bogőran Johansson (Sweden), R. E. Chandler, S. C. Currier, Jr., Jean-François Dumais, T. E. Gantner, H. L. Hiller, R. Hodel, A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), P. R. Meyer, J. V. Michalowicz, Bryce Parry, Nina M. Roy, Frank Siwiec, Arthur Solomon, E. K. van Douwen (Netherlands), R. C. Walker, Albert Wilansky, J. K. Yates, and the proposer.

Editorial Note. The proposer discovered, too late for the problem to be withdrawn, that a solution did indeed exist as example 65, page 87 of the Steen and Seebach text. An example may also be found as # 113, p. 150 of Wilansky, *Topology for Analysis.*

Mean Powers of Prime Divisors

5964 [1974, 412]. *Proposed by C. W. Anderson, University of California, Berkeley*

Define $\text{pow}(n) = (\alpha_1 + \alpha_2 + \cdots + \alpha_k)/k$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. pow$(n)$ measures the average power of a prime in the decomposition of n. Prove that the average value of pow(n) is unity, i.e., show that

$$\frac{1}{N}\sum_{n=1}^{N} \operatorname{pow}(n) = 1 + \frac{c}{\log\log N} + O\left(\frac{1}{[\log\log N]^2}\right);$$

where $c = \Sigma_p\, 1/p\,(p-1)$.

For example, the average power of a prime in the decomposition of all numbers less than a "googol," 10^{100}, is about 1.2. [Received April 18, 1973 — Ed.]

Comment by Jean-Marie DeKoninck, Université Laval, Québec.
A complete answer to this problem can be found in my recent paper, *On sums of quotients of additive functions*, Proc. Amer. Soc., 44 (1974) 35–38. Given any positive integer α, it is there proved that

$$\frac{1}{N}\sum_{..=2}^{N} \operatorname{pow}(n) = 1 + \sum_{i=1}^{\alpha} \frac{c_i}{(\log\log x)^i} + O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right),$$

where $c_1 = \Sigma_p\, 1/p\,(p-1)$ and all the other c_i are computable constants, $N = [x]$.

Similar problems concerning reciprocals of additive functions can be found in the following two papers:

1. On a class of arithmetical functions, J. M. DeKoninck, Duke Math. Journal, 39 (1973), 807–818.
2. On sums of reciprocals of additive functions, J. M. DeKoninck, and Janos Galambos, Acta Arithmetica 25 #2 (1974), 159–164.

Complements of Kernels

5965 [1974, 412]. *Proposed by D. K. Cohoon, University of Minnesota*

Let $\kappa: N - \{0\} \to N \times N$ be a one-to-one correspondence, where $N = \{0, 1, 2, \cdots\}$. Let P denote the set of positive integers. Let V_1 denote the functions valued in the field F which are defined on P but vanish off a finite subset of P. Let V_0 denote the F-valued functions defined on N which vanish off a finite subset of N. Define a linear transformation $B_\kappa: V_1 \to V_0$ by the rule

$$(B_\kappa \psi)(n) = \sum_{m=0}^{\infty} \psi(\kappa^{-1}(m, n))$$

for every n in N. Find the subspaces U of V_1 such that

$$V_1 = \ker(B_\kappa) \oplus U \qquad \text{(direct sum)}.$$

Solution by Brian Wesselink, College of Charleston. Let $P_i = \kappa^{-1}(N \times i)$. Since κ is a one-to-one correspondence, the sets $P_0, P_1, P_2, \cdots$ partition P into a collection of countably infinite sets. We shall show that $U = \bigoplus_i \langle \phi_i \rangle$ where $\langle \phi_i \rangle$ is the subspace of V_1 generated by ϕ_i, and ϕ_i is any element in V_1 with the properties that $\phi_i(p) = 0$ for any $p \in P_j$, $j \neq i$, and $\Sigma_{p \in P_i} \phi_i(p) \neq 0$. We begin by proving the following lemma.

LEMMA: *Let F be a field, let C be a countably infinite set, and let V be the vector space (over F) of all functions $\psi: C \to F$ which vanish off a finite subset of C. If $K = \{\psi \in V: \Sigma_{c \in C} \psi(c) = 0\}$, then $V = K \oplus U$ if and only if $U = \langle \phi \rangle$ where $\langle \phi \rangle$ is the cyclic subspace generated by $\phi \in V - K$.*

Proof of Lemma: If $\phi \in V - K$, then $K \cap \langle \phi \rangle = \{0\}$. To show $V = K + \langle \phi \rangle$, let $\psi \in V$ and let $\Sigma_{c \in C} \psi(c) = f$. If $\Sigma_{c \in C} \phi(c) = g$ $(g \neq 0)$, then $\Sigma_{c \in C}(\psi - (f/g)\phi)(c) = 0$, hence $\psi - (f/g)\phi \in K$. Thus, since $\psi = (\psi - (f/g)\phi) + (f/g)\phi$, we have $V = K + \langle \phi \rangle$ with the uniqueness clear.

Conversely, if $V = K \oplus U$, and if $\phi \in U$, $(\phi \neq 0)$, then, since $\phi \in V - K$, $V = K \oplus \langle \phi \rangle$ and hence $U = \langle \phi \rangle$.

To complete the proof, observe that since

$$(B_\kappa \psi)(n) = \sum_{m=0}^{\infty} \psi(\kappa^{-1}(m, n)) = \sum_{p \in P_n} \psi(p),$$

$\psi \in \ker(B_\kappa)$ if and only if $\Sigma_{p\in P_n}\psi(p)=0$ for each n. Let $V_{1,i}=\{\psi\in V_1\colon \psi(p)=0,\ p\in P_j,\ j\neq i\}$ and let $K_i=V_{1,i}\cap\ker(B_\kappa)$. Then $\ker(B_\kappa)=\bigoplus_i K_i$ and $V_1=\bigoplus_i V_{1,i}$. From the lemma there exist cyclic subspaces U_i of $V_{1,i}$ such that $V_{1,i}=K_i\oplus U_i$. Hence, if $U=\bigoplus_i U_i$, then $V_1=\ker(B_\kappa)\oplus U$.

Conversely, if $V_1=\ker(B_\kappa)\oplus U$, let $U_i=V_{1,i}\cap U$. Then $U=\bigoplus_i U_i$, and since $V_{1,i}=K_i\oplus U_i$, we can appeal to the lemma to show that $U_i=\langle\phi_i\rangle$.

Also solved by the proposer.

Hamiltonian Circuits in Maximal Planar Graphs

5966 [1974, 412]. *Proposed by E. F. Schmeichel, Itasca, Illinois* Does every maximal planar graph have a Hamiltonian circuit?

Solution by M. D. Plummer, Vanderbilt University. It is false that every maximal planar graph has a Hamiltonian circuit. There is an eleven-point counterexample due to C. N. Reynolds. This graph (or more precisely, its planar dual) may be found in H. Whitney, *A theorem on graphs*, Ann. of Math., 32 (1931), 378–390. On the other hand, in this same paper, Whitney proved that if a maximal planar graph G has no separating triangle, then G is indeed Hamiltonian. This result also follows from observing that any maximal planar graph (other than K_4) which contains no separating triangle must be 4-connected. But then the graph must contain a Hamiltonian cycle by a well-known theorem of W. T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc., 82 (1956), 99–116.

The Reynolds' counterexample is "best possible" in the sense that eleven is the smallest number of points that any maximal planar non-Hamiltonian graph can have. This was proved by D. Barnette and E. Jucovič, *Hamiltonian circuits on* 3*-polytopes*, J. Combinatorial Theory, (B), 9 (1970), 54–59.

It is also of interest to know that there even exist maximal planar graphs which contain no Hamiltonian *path*. An example of such a graph with fourteen points may be found in B. Grünbaum, *Convex Polytopes*, Interscience, New York, (1967), p. 357. Moreover, this graph is also a "best possible" counterexample, for P. R. Goodey has proved that the smallest number of points that a maximal planar graph without Hamiltonian paths can have is fourteen; see *Hamiltonian paths on* 3*-polytopes*, J. Combinatorial Theory, (B), 12 (1972), 143–151.

Also solved by Aage Bondesen (Denmark), I. Broere (Republic of South Africa), Ben Manvel, Constantine Roussos, A. J. Schwenk, Richard Steinberg, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Introduction to Ordinary Differential Equations, Second Enlarged Edition with Applications. By Albert L. Rabenstein. Academic Press, New York, 1972. x + 526 pp. $12.75. (Telegraphic Review, March 1972.)

Differential Equations with Applications and Historical Notes. By George F. Simmons. McGraw-Hill, New York, 1972. xvii + 465 pp. \$11.95. (Telegraphic Review, June-July 1972.)

Ordinary Differential Equations. By Otto Plaat. Holden-Day, San Francisco, California, 1971. xi + 295 pp. \$14.95. (Telegraphic Review, August-September 1971.)

Rabenstein's book has a large number of interesting applications, of which I mention particularly a novel economic problem involving periodic supply on page 37. Going beyond Simmons and Plaat, Rabenstein includes a substantial introduction to boundary-value problems within the context of partial differential equations, and also an introduction to orthogonal polynomials. The method of undetermined coefficients is justified by annihilators rather than by the shift theorem, though the latter is proved. The discussion of the Laplace transform states Lerch's uniqueness theorem without proof but does show that $y^{(k)}$ is of exponential type if the forcing function f is. (Many texts forget this important point.) Although the possibility of differentiating a power series is taken for granted, the book establishes convergence of power-series solutions. Something is also said about the phase plane, singular points, and Liapunov functions. The completeness of the set $\{x^n\}$ is not established, but the book does give Sokolnikoff's brief proof for convergence of Fourier series.

Among many praiseworthy details, I mention the use of definite integrals to solve the exact equation. Although most books (including Simmons) use indefinite integrals, it has always seemed to me that Rabenstein's approach gives a cleaner proof of existence. Indeed, the use of indefinite integrals to establish existence for the exact equation is comparable to "solving" $y' = f(x)$ by writing

$$y = \int f(x)\, dx.$$

Since the latter symbol stands for a solution of $y' = f(x)$, it begs the question. The analogous technique in two dimensions is worse, because there are relevant restrictions on the region which come to light only upon introduction of a path of integration.

More to show that I really looked at the book than for any deeper purpose, I mention three very minor objections and one minor one. The proof of commutativity on p. 73 does not make it clear why the coefficients have to be constant; on p. 116–118 the author forgot to say $c > 0$, thus obscuring the notion of steady-state solution; and in the proof of the convolution theorem, absolute convergence is not mentioned. Any competent instructor can fix these with ease.

Perhaps more important is the fact that the "general solution" is defined to be "the set of all solutions" without, however, any discussion of uniqueness, maximal intervals, envelopes, and so forth. The trouble with "all solutions" is that it is so hard to prove you really have them. For example the equation $2y' = 3y^{1/3}$ admits a two-parameter family of solutions through the origin. The author does not always show a full awareness of these troubles, and when he directs students to find the general solution (in his sense) I am not sure he has given them the means to do so in every case. By contrast, Plaat is in full command of these matters and treats them with respect. Simmons defines "general solution" in terms of the initial-value problem, which is also my own preference.

Since Rabenstein stresses applications the following omissions seem to me unfortunate. The important matter of error estimation and stability as regards small changes in the equation is not mentioned, nor is the theory of steady-state solutions developed adequately. The Laplace transform is not applied to systems or to discontinuous and impulse-type inputs, though such applications perhaps represent its main usefulness. The convolution theorem is not applied at all.

Despite these omissions this is an excellent text. It is warmly recommended to all who are open to the idea of including applications in their differential equations course.

A glance at Simmons' preface suggests that he is one of the few textbook-authors equipped with a truly excellent sense of style; and this surmise is confirmed by the rest of the text. Simmons is not a pedestrian author. His work has sparkle — a quality that is perhaps undefinable but is easily recognized. In a way that other books rarely do, this one conveys the feeling that mathematics is a worth-while activity. Few readers could fail to gain a sense of the richness and variety of mathematics, and a sense of its excitement as an intellectual challenge.

In writing the biographies, Simmons has evidently dug deeply into history. Consequently his presentation is not only loaded with interesting facts, but is exceptionally even and well-balanced. The trap in most text-book biographies is that the author just writes down what he happens to know at the time. This lazy approach generally leads to a biased choice of mathematicians, and to a very lopsided view of the achievements of those that are included. Both pitfalls have been avoided with splendid skill here.

The book begins with a number of interesting applications, most of which lead to separable equations. Typical of the author's approach is that in discussing radioactive decay, he gives a clear, readable account of the method of radiocarbon dating. (The point is worth mentioning, because some other texts use the occasion, instead, to engage in futile quibbling about the divisibility of matter, the rigor of the derivation, and so forth.)An unexpected dividend in Chapter 1 is Bernoulli's solution of the brachistochrone problem by an optical analog; the presentation follows Pólya. The examples and problems of Chapter 2 are familiar but interesting. Besides standard topics, Chapter 3 contains a derivation of Kepler's laws. An excellent elementary introduction to oscillation in Chapter 4 is followed by power-series solutions in Chapter 5, where the convergence at regular singular points is established. (Rabenstein omitted this.) Discussion of the special functions in Chapter 6 will be welcome to engineers and physicists, and Volterra's prey-predator equations in Chapter 7 make a smooth transition to the phase plane considered in Chapter 8. Here we find a good discussion of critical points, Liapunov stability, and nonlinear mechanics. The chapter even mentions the Poincaré-Bendixson theorem, and sketches a proof of the theorem of Liénard. The concluding chapters contain a brief introduction to calculus of variations, the Laplace transform, and existence theory. The figures are numerous and clear.

Since tastes differ, I have a few minor objections to this book, too. Chapter 1 is interesting, but it may be too hard; a student who can do the problems could skip quite a bit of the later stuff, which is much easier. (For instance, Agnew's snowplow problem is Problem 1.) The book's refusal to be held up by petty matters leads to occasional glosses which are perhaps unnecessary; for example, there is no reason not to mention the case $z = f(1,z)$ in connection with the homogeneous equation, the danger of introducing extraneous solutions by an integrating factor, and similar things. The general solution for second-order equations is needed on p. 74 without having been defined, so far as I know; in any case, the index was no help. (I am not sure the general solution ever was defined for equations of order n, though Simmons' approach via initial-value problems is the one I approve of.) The book's statement that integrability of $|f|$ implies integrability of f is false; we can have $|f| = 1$ without even having f measurable.

Simmons doesn't do much with undetermined coefficients because he suffers from a common misapprehension: "We are unlikely to care much about finding a single solution," he says, "unless the general solution is at hand." On the contrary — the whole point of the theory of steady-state solutions is that it requires just a particular solution, plus knowledge that the coefficient polynomial is Hurewicz. In applications it is mostly the steady-state solution that you want, not the transient, and hence, you don't have to solve the characteristic equation.

Simmons makes the same omissions I objected to in Rabenstein, except that he does apply the convolution theorem to Abel's integral equation, following an earlier text.

My most serious objection to this book is its studious avoidance of the concept of linearity; the concept of a linear operator occurs first on p. 389. Experience at UCLA indicates that sophomore engineers are well able to understand linear operators in a context of differential equations, and gain

by such a presentation. In the present text the avoidance of linearity exacts a price beyond mere pedagogy: affine transformations are not available in the discussion of singular points, vector notation is not available for the discussion of systems, and so on.

It seems a pity to study differential equations for over 400 pages and never learn about the connection of the subject with linear algebra. On the other hand if that is the price one has to pay for using Simmons, so be it. The book is so good in every other respect that I recommend it highly, and with pleasure.

Plaat's book is a major contribution to the literature both of elementary differential equations and of elementary linear algebra. It has so many special features that I am concerned about the problem of doing it justice in a brief account.

Like other books at this level, Plaat contains a selection of applied problems, of which I mention the Volterra prey-predator equations (also considered by Simmons), some well-chosen examples of coupled systems, and a better treatment of complex solutions and resonance than commonly found in elementary texts.

Chapters 1 and 2 contain the expected introductory material, but with a few innovations. Once Plaat has obtained a differential equation, he interprets it in a way that shows its reasonableness. This sets the tone for the book: the point of view is the modern one of extracting information from the equation even without explicit solution. A remark on p. 21 suggests that Plaat realizes the importance of uniqueness in applied work; the proof of existence for the separable equation is better than anything of the sort in Rabenstein or Simmons; the difficult problem of finding the complete solution is dealt with competently by using the basic existence and uniqueness theorems, and insisting from the outset that a solution be defined on an interval.

Linear equations in Chapter 3 are preceded by a readable discussion of complex numbers. I was pleased to note that Plaat believes the Euler equation for $\exp(ib)$ needs motivation; Simmons and Rabenstein assume it as an axiom. An exceptionally good discussion of $x' = f(x)$ on p. 59 paves the way for the corresponding theory in two variables given later. (Such anticipation of future difficulties is characteristic of this text.) The theory of asymptotic stability is applied to oscillatory phenomena, and the chapter gives an interesting discussion of periodic solutions.

Chapter 4 on linear algebra and its applications introduces the topic which really sets this book apart from others. Although everyone knows that linear algebra is important, it is sometimes hard to show this importance in a convincing way. This pedagogic problem is solved by Plaat's book, which gives a brief, clean development of vector and matrix algebra, starting from scratch, and then applies the theory with great effectiveness to differential equations. As a result the book goes much farther in the theory of systems than any comparably elementary text known to me. I think Plaat would be of great value to those who have had a course in linear algebra, to those who are studying it concurrently, and to those who may study linear algebra at some future date. (For the latter the instructor should, of course, take a correspondingly leisurely pace.)

Further discussion of systems in Chapter 5 shows the connection between the problem of diagonalizing a matrix and the problem of uncoupling a coupled system — an application which alone makes the theory of diagonalization worthwhile. The chapter also gives Liouville's beautiful generalization of Abel's identity, the theory of systems with periodic coefficients, and other topics. Since the theorems are really powerful, one might expect the book to be too hard, but it is not. On the contrary, it is probably easier than Rabenstein and Simmons, which have some pretty tricky special problems.

The discussion of autonomous systems and the phase plane in Chapter 6 is excellent. It resembles that in Simmons, except that the theory of singular points is simplified by the availability of affine transformations, as in Hurewicz, and Plaat mentions the index of a closed curve and the interpretation of integrating factors as density functions, in the spirit of Liouville's theorem in dynamics. Chapter 7 gives a very good account of existence and uniqueness theorems — including a

discussion of error estimation, stability, and dependence on parameters (the latter in a problem). I did not happen to find any mathematical errors, though there are a few misprints.

Some will object because Plaat omits series solutions, but I do not share their opinion. In my view series are of little interest for existence or for computation, and of only slight interest as one way (among several) of introducing the special functions. The real importance of series comes to light only in the complex plane, with the Riemann P-function, asymptotic expansions, and allied topics. These subjects are not accessible in elementary courses. In any case, the things which Plaat proves, and Rabenstein and Simmons do not, more than make up for the omission of series.

In summary, I think Plaat has accomplished the difficult task of writing an accurate book without being pedantic, and of presenting powerful general theories without going beyond the sophomore-junior level. I recommend his book very highly indeed.

I conclude with a brief comparison. Both Rabenstein and Simmons seem to me to be strongly influenced by standard books of the "mathematics for engineers" variety; Simmons, in particular, has taken parts of his text from such books almost word for word. Hence I am not sure they would be suitable for students taking a general applied course concurrently; there might be too much overlap. This objection does not apply to Plaat.

I think Plaat would be better than the others for mathematics majors, and also for majors in theoretical physics, since both groups require command of powerful general tools rather than a multiplicity of detail. Because of its unified structure Plaat would also be better for a short course (one semester, or even one quarter if the students have had some linear algebra). The other books take up too many different topics for one semester, in my opinion.

For students who want a view of the richness and variety of applied differential equations, and who do not get such a view from their other courses, I would recommend Rabenstein and especially Simmons above Plaat. An excellent course could be based on two quarters of Simmons followed by one quarter of Plaat.

Actually all three books are excellent, and the instructor should study them all before making his decision.

R. M. REDHEFFER, University of California, Los Angeles

Calculus — An Introduction to Applied Mathematics. By Harvey P. Greenspan, David J. Benney. McGraw-Hill, New York, 1973. xi + 784 pp. $14.50. (Telegraphic Review, October 1973.)

Does anyone remember Aristotle? In Book II of the *Nicomachean Ethics* he stated that, "Moral virtue is a disposition to choose the mean." Is there, or has there ever been, a more consistently ignored idea than the one contained in this statement? Today the dominant idea is excess. We move from extreme to opposite extreme, passing through moderation, like a pendulum, with maximum velocity. Education, alas, is not immune from the process. The overemphasis on abstraction in undergraduate mathematics of recent times has produced the inevitable "backlash" typified, unfortunately, by this text.

The book is intended as an introduction to the differential and integral calculus from an applied, "practical" point of view. Chapters on limits, differentiation, integration and infinite series are followed by an introduction to vectors (two or three dimensional 'arrows'), partial derivatives, multiple integration, and finally, the main integral theorems of vector calculus. In the preface the authors state that, "Complete rigor is often unneccessary from the scientific viewpoint and may even be counter-productive." Quite true, on this level, but a funny thing happened on our way to the

text. We lost not only "complete rigor," but almost all rigor as well as a great deal of explanatory text. What remains is section after section consisting almost entirely of worked examples, leading even some of my engineering students to complain that there was insufficient "theory" and that they were simply being "fed formulas" as one student remarked on a final examination discuss-the-course question.

In past years at Stevens Institute the one-year sophomore vector calculus sequence was the object of much criticism. A great deal of it — understandable and even partially justified — was centered on what was considered to be an excess of linear algebra. We talked about abstract vector space theory to students who were seeing second and third degree matrix operations for the very first time. In the text being reviewed here, there isn't a matrix to be found anywhere (determinants used for Jacobians are defined without any mention of matrices). Is there anything too rigorous about a matrix, or a transformation of two dimensional space, say, to itself? Indeed, one would be hard pressed to find a more "applicable" mathematical object than a matrix. One example taken directly from the text should illustrate further just how far the pendulum has swung. In discussing directional derivatives we proceed from the equation $df = f_x dx + f_y dy$ directly to the result $df/ds = f_x(dx/ds) + f_y(dy/ds)$ by (you guessed it) "dividing by ds." From abstract function spaces to this in just a few short years. Where, oh where, is that golden mean?

It must be mentioned that the problem sets appearing at the end of each section are excellent. There is a large number of problems ranging from the routine to the extremely difficult and challenging. Many of them are quite fascinating to solve and have proved to be highly instructive to both teacher and student.

E. H. LIPPER, Stevens Institute of Technology

Algebra. By Thomas W. Hungerford. Holt, Rinehart and Winston, New York, 1974. xix + 502 pp. $17. (Telegraphic Review, October 1974.)

This book is a very welcome addition to the many texts in algebra at the beginning graduate level. It is roughly of the same size as Lang's *Algebra* but contains a little less material over all. However, there is more group theory even though Hungerford omits representation theory. In particular there are sections on permutation groups, nilpotent and solvable groups, and normal and subnormal series, all of which Lang relegates to a few exercises. Hungerford does have a concluding chapter on category theory and his presentation is compatible with, while not requiring, categorical language.

Hungerford's exposition is clear enough that an average graduate student can read the text on his own and understand most of it. Each topic is followed by many illustrative examples and almost every section is followed by a long list of exercises of varying degrees of difficulty.

The reviewer objects only to the exclusion of certain topics (included in van der Waerden and Lang) such as representation and valuation theory; and to the inclusion of the introductory chapter in which a student might well become bogged down. However, it is possible to start with chapter 1 and thus avoid the difficulties in the introduction.

In summary, Hungerford's *Algebra* is not a replacement for van der Waerden or Lang. A future research mathematician will benefit from the extra effort required for these classic texts. Nevertheless, Hungerford's book will long stand as a standard in this field. Professor Hungerford is to be complimented for his care in writing such an excellent book.

HUGO S. SUN, California State University, Fresno

Introduction to Probability Models. By Sheldon M. Ross. Academic Press, New York, 1972. xiii + 272 pp. $12.75. (Telegraphic Review, March 1973.)

I used this book in a 3-credit first course in probability for mathematics majors. The only prerequisite was 12 credits in calculus. This requirement allowed people of various backgrounds,

including an economics major, to join the course. I found the prerequisite to be realistic. I chose this text rather than more classical texts such as Kreyszig, Larson or Zehna because this course was a feeder for our course in operations research. Ross's stochastic-processes approach seemed more appropriate for such a purpose than did statistics-oriented texts. The avowed purpose of the text is to get students to "think probabilistically," and this is admirably accomplished.

There are three strong points to this book. The first is the book's extensive answer key, which is unfortunately flawed by the number of errors present. Secondly, conditional expectation is introduced in the third chapter and is used intelligently, consistently and well thereafter. This is an extremely strong point because of the insights gained by the students, both into the nature of the problem and into the concept of expectation.

The third good feature is the ready availability of non-trivial applications of probability. It's nice not to have to tell students "Wait until we get to estimation," or the like. Of course you still face the problem of getting them through basic probability and into continuous distributions, but to be able to investigate the Poisson process really means a lot in terms of student interest. I found that the bull-sessions arising when we talked about model-building really got the students thinking. In fact, several of the students decided that they would take a statistics course to see how one could objectively question assumptions made regarding, say, Poisson arrivals. That's a far healthier attitude toward statistics than the traditional "Oh yeah—statistics comes after probability."

The book has two shortcomings, however. The first is that there are many errors, although these are usually typographical. Others are calculated to reduce the class to bewildered amazement, as on p. 41, where a proof shows that

$$P\{X \leqq a,\ Y \leqq b\} = P\{Y \leqq b\} \cdot P\{X \leqq a\},$$

and concludes that X and Y are continuous.

The other main shortcoming is the super-terseness of the text. Uniformly, the students bemoaned the telegraphic style of the book, especially since the book stresses the intuitive approach. They groaned when they realized the amount of space given in other books to the topics covered by this book in the first 13 pages. (This is Chapter 1 of the text, and comprises elementary probability up to and including Bayes' Formula.)

Similar disenchantment developed with Chapter 2, in which the author defines the expectation of a random variable and proceeds to treat both the discrete and continuous cases, all in four pages. This wouldn't be too bad, except that the immediately preceding eight pages have seen the introduction of the Bernoulli, binomial, geometric, Poisson, uniform, exponential, gamma and normal random variables. (At this point in a course taught from Ross's book, be prepared for almost armed rebellion; do not, under any circumstances, turn your back to the class during this period.)

The students expressed great frustration and dissatisfaction with the book. But they did admit that they have learned quite a bit about probability and model-building. From my standpoint, I have never seen students so familiar with probability after a first course.

In summary, I think that the author succeeds admirably in getting students to "think probabilistically," but at a cost of temporarily great discomfiture. I would definitely recommend the book as a text for a course in applied probability, but I would suggest two things: that the publisher issue a corrected edition and that the teacher be prepared to skip back and forth between the various distributions in chapter 2 and the appropriate models in chapters 5 and 6. To plunge headlong through the text would likely be devastating to the class.

JOHN HICKEY, Hofstra University

An Introduction to Mathematical Ecology. By E. C. Pielou. Wiley, New York, 1969. 294 pp. \$18.25. (Telegraphic Review, October 1970.)

This review is based on a course in mathematical ecology I taught to a mixed group of faculty and students in the Departments of Zoology and Mathematics at the University of Alberta in the school

year 1972–73. Formally, the course was given as one of the offerings of the Zoology Department, so I taught in the role of a mathematician (who was learning ecology) to an audience of biologists (who were trying to learn some relevant mathematics). We met regularly for $1\frac{1}{2}$–2 hours per week, xeroxed copies of the lecture were passed out, and an attempt was made to generate a cross-cultural discussion of the material. We covered all of Pielou with the exception of Part III, Spatial Relations of Two or More Species. The remaining Parts I, II and IV are Population Dynamics, Spatial Patterns in One-Species Populations, and Many-Species Populations, respectively. I had to give the zoologists in the class (faculty, graduate students and undergrads) several quickie math reviews and primers, as the appropriate occasions arose; the necessary statistics I was able to weave more gracefully into the course as it was presented. The major zoological interests in the class were limnology, behaviorism, parasitology and ecology. The steady state attendance was about 10.

In general Pielou has written a useful and interesting book, but I myself found it impossible to present honestly any of the material from Pielou without first having read her major biological, mathematical and statistical references. For the usual two hour lecture I had to read at least five major references in order to make any sense out of Pielou's text, which is in good part a collection of her opinions of certain topics in certain papers. The main problem seems to be the tendency for Pielou to regard mathematical equations as *objets d'art* which sort of spring into life as part of somebody's model. Very little justification or rationalization of the various mathematical models is given. Models are taken too literally as full blown ecological hypotheses, rather than as the *ad hoc* and provocative things they really are. Pielou is rather uncritical of the model derivations per se, but dwells lovingly on their defects as revealed in their properties. In all fairness, this is a problem anyone who tries to put together a book on mathematical ecology is going to face: for the sake of brevity, too much must be taken on authority.

Part I on Population Dynamics is weak; it would have been weak even if written ten years ago. MacArthur is mentioned only once, Leigh once (for a review), while Maynard Smith, R. Levins, E. O. Wilson, Pianka and Kerner are not referenced at all, along with a large cast of many more distinguished ecologists of a mathematical bent, including Andrewartha and Birch. However these gentlemen did make several "appearances" in the course I actually taught.

Part II, Spatial Patterns, is basically dull — lots of statistical fitting of clumping phenomena, with little biological explanation. An overall summary and comparison of the appropriateness of the various distributions is needed. Mosaics are mentioned only in terms of sampling; their direct use (as in behaviorism) is ignored. However, Chapter II in this section on the Fokker-Plank diffusion equation is quite well done, but unfortunately with no reference to the first use of diffusion in biology in genetics, as by Wright, Haldane and Fisher. The ecological examples are amusing, e.g., Skellam's muskrats showing $\sqrt{\text{area}} \propto$ time behavior, and Clement Reid's oak invasion of England with the help of rooks.

Pielou does a good job with Part IV, Many Species Populations, with her treatment of the Shannon-Weaver formula as an index of ecological diversity, and her fairly thorough discussion of ordination techniques, including, e.g., the popular methods of Orloci.

In my opinion it would be difficult for a course to be taught out of this book by the average biologist with an "interest" in mathematics. The job pretty much has to be done by a bona fide mathematical ecologist or by a critical mathematician who has read in ecology and is willing to spend an enormous amount of time checking out Pielou's references. As a final remark, it must be said that despite the drawbacks of this textbook, it does touch on many of the diverse topics which come under the heading of "mathematical ecology" today. It should be considered as a useful supplement, (say along with Robert May's new book) to a course for which at the present time there really is no complete text book.

THOMAS ROGERS, University of Alberta

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Pubishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T(13-14). *Mathematics for College Students: Elementary Concepts, Second Edition*. A. William Gray, Otis M. Ulm. Glencoe Pr, 1975, 352 pp, $7.95 (P). General introduction to mathematics for a non-majors course. Topics: logic, sets, real numbers, algebra, geometry, probability and statistics, introduction to computers. Appendices on BASIC and Student Research Projects. Good references at the end of each chapter. Perhaps too much algebra for a standard "math for poets" course.PJM

GENERAL, T(13: 1). *A Survey of Mathematics*. Mario F. Triola. Cummings, 1975, 403 pp, $10.95. Math for liberal arts students, including number systems, probability, statistics, computers (APL, FORTRAN and BASIC). Nice illustrations. No references for further reading. PJM

GENERAL, S(8-14). *How to Count Like a Martian*. Glory St. John. Henry Z. Walck, Inc., 1975, 66 pp, $6.95. A series of informal tales beginning and ending with a "mysterious message from Mars", with Egyptian, Mayan, Chinese, ... versions in between. A fun way to look at alternative number systems. LAS

GENERAL, L**. *Patterns in Nature*. Peter S. Stevens. Little, Brown, 1974, 240 pp, $10. Visual pattern and form in the natural world "emerge from the working and reworking of only a few formal themes." A beautiful, entirely non-mathematical, survey of these few themes--spirals, branching, bubble clusters, close packings, etc.,--illustrated with hundreds of telling photographs and diagrams. A rich warp into which much elegant mathematics could be woven. LAS

GENERAL, S. *The Calculating Book, Fun and Games with your Pocket Calculator*. James T. Rogers. Random House, 1975, 80 pp, $2.95 (P). Assortment of visual display tricks, number puzzles and demonstrations of properties of numbers to be done on a pocket calculator. Fun for the novice but not likely to impress anyone with much mathematics background. RSK

GENERAL, S. *The Slide Rule, Electronic Hand Calculator, and Metrification in Problem Solving, Third Edition*. George C. Beakley, H.W. Leach. Macmillan, 1975, v + 234 pp, $4.95 (P). Second edition, entitled *The Slide Rule and Its Use in Problem Solving*, published in 1969. New chapter on hand calculators describes the operation of several of the most popular ones. Problems emphasize basic physical principles. Unfortunately, slide rule material still is emphasized. RSK

GENERAL, S(13). *Success in Mathematics*. Walter Van Stigt. Transatlantic Arts, 1974, xx + 609 pp, $5.95 (P). Part of the British "Success Studybook" series; designed for self-study. Contains topics ranging from arithmetic through a brief introduction to calculus. RSK

BASIC, T(13:1). *Basic Mathematical Skills: A Text Workbook, Second Edition*. Loyce J. Gossage. McGraw, 1975, xiii + 296 pp, $7.95 (P). Revision of 1968 edition. Arithmetic for general business, clerical students. LH

BASIC, T(7-13). *Essentials of Business Mathematics*. W. Alton Parish, William L. Kindsfather. Rinehart Pr, 1975, xi + 403 pp, $10.95 (P). Review of arithmetic followed by a lot of detailed business applications including a tiny bit of statistical vocabulary. The presentation of a significant amount of business vocabulary may help many students use their arithmetic better. JAS

BASIC, T?(13:1). *Modern Shop Mathematics*. Edward Kratfel. Reston, 1975, x + 279 pp, $10.95. Divided into four main sections--basic mathematics, plane geometry, solid geometry, and basic trigonometry. Emphasis is on practicality, but little is done to make the mathematical steps meaningful. RSK

BASIC, T(13: 1). *Elementary Algebra: A Worktest*. Vivian Shaw Groza. Saunders, 1975, xiii + 728 pp, $10.95 (P); *Instructor's Guide*, 60 pp, (P). Each chapter states objectives, has a pretest and a posttest. Development of theory is minimized. Includes chapters on basic operations, sets, axioms, integers, linear equations and systems, polynomials, factoring, algebraic functions, graphs and quadratic equations. CEC

PRECALCULUS, T(13:1). *Trigonometry*. Harley Flanders, Justin J. Price. Acad Pr, 1975, xii + 235 pp, $10.50. A short text on functions and graphs, trigonometric functions, exponentials and logarithms. Includes a discussion of solutions with a pocket calculator. LLK

PRECALCULUS, T(13: 2). *College Algebra and Trigonometry, Second Edition*. Margaret F. Willerding. Wiley, 1975, xiv + 613 pp, $12.50. More algebra review than in first edition (TR, May 1971). Functions, complex numbers, series, permutations, probability in addition to basic algebra and trigonometry. LH

PRECALCULUS, T(13: 1). *Algebra and Trigonometry, Third Edition*. Paul K. Rees, Fred W. Sparks, Charles Sparks Rees. McGraw, 1975, ix + 566 pp, $12.95. A good precalculus text with the new edition containing several sections leading to and on linear programming. LLK

PRECALCULUS, T(13: 1). *Algebra*. Harley Flanders, Justin J. Price. Acad Pr, 1975, xiii + 310 pp, $11.50. A short text on algebra topics with applications. Up to date: includes discussion of solutions with a pocket calculator. LLK

EDUCATION, P, L*, *Mathematical Talent: Discovery, Description, and Development.* Ed: Julian C. Stanley, Daniel P. Keating, Lynn H. Fox. John Hopkins U Pr, 1974, xvii + 215 pp, $10; $2.95 (P). Detailed record and analysis of a program centered at Johns Hopkins to identify mathematically precocious youth. Full of interesting nuggets buried in a mass of statistics, e.g., sex differentials are very strong even in grade school; sibling position is irrelevant, and "mean scores on all three screening tests increase as liking for school decreases." LAS

EDUCATION, S*(16-17), P, L, *Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies.* Max A. Sobel, Evan M. Maletsky. P-H, 1975, xii + 240 pp, $11.95. Excellent source of teaching aids, laboratory techniques, games, and teaching strategies for both elementary and secondary level. Possible supplementary text for pre-service and in-service mathematics education courses. Includes exercises and good bibliographies. Recommended. PSJ

EDUCATION, L, *A Rhythmic Approach to Mathematics.* Edith L. Somervell. NCTM, 1975, 67 pp, $5.50. Fifth in NCTM's "Classics" reprint series. An unaltered reprint of an unusual monograph first published in 1906 in London, it is devoted to the use of "sewing cards" to illustrate exotic curves by creating their envelopes out of colored string. LAS

HISTORY, P, L*, *Lebesgue's Theory of Integration: Its Origins and Development.* Thomas Hawkins. Chelsea, 1975, xv + 227 pp, $9.50. Reprint of the 1970 U. Wisconsin original edition, reviewed here in April 1971. LAS

HISTORY, P, L**, *Ernst Eduard Kummer: Collected Papers, Volume I: Contributions to Number Theory.* Ed: André Weil. Springer-Verlag, 1975, viii + 957 pp, $40.20. E. Lampe's biographical notice, K. Hensel's *Festschrift*, letters between Kummer and his mother, and between Kummer and Kronecker, plus Kummer's number theory papers; introduced, in English, by the editor. LAS

HISTORY, P, *Collected Papers of Hans Rademacher.* Ed: Emil Grosswald. MIT Pr, 1974. *V. I:* xix + 692 pp; *V. II:* xxi + 638 pp, $75 set. These two volumes contain all the papers published by Hans Rademacher, either alone or as a joint author, essentially in chronological order. The editor has provided notes for each paper and has contributed a biographical sketch. CEC

HISTORY, P?, *Sociology of Mathematics and Mathematicians--A Prolegomenon.* H. Fang, K.P. Takayama. Paideia Pr, 1975, 364 pp, $15. A diffuse, idiosyncratic, and prolix statement, purportedly on the nature of mathematics and the social context in which it thrives. In fact, it consists mostly of personal assertions of historical and social generalizations with very little connection to mathematics (only about 5% of the bibliography deals with mathematics), all expressed in a tiresome, unidiomatic style. The author's objectives are grandiose, and his effort enormous; but the bearing of his work on the discipline of mathematics is insignificant. LAS

COMBINATORICS, P, *Infinite and Finite Sets.* Ed: A. Hajnal, R. Rado, Vera T. Sós. North-Holland, 1975. *V. I:* 604 pp; *V. II:* 472 pp; *V. III:* 472 pp, $145.95 set. Nearly one hundred papers from a July 1973 colloquium in honor of Paul Erdös' 60th birthday. LAS

COMBINATORICS, P, *Lecture Notes in Mathematics-452: Combinatorial Mathematics III.* Ed: Anne Penfold Street, W.D. Wallis. Springer-Verlag, 1975, ix + 233 pp, $9.90 (P). Proceedings of the third Australian conference held at the University of Queensland in May 1974. JAS

LINEAR ALGEBRA, T(16-18: 2), S, P, L, *Linear Algebra, Fourth Edition.* Werner Greub. Grad. Texts in Math., V. 23. Springer-Verlag, 1975, xvii + 451 pp, $18.80. Changes from the 1966 *Third Edition* include new material on the classical adjoint, quaternions, and associative division algebras, as well as clarification of the final two chapters. LAS

ALGEBRA, T(18: 2), P, *Infinite-dimensional Lie algebras.* Ralph K. Amayo, Ian Stewart. Noordhoff, 1974, xi + 425 pp, Dfl. 119. Exploits the surprising depth of analogy existing between infinite-dimensional Lie algebras and infinite groups. The coverage is comprehensive. Lists 44 unsolved problems. I-CH

ALGEBRA, P, *Lecture Notes in Mathematics-414: Unipotent Algebraic Groups.* T. Kambayashi, M. Miyanishi, M. Takeuchi. Springer-Verlag, 1974, 165 pp, $8.20 (P). A report on the authors joint investigations of both the group-theoretical and the geometric structures of unipotent algebraic groups defined over an arbitrary ground field. A thorough bibliography is included. CEC

COMPLEX ANALYSIS, P, *Lecture Notes in Mathematics-432: Holomorphiegebiete, pseudokonvexe Gebiete und das Levi-Problem.* Rolf Peter Pflug. Springer-Verlag, 1975, vi + 210 pp, $9.90 (P).

COMPLEX ANALYSIS, T(16: 2), *Complex Variables for Scientists and Engineers.* John D. Paliouras. Macmillan, 1975, xii + 371 pp, $12.95. The reason for the title is unclear. The principal feature that distinguishes the text from several other introductory texts is the placement of proofs for theorems in appendices following each chapter. Few examples of applications are given in spite of the title. The book is carefully written and edited and is a possible choice for any audience. TAV

COMPLEX ANALYSIS, P, *The Theory of Extremal Problems for Univalent Functions of Class S.* K.I. Babenko. Proc. of Steklov Inst. of Math., No. 101. AMS, 1975, iii + 327 pp, $46.40. The class S, consisting of univalent functions of the form $\Sigma a_n z^{n+1}$, $a_0=1$, is studied using variational methods. Particular emphasis is given to the use of the second variation of a functional and to the smoothness of the boundary of the coefficient region. A meager bibliography. TAV

COMPLEX ANALYSIS, T(15-16: 1), *Functions of Complex Variables, An Introduction.* Zane C. Motteler. Intext, 1975, xi + 307 pp, $10. A text to be considered for an undergraduate course. Unusual in providing a substantial application to two-dimensional fluid flow. Each section has "exercises, problems, and proofs." RBK

COMPLEX ANALYSIS, P. *Theta Functions with Applications to Riemann Surfaces.* Harry E. Rauch, Hershel M. Farkas. Williams & Wilkins, 1974, xii + 232 pp, $15.50. A research monograph and reference on Riemann's first order g-variable theta functions with integer characteristics, and their application to the problem of the moduli of algebraic surfaces. Treatment is classical, concrete, and computational, and deals with some of the deepest parts of algebraic Riemannian surface theory. RBK

COMPLEX ANALYSIS, T(18: 1), P. *Introduction to the Theory of Entire Functions of Several Variables.* L.I. Ronkin. Trans. Math. Mono., V. 44. AMS, 1974, vi + 273 pp, $28.50. Subharmonic and plurisubharmonic functions; growth of entire functions of several variables; distribution of zeros of entire functions of several variables. Bibliography, index. RBK

DIFFERENTIAL EQUATIONS, P. *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions.* T. Yoshizawa. Appl. Math. Sci., V. 14. Springer-Verlag, 1975, vii + 233 pp, $9.50 (P). Intended as an introduction to stability theory. Discusses stability theory by Liapunov's second method, stability properties in almost periodic systems; existence of (almost) periodic solutions in (almost) periodic systems. SG

DIFFERENTIAL EQUATIONS, P. *Boundary Value Problems for Differential Equations, III.* Ed: V.P. Mihaĭlov. Proc. of Steklov Inst. of Math., No. 126. AMS, 1975, vi + 256 pp, $30.60 (P).

DIFFERENTIAL EQUATIONS, T(17-18: 1), S, P. *Introduction to Topological Dynamics.* K.S. Sibirsky. Trans: Leo F. Boron. Noordhoff, 1975, ix + 163 pp, Dfl. 55. A respectable well-edited American edition within the fairly general framework of metric spaces, this book presents the fundamentals of the topological theory of dynamical systems. Also touches upon certain topics in depth namely, recurrent motions, almost periodic motions and generalized dynamical systems. I-CH

DIFFERENTIAL EQUATIONS, P. *Analysis on Lie Groups and Homogeneous Spaces.* Sigurdur Helgason. CBMS, No. 14. AMS, 1972, vi + 64 pp, $4 (P). Notes based on series of lectures at a CBMS Regional Conference at Dartmouth, August 1971. General theme is the treatment of invariant differential equations by separation of variables techniques. Treats spherical functions on symmetric spaces, conical distributions on the space of horocycles and central eigendistributions and characters. Bibliography. RBK

DIFFERENTIAL EQUATIONS, T(18: 2). *Ecuaciones Diferenciales Ordinarias: Teoría de Estabilidad y Control.* M. De Guzman. Editorial Alhambra, 1975, x + 300 pp, (P). An expository text with historical notes based on courses given at the Universidad Complutense de Madrid. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-446: Partial Differential Equations and Related Topics.* Ed: Jerome A. Goldstein. Springer-Verlag, 1975, iv + 389 pp, $13.80 (P). Lectures, partially expository, partially research, from the special program in partial differential equations during the spring of 1974 at Tulane University. JAS

NUMERICAL ANALYSIS, T(17), P. *Essentials of Padé Approximants.* George A. Baker, Jr. Acad Pr, 1975, xi + 306 pp, $26. A very thorough development. Algebraic properties. Relation to other approximants. Numerical methods. Convergence theory. Stieltjes and Polya series. Extensions and applications. Good bibliography. RWN

NUMERICAL ANALYSIS, P. *Studies in Numerical Analysis: Papers in Honour of Cornelius Lanczos.* Ed: B.K.P. Scaife. Acad Pr, 1974, xxii + 333 pp, $13. 19 papers. Several are historical or tutorial. Most are elaborations and extensions of some inherently clever contribution of Lanczos. Among the authors are Synge, Householder, Wilkinson and Schoenberg. Examples of the topics are conservation laws, the Tau method, eigenvalue problems, Chebyshev series, harmonic analysis and the hypercircle method. RWN

FUNCTIONAL ANALYSIS, S(17-18), P. *Lecture Notes in Mathematics-435: Representations of Commutative Semitopological Semigroups.* Charles F. Dunkl, Donald E. Ramirez. Springer-Verlag, 1975, vi + 181 pp, $8.60 (P). Studies representations of semitopological semigroups in structures native to harmonic analysis (i.e., algebra of bounded operators on a Hilbert space). Many fresh results. LH

FUNCTIONAL ANALYSIS, P. *Introducere în Teoria Operatorilor Liniari.* V.I. Istrăţescu. Editura Academiei Romania, 1975, 477 pp. A self-contained introduction to the theory of operators in Hilbert and Banach spaces together with some recent results on non-normal operators. JAS

FUNCTIONAL ANALYSIS, T(18: 1), P. *Lecture Notes in Mathematics-416: Pseudo Differential Operators.* Michael Taylor. Springer-Verlag, 1974, iv + 155 pp, $7.40 (P). Basically these notes develop one tool, the calculus of pseudo-differential operators, and apply it to obtain interior regularity results for elliptic and hypoelliptic operators. In doing so, the author freely employs methods from the Fourier transform, distribution theory and Sobolev spaces without comment. I-CH

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-434: Besov Spaces and Applications to Difference Methods for Initial Value Problems.* Philip Brenner, Vidar Thomée, Lars B. Wahlbin. Springer-Verlag, 1975, 154 pp, $7.80 (P). With material simplified and extracted from authors' recent research publications, these notes present certain Fourier techniques for analyzing finite difference approximations to initial value problems for linear PDE's with constant coefficients. The main tools used are Fourier multipliers, the Carlson-Beurling inequality and van der Corput's lemma. Many results are expressed in terms of norms in Besov spaces. I-CH

FUNCTIONAL ANALYSIS, P. *Convolution Equations and Projection Methods for Their Solution.* I.C. Gohberg, I.A. Fel'dman. Trans: Math. Mono., V. 41. AMS, 1974, ix + 261 pp, $27. Operational calculus for solving Wiener-Hopf integral equations, discrete and difference analogues, pair equations, singular integral equations on the circle, etc. Appendix on asymptotics of solutions of homogeneous convolution equations. RBK

FUNCTIONAL ANALYSIS, T*(17-18: 1), S, P. *An Introduction to Invariant Imbedding.* R. Bellman, G.M. Wing. Wiley, 1975, xv + 250 pp, $18.95. An excellent book by authors well-known for the development of the invariant imbedding, a perturbation method which claims its novelty and fame basically by perturbing non-classical variables. After minimum coverage of 'tool' methods, this well-organized book leads readers to such applications as random walk, wave propagation, transport theory and radiative transfer, eigenvalue problems, integral equations. I-CH

ANALYSIS, P. *The Dimension of Spaces of Automorphic Forms on a Certain Two-Dimensional Complex Domain.* Leslie Cohn. Memoirs, No. 158. AMS, 1975, iii + 96 pp, $3.30 (P). Using the Selberg trace formula, the author obtains results relating to the dimension of certain spaces of automorphic forms. The spaces considered differ from previously studied spaces in that the Fourier coefficients of the forms are non-constant, being theta functions. TAV

ANALYSIS, P*. *Lecture Notes in Mathematics-457: Fractional Calculus and Its Applications.* Ed: Bertram Ross. Springer-Verlag, 1975, vi + 381 pp, $13.80 (P). Proceedings of an international conference at New Haven in June 1974. This was the first conference on a rather new aspect of analysis wherein derivatives of fractional order are studied and applied. Includes introductory expository material. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-436: Abelian Harmonic Analysis, Theta Functions and Function Algebras on a Nilmanifold.* Louis Auslander, Richard Tolimieri, with the assistance of H.E. Rauch. Springer-Verlag, 1975, 99 pp, $7.80 (P).

ANALYSIS, T(16: 1), S, L. *Asymptotic Analysis.* J.D. Murray. Oxford U Pr, 1974, vii + 140 pp, $11.25. Introduction to methods for obtaining analytical approximations to functions defined by integrals or as solutions to differential equations. Healthy philosophy is that no method, rigorous or heuristic, is scorned. Asymptotic expansions; Laplace's method for integrals; method of steepest descents; method of stationary phase; transform integrals; differential equations. RBK

ANALYSIS, P. *Proceedings of the Seminar on Random Series, Convex Sets and Geometry of Banach Spaces.* Aarhus U, 1975, vi + 222 pp, (P). Papers from an October 1974 conference in memory of Edgar Asplund; includes an Asplund bibliography. LAS

ANALYSIS, P. *Integration on Locally Compact Spaces.* N. Dinculeanu. Noordhoff, 1974, xv + 626 pp, Dfl. 195. A continuation of author's *Vector Measures.* Follows Bourbaki point of view in which vector measure is linear operation on space of continuous vector functions with compact support taking their values in a Banach space. Includes measures defined by densities; sums and images of measures; measures on locally compact groups; spaces of vector fields. RBK

ANALYSIS, T(16-17: 1, 2). *Reelle Analysis einer Veränderlichen.* Karl Kiesswetter. Hochschultaschenbücher, B. 269. Bibliographisches Inst, 1975, 316 pp, (P). Basics of abstract real analysis beginning with a fairly extensive study of the real numbers. Straightforward use of topology but includes more than usual on polynomial approximation. JAS

GEOMETRY, S*(13-15), L. *Taxicab Geometry.* Eugene F. Krause. A-W, 1975, viii + 88 pp, $1.98 (P). A do-it-yourself guidebook to explore elementary properties of geometry with the L^1 metric, designed for enrichment in high school or college geometry courses. A rich, unknown territory for mathematical Darwins, replete with square circles, equilateral right triangles, and many other exotic geometric forms. LAS

GEOMETRY, T(12), L. *Geometry.* Harold R. Jacobs. Freeman, 1974, xii + 701 pp, $9. Truly an unusual high school textbook covering standard Euclidean plane and solid geometry plus material on transformations, symmetry and non-Euclidean geometry. Carefully done with a sense of humor; puzzles and applications to entice and convince students of the relevance of the subject to many fields. SS

GEOMETRY, T(16-18), P. *Tensors, Differential Forms, and Variational Principles.* David Lovelock, Hanno Rund. Wiley, 1975, xi + 364 pp, $21.95. Intended for physicists, applied mathematicians, engineers. Topics include tensor analysis on manifolds, differential forms, invariance principles in the calculus of variations, Riemannian spaces (including those with indefinite metrics), physical field theories. Approach is "old-fashioned": tensors described in terms of components rather than as elements of tensor algebras or bundles; however a useful appendix reconciles the classical and modern viewpoints. Many exercises. An excellent book. SG

PROBABILITY, P. *Lecture Notes in Mathematics-426: Symmetric Markov Processes.* Martin L. Silverstein. Springer-Verlag, 1974, x + 287 pp, $11.50 (P). Presumes a familiarity with martingales. The treatment is abstract and thorough. Tough sledding for all but the expert. TAV

PROBABILITY, P. *The General Point Process: Applications to Structural Fatigue, Bioscience, and Medical Research.* V.K. Murthy. Appl. Math. and Comp., No. 5. A-W, 1974, xix + 604 pp, $22.50; $13.50 (P). Develops the theory of the general point process, a generalization of a renewal process in which the true starting time is unknown, and provides some applications of it. Good set of references; no index. RSK

PROBABILITY, P. *Theory of Stochastic Processes, No. 1.* Ed: I.I. Gikhman. Trans: D. Louvish. Halsted Pr, 1974, v + 157 pp, $23.50. A collection of papers, all at a reasonably sophisticated level. The unifying thread seems to be limit theorems for processes. TAV

PROBABILITY, T*(16-17: 1, 2). *Introduction to Probability and Statistics, Part I: Probability.* Narayan C. Giri. Statistics, V. 7. Dekker, 1974, viii + 260 pp, $13.75. Good theoretical introduction, presuming a strong calculus background. First three chapters include mostly standard material and may be used for a one-semester course. Remaining two chapters are more specialized, containing material on stochastic convergence and limit theorems, and further information on distributions,

particularly non-central and multivariate ones. Appendix contains matrix algebra needed for the last chapter. RSK

PROBABILITY, T(15:1), S. *Lecture Notes in Biomathematics-3: Stochastic Population Theories.* Donald Ludwig. Springer-Verlag, 1974, vi + 108 pp, $7.40 (P). Little background in probability or differential equations is needed to understand the development of the stochastic models presented. Topics include birth and death processes, branching and epidemic models. The treatment is basically intuitive, with references given for proofs and extensions of the theory. A very nice introduction to stochastic modelling. TAV

STATISTICS, T?(15-16: 1), P. *Techniques Statistiques. Moyens rationnels de choix et de décision.* Georges Parreins. Dunod, 1974, xviii + 303 pp, 98F (P). Écrit pour les praticiens, ce livre évite une rigueur trop grande mais précise les hypothèses restrictives des tests statistiques. Le lecteur doit connaître le calcul. Contenu: calcul des probabilités, enchantillonnage, tests d'hypothèse, contrôle industriel, analyse de la variance, liaisons, mesures, et calculs numériques. Pas d'exercises. PJC

STATISTICS, T(17-18), S, L. *Theoretical Statistics.* D.R. Cox, D.V. Hinkley. Chapman, 1974, xii + 511 pp, $18. Presumes background in probability and statistics, including the theory of the linear model. Emphasizes general concepts in the areas of hypothesis testing, interval and point estimation, asymptotic theory, Bayesian methods and decision theory. Uses many (non-numerical) examples and concludes each chapter with a good set of bibliographic notes. RSK

STATISTICS, S(13). *Chance and Choice: Practical Probability and Statistics.* J.A. D'Arcy. Transatlantic Arts, 1968, 111 pp, $3.95 (P). Published by Thames and Hudson in "The General Studies Library" and now being distributed by Transatlantic Arts. Written for sixth form students as part of a general studies program, it introduces selected topics, such as game theory and Markov chains, at an elementary level. Includes exercises. RSK

STATISTICS, T(13: 1). *Vital Statistics.* Michael Orkin, Richard Drogin. McGraw, 1975, x + 388 pp, $11.95. Designed for students with minimal mathematics backgrounds. Beginning chapters cover the usual material in an elementary manner. Later chapters contain some nonstandard material, such as nonparametric tests for multiple comparisons and a method of estimating sensitive parameters, and more mathematical topics, such as permutations and combinations, conditional probability, expectation and Chebyshev's theorem. RSK

STATISTICS, T(13-14: 1, 2). *Basic Ideas of Statistics.* Bernard W. Lindgren. Macmillan, 1975, x + 352 pp, $10.95. An attractive and sound treatment of the basic notions of statistics for students with no calculus. FLW

STATISTICS, T?(13). *Basic Statistics for Librarians.* I.S. Simpson. Shoe String Pr, 1975, 113 pp, $8. The author claims that librarians need a basic understanding of statistical techniques. It is doubtful that they could get such an understanding from this brief, sketchy and incomplete treatment. TAV

STATISTICS, T(13-14: 1, 2). *Introduction to Probability and Statistics, Fourth Edition.* William Mendenhall. Duxbury Pr, 1975, xiii + 460 pp, $12.95. This new edition of this sound treatment has additional problems and more projects involving real data. (The third edition was reviewed in the Monthly in February 1973.) FLW

STATISTICS, T(15-17: 1/2), S, L. *Elements of Sampling Theory.* Vic Barnett. Crane, Russak, 1974, xiii + 152 pp, $5.50 (P). A short, direct, treatment at an intermediate level of sample survey methods. FLW

STATISTICS, T(17: 1, 2). *Mathematical Statistics.* S. Varadhan. Courant Inst, 1974, x + 287 pp, $7.50 (P). Lecture notes from a 1973-74 course. Theoretical introduction, assuming a strong mathematics background. Touches briefly on Gauss-Markov theory, nonparametric methods, decision theory and sequential procedures in the final 86 pages. RSK

STATISTICS, P**. *Index to Statistics and Probability: Permuted Titles.* Ian C. Ross, John W. Tukey. R&D Pr, 1975. 2 volumes: *A--Microbiology*, xxviii + 796 pp; *Microclimatic--Z*, 792 pp, $105 set. Volumes 3 and 4 in the Information Access Series, of which Volumes 2-6 comprise the *Index to Statistics and Probability*, covering the literature through 1966 (Volume 2, TR June-July 1974; Volume 5, TR October 1975). Contains titles of all source items for the *Index*, listed alphabetically by each key word in the title, in a unique format which facilitates rapid scanning. RSK

STATISTICS, T(17-18: 2), P. *Introduction to Mathematical Statistics.* Leopold Schmetterer. Trans: Kenneth Wickwire. Gund. math. Wissenschaften, B. 202. Springer-Verlag, 1974, vii + 502 pp, $50.90. Revised English edition of the author's 1966 German work. Highly theoretical and detailed introduction--no problems and very few examples. Good set of references. Note price! RSK

STATISTICS, T(16-18: 2), S*, P. *Statistical Techniques in Simulation.* Jack P.C. Kleijnen. Statistics, V. 9. Dekker. *Part I*, 1974, xiii + 285 pp, $17.75; *Part II*, 1975, xv + 468 pp, $29.50. This two-part book fills the gap existing between the practice of simulation and the theory of statistical design and analysis. Independent of one another, all chapters contain exercises and references to the literature for further study. Coverage ranges from general statistical aspects of simulation, through variance reduction techniques that may have wide applicability in complicated simulation experiments, to experimental designs relevant in simulation and the effects of sample size or reliability. The book ends with an illuminating case study (a Monte Carlo experiment with the multiple ranking procedure). Designed for readers who have only a basic knowledge of mathematical probability and statistics, a good book indeed. I-CH

STATISTICS, T(13: 1), S*, *Understanding and Using Statistics, Basic Concepts.* Marty J. Schmidt. Heath, 1975, 361 pp, $5.95 (P). Divided into two main sections--descriptive statistics, including correlation and regression, and inferential statistics. Emphasizes fundamental concepts and interpretations rather than methods and techniques, so coverage is somewhat limited. However, what is covered is presented very well and would be good supplementary reading. RSK

STATISTICS, T*(13: 1), *Statistics, Meaning and Method.* Lawrence L. Lapin. HarBrace J, 1975, xiii + 591 pp, $12.95. Well-written text with many interesting examples and an attractive format. Includes chapters on analysis of variance, nonparametric statistics, and multiple regression and correlation. RSK

STATISTICS, T*(16: 2), *An Introduction to Mathematical Statistics, Third Edition.* H.D. Brunk. Xerox, 1975, xiv + 512 pp, $13.50. Major change is the replacement of a chapter on decision theory with two new chapters on Bayesian inference about binomial and normal parameters. A sound text, with many optional sections to provide flexibility. RSK

STATISTICS, T(14: 1), *Statistics for Technology.* Christopher Chatfield. Chapman & Hall, 1975, 359 pp, $6.95 (P). Paperback reprint of a 1970 Penguin book (TR, November 1970). Brief section on descriptive statistics, followed by a section on theory, which contains the usual topics through regression and correlation, and a final section on applications, including design and analysis of experiments, quality control and life testing. RSK

COMPUTER SCIENCE, P, *RPG for IBM Systems/360, 370, and System/3.* Richard F. Loschetter. P-H, 1975, ix + 452 pp, $20.50.

COMPUTER SCIENCE, T(13: 1), *Standard COBOL, Second Edition.* Mike Murach. SRA, 1975, xii + 433 pp, $9.95 (P). A modular approach excluding sort/merge, segmentation and report writer. RWN

COMPUTER SCIENCE, T(13: 1), *Data Processing.* Mike Murach. SRA, 1975, x + 416 pp, $9.95. The business data processing portion of *Business Data Processing and Computer Programming*, SRA, 1973. RWN

COMPUTER SCIENCE, T(14-15), L, *Introduction to the Theory of Computing.* Charles A. Kapps, Samuel Bergman. Merrill, 1975, viii + 344 pp, $14.95. Mostly discrete structures for an introductory course. Includes cardinality, Boolean algebra, logic, number theory, representations of numbers, graph theory, groups and computability. Some applications. Exercises. No bibliography. RWN

COMPUTER SCIENCE, P, *Computer Graphics, Techniques and Applications.* Ed: R.D. Parslow, R.W. Prowse, R. Elliot Green. Plenum Pr, 1975, xiii + 233 pp, $6.95 (P). Paperback edition of a 1969 publication containing papers from a 1968 conference in Uxbridge, England. Almost ancient history in this rapidly evolving field. LAS

COMPUTER SCIENCE, T(13-14: 1), S, *SNOBOL, An Introduction to Programming.* Peter R. Newsted. Hayden, 1975, 152 pp, $5.25 (P). An introductory text. Easy to read. Advanced features and details are relegated to appendixes. Exercises. RWN

COMPUTER SCIENCE, T**(16-18: 1), S, P, L, *The Design and Analysis of Computer Algorithms.* Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. A-W, 1974, x + 470 pp, $17.95. The emphasis is on efficiency--both practically and asymptotically. High-level descriptions of algorithms. First establishes several models of computation and introduces data structures. Algorithms for searching, sorting, integer and polynomial arithmetic, for manipulating matrices, strings and graphs. Common programming techniques. Establishes lower bounds on complexity. Discusses NP-complete problems. Excellent range of exercises. Bibliographic notes. RWN

COMPUTER SCIENCE, T(13: 1), *RPG With Business and Accounting Applications.* Stanley E. Myers. Reston, 1974, xvii + 462 pp, $13.95; $9.95 (P).

COMPUTER SCIENCE, S(15-18), P, L, *Stochastic Automata: Constructive Theory.* A.A. Lorenz. Trans: D. Louvish. Halsted Pr, 1974, vii + 174 pp, $25. Summary of author's research. Illustrates practical application of constructive mathematics. Considers finite and infinite automata. LH

COMPUTER SCIENCE, S(16), *Information, Computers, and System Design.* Ira G. Wilson, Marthann E. Wilson. Krieger, 1974, xx + 341 pp, $14.75. A reprint of the 1965 edition plus corrections. A discourse of the basics of system design. Good examples. Computational methods. Cost and reliability. RWN

COMPUTER SCIENCE, T(13), *Introduction to Computer Science, Short Edition.* C.W. Gear. SRA, 1973, xi + 338 pp, $9.50. The first seven chapters of the 1973 edition (TR, August 1973). Excludes the last two chapters on non-numerical applications and numerical methods. RWN

COMPUTER SCIENCE, P, *Lecture Notes in Computer Science-22: Formal Aspects of Cognitive Processes.* Ed: Thomas Storer, David Winter. Springer-Verlag, 1975, 214 pp, $9.50 (P). A representative selection of papers--ranging widely over sociology, linguistics, neurophysiology, mathematical logic--from an interdisciplinary conference at Ann Arbor, in March 1972. LAS

COMPUTER SCIENCE, T*(14-16: 1), L*, *Data Structures: Theory and Practice, Second Edition.* A.T. Berztiss. Comp. Sci. and Appl. Math. Acad Pr, 1975, xv + 586 pp, $15.96. Contains many additions and improvements to the well-accepted first edition (TR, April 1972). 200 new exercises. Good bibliography. Maintains its excellent balance between theory and practice. RWN

COMPUTER SCIENCE, P, *Structuri De Date si Sisteme Operative.* Teodor Rus. Editura Academiei Romania, 1974, 368 pp, Lei 24,50 (P).

SYSTEMS THEORY, S(16-18), P, L, *General Systems Theory: Mathematical Foundations.* M.D. Mesarovic, Yasuhiko Takahara. Math. in Sci. and Eng., V. 113. Acad Pr, 1975, xii + 268 pp, $20.

First steps in a "program aimed at formalizing all major systems, concepts and the development of an axiomatic and general theory of system." FLW

SYSTEMS THEORY, S(15-18), P. *Differential Games and Control Theory*. Ed: Emilio O. Roxin, Pan-Tai Liu, Robert L. Sternberg. Lect. Notes in Pure and Appl. Math., V. 10. Dekker, 1974, x + 412 pp, $24.50 (P). The subsidiary papers presented at the regional conference held at the University of Rhode Island, Kingston, June 1973. LH

SYSTEMS THEORY, T*(16-17: 2), S, P. *Introduction to Control Theory*. O.L.R. Jacobs. Oxford U Pr, 1974, xii + 365 pp, $21.75. In three major parts. I and II: deterministic linear (resp. non-linear) systems described by equations with (resp. without) analytic solutions and III: probabilistic systems with uncertainty. Features uniform treatment of both continuous-time and discrete-time systems and emphasizes dynamic programming and Bayes's rule as foundations for optimal and stochastic control theory. I-CH

SYSTEMS THEORY, T(18: 2), P. *Lecture Notes in Economics and Mathematical Systems-101: Linear Multivariable Control: A Geometric Approach*. W. Murray Wonham. Springer-Verlag, 1974, x + 344 pp, $12.30 (P). Treats familiar system concepts (controllability, observability, etc.) as geometric properties of distinguished state subspaces. Presents a geometric approach to the structural synthesis of multivariable control systems that are linear, time-invariant, and of finite dynamic order. Numerical examples and APL-program-tested computational procedures appear in the exercises. Main text tends to be theoretical and research-oriented. I-CH

APPLICATIONS (ARCHITECTURE), S*(13-16), P*, L*. *The Geometry of Environment: An Introduction to Spatial Organization in Design*. Lionel March, Philip Steadman. MIT Pr, 1974, 360 pp, $12.50 (P). Addressed to students and potential students of architecture, this splendid book introduces contemporary concepts of mathematics in an effort to comprehend the geometrical relationships in physical and spatial arrangements of buildings. Transformations and symmetry groups in the plane, matrices and vectors, modular spaces, mathematical description of shape, networks and graph theory, computer techniques in spatial allocation. In all cases the authors are very careful to provide illustrations from architectural practice for mathematical statements. PJC

APPLICATIONS (BIOLOGY). *The Mathematical Theory of the Dynamics of Biological Populations*. Ed: M.S. Bartlett, R.W. Hiorns. Acad Pr, 1973, xii + 347 pp, $21.50. Proceedings of a September 1972 Oxford conference. In five parts: population processes in time, in space, genetics, simulation and community structure. LAS

APPLICATIONS (BIOLOGY), P. *Mathematical Models for Cell Rearrangement*. Ed: G.D. Mostow. Yale U Pr, 1975, ix + 271 pp, $17.50. Translations of I.I. Pyatetskii-Shapiro's papers from Moscow State University, together with related papers by others, on mathematical models for kinetic and morphogenic features of cell sorting. LAS

APPLICATIONS (BIOLOGY), S**(15-18), P***, L***. *Structural Stability and Morphogenesis: An Outline of a General Theory of Models*. René Thom. Trans: D.H. Fowler. Benjamin, 1975, xxv + 348 pp, $13.50 (P); $22.50. Updated English translation of the 1972 French edition, "one of the most original contributions to the methodology of thought...since the first stirrings of quantum and relativity theories." A masterful blending of mathematical theory (primarily the classification of elementary catastrophes in differential topology), scientific detail (largely in developmental biology), and philosophic speculation concerning the succession of form. "The universe as we see it is a ceaseless creation, evolution, and destruction of forms, and the purpose of science is to foresee this change of form and, if possible, explain it." LAS

APPLICATIONS (BIOLOGY), P. *Mathematical Biofluiddynamics*. Sir James Lighthill. SIAM, 1975, ix + 281 pp, $21.50. Lecture notes on external (acquatic locomotion, animal flight, etc.) and internal (respiratory flow, blood flow, etc.) biofluiddynamics. SG

APPLICATIONS (BIOLOGY), P. *Mathematical Aspects of Heart Physiology*. Charles S. Peskin. New York U, 1975, iv + 278 pp, $7 (P). Differential equations are used to model various aspects of the cardiac functions, e.g., value action, murmurs, rate and rythm. Useful as an example of modelling techniques. TAV

APPLICATIONS (BUSINESS), T*(13-15: 2). *College Mathematics with Business Applications, Second Edition*. John E. Freund. P-H, 1975, xi + 667 pp, $13.95. Practical introduction to most quantitative concepts used in management. Easy to read. Assumes high school algebra. Includes lienar programming, calculus, decision theory, simulation. LH

APPLICATIONS (BUSINESS), T(14-16: 1), S. *Quality Control*. Richard C. Vaughn. Iowa St U Pr, 1974, xiv + 233 pp, $8.95. Blends elementary (freshman background) quantitative aspects with discussion of law and management problems. Covers management forecasting, control of errors, reliability, single and multiple acceptance sampling systems. LH

APPLICATIONS (CHEMISTRY), P. *The Mathematics of Diffusion, Second Edition*. J. Crank. Oxford U Pr, 1975, ix + 414 pp, $39. Deals mainly with mathematical aspects of diffusion, a process caused by random molecular motions. Has little mention of molecular mechanisms. Describes the diffusion processes in terms of solutions of the differential equations. Besides extensive revision, the second edition contains two entirely new chapters, one on anomalous diffusion, the other on diffusion in heterogeneous media. I-CH

APPLICATIONS (CHEMISTRY), P. *Reduced Density Operators with Applications to Physical and Chemical Systems-II*. Ed: R.M. Erdahl. Pure and Appl. Math., No. 40. Queen's U, 1974, x + 235 pp, $10 (P). Proceedings of the fourth RDO conference at Queen's University held in conjunction with a meeting of theoretical chemists in Ottawa in June 1974. LAS

APPLICATIONS (CONTROL THEORY), P, *Lecture Notes in Economics and Mathematical Systems-107: Control Theory, Numerical Methods and Computer Systems Modelling*. Ed: A. Bensoussan, J.L. Lions. Springer-Verlag, 1975, viii + 757 pp, $22.80 (P). Proceedings of a 1974 International Symposium at Rocquencourt sponsored by the Institut de Recherche d'Informatique et d'Automatique. LAS

APPLICATIONS (ECONOMICS), S(16-18), P, L, *Public Goods and Decentralization: The Duality Approach in the Theory of Value*. P.H.M. Ruys. Tilburg U Pr (Distr. by Academic Book Services, Groningen, The Netherlands), 1974, x + 236 pp, $23. Price theory for an economy with public and private goods. Equilibrium for decentralized economy. Requires mathematical sophistication--duality of sets, multifunctions, convex algebra. LH

APPLICATIONS (ECONOMICS), S*(15-18), P, L, *Interrelated Macro-Economic Systems*. P.J.L.M. Peters. Tilburg U Pr (Distr. by Academic Book Services, Groningen, The Netherlands), 1974, xv + 170 pp, $20. Concerned with multi-regional economic systems. Foreign variables not independent of domestic events. Simple mathematically--linear difference equations. LH

APPLICATIONS (ECONOMICS), S(14-16), P, L, *Cost of Living Index Numbers: Practice, Precision, and Theory*. Kali S. Banerjee. Statistics, V. 11. Dekker, 1975, xiii + 179 pp, $13.75. Practical and statistical aspects of these important index numbers. FLW

APPLICATIONS (ECONOMICS), P, *Lecture Notes in Economics and Mathematical Systems-89: Estimation of Product Attributes and Their Importances*. J.P. Wallace, A. Sherret. Springer-Verlag, 1973, v + 94 pp, $6.20 (P). A study of factors affecting choices in the transportation industry. LH

APPLICATIONS (ECONOMICS), P, *Optimal Investment Planning: A Reappraisal of Mahalonobis-Fel'dman Strategy*. R.K. Das. Rotterdam U Pr, 1974, xi + 146 pp, $19.80. With the background problem of investment planning in an economy with abundant labor but scarce capital resources, this book analyzes a non-linear optimization model qualitatively and numerically, using the techniques of variational calculus and optimal control theory. The author then interprets his results in economic terms. I-CH

APPLICATIONS (ECONOMICS), T(18: 1), P, *Analysis and Control of Dynamic Economic Systems*. Gregory C. Chow. Wiley, 1975, xv + 316 pp, $19.95. In the Wiley Series in Probability and Mathematical Statistics. Assumes background in econometric methods and matrix algebra. "Presents a set of related techniques for analyzing the properties of dynamic stochastic models [in discrete time] in economics and for applying these models in the determination of quantitative economic policy." RSK

APPLICATIONS (ECONOMICS), T(18: 2), *Econometrics: Statistical Foundations and Applications*. Phoebus J. Dhrymes. Springer-Verlag, 1974, xiv + 592 pp, $14.80 (P). "Corrected" reprint of the 1970 Harper and Row edition (TR, August-September 1972). RSK

APPLICATIONS (ENGINEERING), T?(15-16), P, L, *Aspects Modernes de la Fiabilité*. Dinkar Mukhedkar, Pierre Bretault, Gérard Sevestre. Pr U Montreal, 1974, 285 pp, $9.75 (P). S'addresse aux ingénieurs en fiabilités. Rappels mathématiques précèdent probabilité et statistiques developpés à un niveau suffisant pour les applications traitées: fiabilité des ensembles non-réparables, des systèmes complexes en fonction du temps, des systèmes de distribution d'énergie électrique, et des réseaux. Peu d'exemples, pas d'exercises. PJC

APPLICATIONS (ENGINEERING), P, *Random Processes, II: Poisson and Jump-Point Processes*. Ed: Anthony Ephremides. Halsted Pr, 1975, x + 352 pp, $24. One of the Benchmark series in EE offering an "expert's selection of the critical papers on a given topic as well as his views on its structure, development and present status." This volume contains 14 papers, all recent, each with the editor's comments. LAS

APPLICATIONS (ENGINEERING), T(16), *Computer-Aided Control System Design*. H.H. Rosenbrock. Acad Pr, 1974, xi + 230 pp, $19.75. Extends frequency response methods to multivariable regulators. Some of the methods are best suited for interactive computer graphics. Problems. RWN

APPLICATIONS (ENGINEERING), T(16-17), *Error Coding for Arithmetic Processors*. T.R.N. Rao. Acad Pr, 1974, xiv + 216 pp, $16.50. Arithmetically invariant codes for fault-tolerant computing. Includes background from algebra and number theory. Special arithmetic codes. Error correction.RWN

APPLICATIONS (FLUID DYNAMICS), P, *Numerical Methods in Fluid Dynamics*. Ed: C.A. Brebbia, J.J. Connor. Crane, Russak, 1974, 571 pp, $31.50. Twenty-eight papers on aerodynamics and hydrodynamics from the International Conference held at the University of Southampton, England, September 26-28, 1973. Strong emphasis on variational methods such as finite elements. I-CH

APPLICATIONS (HISTORY), P*, *Historian's Guide to Statistics: Quantitative Analysis and Historical Research*. Charles M. Dollar, Richard J. Jensen. Krieger, 1974, ix + 332 pp, $12. Reprint of a 1971 Hilt, Rinehart and Winston book. "A practical guide to the use of quantitative methods and computers in historical research." Extensive bibliography of sources of quantitative historical data and other pertinent references. RSK

APPLICATIONS (INFORMATION THEORY), T(15-16: 1), L, *Informationstheorie*. F. Topsøe. Teubner, Stuttgart, 1973, 88 pp, (P). Translated from the 1973 Danish edition. An introduction via entropy functions for moderately sophisticated students. JAS

Reviewers Whose Initials Appear Above

Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washingtion, D. C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Professor R. N. Bradt, The University of Kansas, represented the Association at the inauguration of Dr. G. C. Walker as President of Baker University on April 12, 1975.

Professor Robert Hogg, University of Iowa, represented the Association at the inauguration of Dr. Philip B. Secor as President of Cornell College on April 27, 1975.

Dr. Jack Alanen, formerly with SUNY at Buffalo, has been appointed Senior Staff Member, Informatics Group, at the University of Utrecht in The Netherlands.

Assistant Professor M. E. Camburn, Mercyhurst College, has been promoted to Associate Professor.

Dr. S. I. Gass has been appointed Professor and Chairman of the Faculty in Management Science and Statistics of the School of Business and Management, University of Maryland, College Park. He was formerly Vice President of the Mathtech Division of MATHEMATICA, Inc.

Dr. P. J. Hilton, a Fellow at the Battelle Seattle Research Center and Professor of Mathematics at Case Institute of Technology, has been awarded the Silver Medal for 1974 by the University of Helsinki, Finland, in recognition of his outstanding contributions to theoretical mathematics research and in appreciation for his collaboration with Finnish mathematicians.

Assistant Professor J. R. Hubbard, Columbus College, has been appointed Assistant Professor at Lycoming College.

Dr. E. H. Lieb, M. I. T., has been appointed Professor of Mathematical Physics at Princeton University, where he is currently teaching as a Visiting Faculty member while on leave from M. I. T.

Professor Bill Watson, Universidad Simon Bolivar, Sartenejas, Venezuela, has been appointed Assistant Professor at Case Western Reserve University.

Assistant Professor R. J. Weaver, Mount Holyoke College, has been promoted to Associate Professor.

Dr. William C. Chewning, Jr., University of South Carolina, died on March 23, 1975, at the age of 29. He was a member of the Association for six years.

Mr. Frank A. Downing, Portsmouth, Virginia, died on January 12, 1975. He was a member of the Association for forty-two years.

Dr. Dennis M. Nead, University of New Orleans, died on April 11, 1975, at the age of 57. He was a member of the Association for fourteen years.

VISITING LECTURER PROGRAM IN STATISTICS

The VISITING LECTURER PROGRAM IN STATISTICS is continuing into its thirteenth successive year. This year's program again is available to Canadian schools. The program is sponsored jointly by the principal statistical organizations in North America, the American Statistical Association, the Biometric Society, and the Institute of Mathematical Statistics. Partial support is also provided by International Business Machines Corporation. Leading teachers and research workers in statistics — from universities, industry, and government — have agreed to participate as lecturers. Lecture topics include subjects in experimental and theoretical statistics, as well as in such related areas as probability theory, information theory and stochastic models in the physical, biological and social sciences.

The purpose of the program is to provide information to students and college faculty about the nature and scope of modern statistics, and to provide advice about careers, graduate study, and college curricula in statistics. Inquiries should be addressed to: H. T. David, Visiting Lecturer Program in Statistics, Department of Statistics, Iowa State University, Ames, Iowa 50010.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Indiana University-Purdue University at Indianapolis, on Saturday, November 30, 1974, with approximately 60 persons in attendance. The Chairman of the Section, Maynard Thompson of Indiana University, presided.

S. Abhyanker of Purdue University gave the invited address, "Historical Ramblings in Algebraic Geometry and Related Algebra."

The following papers were presented:

1. *Cohomology with multiple-valued functions applied to fixed point theory*, by T. J. Worosz, Wabash College.
2. *On a new definition of the Lebesgue integral*, by Herman Rubin, Purdue University.
3. *On the improvement of convergence of certain series*, by A. K. Naghdi, IUPUI.
4. *Giffen's paradox or an exercise in multivariate calculus*, by David Bash, Purdue University, Fort Wayne.
5. *Irreducible factors of various polynomials*, by Clark Kimberling, University of Evansville.
6. *Abstract algebra and finite-state machines*, by Judith Gersting, IUPUI.
7. *Distribution theory — an example of the evolution of a mathematical concept*, by John Synowiec, Indiana University Northwest.
8. *On the transfer homomorphism*, by Larry Schiefelbusch, Indiana University Northwest.
9. *Extensions of Haar measure to a nonmeasurable subgroup of finite index*, by H. L. Peterson, Indiana University Northwest.
10. *Rest points, tangent circles, and the rational number line*, by R. T. Hood, Franklin College.

During the lunch break, the film, "Regular Homotopies in the Plane," was shown.

D. E. WILSON, *Secretary*

APRIL MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Purdue University, Fort Wayne, on Saturday, April 26, 1975, with approximately 45 persons in attendance. The Chairman of the Section, Maynard Thompson of Indiana University, presided.

The invited addresses were:

1. *Some new thoughts on teaching statistics*, by Meyer Dwass, Northwestern University.
2. *Langford sequences*, by D. P. Roselle, Virginia Polytechnic Institute.

The following papers were presented:

1. *Local degree computation by methods direct and devious*, by D. R. McCarthy, Purdue University, Fort Wayne.
2. *Networks as models for human behavior*, by R. D. Ringeisen, Purdue University, Fort Wayne.

L. J. Cote, Purdue University, awarded prizes to Laura Chihara, Highland High School, and R. A. Dwyer, Yorktown High School, for solving problems appearing in the Indiana School Mathematics Journal.

During the lunch break, the film, "Unsolved Problems (Victor Klee)," was shown.

At the business meeting, memberships in the MAA were awarded to J. H. Boyd III, Indiana University, and R. S. Gumerlock, Notre Dame, in recognition of their performances on the Putnam examination.

P. T. Mielke, Wabash College, gave the Governor's report, and Maynard Mansfield, Purdue University, Fort Wayne, was introduced as the new Governor of the Section.

Earl McKinney, Ball State University, as chairman of the Nominating Committee, presented the following slate of officers for 1975–76 (which was unanimously approved): Chairman, R. T. Hood, Franklin College; Vice-Chairman, M. C. Gemignani, Indiana University-Purdue University at Indianapolis; Secretary-Treasurer, D. E. Wilson, Wabash College.

D. E. WILSON, *Secretary*

APRIL MEETING OF THE IOWA SECTION

The 62nd regular meeting of the Iowa Section of the MAA was held at Iowa State University, Ames, on April 19, 1975. Chairman Donald Pilgram presided. Total attendance was 59, including 44 members of the section.

Following the contributed papers and invited lectures, Governor Muller's report was given and the business session held. The Visiting Lecture Program conducted by the Section was discussed and it was decided to continue the program on an informal basis.

James Cornette, Iowa State University, was elected Chairman-Elect, and B. E. Gillam, Drake University, was elected Secretary-Treasurer.

The program, arranged by Lawrence Hart, consisted of the following:

1. *Patterns in problem solving: a report on an NSF/AAAS Chautauqua-type short course*, by Joseph Schaefer, Loras College.
2. *Elementary proofs of Phragmen-Brouwer and related topological properties in euclidean spaces*, by Don Sanderson, Iowa State University.
3. *Unknotting knots and links in E^3 by maps*, by Howard Lambert, University of Iowa.
4. *How can articulation between two-year and four-year colleges be improved*?, by Merlin Fischer, Iowa Central Community College.
5. *Boundary value problems and functions of matrices*, by Walter Will, Decorah.
6. (Invited Lecture) *Problems in the mathematics curriculum and student placement*, by Betty Hinman, Downtown College, University of Houston.
7. *The bicentennial puzzle*, by D. Greenwell, Iowa State University.
8. *Historical development as a guide to mathematics education*, by J. L. Cornette, Iowa State University.
9. (Invited Lecture) *Optimal strategies for red-black casinos*, by Stuart Klugman, University of Iowa.

B. E. GILLAM, *Secretary-Treasurer*

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held April 26, 1975, at Madison College, Harrisonburg, Virginia. Seventy-six persons attended of whom sixty-six were members of the Association. Professor Geraldine Coon, chairman of the Section, presided.

During the morning there was a session of contributed papers and a short business meeting. In the business meeting Professor Richard Molloy, U.S. Naval Academy, was elected Vice-Chairman for Membership.

Following lunch, there were two additional sessions of contributed papers and a panel discussion. The members of the panel were Hilda R. Findley-Knier, Washington, D. C., R. M. Davis, Northern Virginia Community College, Jack Nachman, University of Maryland, and C. W. S. Ziegenfus, Madison College. Their topic was "Content and Presentation of Lower Division Mathematics."

The contributed papers presented were:

1. *Some computer generated non-singular matrices over the ring $Z/(m)$*, by W. M. Sanders, Madison College.
2. *Starlike and spiral-like functions*, by J. N. Walbert, Aberdeen, Maryland.
3. *Close-to-convex functions*, by R. J. Shores, Lynchburg College.
4. *Partial orderings of the group of integers*, by Russel Belding, U. S. Naval Academy.
5. *N-associative groupoids*, by W. P. Wardlaw, U. S. Naval Academy.
6. *Accelerating mathematics instruction for the mathematically talented*, by W. C. George, The Johns Hopkins University.
7. *Mathematical education for gifted female adolescents*, by Lynn H. Fox, The Johns Hopkins University.
8. *Casting out cultural artifacts in the pre-calculus curriculum*, by Hilda R. Findley-Knier, Washington, D.C..
9. *Time-shared interactive computer-controlled information television (TICCIT)*, by R. M. Davis, Northern Virginia Community Collge.
10. 2-*metric properties of* 2-*normed lattices*, by Ann Miller, University of Maryland, Eastern Shore.
11. *The pencil balanced on its point does not topple over*, by W. A. Barwick, College Park, Maryland.
12. *Introduction to geometric programming*, by Patrick Hayes, Analytical Services.
13. *Partitioning the reals into two additive subsets*, by George Crofts, Virginia Polytechnic Institute and State University.
14. *Several new space-filling polyhedra*, by Michael Goldberg, Washington, D.C..
15. *History of topological spaces and separation*, by C. E. Aull, Virginia Polytechnic Institute and State University.

J. M. SMITH, *Secretary*

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirty-fourth annual meeting of the Metropolitan New York Section of the MAA was held at Brooklyn College, CUNY, on April 20, 1975. One hundred and thirty-one persons registered.

Professor Israel Rose of Lehman College, CUNY, Chairman of the Section, presided at the morning session which began with the business meeting.

Professor Meyer Jordan of Brooklyn College, CUNY, Sectional Governor, reported on major items before the Board of Governors in the past year. In particular he noted that Professor William Zlot was elected Sectional Governor as of July 1, 1975. Committee chairmen then gave their reports as follows: Professor Erwin Just, Bronx Community College, CUNY, reported on the Section Committee for Two-Year Colleges; Professor William Zlot, New York University, reported on the Mathematics Services Committee; Professor Howard Kleiman, Queensborough Community College, CUNY, reported on the Speakers' Bureau; and Dr. Harry Ruderman, Hunter College High School, reported on the Greater Metropolitan New York Math Fair, and the MAA High School Contest.

The Section Award to the highest regional scorer in the Putnam Competition was presented to John P. Matelski of Columbia University.

The following Section officers were elected for 1975–77: Chairman, Gerald Freilich, Queens College, CUNY; Vice-Chairman for Four-Year Colleges, Maurice Nadler, Pace University; Vice-Chairman for Two-Year Colleges, Helen Siner, Staten Island Community College, CUNY; Vice-Chairman for High Schools, Harry Ruderman, Hunter College High School.

The morning session continued with the following lectures:

1. *Convex sets and proper values*, by Alan Hoffman, Graduate Center, CUNY, and IBM, Yorktown Heights.
2. *A significant but forgotten squabble on the foundations of modern probability theory*, by Albert Novikoff, Courant Institute, NYU.

Professor Gerald Freilich, Vice-Chairman for Four-Year Colleges, presided at the afternoon session which began with a panel discussion on *Applications of Group Theory.* Robert Bumcrot, Hofstra University, spoke on *Groups and Finite Geometries*; Leon Landovitz, Graduate Center, CUNY, spoke on *Some Uses of Groups in Theoretical Physics*; and Marvin Tretkoff, Stevens Institute of Technology, spoke on *Group Theory and Ornamentation.*

The following student and faculty papers were then given in parallel sessions:

1. *Evaluation of* $\int_0^\infty e^{x^2}\,dx$, by B. Lederman, Math Fair Winner, South Central H. S.
2. *A nonmeasurable subset of the reals*, by W. Page, New York City Community College, CUNY.
3. *A modern approach to Stokes' theorem*, by P. Huang, student, Princeton University.
4. *The Poisson transform*, by H. J. Hindin, Hunter College, CUNY.
5. *Practical numbers*, by H. Heller, Math Fair Winner, Cardoza H. S.
6. *Initially and terminally extendable sets of integers*, by I. Wolf, Brooklyn College, CUNY.
7. *Formalization of number theory by reduction of all meaningful statements to symbol strings*, by J. Propp, Math Fair Winner, Great Neck North H. S.
8. *Are there an infinite number of primes in the sequence* $10^n + 1$?, by M. Avidon, Math Fair Winner, Lafayette H. S.
9. *A solution to the baseball pool problem via coding theory*, by S. Cohen, New Jersey Institute of Technology.
10. *Binomial identities derived from games of chance*, by H. Schwesky, student, Queens College, CUNY.
11. *The small college math program*, by S. d'Ambra, Brooklyn College, CUNY.
12. *A mathematics teaching philosophy*, by C. Sutton, Manhattan Community College, CUNY.
13. *A theorem on two tetrahedra*, by L. Gerber, St. John's University, Jamaica, N.Y.
14. *How not to lose a root of a quadratic equation*, by A. Todd, Brooklyn College, CUNY.
15. *Balancing chemical equations mathematically*, by G. Gruber, student, Brooklyn College, CUNY.

RORA IACOBACCI, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The annual Spring meeting of the Missouri Section of the MAA was held at Missouri Western State College, St. Joseph, Missouri, on April 18 and 19, 1975. A total of 90 registered for this meeting.

The officers for 1975–76 are as follows: Chairman, J. R. Downing, Southwest Missouri State University; Vice-Chairman, F. W. Wilke, University of Missouri-St. Louis; Secretary-Treasurer, H. K. Stumpff, Central Missouri State University; Past-Chairman, Jerry Wilkerson, Missouri Western State College.

Except for invited addresses, dual sessions were conducted throughout the meeting. The following presentations were made at the Friday afternoon session:

1. *Examples of problem solving*, by Richard Friedlander, University of Missouri-St. Louis.
2. *An experimental project to increase women in the sciences*, by Barbara A. Currier, University of Missouri-Kansas City.
3. *nth root groups*, by R. E. Kennedy, Central Missouri State University.
4. *Fixed point theorems*, by Troy Hicks, University of Missouri-Rolla.
5. *Every finite group is the automorphism group of some finite orthomodular lattice*, by Gerald Schrag, Central Missouri State University.
6. *On the nature of applied mathematics* (invited address), by H. O. Pollak, President of MAA, Bell Laboratories.
7. *Planetarium Show and Geometry Film show.*
8. *Instructional materials on applied mathematics* (invited banquet speaker), by Maynard Thompson, Indiana University.

The Saturday morning session consisted of a Sectional Business meeting and the following presentations:

9. *Patterns of wrong response in elementary calculus*, by C. A. Johnson, University of Missouri-Rolla.
10. *An attempt to answer the question: Should students be required to earn C or better in prerequisite mathematics courses?*, by J. W. Joiner, University of Missouri-Rolla.
11. *An application of Pfaffians to a physical problem: the Dimer problem*, by Carolyn T. MacDonald, University of Missouri-Kansas City.
12. *Bernstein's theorem in a DSC-POLA*, by Edward Davenport, Central Missouri State University.
13. *The limits of quantitative methods in history* (invited address), by R. W. Fogel, Professor of Economics, The University of Chicago.

H. K. STUMPFF, *Secretary-Treasurer*

APRIL MEETING OF THE NEBRASKA SECION

The fifty-first annual meeting of the Nebraska Section of the MAA was held on April 19, 1975, at Nebraska Wesleyan University in Lincoln. There were 65 persons present of whom 40 were members of the Association.

Officers for 1975–76 were elected as follows: Chairman, Professor L. M. Larsen Kearney State College; Chairman Elect, Professor S. D. Luke, Nebraska Wesleyan University; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska-Lincoln; Contest Chairman, Professor L. J. Stephens, University of Nebraska-Omaha. Professor R. D. Anderson represented the MAA and brought greetings to the group.

The following papers were presented:

1. *An inner-connection system between a mini-computer and a large time share computer network*, by Charles Sedlacek, University of Nebraska-Omaha.
2. *A combinatorial algorithm for solving the multiple assignment problem*, by C. A. Combs, Jr., University of Nebraska-Omaha.
3. *Adapt—a classroom response to Piaget's theory of intellectual development*, by M. C. Thornton, University of Nebraska-Lincoln.
4. *Singular integrals*, by C. P. Downey, University of Nebraska-Omaha.
5. *The behavior of a function of a random number*, by Y. L. Tong, University of Nebraska-Lincoln.
6. *The twenty-sixth annual high school mathematics examination*, by H. M. Cox, University of Nebraska-Lincoln.
7. *Mathematics contest honor roll students*, by H. M. Cox, University of Nebraska-Lincoln (by title).
8. *The Nebraska-South Dakota annual high school mathematics contest*, by L. J. Stephens, University of Nebraska-Omaha.
9. *Summability of power series by means of product $B \cdot S(r, t)$ method of summability transform*, by S. D. Luke, Nebraska Wesleyan University.
10. (Invited Address) *Infinite dimensional intuition*, by R. D. Anderson, Louisiana State University.
11. *A new method for finding the parameters in single and double acceptance sampling plans*, by L. J. Stephens, University of Nebraska-Omaha.
12. *The latitude of forms: a fourteenth century precursor of calculus*, by Neil Smith, University of Nebraska-Omaha.
13. *The equivalence of the euclidean parallel postulate and the Pythagorean theorem*, by J. S. Downing, University of Nebraska-Omaha.

14. *Alternatives to Taylor's theorem in proving analyticity*, by J. A. Eidswick, University of Nebraska-Lincoln.
15. (Invited Address) *Distributions and their uses*, by G. H. Meisters, University of Nebraska-Lincoln.
16. *What is spectral analysis?*, by C. F. Masters, Doane College.
17. *Branching, bounding, and stopping*, by G. F. Haddix, University of Nebraska-Omaha.

H. M. Cox, *Secretary-Treasurer*

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The fifty-eighth Annual Meeting of the Rocky Mountain section of the MAA was held on the campus of Mesa College, April 11 and 12, 1975. There were 108 registrants including D. C. Benson, Governor of the Section, and J. C. Davis, Chairman of the Section. The invited address was presented by Professor Ivan Niven, University of Oregon, First Vice President of the Association. Dr. Carl Wahlberg, Acting President of Mesa College, welcomed the Section after the banquet Friday evening.

The following papers were contributed and read on the program:

1. *Contractive mappings on complete metric spaces*, By John Ausink and Douglas James, USAF Academy.
2. *Survey of approximations to pi*, by David Ballew, South Dakota School of Mines and Technology.
3. *Lower math and what you can do with it*, by R. J. Bitts, Arapahoe State College.
4. *Accessibility of the boundary of a plane region*, by C. E. Burgess, University of Utah.
5. *An upper bound on the dimension of the space of solutions A of the matrix equation*, by Bruce Collings, Brigham Young University.
6. *Introductory statistics: a module-course demanding student responsibility and student performance*, by W. P. Cooke, University of Wyoming.
7. *What are the opportunities for a B.S. degree with a math major*, by Allan David, University of Utah.
8. *Exploiting some ideas in computing to teach mathematics*, by W. S. Dorn, Denver University.
9. *Limiting distributions for immigration branching processes with decomposable mean matrix: the strictly critical case*, by J. H. Foster, Denver University.
10. *Applying the "shepherd's principle,"* by R. A. Gibbs, Fort Lewis College.
11. *A generalization of certain corresponding continued fractions*, by John Gill, Southern Colorado State College.
12. *Existence, uniqueness and continuability of solutions of second order functional differential equations*, by Gary Grefsrud, Fort Lewis College.
13. *Characterizations of strictly convex normed linear spaces*, by Stan Gudder, Denver University.
14. *Models and misfits*, by J. R. Hanna, University of Wyoming.
15. *Geometry as mathematics: an approach to undergraduate geometry*, by Z. R. Hartvigson, Colorado University at Denver.
16. *Matric analogues of number theory*, by E. E. Hasz, Metropolitan State College.
17. *Las Vegas: springboard to mathematics study — fact or fiction?*, by L. S. Johnson, Fort Lewis College.
18. *For to think metric: an operational model for teacher re-education*, by Bob Kansky, University of Wyoming.
19. *Dividing the area of a circle into two parts in the ratio $s:t$ by certain curves*, by Hung C. Li, Southern Colorado State College.
20. *Mastery learning — consider it*, by M. D. McClenahan, University of Wyoming.
21. *On sigma-ideals of sets*, by C. G. Mendez, Metropolitan State College.
22. *Projectile targets revisited, or — can I hit it twice?*, by Roger Opp, South Dakota School of Mines and Technology.
23. *Programmed instruction*, by L. M. Orman, Southern Colorado State College.
24. *Mini-courses for mathematics education*, by A. D. Porter, University of Wyoming.
25. *The generalization of a Putnam problem*, by W. C. Ramaley, Fort Lewis College.
26. *Ulm's theorem without addition*, by Laurel Rogers, Colorado University.
27. *Watson's lemma in several variables*, by D. M. Rognlie, South Dakota School of Mines and Technology.
28. *Estimates for factorial effects in constrained randomization models*, by R. L. Schwaller, South Dakota School of Mines and Technology.
29. *Plane coloring problems*, by Leslie Shader, University of Wyoming.
30. *Mathematical analysis of computer system performance*, by Don Warner, Mesa College.
31. *A qualitative study of ordinary differential equations at the U. S. Air Force Academy*, by Major Williams, USAF Academy.

In addition to the above papers, a textbook exhibit was presented with the assistance of numerous publishers.

D. J. Sterling, *Secretary*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at Glendale Community College, Glendale, Arizona, on April 11 and 12, 1975. Twenty-seven members and twenty-five guests registered their attendance.

The invited speaker was the retiring national secretary, H. L. Alder, University of California, Davis. He gave a talk after the banquet on Friday night entitled, "Mathematics for Liberal Arts Students," and also spoke the next morning on "Recent Developments in the Theory of Partition Identities."

The following papers were contributed:

1. *On normal radical extensions of the rationals*, by W. Y. Velez, University of Arizona.
2. *Differentiable points of the generalized Cantor function*, by T. P. Dence, New Mexco State University.
3. *Cosine triples*, by C. G. Moore, Northern Arizona University.
4. *A problem in probability involving Fibonacci numbers*, by J. H. Butchart, Northern Arizona University.
5. *Examples of non-commutative C* algebras*, by M. L. Morgan, New Mexico Institute of Mining and Technology.
6. *The axiomatics of diagrams*, by R. A. Knoebel, New Mexico State University.
7. *Working with the metric system*, by R. M. Lutz, Western New Mexico University.
8. *Coordinate free construction of finite projective geometries*, by A. Swimmer, Arizona State University.
9. *Surficial embedments of structures*, by D. Soderberg, Phoenix, Arizona.
10. *What's a field?*, by F. Richman, New Mexico State University.
11. *Kronecker's theorem revisited*, by R. Mines, New Mexico State University.

The Chairman of the session on contributed papers was R. D. Meyer of Northern Arizona University. Betty Field of Glendale Community College acted as meeting coordinator.

A. SWIMMER, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The fifty-fourth annual meeting of the Illinois Section was held on the campus of Rockford College, Rockford, Illinois, Friday and Saturday, May 9–10, 1975, with 110 members and guests in attendance. Chairman Robert Bryan of Knox College presided. Dr. G. E. Wesner, Vice-president and Dean of the College, welcomed ISMAA on behalf of the college.

During the two-day session the following lectures were given:

Ramanujan's Notebook, by Bruce Berndt, University of Illinois.

An approach to metric education for elementary teachers, and *A summer computer institute for high school juniors*, by Warren Burstrom, Highland Community College.

Elegance and applications in complex variables, by Linda Sons, Northern Illinois University.

Individualized mathematics programs-complete freedom, by Carole Bauer, Triton College.

A non-synthetic approach to transformation geometry, by Richard Millman, Southern Illinois University—Carbondale.

Is set theory really necessary? by Mary Ellen Rudin, University of Wisconsin-Madison.

Dr. A. B. Willcox, Executive Director of MAA, was an honored guest and banquet speaker at the Friday evening dinner. His topic was "Some Bridges to and from Mathematics" and was enjoyed by all in attendance.

The annual business meeting was held on Friday afternoon and officers for 1975–1976 include: Chairman: Professor J. M. Laible, Eastern Illinois University; 1st Vice-chairman: Professor Walter McCurdy, Bradley University; 2nd Vice-chairman: Professor Darrell Clevidence, Carl Sandburg College.

H. C. SAAR, *Secretary-Treasurer*

MAY MEETING OF THE OHIO SECTION

The Ohio Section of the MAA held its annual Spring meeting at Bowling Green State University, May 2 and 3, 1975. One hundred-ninety people attended the meeting. Chairman Louis Green presided; Richard Little was Program Chairman.

The following invited addresses were presented: *Some Bridges to and from Mathematics*, by Dr. A. B. Willcox, Executive Director, MAA; *Computer Evaluation of Functions*, by Dr. W. J. Cody, Argonne National

Laboratories; and *Numerical Approximation for Data Driven Ocean Circulation Models*, by Professor George Fix, University of Michigan.

The following contributed papers were presented:

On the parity of a permutation, by L. D. Rodabaugh, University of Akron.

New problems in Markov matrices arising fom recent results in ergodic and information theory, by P. C. Shields, University of Toledo.

An application of geography to mathematics, by V. F. Rickey, Bowling Green State University.

The ring of quotients of a Boolean ring, by H. L. Putt, Bowling Green State University.

Logic, form, and simplicity, by A. A. Johanson, University of Toledo.

Review of computerized instruction in mathematics at the University of Akron, by J. J. Hirschbuhl and P. J. Gingo, University of Akron.

Stochastic processes with independent increments taking values in a Hilbert space, by J. L. Spielman, Bowling Green State University.

Stochastic formulation in the solution of a partial differential equation-iterative vs. exact solutions, by P. J. Gingo, University of Akron.

Manipolds, manitolds, and manilolds, by Janet D. Blair, Bowling Green State University.

Two arithmetical functions associated with K-free and K-full integers, by D. Suryanarayana and R. Sita Rama Chandra Rao, University of Toledo.

Assignments with limited duplication, by S. F. Barger, Youngstate State University.

Computer graphics from the Carleton workshop, by Clifford Long, Bowling Green State University.

Connectedness: topological and uniform, by H. L. Bentley, University of Toledo.

Three non-linear oscillators in resonance, by Martin Kummer, University of Toledo.

Optimization, numerical analysis, and the banana function, by Junior Stein, University of Toledo.

Fluid flow in curved tubes, by H. W. Vayo, University of Toledo.

The program also included the following swap sessions: *Effective Learning with Compact Video-cassettes*, led by Professor Judah Rosenblatt, Case-Western Reserve University; *A Proposed Resolution from the Department of Mathematics of Miami University*, led by Professors Robert Bullock and Fred Gass, Miami University; *The Minicalculator in the Mathematics Classroom*, led by Professor H. P. Allen, The Ohio State University; and *A Summary of the U. S. Metric Study on Education*, led by Professor Louis Ross, University of Akron.

Also on the agenda were the annual business meeting of the Section, a meeting of the Executive Board of the Section, and meetings of the ad hoc committees: Committee on Cooperation between Colleges and Universities, Committee on Curriculum, and Committee on Teacher Training and Certification.

The Officers for 1975–76 àre: R. G. Laatsch, Miami University, Chairman; J. A. Murtha, Marietta College, Chairman-Elect; Gus Mavrigian, Youngstown State University, Secretary-Treasurer; R. S. Varga, Kent State University, Program Chairman; Marion Wetzel, Denison University and J. H. Carney, Lorain County Community College, Program Committee.

R. H. ROLWING, *Secretary-Treasurer*

MAA WORKSHOP ON MODULES IN APPLIED MATHEMATICS

The Mathematical Association of America will sponsor a four week College Faculty Workshop on Modules in Applied Mathematics. This activity is supported by a grant from the National Science Foundation, and is scheduled to meet at Cornell University in Ithaca, New York, from July 26 to August 20, 1976. About twenty-four participants and six staff members will prepare educational materials suitable for courses at various undergraduate levels and illustrating a variety of different uses of mathematics. The goal is to produce a collection of relatively self-contained units demonstrating current, relevant and interesting applications which involve all aspects of the applied mathematical methodology. These units can be embedded in existing courses or grouped in different ways to form the basis for new courses on modeling. Many of the modules will be designed for a more open-ended or student-oriented mode which may include students working in rather self-directed project groups. Anyone interested in additional information should contact the workshop director: Professor W. F. Lucas, 334 Upson Hall, Cornell University, Ithaca, NY 14853.

The deadline for applications to participate is December 31, 1975.

REPORT OF THE TREASURER FOR THE YEARS 1973 AND 1974

Herewith is a summary of the report of the Treasurer of the Association for the years 1973 and 1974. In this summary, all entries have been rounded to the nearest dollar; therefore sums of entries may differ from the entered total. The full report has been approved by the Finance Committee and accepted by a vote of the Board of Governors. Any member of the Association who wishes to have a copy of the full report may obtain one by writing to the Washington Office of the Association.

Assets	*Dec. 31, 1972*	*Dec. 31, 1973*	*Dec. 31, 1974*
Cash	$ 67,711	$ 47,685	$ 53,222
Securities (at cost), unrestricted	213,954	213,954	235,818
Securities (at cost), restricted	151,543	151,010	151,010
Accounts Receivable	20,657	26,191	41,470
Furniture and Equipment	18,847	17,808	17,328
Prepaid Expenses	4,065	4,850	33,285
Two-Year College Mathematics Journal	. ...	. ...	26,800
Total Assets	$ 476,778	$ 461,498	$ 558,934
Liabilities			
Accounts Payable	$ 15,957	$ 33,648	$ 22,053
Unearned Income			
Dues	142,515	199,910	199,790
Subscriptions	49,779	52,162	87,449
Other	1,596	1,573	6,421
Prindle-Weber-Schmidt Advertising Account	. ...	. ...	26,800
NSF Fund	27,460	12,147	18,736
High School Contest Fund	27,306	27,099	24,215
Total Liabilities	$ 264,614	$ 326,540	$ 385,464
Assets minus Liabilities (Net Worth, including restricted funds)	$ 212,164	$ 134,958	$ 173,470

Operating Income	*1973*	*1974*
Dues	$ 227,548	$ 298,266
Publications	171,740	215,060
Dividends and Interest	19,625	21,540
Gift Establishing the Dolciani Fund	. ...	20,000
Other Contributions	20,258	16,768
Registrations Fees	6,653	9,193
Indirect Costs — NSF	23,764	20,758
Indirect Costs — High School Contest	5,788	6,000
Miscellaneous	1,358	3,155
Total Operating Income	$ 476,734	$ 610,739

Operating Expenditures	*1973*	*1974*
Salaries	$ 214,636	$ 229,615
Office Expense	101,922	108,306
Publications	160,532	167,689
Travel	46,127	33,675
Taxes	9,950	10,890
Dues and Contributions	18,448	19,902
Awards and Grants	2,325	2,150
Total Operating Expenditures	$ 553,940	$ 572,228
Operating Income over (under) Operating Expenditures	($ 77,206)	$ 38,512

LEONARD GILLMAN, *Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Valparaiso University, Valparaiso, November 15, 1975.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., November 22, 1975.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Simmons College, Boston, Mass., November 29, 1975.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

OHIO, Otterbein College, Westerville, November 7–8, 1975.

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Franklin and Marshall College, Lancaster, November 22, 1975.

ROCKY MOUNTAIN, Fort Lewis College, Durango, Colorado, May 1–2, 1976.

SEAWAY, College of St. Rose, Albany, NY, April 30-May 1, 1976.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A&M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 22–27, 1976.

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MU ALPHA THETA

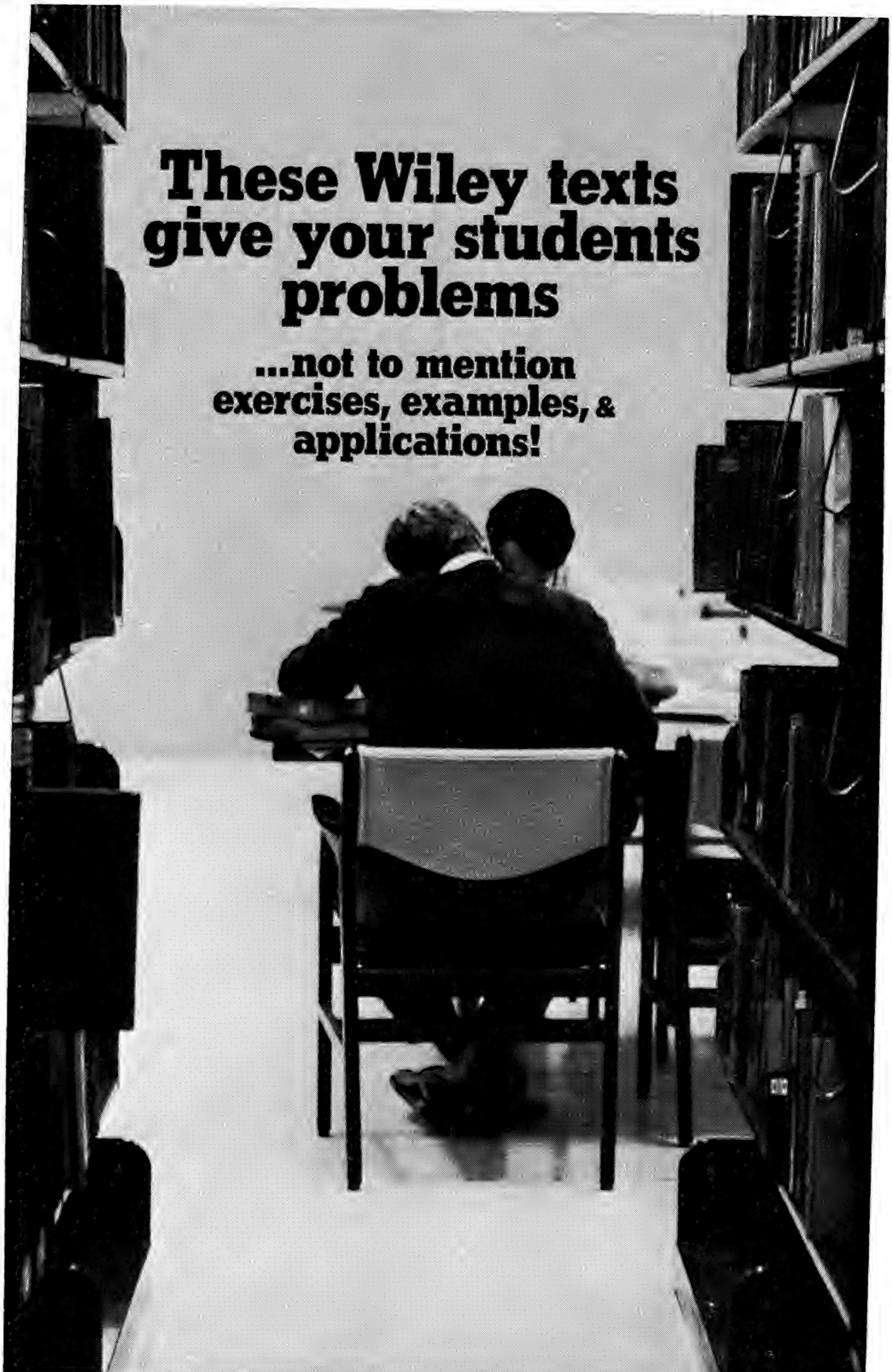
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, MGM Grand Hotel, Las Vegas, November 17–19, 1975.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton O'Hare, Chicago, November 6–8, 1975.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Sheraton-Palace Hotel, San Francisco, December 3–5, 1975 (SIAM-SIGNUM 1975 Fall Meeting).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A continuation of NML 5 containing problems and solutions from the Annual High School Mathematics Contests for the period 1961-1965.

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by W. G. Chinn and N. E. Steenrod
The development of topology and some of its simple applications. The power and adaptability of topology are demonstrated in proving so-called existence theorems.

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The purpose of this book is to revisit elementary geometry, using modern techniques such as transformations, inversive geometry, and projective geometry to facilitate geometric understanding and link the subject with other branches of mathematics.

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by Oystein Ore
A highly readable introduction to number theory that explains why the properties of numbers have always held such fascination for man. Problems, with solutions, allow students to find number relationships of their own.

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by I. M. Yaglom translated by Allen Shields
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by Abraham Sinkov
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INGENUITY IN MATHEMATICS (NML-23)
by Ross Honsberger
Nineteen independent essays that reveal elegant and ingenious approaches used in thinking about such topics in elementary mathematics as number theory, geometry, combinatorics, logic and probability.

GEOMETRIC TRANSFORMATIONS III (NML-24)
by I. M. Yaglom translated by Abe Shenitzer
This part of Yaglom's work (sequel to NML-08 and NML-21) treats affine and projective transformations, and introduces the reader to non-Euclidean geometry in a supplement to hyperbolic geometry. As in the previously published parts of his work, Yaglom focuses on problems and their detailed solutions and keeps the text brief and simple.

THE MAA PROBLEM BOOK III (NML-25)
A continuation of NML-05 and NML-17, containing problems and solutions from the Annual High School Mathematics Contests for the period 1966-1972.

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THE AMERICAN MATHEMATICAL MONTHLY

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PROBLEMS SOLVED

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ANATOMY OF THE ORDINARY DIFFERENTIAL EQUATION

W. T. REID

1. Introduction. The study of ordinary differential equations dates from the time of the development of calculus during the last third of the seventeenth century. Indeed, some authors, (see, for example, Ince [**37**, p. 529]), have assigned the genesis of this subject to November 11, 1675, when Leibnitz wrote the relation $\int y dy = \frac{1}{2}y^2$, which may properly be considered the solution of a simple differential equation. As one of the basic types of functional equations that has wide occurrence in the study of phenomena, both within and without the domain of mathematics per se; it is the purpose of the present discussion to examine critically some of the component aspects of ordinary differential equations, and hence the use of the term "anatomy" in this purely mathematical context.

The main theme of the following discussion is the study of the genesis of the correlated concepts of a differential equation and a solution. Beginning with the more intuitive concept of solution, through the equivalence of an initial value problem for a differential equation and an integral equation, one is led quite naturally in Sections 2 and 3 to the consideration of differential equations involving functions which satisfy hypotheses of "Carathéodory type." The study of differential equations in this context initiates the idea of solution in the sense of a Schwartz distribution, and interrelations with the fundamental lemma of the calculus of variations. In turn, this variational association is a precursor to the consideration in Section 4 of a type of generalized differential equation that is equivalent to a Riemann-Stieltjes integral equation, and for which one has direct extensions of the classical Sturmian theory for real second order linear differential equations. Section 5 is devoted to a brief survey of generalized differential equations of so-called "contingent" type, which within recent years have been studied extensively in connection with problems of "control theory." The latter part of this section is concerned with a technique of Filippov, that is one of the very few methods for treating a differential equation $y'(t) = f(t, y(t))$ wherein the function f is discontinuous in y. Finally, in this section there is described a particular simple optimal control problem involving approximation by monotone functions, and whose extremizing function is the solution in the Filippov sense of a differential system. Section 6 is concerned with a description of some results on differential equations in abstract spaces, notably in Banach spaces.

The class of problems considered in the first five sections may be described loosely as belonging to two categories. A major portion of the discussion is concerned with the "Cauchy initial value problem," dealing with the existence and uniqueness of solution of a differential equation passing through given initial data. A historical prototype of this category of problems is the determination of trajectories of a dynamical system assuming at an initial time prescribed position and momentum. The second category is that of boundary problems wherein the restraints involve solution elements at more than one value. A classic example of this category is that of determining trajectories of a dynamical system which pass through two prescribed positions at different times. Another such example is that of specifying the set of eigenvalues $\lambda_n = n^2\pi^2/L^2$, $(n = 1, 2, \cdots)$ and associated orthonormalized eigenfunctions $y_n(x) = \sqrt{2/L}\sin\sqrt{\lambda_n}\,x$ of the two-point boundary problem

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = 0 = y(L).$$

This particular boundary problem occurs in the solution of the one-dimensional wave equation $\partial^2 u/\partial t^2 = a^2 \partial^2 u/\partial x^2$ in the problem of the transverse vibrations of a string with fixed ends. In the general setting of Section 6 the above described class of problems is no longer subject to strict dichotomy. In particular, whenever the Banach space is a space X of numerically-valued functions, or of classes of functions defined on a given abstract set S, then depending upon the particular space X various boundary conditions may be prescribed for all considered elements of X, and Cauchy initial value problems frequently involve in some fashion the solution of a partial differential equation.

In its entirety, the present discussion is limited to deterministic differential equations. For treatment of the important fields of stochastic differential equations and functional differential equations, the reader is referred to such works as Itô [**38**], Balakrishnan [**3**], Arnold [**1**], McShane [**52**], Warga [**63**], Hale [**29**], and their Bibliographies.

Although the appended Bibliography is fairly extensive, it is not intended to be highly comprehensive, and there is no pretense that all significant aspects of the subject are represented therein. Undoubtedly, the author has overlooked some very relevant papers of which he is cognizant, and unfortunately others of which he is not aware. To the authors of all such papers, regrets are extended herewith.

2. Existence theorems. One of the best-known existence theorems for ordinary differential equations is due to Peano [**54**], and states that if $f\colon D \to \mathbf{R}$ is a real-valued continuous function on an open domain D of the real (t, y)-plane, and $(\tau, \eta) \in D$, then there exists an $\varepsilon > 0$ such that on the interval $[\tau - \varepsilon, \tau + \varepsilon]$ there exists a solution $y = y(t)$ of the differential equation

$$y'(t) = f(t, y(t)) \tag{2.1}$$

which passes through the initial point (τ, η); that is,

$$y(\tau) = \eta, \tag{2.2}$$

and has graph $\{(t, y(t))\colon t \in [\tau - \varepsilon, \tau + \varepsilon]\}$ in D. In many elementary texts on differential equations this result is stated, although frequently the reader is referred to more advanced books for proof. A somewhat more sophisticated result is that each such solution admits extension to a solution y with maximal interval of existence (a, b), and that the graph of such an extended solution tends to the boundary of D as $t \to a$ and as $t \to b$, (see, for example, Hartman [**31**; §§2, 3 of Ch. II] or Reid [**59**; §3 of Ch. I]).

Now these results are in accord with the intuitive idea of "solution of a differential equation," obtained by plotting an aggregate of line segments through chosen points (t_0, y_0) with slope $f(t_0, y_0)$, and envisioning a solution as an enveloping curve of such lineal elements. However, the student in a beginning course on differential equations is soon confronted with examples showing that the solution of (2.1) passing through an initial point (τ, η) need not be unique. In some examples there is a one-sided uniqueness, as for $y' = (3/2)y^{1/3}$; in this example a solution passing through an initial point $(\tau, 0)$ is uniquely specified as $y(t) \equiv 0$ on $(-\infty, \tau]$, but on $[\tau, \infty)$ the solution may be $y(t) \equiv 0$, or $y(t) \equiv 0$ on (τ, c) where $\tau \leqq c < \infty$, and $y(t) = (t - c)^{3/2}$ for $c \leqq t < \infty$. In other examples, such as $y' = 3y^{2/3}$, for an initial point $(\tau, 0)$ there is non-uniqueness of solution on both $(-\infty, \tau]$ and $[\tau, \infty)$. It is to be noted that examples have been given of functions f which are continuous throughout the whole (t, y)-plane, and such that every point of the plane fails to be a point of even one-sided uniqueness. The first such example was given by Lavrientieff [**45**]; a greatly simplified example of this phenomenon has been given by Hartman [**30**; **31**, §5 of Ch. II]. Thus in order to insure local uniqueness of solution of a differential equation (1.1) some condition besides continuity is required, and as is well known this property is assured by certain types of conditions restricting the behavior of f as a function of y. In particular, one of the most frequently used conditions is that of a Lipschitz condition. A function $f\colon D \to \mathbf{R}$ is said to be *locally Lipschitzian* in y if for each $(\tau, \eta) \in D$ there exists a neighborhood $\bar{N}_\delta(\tau, \eta) = \{(t, y)\colon |t - \tau| \leqq \delta, |y - \eta| \leqq \delta\}$ of (τ, η) in D, and a (Lipschitz) constant k such that

$$|f(t, y) - f(t, z)| \leqq k|y - z|, \quad \text{for} \quad (t, y) \quad \text{and} \quad (t, z) \quad \text{in} \quad \bar{N}_\delta(\tau, \eta). \tag{2.3}$$

In the above comments attention has been restricted to continuous functions f, and one may rightly ask if such a condition is really pertinent for the topic under consideration. After all, given a function $f\colon D \to \mathbf{R}$, should not a "solution" of (2.1) passing through a point $(\tau, \eta) \in D$ be merely a function $y\colon I \to \mathbf{R}$, where I is a non-degenerate interval containing τ, with graph in D and such that

at each $t \in I$ the function y has a derivative $y'(t)$ which satisfies (2.1)? Using this criterion, however, one soon discovers that certain "pathologies" may exist. For example, let f be defined on the whole (t, y)-plane as $f(t, y) = a(t)y$, where

$$a(t) = 2/t^3 \quad \text{for} \quad t > 0; \quad a(t) = 0 \quad \text{for} \quad t \leqq 0. \tag{2.4}$$

If $y_0(t) = \exp\{-1/t^2\}$ for $t > 0$, $y_0(t) = 0$ for $t \leqq 0$, then $y_0(t)$ is continuous and has continuous derivatives of all orders on $(-\infty, \infty)$. Also, for arbitrary constants k the function $y(t) = ky_0(t)$, $-\infty < t < \infty$, is a solution of the corresponding differential equation (2.1). Thus $(0,0)$ is a point of local non-uniqueness, and each of the above described solutions has maximal interval of existence equal to $(-\infty, \infty)$, so as far as these solutions are concerned, no phenomenon is exhibited that has not occurred in the earlier examples. On the other hand, for $\tau < 0$ and $\eta \neq 0$ there is a unique solution passing through the point (τ, η), namely $y(t) \equiv \eta$; however, the maximal interval to which this solution may be extended is $(-\infty, 0]$, so solutions of this equation fail to have the extensibility property mentioned above for the case of continuous f. It is to be remarked that for the function $f(t, y) = a(t)y$ defined by (2.4) the only points of the plane at which f fails to be both continuous and locally Lipschitzian are the points on the line $t = 0$. An extension of the above example is given by the system

$$\begin{aligned} v'(t) &= c(t)u(t) - a(t)v(t), \\ u'(t) &= a(t)u(t) + b(t)v(t) \end{aligned} \tag{2.5}$$

in (t, u, v)-space, where the function $a(t)$ is defined by (2.4), $b(t) \equiv 0$, while $c(t)$ is a continuous function on $(-\infty, \infty)$ which is identically zero on $[0, \infty)$. One may verify that the only solutions of (2.5) existing on an open interval containing $\tau = 0$ are of the form $u(t) = ky_0(t)$, $v(t) \equiv 0$. It is to be emphasized that these comments do not mean that this definition of "solution" of (2.1) should be discarded, but only that when this definition is used one need be careful to either prove the existence of solutions on a given interval, or to specifically postulate the existence of such solutions. For a discussion of a treatment of results corresponding to some of those of the classical Sturmian theory under such a definition of solution, the reader is referred to Diaz and McLaughlin **[21]**.

For simplicity in the above discussion, consideration of specific differential equations has been limited to a single scalar equation, except for the above system (2.5). This practice will be continued, although the discussion of the following sections is equally valid in finite dimensional spaces.

3. Differential versus integral equation. Returning to the case of a continuous $f\colon D \to \mathbf{R}$, it is to be recalled that at an early stage in considering equation (2.1) it is noted that if $y(t)$, $t \in [\tau - \varepsilon, \tau + \varepsilon]$, is a continuous function whose graph is in D, and which has at each t on this interval a derivative $y'(t)$ which satisfies (2.1), then $f(t, y(t))$ is continuous and for points t_1, t_2 of $[\tau - \varepsilon, \tau + \varepsilon]$ we have

$$y(t_2) = y(t_1) + \int_{t_1}^{t_2} f(s, y(s))\, ds. \tag{3.1}$$

In particular, if this solution passes through the point (τ, η), then

$$y(t) = \eta + \int_{\tau}^{t} f(s, y(s))\, ds, \quad \text{for} \quad t \in [\tau - \varepsilon, \tau + \varepsilon]. \tag{3.2}$$

Now the integral of (3.2) is meaningful if $y\colon [\tau - \varepsilon, \tau + \varepsilon] \to \mathbf{R}$ is any continuous function with graph in D, since for such y the function $f(t, y(t))$ is continuous on $[\tau - \varepsilon, \tau + \varepsilon]$. Indeed, the integral $\int_\tau^t f(s, y(s))\, ds$ defines a continuously differentiable function on $[\tau - \varepsilon, \tau + \varepsilon]$, whose derivative is equal to $f(t, y(t))$ at each t on this interval. Moreover, if such a y satisfies (3.2) then y must be continuously differentiable on this interval and $y(\tau) = \eta$. That is, whenever f is a continuous function on D the condition that $y(t)$ be a continuously differentiable function on $[\tau - \varepsilon, \tau + \varepsilon]$

which satisfies (2.1) and the initial condition $y(\tau)=\eta$ is equivalent to the condition that y be a continuous function on $[\tau-\varepsilon, \tau+\varepsilon]$ with graph in D, and which satisfies the integral equation (3.2).

Now if $\square=\{(t,y): |t-\tau|\leqq\varepsilon, |y-\eta|\leqq\delta\}\subset D$, the continuity of f on D implies the existence of constant μ such that $|f(t,y)|\leqq\mu$ for $(t,y)\in\square$, and if $\varepsilon_0=\text{Min}\{\varepsilon,\delta/\mu\}$ then for $y:[\tau-\varepsilon_0,\tau+\varepsilon_0]\to\mathbf{R}$ a continuous function with graph in $\square$ we have that $|\int_\tau^t f(s,y(s))\,ds|\leqq\delta$ for $t\in[\tau-\varepsilon_0,\tau+\varepsilon_0]$. Consequently, if $\mathfrak{M}_\square$ denotes the class of continuous functions $y:[\tau-\varepsilon_0,\tau+\varepsilon_0]\to\mathbf{R}$ then

$$z(t)=\eta+\int_\tau^t f(s,y(s))\,ds,\quad t\in[\tau-\varepsilon_0,\tau+\varepsilon_0], \tag{3.3}$$

is also a function of the class $\mathfrak{M}_\square$, so that the condition that $y(t)$, $t\in[\tau-\varepsilon_0,\tau+\varepsilon_0]$, be a solution of (2.1) is equivalent to the condition that y be a "fixed point" of the transformation $T:\mathfrak{M}_\square\to\mathfrak{M}_\square$ defined by $(T_y)(t)=z(t)$ as given by (3.3). It is to be emphasized that the proof of an existence theorem for (2.1) subject to the initial condition (2.2) depends upon properties of the thus defined transformation T on a class of functions large enough to include solutions of this equation passing through (τ,η), but in general includes a much larger class of functions.

The equivalence of the concept of a solution of (2.1), (2.2) with the concept of a solution of the integral equation (3.2) immediately raises the possibility of extending the concept of a solution by way of this integral equation. This was done by Carathéodory [**12**; Ch. XI], by considering the integral to be in the Lebesgue sense, so that the validity of equation (2.1) provides merely the local absolute continuity of $y(t)$ and the existence almost everywhere of a derivative $y'(t)$ which satisfies (2.1). The consideration of solutions of (2.1) in this sense may then be carried out under the following hypotheses, which are called "conditions of Carathéodory type": *$f: D\to\mathbf{R}$ is continuous in y for fixed t, locally measurable in t for fixed y, and for $(t_0,y_0)\in D$ there exists a rectangular neighborhood*

$$\bar{N}_\delta(t_0,y_0)=\{(t,y): |t-t_0|\leqq\delta, |y-y_0|\leqq\delta\}\subset D$$

and a Lebesgue integrable function $\lambda(t)$ on $[t_0-\delta,t_0+\delta]$ such that $|f(t,y)|\leqq\lambda(t)$ for $(t,y)\in\bar{N}_\delta(t_0,y_0)$. Among other advantages, certain types of discontinuities in $f(t,y)$ are automatically taken care of. For example, if $f(t,y)=a(t)y$, where the function $a(t)$ is Lebesgue integrable on arbitrary compact intervals of the real line, then the general solution of $y'(t)=a(t)y(t)$ is $y(t)=k\exp\{\int_\tau^t a(s)\,ds\}$, so that there is local uniqueness of solution and extensibility of solutions to the whole real line.

Now the realization that under the Carathéodory hypotheses a solution y of (2.1) on an interval $[a,b]$ is an absolutely continuous function such that $r(t)=y(t)-\int_a^t f(s,y(s))\,ds$ is identically equal to a constant, which must necessarily be $y(a)$, enables one to present an integral criterion equivalent to this condition. Let $\mathfrak{D}_0^1[a,b]$ denote the class of real-valued functions $\phi:[a,b]\to\mathbf{R}$ which are continuous, have piecewise continuous derivatives on this interval, and $\phi(a)=0=\phi(b)$. The simplest form of the so-called Fundamental Lemma of the Calculus of Variations, (see, for example, Bliss [**5**; Ch. I, §§2,3]), then states that the constancy of r on $[a,b]$ is equivalent to the integral condition $\int_a^b\phi'(s)r(s)ds=0$ for all $\phi\in\mathfrak{D}_0^1[a,b]$, and in turn an integration by parts yields the equivalent condition

$$J_1[\phi]\equiv\int_a^b\{\phi'(s)y(s)+\phi(s)f(s,y(s))\}\,ds=0\quad\text{for}\quad\phi\in\mathfrak{D}_0^1[a,b]. \tag{3.4}$$

That is, y is a solution of (2.1) on an interval $[a,b]$ if and only if $J_1[\phi]$ is equal to zero for all "test functions" $\phi\in\mathfrak{D}_0^1[a,b]$. This class of test functions is much larger than actually needed to establish that y is a solution of (2.1). For a given positive integer m, let $\mathfrak{C}_0^m(a,b)$ denote the class of functions $\phi:(a,b)\to\mathbf{R}$ which have continuous derivatives of the first m orders, and the support of ϕ, defined as the closure of the set $\{t: t\in(a,b),\ \phi(t)\neq 0\}$, is a compact subset of (a,b). By a relatively simple

extension of the argument yielding the Fundamental Lemma, (see, for example, Reid [**59**; Ch. III, §2, Probs. 1–5]), one may show that y is a solution of (2.1) if and only if y is a continuous function for which there exists a positive integer m such that $J_1[\phi]=0$ for all $\phi \in \mathfrak{C}_0^m(a,b)$. Indeed, it is not difficult to show that y is a solution of (2.1) on $[a,b]$ if and only if $J_1[\phi]=0$ for all ϕ belonging to $\mathfrak{C}_0^\infty(a,b)$, the class of functions $\phi\colon (a,b)\to \mathbf{R}$ which have continuous derivatives of all orders on $[a,b]$, and whose support is a compact subset of the open interval (a,b).

Now for K an arbitrary compact subset of (a,b), let $\mathfrak{D}_K$ denote the class of functions $f\in \mathfrak{C}_0^\infty(a,b)$ whose support lies in K, and for $f\in\mathfrak{D}_K$ and $m=0,1,\cdots$ let $p_{K,m}(f)$ denote the supremum of $|f^{[s]}(t)|$ for $s\leqq m$, $t\in K$. Then the collection $p_{K,m}(f)$ is a family of semi-norms on $\mathfrak{C}_0^\infty(a,b)$, and the family of continuous linear functionals on the topological space $\mathfrak{C}_0^\infty(a,b)$, as topologized by this family of semi-norms, is the family of *distributions* in the sense of L. Schwartz [**62**]. In particular, if $\xi\colon (a,b)\to\mathbf{R}$ is Lebesgue integrable then $T_\xi\phi=\int_a^b\phi(s)\xi(s)\,ds$ defines a distribution. Also, $S_\xi\phi=\int_a^b\phi'(s)[-\xi(s)]\,ds$ defines a distribution, whose value is equal to $\int_a^b\phi(s)\xi'(s)\,ds$ if ξ is locally absolutely continuous on (a,b). Consequently, on the space of distributions S_ξ is called the *derivative* of the distribution T_ξ and we write $S_\xi=T_\xi^{[1]}$. In general, for $m=1,2,\cdots$ the integral $\int_a^b\phi^{[m]}(s)[(-1)^m\xi(s)]\,ds$ defines a distribution, which is called the mth derivative of T_ξ and denoted by $T_\xi^{[m]}$. Also, if g has continuous derivatives of all orders on $[a,b]$, then $g\phi\in\mathfrak{C}_0^\infty(a,b)$ whenever $\phi\in\mathfrak{C}_0^\infty(a,b)$, and if T is a distribution then $T(g\phi)$ defines a distribution T_1 such that $T_1\phi=T(g\phi)$ for $\phi\in\mathfrak{C}_0^\infty[a,b]$, and we write $T_1=gT$.

In view of the above discussion, we have that y is a solution of (2.1) on $[a,b]$ if and only if $T_{f(\cdot,y(\cdot))}=T_{y(\cdot)}^{[1]}$. As a further comment on the relation of the concept of a solution to the theory of distributions, consider the linear differential expression, or formal differential operator, $L[u]=\sum_{j=0}^m a_j(t)u^{[j]}(t)$, where each coefficient function a_j has continuous derivatives of all orders and $a_m(t)\neq 0$ on (a,b). The formal adjoint differential expression $L^*[v]=\sum_{j=0}^m(-1)^j\{a_j(t)v(t)\}^{[j]}$ then satisfies the Green identity

$$vL[u]-L^*[v]u=\{B[u;v]\}',$$

where B is a bilinear form in $u,u',\cdots,u^{[m-1]},v,v',\cdots,v^{[m-1]}$, and from this it follows (see, for example, [**59**; Ch. III, §3]), that u is a function of class $\mathfrak{C}_0^m(a,b)$ which satisfies $L[u]=0$ on this interval if and only if u is a continuous function on (a,b) such that $\int_a^b L^*[v]u\,dt=0$ for all $v\in\mathfrak{C}_0^m(a,b)$ and in turn this condition is equivalent to the vanishing of this integral for all $v\in\mathfrak{C}_0^\infty(a,b)$. That is, u is a solution of $L[u]=0$ on $[a,b]$ if and only if u is continuous on $[a,b]$ and $\sum_{j=0}^m(-1)^j\int_a^b[a_j(s)\phi(s)]^{[j]}u(s)\,ds=0$ for $\phi\in\mathfrak{C}_0^\infty(a,b)$, which in terms of distributions may be stated as follows: *If $u\colon[a,b]\to\mathbf{R}$ is continuous, then u is a solution of $L[u]=0$ on $[a,b]$ if and only if the distribution $T=T_u$ satisfies the distributional differential equation $\sum_{j=0}^m a_jT^{[j]}=0$.*

4. Sturmian theory for a type of generalized differential equation. Suppose that r and p are piecewise continuous real-valued functions on $[a,b]$, and $r(t)$ is positive on this interval. A function u is said to be a solution of the differential equation

$$L[u]\equiv -[r(t)u'(t)]'+p(t)u(t)=0 \tag{4.1}$$

on $[a,b]$ if $u(t)$ is absolutely continuous and there exists an absolutely continuous function $v(t)$ such that almost everywhere on $[a,b]$ we have $r(t)u'(t)=v(t)$ and $-v'(t)+p(t)u(t)=0$. That is, u is called a solution of (4.1) if this function and the associated function v is a solution in the Carathéodory sense of the system

$$-v'(t)+p(t)u(t)=0,\quad u'(t)-[1/r(t)]v(t)=0. \tag{4.2}$$

Now it is well known, (see, for example, [**59**; Ch. V, VI]), that much of the qualitative nature of solutions of (4.1) results from the fact that this equation is the so-called "Euler differential equation"

for the integral

$$I[\eta]=\int_a^b \{r(t)[\eta'(t)]^2+p(t)\eta^2(t)\}\,dt, \tag{4.3}$$

which is quadratic on the class $\mathfrak{D}$ of functions $\eta\colon [a,b]\to \mathbf{R}$ which are absolutely continuous and whose derivative is square integrable in the sense of Lebesgue on $[a,b]$. Indeed, the fundamental properties of such solutions persist when the above hypotheses on r and p are replaced by the condition that r, $1/r$ and p are measurable and essentially bounded on $[a,b]$, and u is called a solution of (4.1) on $[a,b]$ when there exists a corresponding v such that (u,v) is a solution of (4.2) in the Carathéodory sense. Now suppose that r and p satisfy these latter conditions, and m is a real-valued function of bounded variation on $[a,b]$. Then

$$I_1[\eta]=\int_a^b \{r(t)[\eta'(t)]^2+p(t)\eta^2(t)\}\,dt+\int_a^b dm(t)\,\eta^2(t) \tag{4.4}$$

is also a quadratic functional on the class $\mathfrak{D}$, and the corresponding functional equation corresponding to the Euler differential equation for (4.3) may be written as

$$-dv(t)+p(t)u(t)\,dt+[dm(t)]u(t)=0,\quad u'(t)-[1/r(t)]v(t)=0, \tag{4.5}$$

where now a solution (u,v) of (4.5) on $[a,b]$ is by definition an absolutely continuous function u and a function of bounded variation v such that $u'(t)-[1/r(t)]v(t)=0$ almost everywhere on $[a,b]$, while the other equation of (4.5) is understood to hold as the Riemann-Stieltjes integral equation

$$v(t)=v(\tau)+\int_\tau^t p(s)u(s)\,ds+\int_\tau^t u(s)\,dm(s),\quad \text{for}\quad (t,\tau)\in[a,b]\times[a,b]. \tag{4.6}$$

Such generalized differential equations have been considered by various authors, (see, for example, Feller [**24**], Guggenheimer [**28**], Kac and Krein [**39**], Reid [**57**], and references cited in these papers). Also, as shown in [**57**] for a vector system in n-dimensional vector functions, which contains (4.5) as a special case for $n=1$, the classical comparison and oscillation theorems of the Sturmian theory, (see, for example, [**59**; Chs. V, VI]), hold for the generalized differential system (4.5). Such generalized differential equations include as special instances certain differential systems with "interface" conditions at a finite number of points, and also certain systems of second order difference relations. In particular, suppose $m(t)=s(t)+\int_a^t q(s)\,ds$, where the function q is bounded and Lebesgue integrable on $[a,b]$, and $s(t)$ is a step function; that is, there is a finite sequence of points $a=t_0<t_1<\cdots<t_k=b$ such that $s(t)$ has the constant value s_α on the open interval $(t_{\alpha-1},t_\alpha)$, $(\alpha=1,\cdots,k)$. In this case (u,v) is a solution of (4.5) on $[a,b]$ if and only if (u,v) is a solution of the ordinary differential system

$$-v'(t)+[p(t)+q(t)]u(t)=0,\quad u'(t)-[1/r(t)]v(t)=0, \tag{4.7}$$

on each subinterval $(t_{\alpha-1},t_\alpha)$, while the right- and left-hand limits of these functions satisfy the interface conditions

$$\begin{aligned} &u(t_\beta)=u(t_\beta^-)=u(t_\beta^+),\\ &v(t_\beta)-v(t_\beta^-)=[s(t_\beta)-s(t_\beta^-)]\,u(t_\beta),\\ &v(t_\beta^+)-v(t_\beta)=[s(t_\beta^+)-s(t_\beta)]\,u(t_\beta),\quad (\beta=0,1,\cdots,k), \end{aligned} \tag{4.8}$$

with the understanding that at $t_0=a$ and $t_k=b$ only the appropriate one-sided relations of (4.8) are to be considered. In particular, if $r(t)$ is equal to a positive constant r_α on $(t_{\alpha-1},t_\alpha)$, and $q(t)\equiv -p(t)$, then (u,v) is a solution of (4.7), (4.8) on $[a,b]$ if and only if u is the continuous polygonal function whose graph joins the successive points $(t_\beta,u(t_\beta))$, $(\beta=0,1,\cdots,k)$, and the

values $u(t_\beta)$ satisfy the linear second order difference relations

$$r_{j+1}\frac{u(t_{j+1})-u(t_j)}{t_{j+1}-t_j}-r_j\frac{u(t_j)-u(t_{j-1})}{t_j-t_{j-1}}-\sigma_j u(t_j)=0,\quad (j=1,\cdots,k-1),$$

where $\sigma_j = s(t_j^+)-s(t_j^-)$, $(j=1,\cdots,k-1)$, while $v(t)=r_\alpha[u(t_\alpha)-u(t_{\alpha-1})]/[t_\alpha-t_{\alpha-1}]$ on $(t_{\alpha-1},t_\alpha)$, $v(t_\alpha)=[s(t_\alpha)-s(t_\alpha^-)]u(t_\alpha)+r_\alpha[u(t_\alpha)-u(t_{\alpha-1})]/[t_\alpha-t_{\alpha-1}]$ for $\alpha=1,\cdots,k$, and $v(a)=[s(a)-s(a^+)]u(a)+r_1[u(t_1)-u(a)]/[t_1-a]$.

The fact that solutions of a system (4.5) possess properties analogous to those of solutions of systems of ordinary differential equations (4.2) is not surprising since one may show that (u,v) is a solution of (4.5) if and only if $(\hat{u},\hat{v})=(u,v-mu)$ is a solution of the ordinary differential system

$$-\hat{v}'+[p(t)-m^2(t)/r(t)]\hat{u}-[m(t)/r(t)]\hat{v}=0,$$
$$\hat{u}'-[m(t)/r(t)]\hat{u}-[1/r(t)]\hat{v}=0.$$

No further discussion of such generalized differential systems will be presented here, and for the theory of such equations in the more general vector case the reader is referred to results of [**57**; 59, Ch. VII], and references there given for related literature. In passing, it is to be noted that recently one of the author's students, William Denny, II, in [**19**], has extended earlier results of the author in the n-dimensional vector case to a class of pairs of Riemann-Stieltjes integral equations which are not reducible to a system of ordinary differential equations. In particular, for $n=1$ Denny's results provide such extension to systems

$$-dv(t)+p(t)u(t)\,dt+[dm(t)]u(t)=0,\quad du(t)-[dh(t)]v(t)=0,$$

where p and m are functions satisfying the conditions given above and $h(t)$ is a continuous monotone non-decreasing function on $[a,b]$.

5. Contingent equations. Let Δ be an open region in $(1+m)$-dimensional (t,y) space, and $F\colon \Delta\to\mathcal{P}(\mathbf{R}^m)$ be a function whose values are non-empty compact sets in real m-dimensional space $\mathbf{R}^m$. If $(\tau,y^0)\in\Delta$, the system

$$y'(t)\in F(t,y(t)),\qquad y(\tau)=y^0, \tag{5.1}$$

is an "initial value problem" which reduces to the ordinary differential equation case whenever $F(t,y)$ is a single point in $\mathbf{R}^m$ for each $(t,y)\in\Delta$. A solution of (5.1) is by definition an absolutely continuous m-dimensional vector function $y\colon I\to\mathbf{R}^m$, where I is a non-degenerate interval containing τ, and such that almost everywhere on I the derivative vector $y'(t)$ belongs to the set $F(t,y(t))$. This type of generalized differential equation arises in various manners. For example, if $r\colon I\times\mathbf{R}^m\to\mathbf{R}$ and $h\colon I\times\mathbf{R}^m\to\mathbf{R}^m$ are given, where I is a non-degenerate interval on $\mathbf{R}$, and $F(t,y)=\{z\colon z\in\mathbf{R}^m, |z-h(t,y)|\leqq r(t,y)\}$, then a function $y\colon I\to\mathbf{R}^m$ is a solution of (5.1) if and only if $y(t)$ is absolutely continuous on I, with $y(\tau)=y^0$, and the inequality $|y'(t)-h(t,y)|\leqq r(t,y)$ holds almost everywhere on I. Equations of the form (5.1) appear in the theory of control systems. One of the simplest such occurrences may be described by a differential system

$$y'(t)=f(t,y(t),u(t)),\qquad y(\tau)=y^0, \tag{5.2}$$

where the "control" function $u(t)$ may be chosen as an arbitrary measurable k-dimensional vector function with value at time t in a given set $U(t)\subset\mathbf{R}^k$. Whenever the function $f(t,y,u)$ and the sets $U(t)$ satisfy certain conditions, the subsets $F(t,y)=\{f(t,y,u)\colon u\in U(t)\}$ are non-empty compact subsets of $\mathbf{R}^m$ and the concept of a solution pair $y(t)$, $u(t)$ of (5.2) reduces to the above defined solution of (5.1). For general references on optimal control theory and the calculus of variations, the reader is referred to Hestenes [**34**], Lee and Markus [**46**], and Young [**67**].

Recent advances in the mathematical theory of control have provided great impetus to the study of equations of the form (5.1). It is to be noted, however, that problems equivalent to the one

formulated above have been the object of study for four decades. In the early 1930's, Marchaud [**49**; **50**] and Zaremba [**68**] generalized the notion of an ordinary differential equation by considering at an arbitrary point in m-dimensional space an aggregate of possible tangent directions defining a family of trajectories. The aggregate of possible tangent directions was termed "contingent," and the corresponding equation a "contingent equation." If $y(t)$, $a \leqq t \leqq b$, is an m-dimensional vector function, then for $t_0 \in [a, b]$ the "contingent" or "contingent derivative" of $y(t)$ at t_0 is defined as the set of all m-dimensional vectors v for which there exists a sequence of values $t_n \in [a, b]$ distinct from t_0, convergent to t_0, and such that

$$\lim_{n \to \infty} (1/[t_n - t_0])\{y(t_n) - y(t_0)\} = v;$$

the contingent derivative of $y(t)$ at t_0 is denoted by $(D^*y)(t_0)$. If Δ is a set in $(1+m)$-dimensional (t, y)-space, and $F\colon \Delta \to \mathscr{P}(\mathbf{R}^m)$ is a function whose values are sets in real m-dimensional space $\mathbf{R}^m$, then

$$(D^*y)(t) \subset F(t, y(t)), \qquad t \in [a, b], \tag{5.3}$$

is a contingent equation, and a solution of such an equation is a function $y(t)$, $t \in [a, b]$, whose contingent at each point $t_0 \in [a, b]$ lies in the given set $F(t_0, y(t_0))$. In order to establish results on the existence and properties of such solutions, one needs to require certain conditions on the set-valued function F, just as for the classical differential equation (2.1) certain conditions are imposed on the point-valued function f.

If y and z are two points of a metric space $\mathfrak{M}$, let $d(y, z)$ denote the distance between y and z. If S is a set in $\mathfrak{M}$ and $y \in \mathfrak{M}$, then the distance $d(y, S)$ between y and S is defined as the infimum of $d(y, z)$ for $z \in S$. For two sets S_1, S_2 of $\mathfrak{M}$ the "separation of S_1 from S_2" is defined as the supremum of $d(y, S_2)$ for $y \in S_1$, and denoted by $d^*(S_1, S_2)$. Clearly in general $d^*(S_1, S_2)$ and $d^*(S_2, S_1)$ are not equal. However, the greater of these two values is a symmetric function of sets; it is called the Hausdorff distance between S_1 and S_2, and denoted by $d(S_1, S_2)$. The aggregate of all non-empty compact subsets of $\mathbf{R}^m$, with topology induced by the Hausdorff metric, is a complete metric space, (see, for example, Dieudonné [**23**; p. 58, problem 3]). If S is a subset of $\mathfrak{M}$, and $\delta > 0$, then the δ-neighborhood $N_\delta(S)$ is the set of all $x \in \mathfrak{M}$ satisfying $d(x, S) < \delta$. In general, if $\Gamma\colon T \to \mathscr{P}(\mathfrak{M})$ is a mapping from a topological space T to the subsets of $\mathfrak{M}$, and $\tau_0 \in T$, then Γ is said to be continuous at τ_0 if for each $\delta > 0$ there is some neighborhood $N(\tau_0)$ such that $d(\Gamma(\tau), \Gamma(\tau_0)) < \delta$ if $\tau \in N(\tau_0)$; Γ is said to be upper {lower} semi-continuous at τ_0 if for each $\delta > 0$ there is some neighborhood $N_1(\tau_0)$ such that $d^*(\Gamma(\tau), \Gamma(\tau_0)) < \delta$, $\{d^*(\Gamma(\tau_0), \Gamma(\tau)) < \delta\}$ if $\tau \in N_1(\tau_0)$. Clearly Γ is continuous at τ_0 if and only if it is both upper and lower semi-continuous at τ_0. Moreover, if $\mathfrak{M}$ is a region in $(1+m)$-dimensional (t, y)-space and for each $(t, y) \in \mathfrak{M}$ the corresponding Γ is a singleton set, denoted by $f(t, y)$, then clearly upper semi-continuity of Γ at a particular value (t_0, y^0) is equivalent to continuity of the vector function f at (t_0, y^0).

Now suppose that $(\tau, y^0) \in [a, b] \times \mathbf{R}^m$, and that on some closed neighborhood $\bar{N}$ of (t_0, y^0) there is defined a mapping $F\colon \bar{N} \to K^m$, where K^m denotes the collection of non-empty, compact and convex subsets of $\mathbf{R}^m$. In [**68**] Zaremba showed that if F is such a mapping that is upper semi-continuous in (t, y) on $\bar{N}$, then there exists at least one solution of (5.1) satisfying the initial condition $y(\tau) = \eta$, and this solution may be continued until reaching the boundary of $\bar{N}$. That is, Zaremba generalized the basic Peano existence and continuation theorems to contingent equations.

The early work of Marchaud and Zaremba was highly geometric in nature. Under the assumption that the sets $F(t, y)$ are compact, however, a solution of the contingent equation (5.3) is locally Lipschitzian, and therefore a relatively smooth vector function. Indeed, as Ważewski pointed out in [**64**], under the assumed hypotheses of Zaremba's theorem, the condition that $y(t)$ be continuous and satisfy $(D^*y)(t) \subset F(t, y(t))$ for all t on the compact interval $[a, b]$ is equivalent to the condition that y be absolutely continuous and $y'(t) \in F(t, y(t))$ almost everywhere on $[a, b]$. Consequently,

most of the work subsequent to Ważewski's paper has been in character more analytical than geometric, and largely concerned with generalizations of known results for the more classical ordinary differential equations. No attempt is made to survey the extensive literature of this area in recent years, but a partial list of worthwhile references is presented by the papers of Filippov [**25**, **26**], Roxin [**61**], Castaing [**13**], Kikuchi [**40**], Bridgland [**8**], Hermes [**33**], Davy [**18**], and Warga [**63**, Chapter VI] of the appended Bibliography.

In closing this section, however, attention will be directed to the method presented in Filippov [**25**] for the determination of a specific type of set function $F(t,y)$. Suppose that $f(t,y)$ is a real-valued function defined almost everywhere on an open or closed domain Q in $(1+m)$-dimensional (t,y)-space, and that if $(t_0,y^0)\in Q$ then the section $Q_{t_0}=\{y\colon (t_0,y)\in Q\}$ is such that for an arbitrary δ-neighborhood $N_\delta(y^0)$ of y^0 the set $Q_{t_0}\cap N_\delta(y^0)$ has positive m-dimensional Lebesgue measure. Moreover, suppose that f is a Lebesgue measurable function of (t,y) on Q, and that for arbitrary closed subsets S of Q there exists a locally Lebesgue integrable function $\lambda_S(t)$ such that $|f(t,y)|\leqq \lambda_S(t)$ almost everywhere on S. Now for $(t,y)\in Q$, let $K\{f(t,y)\}$ denote the smallest convex closed set of $\mathbf{R}^m$ containing essentially all the values of the vector function $f(t,y)$ as y ranges over almost all the δ-neighborhoods of the value y; specifically,

$$K\{f(t,y)\} = \bigcap_{\delta>0}\ \bigcap_{\text{meas } S=0} \overline{\text{co}}\, f(t, N_\delta(y)-S), \tag{5.4}$$

where $\overline{\text{co}}$ denotes the closed convex hull of the designated set. A vector function $y(t)$ defined on a non-degenerate interval I, is then called a *solution in the sense of Filippov* [**25**] of

$$y'(t)=f(t,y(t)), \qquad t\in I, \tag{5.5}$$

if y is locally absolutely continuous on I and

$$y'(t)\in K\{f(t,y(t))\}, \quad \text{for } t \text{ almost everywhere on } I. \tag{5.6}$$

As a simple example illustrating solutions in the sense of Filippov, for $m=1$ let $f(t,y)=-1$ for $|y|\leqq 1$, $f(t,y)=1$ for $|y|>1$. Then $K\{f(t,y)\}=f(t,y)$ if $y\neq -1,1$, while $K\{f(t,1)\}$ and $K\{f(t,-1)\}$ are each equal to the closed interval $[-1,1]$. Now if $y_0(t)$ is the solution of (5.5) in the classical sense which passes through the point $(0,0)$, then its maximal interval of existence is $[-1,1]$, and $y_0(t)=-t$ on this interval. When "solution" is taken in the Filippov sense however, the solution $y(t)$ of this equation passing through $(0,0)$ has maximal interval of existence $(-\infty,\infty)$, with $y(t)=y_0(t)=-t$ on $[-1,1]$, $y(t)=1$ on $(-\infty,-1]$, and $y(t)=-1$ on $[1,\infty)$. Moreover, it is clear that the solution in the Filippov sense passing through the point $(-1,1)$ is not unique. Besides the solution $y(t)$ already specified, for $c\geqq -1$ there are also solutions $y_c^+(t)$, $y_c^-(t)$ and $y_\infty(t)$ given by: $y_c^+(t)=1$ for $t\leqq c$, $y_c^+(t)=1-c+t$ for $t\geqq c$; $y_c^-(t)=1$ for $t\leqq c$, $y_c^-(t)=1+c-t$ for $c\leqq t\leqq c+2$, $y_c^-(t)=-1$ for $t\geqq c+2$; $y_\infty(t)\equiv 1$ for $-\infty<t<\infty$. In the Filippov sense the maximal solution passing through $(-1,1)$ is $y_{-1}^+(t)$, the minimal solution is $y_{-1}^-(t)$, and the right-hand funnel of solutions passing through $(-1,1)$ fills out the area bounded by y_{-1}^+ and y_{-1}^-.

Another simple example in which one may readily determine the totality of solutions in the Filippov sense, and verify the analogues of known results in the classical case, is afforded by $m=1$ and $f(t,y)=1$ for $t^2+y^2<1$, $f(t,y)=-1$ for $t^2+y^2>1$.

In particular, Filippov's method may be used to establish results on the existence, uniqueness, and continuation of solutions for certain cases in which solutions of the differential equation encounter a manifold of discontinuities of $f(t,y)$ infinitely many times in an arbitrary neighborhood of certain points, and hence the customary "piecing together of solutions" is not appropriate for the derivation of these results. As pointed out in Filippov [**25**, p. 199], this situation can arise for the differential equation

$$\ddot{x}(t)+b\dot{x}(t)+k\,\text{sgn}\,\dot{x}(t)+cx(t)=e(t),$$

where b, c, k are positive constants and $e(t)$ is a continuous periodic function, which describes forced oscillation of a system involving the simultaneous occurrence of viscous and dry friction.

The author's personal interest in differential equations of Filippov type was heightened by their occurrence in the study [**58**] of a simple optimal control problem involving approximation by monotone functions. Indeed, his concern with the general area dates from a much earlier consideration of a related class of problems with Brunk and Ewing [**10**], a discussion which in turn arose as an extension of an earlier work of these authors with Ayer and Silverman [**2**] on a problem in maximum-likelihood estimation. It is to be remarked that reference [**10**] is an abstract; its specific results were subsumed in the later work of Brunk, Ewing and Utz [**11**]. In the subsequent years there has occurred a considerable literature dealing with statistical inference in the presence of order restrictions, and for this area the reader is referred to [**4**].

Let $h(t)$ be a given bounded measurable function on $[a, b]$, and for $k > 0$ denote by $\mathfrak{M}_k$ the class of monotone non-increasing functions y on $[a, b]$ satisfying $0 \leqq y(t_1) - y(t_2) \leqq k(t_2 - t_1)$ for $a \leqq t_1 < t_2 \leqq b$. A particular case of the problems considered in [**58**] is that of minimizing the integral functional

$$J[y] = \frac{1}{2}\int_a^b [y(s) - h(s)]^2\, ds$$

in the class $\mathfrak{M}_k$. There is a unique solution y to this problem which is characterized by the existence of a function z such that y, z is a solution in the Filippov sense of the differential system

$$z' = y, \quad y' = g_k(t, z), \quad t \in [a, b],$$
$$z(a) = 0, \quad z(b) = H(b),$$

where $g_k(t, z) = 0$ if $z > H(t)$, $g_k(t, z) = -k$ if $z < H(t)$, and $H(t) = \int_a^t h(s)\, ds$ for $t \in [a, b]$. Recently [**60**], the author has extended certain results of [**58**] to a corresponding problem wherein the monotoneity condition is required of a regular linear differential operator of higher order.

6. Differential equations in abstract spaces. Within recent years the study of differential equations in abstract spaces has been a very active topic. Consistent with the historical development of functional analysis, much of the earlier work dealt with differential equations in special infinite dimensional spaces, and, in particular, with equations in classical coordinatized Hilbert space. Specific mention of individual papers in this area will be limited to only one, and that for a purely personal reason. The author's dissertation [**56**] was written in this area, and under the direction of Professor Ettlinger, to whom the present paper is dedicated.

In extending the study of differential equations from finite dimensional spaces to infinite dimensional abstract spaces, one possible procedure is to restrict the space or function in such a manner that one may establish results which are analogues of certain results of the more elementary theory. For example, Phillips [**55**; §9] applied established results on integration in abstract spaces to prove a Carathéodory type existence theorem for a differential equation

$$y'(t) = f(t, y(t)) \tag{6.1}$$

in $[0, 1] \times \mathfrak{M}$, where $\mathfrak{M}$ is a linear convex topological space which is sequentially complete and satisfies the first countability axiom, while the map of $[0, 1] \times \mathfrak{M}$ under f lies in a compact subset of $\mathfrak{M}$.

For a general introduction to differential equations in abstract spaces the reader is referred to Hille and Phillips [**35**, Ch. III, §3, 4], MacNerney [**47**, **48**], Dieudonné [**23**, Ch. X], Bourbaki [**6**], and Ladas and Lakshmikantham [**43**, Ch. 5]. A major portion of the work extending classical results for differential equations to abstract spaces has been in the context of Banach spaces, and further discussion of this area will be limited to such cases.

If $\mathbf{B}$ is a Banach space and f: $[a, b]\times\mathbf{B}\to\mathbf{B}$ is a continuous function $f(t, y)$ which satisfies at each point of $[a, b]\times\mathbf{B}$ a local Lipschitz condition in y, then one can show that for each $(\tau, \eta)\in[a, b]\times\mathbf{B}$ there is an interval I containing τ on which there is uniquely determined a function $y(t)$ which is strongly differentiable, satisfies the differential equation (6.1) for $t\in I$, and also the initial condition

$$y(\tau)=\eta; \tag{6.2}$$

moreover, this solution is extensible in a unique manner to a maximal interval of existence, $\hat{I}$. However, in the case of infinite dimensional $\mathbf{B}$ it is possible that $\hat{I}$ is not the whole of $[a, b]$, and that the norm of $y(t)$ does not tend to ∞ as t tends to an endpoint of $\hat{I}$. Also, under the assumption that f is merely continuous, one is not assured of the existence of at least one solution passing through a given initial point (τ, η). Dieudonné **[22]** has given examples of these possibilities, where in each of his examples $\mathbf{B}$ is (c_0), the Banach space of sequences of real numbers convergent to zero, and endowed with the supremum norm. Yorke **[66]** has given an example of a differential equation (6.1) in Hilbert space l^2 with f continuous, and for which the local Peano existence theorem does not hold; a simplification of this latter example, using the Hilbert space $\mathfrak{L}^2(0,\infty)$ instead of l^2, is presented in the paper **[44]** by Lasota and Yorke.

In view of these counterexamples, a natural question is that of asking whether or not there exist infinite dimensional Banach spaces on which Peano's theorem holds, or whether the validity of a Peano type theorem is a characterization of finite dimensionality. A partial answer to this question has been provided by Cellina **[15]**, who showed that if $\mathbf{B}$ is a non-reflexive Banach space then there exists a continuous f: $(-\infty,\infty)\times\mathbf{B}\to\mathbf{B}$ such that the initial value problem $y'(t)=f(t, y(t))$, $y(0)=0$, admits no solution on any non-degenerate interval containing $t=0$.

Again, for a differential equation (6.1) in a Banach space there have been established existence theorems of Peano type when the involved function f satisfies certain conditions beyond continuity, and especially when the additional conditions involve a type of uniform continuity, or a condition on the image of $[a, b]\times\mathbf{B}$ similar to that mentioned above for the result of Phillips **[55]**. Papers of the accompanying Bibliography in this category are those of Corduneanu **[17]**, and Bownds and Diaz **[7]**. In a related area is the paper **[14]** of Cellina, wherein hypotheses involve the Kuratowski measure of noncompactness.

In the case of infinite dimensional spaces there is of course the possibility of considering the concepts of continuity and differentiability in various topologies, and in this regard diverse results have been obtained. For a Hilbert space $\mathbf{H}$, let $\mathbf{H}_w$ denote $\mathbf{H}$ with its weak topology and suppose that f: $[0,\infty)\times\mathbf{H}\to\mathbf{H}$ is weakly continuous in the sense that it is continuous as a map from $[0,\infty)\times\mathbf{H}_w$ into $\mathbf{H}_w$. Browder **[9**; Th. 7, p. 518**]** has shown that if $\mathbf{H}$ is a Hilbert space, and f is a weakly continuous mapping of $[0,\infty)\times\mathbf{H}$ into $\mathbf{H}$, then for each $r>0$ there exists an $\alpha(r)>0$ such that if $\eta\in\mathbf{H}$ with $|\eta|<r$ then there exists a strongly $\mathfrak{C}^1$-solution of (6.1) on $0\leqq t<\alpha(r)$ with $y(0)=\eta$. Medeiros **[51]**, and Diaz and Weinacht **[20]** have discussed further properties of solutions of such an equation, including uniqueness criteria of "Nagumo" and "Osgood" type.

Let $\mathbf{B}$ be a separable, reflexive Banach space, and $\mathbf{B}_w$ the space $\mathbf{B}$ with the weak topology. Chow and Schuur **[16]** have shown that if f: $(0;1)\times\mathbf{B}_w\to\mathbf{B}_w$ is continuous and bounded in norm on a neighborhood of a given point (τ, η) of $(0,1)\times\mathbf{B}_w$, then there exists a solution of (6.1) on some subinterval Δ of $(0, 1)$ containing τ such that $y(t)$ is absolutely continuous on Δ and $y'(t)=f(t, y(t))$ almost everywhere on Δ, where $y'(t)$ is the strong limit of $(1/h)[y(t+h)-y(t)]$ as $h\to 0$. Fitzgibbon **[27]** has considered the autonomous differential equation

$$y'(t)+Ay(t)=0, \qquad y(0)=\eta, \tag{6.3}$$

where A is a weakly continuous, possibly nonlinear, operator mapping a reflexive Banach space $\mathbf{B}$ into itself, and has shown that if the domain of A is all of $\mathbf{B}$ then there is a finite interval $[0, T)$ on which (6.3) has a strong solution, where a strong solution is a function $y(t)$ which is Lipschitz continuous on each compact subinterval of $[0, T)$, has a strong derivative at each point, and

$y'(t)+Ay(t)=0$ almost everywhere on $[0,T)$. If in addition A is accretive, (i.e., $\|y+\lambda Ay-(z+\lambda Az)\| \geqq \|y-z\|$ for all $\lambda \geqq 0$ and y, z belonging to the domain of A), then (6.3) is shown to have a unique strong global solution on $[0,\infty)$. For a Banach space that is not assumed to be either separable or reflexive, Knight [**41**] has announced for the initial value problem (6.1), (6.2) some results on the local existence of pseudo-solutions, (i.e., absolutely continuous functions possessing a pseudo-derivative in the sense of Pettis), and where his hypotheses require, in particular, that for each strongly absolutely continuous function y: $[0,b]\to B$ the composite function $f(t,y(t))$ is Pettis integrable on $[0,b]$.

Finally, it is to be remarked that various aspects of differential equation theory have been studied in the context of a $\mathbf{B}^*$-algebra. For general discussion in this area the reader is referred to Hille and Phillips [**35**, Ch. IV] and Hille [**36**, Ch. 4]. In particular, in this setting there have been established certain generalizations of some of the oscillation phenomena of the classical Sturmian theory. References of the accompanying Bibliography in this category are those of Hille [**36**, §9.6], Heimes [**32**], Kreith and Benson [**42**], Williams [**65**], and Noussair [**53**].

7. Epilogue. The preceding discussion certainly does not provide an answer to the question, "What is a differential equation?' Moreover, there is no pretense that the description of the anatomy of such equations is in any sense complete. Rather, there have been presented various definitions of "differential equation," and different possible meanings for the term "solution of a differential equation."

Indeed, one is reminded of the following passage from Lewis Carroll's *Through the Looking-glass, and What Alice Found There.*

"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean — neither more nor less." "The question is," said Alice, "whether you can make words mean so many different things." "The question is," said Humpty Dumpty, "which is to be the master — that's all."

In the opinion of the present author, in the future there will be many different kinds of "differential equations," many different possible meanings for "solution of a differential equation," and, we trust, many masters of the subject.

An expanded version of an Invited Address to the Texas Section of the MAA, April 5, 1974, and dedicated to Professor H. J. Ettlinger. A preliminary version of the paper was given at the Mathematics Colloquium of the Oklahoma State University, Stillwater, Oklahoma, on November 1, 1973, under the title "What is a differential equation?" Thanks are hereby extended to the referee, who suggested elaboration of certain aspects of the original manuscript.

References

1. L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley-Interscience, New York, 1974.

2. M. Ayer, H. D. Brunk, G. M. Ewing, W. T. Reid, and E. Silverman, An empirical distribution function for sampling with incomplete information, Ann. Math. Statist., 26 (1955) 641–647.

3. A. V. Balakrishnan, Stochastic Differential Systems I, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, New York, 1973.

4. R. E. Barlow, D. J. Bartholomew, J. M. Bremner, H. D. Brunk, Statistical Inference under Order Restrictions, Wiley, New York, 1972.

5. G. A. Bliss, Lectures on the Calculus of Variations, Univ. of Chicago Press, 1946.

6. N. Bourbaki, Éléments de Mathématique, Livre IV, Fonctions d'une variable réelle, Ch. IV, Équations Différentielles, Act. Sci. Ind., 1132, Hermann, Paris, 1961.

7. J. M. Bownds and J. B. Diaz, Euler-Cauchy polygons and the local existence of solutions to abstract ordinary differential equations, Funkcial. Ekvac., 15 (1972) 193–207.

8. T. F. Bridgland, Jr., Contributions to the theory of generalized differential equations, I; II, Math. Systems Theory, 3 (1969) 17–50; 156–165.

9. F. E. Browder, Non-linear equations of evolution, Ann. Math., 80 (1964) 485–523.

10. H. D. Brunk, G. M. Ewing, and W. T. Reid, The minimum of a certain definite integral suggested by the maximum likelihood estimate of a distribution function, (Abstract), Bull. Amer. Math. Soc., 60 (1954) 535.

11. H. D. Brunk, G. M. Ewing, and W. R. Utz, Minimizing integrals in certain classes of monotone functions, Pacific J. Math., 7 (1957) 833–847.

12. C. Carathéodory, Vorlesungen über reelle Funktionen, Teubner, Leipzig; second edition, 1927.

13. C. Castaing, Sur les équations différentielles multivoques, C. R. Acad. Sci. Paris, Ser. A-B, 263 (1966) A63–A66.

14. A. Cellina, On the existence of solutions of ordinary differential equations in Banach spaces, Funkcial. Ekvac., 14 (1971) 129–136.

15. ———, On the non-existence of solutions of differential equations in nonreflexive spaces, Bull. Amer. Math. Soc., 78 (1972) 1069–1072.

16. S. Chow and J. D. Schuur, An existence theorem for ordinary differential equations in Banach spaces, Bull. Amer. Math. Soc., 77 (1971) 1018–1020.

17. C. Corduneanu, Equazioni differenziali negli spazi di Banach-teoremi di esistenza e prolungabilita, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mit. Nat., (8) 23 (1957) 226–230.

18. J. L. Davy, Properties of the solution set of a generalized differential equation, Bull. Australian Math. Soc., 6 (1972) 379–398.

19. W. F. Denny, II, A linear Riemann-Stieltjes integral equation system, (Dissertation, Univ. of Oklahoma, 1974).

20. J. B. Diaz and R. J. Weinacht, On nonlinear differential equations in Hilbert spaces, Applicable Analysis, 1 (1971) 31–41.

21. J. B. Diaz and Joyce R. McLaughlin, Sturm comparison and separation theorems for linear, second order self-adjoint ordinary differential equations and for first order systems, Applicable Analysis, 2 (1972) 355–376.

22. J. Dieudonné, Deux exemples singuliers d'équations différentielles, Acta, Sci. Math. Szeged, (1950) 38–40.

23. ———, Foundations of Modern Analysis, Academic Press, New York, 1960.

24. W. Feller, Generalized second order differential equations and their lateral conditions, Illinois J. Math., 1 (1957) 459–504.

25. A. F. Filippov, Differential equations with discontinuous right-hand side, Amer. Math. Soc. Transl., (2) 42 (1964) 199–231.

26. ———, The existence of solutions of generalized differential equations, Math. Notes of the Academy of Sciences of the USSR (translation of Matematicheskie Zametki), 10 (1971) 608–611.

27. W. E. Fitzgibbon, Weakly continuous accretive operators, Bull. Amer. Math. Soc., 79 (1973) 473–474.

28. H. Guggenheimer, Geometric theory of differential equations. I: Second order differential equations, SIAM J. Math. Analysis, 2 (1971) 233–241.

29. J. Hale, Functional Differential Equations, Springer Verlag, New York, 1971.

30. P. Hartman, A differential equation with nonunique solutions, this MONTHLY, 70 (1963) 255–259.

31. ———, Ordinary Differential Equations, Wiley, New York, 1964.

32. K. A. Heimes, Two-point boundary problems in Banach spaces, J. Differential Eqs., 5 (1969) 215–225.

33. H. Hermes, The generalized differential equation $\dot{x} \in R(t, x)$, Advances in Math., 4 (1970) 149–169.

34. M. R. Hestenes, Calculus of Variations and Optimal Control Theory, Wiley, New York, 1966.

35. E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., XXXI, 1957.

36. E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Mass., 1969.

37. E. L. Ince, Ordinary Differential Equations, Longmans Green and Co., London, 1927.

38. K. Itô, On Stochastic Differential Equations, Mem. Amer. Math. Soc., No. 4, Providence, R.I., 1951.

39. I. S. Kac and M. G. Krein, On the spectral function of the string, Amer. Math. Soc. Transl. (2), 103 (1974) 19–102.

40. N. Kikuchi, On contingent equations satisfying the Carathéodory type condition, Publ. RIMS, Kyoto Univ., Sec. A, 3 (1968) 361–371.

41. W. J. Knight, Existence of solutions of differential equations in Banach space, Bull. Amer. Math. Soc., 80 (1974) 148–149.

42. K. Kreith and D. C. Benson, On abstract Prüfer transformations, Proc. Amer. Math. Soc., 26 (1970) 137–140.

43. G. Ladas and V. Lakshmikantham, Differential Equations in Abstract Spaces, Academic Press, New York, 1972.

44. A. Lasota and J. A. Yorke, The generic property of existence of solutions of differential equations in Banach space, J. Differential Eqs., 13 (1973) 1–12.

45. M. Lavrientieff, Sur un équation différentielle du premier ordre, Math. Z., 23 (1925) 197–209.

46. E. B. Lee and L. Markus, Foundations of Optimal Control Theory, Wiley, New York, 1967.

47. J. S. MacNerney, Stieltjes integrals in linear spaces, Ann. Math., 61 (1955) 354–367.

48. ———, Integral equations and semigroups, Illinois J. Math., 7 (1963) 148–173.

49. A. Marchaud, Sur les champs des demi-droites et les équations différentielles du premier order, Bull. Xoc. Math. France, 63 (1934) 1–28.

50. ———, Sur les champs continus de demi-cônes convexes et leurs intégrales, Compositio Math., 3 (1936) 89–127.

51. L. A. Medeiros, On nonlinear differential equations in Hilbert spaces, this MONTHLY, 76 (1969) 1024–1027.

52. E. J. McShane, Stochastic Calculus and Stochastic Models, Academic Press, New York, 1974.

53. E. S. Noussair, Differential equations in Banach spaces, Bull. Australian Math. Soc., 9 (1973) 219–226.

54. G. Peano, Sull' integrabilita delle equazioni differenziali del primo ordine, Atti R. Accad. Sci. Torino, 21 (1885–1886) 677–685.

55. R. S. Phillips, Integration in a convex linear topological space, Trans. AMS, 47 (1940) 114–145.

56. W. T. Reid, Properties of solutions of an infinite system of ordinary linear differential equations of the first order with auxiliary boundary conditions, (Dissertation, Univ. of Texas, 1929), Trans. Amer. Math. Soc., 32 (1930) 284–318.

57. ———, Generalized linear differential systems, J. Math. Mech., 8 (1959) 705–726.

58. ———, A simple optimal control problem involving approximation by monotone functions, J. Optimization Theory and Appl., 2 (1968) 365–377.

59. ———, Ordinary Differential Equations, Wiley, New York, 1971.

60. ———, A class of optimal control problems involving higher order isotonic approximation, Proc. International Conference on Differential Equations, held at the University of Southern California, September 3–7, 1974, Academic Press, 1975, 651–673.

61. E. Roxin, On generalized dynamical systems defined by contingent equations, J. Differential Eqs., 1 (1965) 188–205.

62. L. Schwartz, Théorie des Distributions I, II. Act. Sci. Ind., 1091, 1122, Hermann, Paris, 1951.

63. J. Warga, Optimal Control of Differential and Functional Equations, Academic Press, New York, 1972.

64. T. Ważewski, Systèmes de commande et équations au contingent, Bull. Acad. Polon., Sci., 9 (1961) 151–155.

65. M. Williams, Oscillation phenomena for linear differential systems in a B*-algebra, (Dissertation, Univ. of Oklahoma, 1971).

66. J. A. Yorke, A continuous differential equation in Hilbert space without existence, Funkcial. Ekvac., 13 (1970) 19–21.

67. L. C. Young, Lectures on the Calculus of Variations and Optimal Control Theory, Saunders, Philadelphia, 1969.

68. S. C. Zaremba, Sur les équations au paratingent, Bull. Sci. Math., (2) 60 (1936) 139–160.

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REBELLION

A double dactyl by B. F. WYMAN

Higgledy-Piggledy
William R. Hamilton
Stood on a bridge when he
Found out the rule.

"Multiplication is
Anticommutative!
Down with the precepts they
Taught us in school!"

PERIOD THREE IMPLIES CHAOS

TIEN-YIEN LI AND JAMES A. YORKE

1. Introduction. The way phenomena or processes evolve or change in time is often described by differential equations or difference equations. One of the simplest mathematical situations occurs when the phenomenon can be described by a single number as, for example, when the number of children susceptible to some disease at the beginning of a school year can be estimated purely as a function of the number for the previous year. That is, when the number x_{n+1} at the beginning of the $n+1$st year (or time period) can be written

$$x_{n+1} = F(x_n), \tag{1.1}$$

where F maps an interval J into itself. Of course such a model for the year by year progress of the disease would be very simplistic and would contain only a shadow of the more complicated phenomena. For other phenomena this model might be more accurate. This equation has been used successfully to model the distribution of points of impact on a spinning bit for oil well drilling, as mentioned in [**8, 11**], knowing this distribution is helpful in predicting uneven wear of the bit. For another example, if a population of insects has discrete generations, the size of the $n+1$st generation will be a function of the nth. A reasonable model would then be a generalized logistic equation

$$x_{n+1} = rx_n[1 - x_n/K]. \tag{1.2}$$

A related model for insect populations was discussed by Utida in [**10**]. See also Oster *et al* [**14, 15**].

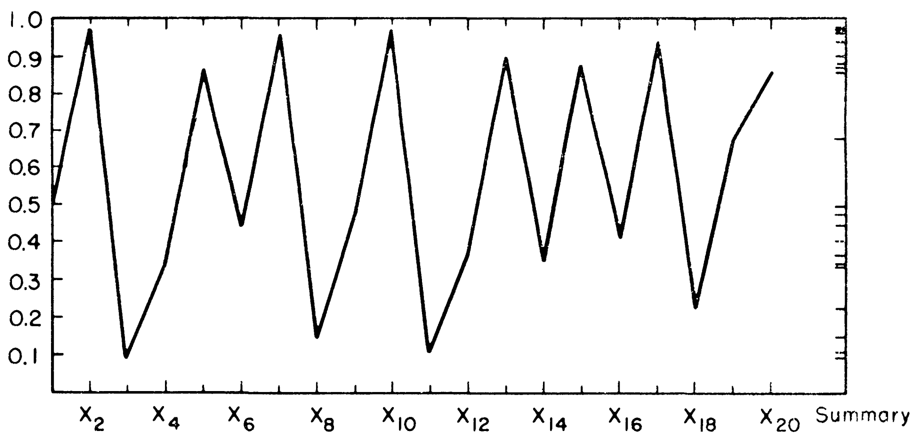

FIG. 1. For $K = 1$, $r = 3.9$, with $x_1 = .5$, the above graph is obtained by iterating Eq. (1.2) 19 times. At right the 20 values are repeated in summary. No value occurs twice. While $x_2 = .975$ and $x_{10} = .973$ are close together, the behavior is not periodic with period 8 since $x_{18} = .222$.

These models are highly simplified, yet even this apparently simple equation (1.2) may have surprisingly complicated dynamic behavior. See Figure 1. We approach these equations with the viewpoint that irregularities and chaotic oscillations of complicated phenomena may sometimes be understood in terms of the simple model, even if that model is not sufficiently sophisticated to allow accurate numerical predictions. Lorenz [**1–4**] took this point of view in studying turbulent behavior in a fascinating series of papers. He showed that a certain complicated fluid flow could be modelled

by such a sequence $x, F(x), F^2(x), \cdots$, which retained some of the chaotic aspects of the original flow. See Figure 2. In this paper we analyze a situation in which the sequence $\{F^n(x)\}$ is non-periodic and might be called "chaotic." Theorem 1 shows that chaotic behavior for (1.1) will result in any situation in which a "population" of size x can grow for two or more successive generations and then having reached an unsustainable height, a population bust follows to the level x or below.

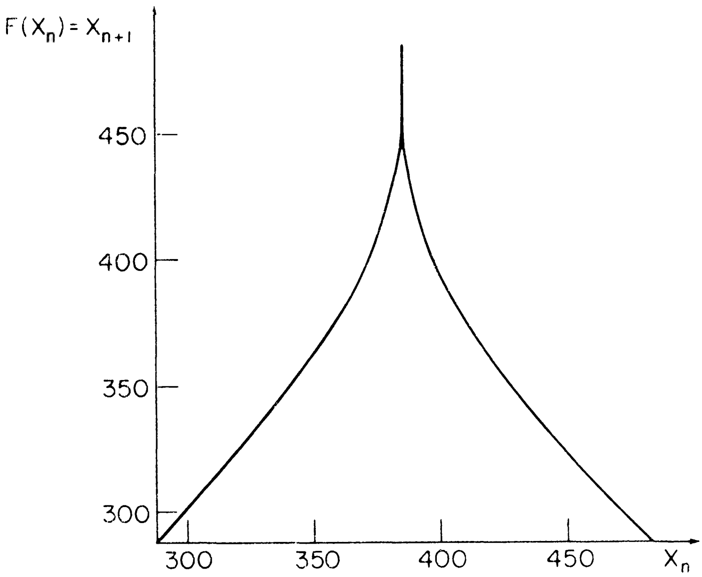

FIG. 2. Lorenz [1] studied the equations for a rotating water-filled vessel which is circularly symmetric about its vertical axis. The vessel is heated near the rim and cooled near its center. When the vessel is annular in shape and the rotation rate high, waves develop and alter their shape irregularly. From a simplified set of equations solved numerically, Lorenz let X_n be in essence the maximum kinetic energy of successive waves. Plotting X_{n+1} against X_n, and connecting the points, the above graph is obtained.

In section 3 we give a well-known simple condition which guarantees that a periodic point is stable and then in section 4 we quote a result applicable when F is like the one in Figure 2. It implies that there is an interval $J_\infty \subset J$ such that for almost every $x \in J$, the set of limit points of the sequence $\{F^n(x)\}$ is J_∞.

A number of questions remain unanswered. For example, is the closure of the periodic points an interval or at least a finite union of intervals? Other questions are mentioned later.

Added in proof. May has recently discovered other strong properties of these maps in his independent study of how the behavior changes as a parameter is varied [17].

2. The main theorem. Let $F: J \to J$. For $x \in J$, $F^0(x)$ denotes x and $F^{n+1}(x)$ denotes $F(F^n(x))$ for $n = 0, 1, \cdots$. We will say p is a **periodic point with period** n if $p \in J$ and $p = F^n(p)$ and $p \neq F^k(p)$ for $1 \leqq k < n$. We say p is **periodic** or is a **periodic point** if p is periodic for some $n \geqq 1$. We say q is **eventually periodic** if for some positive integer m, $p = F^m(q)$ is periodic. Since F need not be one-to-one, there may be points which are eventually periodic but are not periodic. Our objective is to understand the situations in which iterates of a point are very irregular. A special case of our main result says that if there is a periodic point with period 3, then for each integer $n = 1, 2, 3, \cdots$, there is a periodic point with period n. Furthermore, there is an uncountable subset of points x in J which are not even "asymptotically periodic."

THEOREM 1. *Let J be an interval and let $F: J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy*

$$d \leqq a < b < c \text{ (or } d \geqq a > b > c).$$

Then

T1: *for every $k = 1, 2, \cdots$ there is a periodic point in J having period k.*

Furthermore,

T2: *there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:*

(A) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{2.1}$$

and

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0. \tag{2.2}$$

(B) *For every $p \in S$ and periodic point $q \in J$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

REMARKS. Notice that if there is a periodic point with period 3, then the hypothesis of the theorem will be satisfied.

An example of a function satisfying the hypotheses of the theorem is $F(x) = rx[1 - x/K]$ as in (1.2) for $r \in (3.84, 4]$ with $J = [0, K]$ and for $r > 4$, $F(x) = \max\{0, rx[1 - x/K]\}$ with $J = [0, K]$. See [2] for a detailed description of iterates of this function for $r \in [0, 4)$. The case $r = 4$ is discussed in [**6, 7, 12**].

While the existence of a point of period 3 implies the existence of one of period 5, the converse is false. (See Appendix 1).

We say $x \in J$ is **asymptotically periodic** if there is a periodic point p for which

$$F^n(x) - F^n(p) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.3}$$

It follows from (B) that the set S contains no asymptotically periodic points. We remark that it is unknown what the infimum of r is for which the equation (1.2) has points which are not asymptotically periodic.

Proof of Theorem 1. The proof of T1 introduces the main ideas for both T1 and T2. We now give the proof of T1 with necessary lemmas and relegate the tedious proof of T2 to Appendix 2.

LEMMA 0. *Let $G: I \to R$ be continuous, where I is an interval. For any compact interval $I_1 \subset G(I)$ there is a compact interval $Q \subset I$ such that $G(Q) = I_1$.*

Proof. Let $I_1 = [G(p), G(q)]$, where $p, q \in I$. If $p < q$, let r be the last point of $[p, q]$ where $G(r) = G(p)$ and let s be the first point after r where $G(s) = G(q)$. Then $G([r, s]) = I_1$. Similar reasoning applies when $p > q$.

LEMMA 1. *Let $F: J \to J$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subset J$ and $I_{n+1} \subset F(I_n)$ for all n. Then there is a sequence of compact intervals Q_n such that $Q_{n+1} \subset Q_n \subset I_0$ and $F^n(Q_n) = I_n$ for $n \geqq 0$. For any $x \in Q = \cap Q_n$ we have $F^n(x) \in I_n$ for all n.*

Proof. Define $Q_0 = I_0$. Then $F^0(Q_0) = I_0$. If Q_{n-1} has been defined so that $F^{n-1}(Q_{n-1}) = I_{n-1}$, then $I_n \subset F(I_{n-1}) = F^n(Q_{n-1})$. By Lemma 0 applied to $G = F^n$ on Q_{n-1} there is a compact interval $Q_n \subset Q_{n-1}$ such that $F^n(Q_n) = I_n$. This completes the induction.

The technique of studying how certain sequences of sets are mapped into or onto each other is often used in studying dynamical systems. For instance, Smale uses this method in his famous "horseshoe example" in which he shows how a homeomorphism on the plane can have infinitely many periodic points [**13**].

LEMMA 2. *Let $G: J \to R$ be continuous. Let $I \subset J$ be a compact interval. Assume $I \subset G(I)$. Then there is a point $p \in I$ such that $G(p) = p$.*

Proof. Let $I = [\beta_0, \beta_1]$. Choose $\alpha_i (i = 0, 1)$ in I such that $G(\alpha_i) = \beta_i$. It follows $\alpha_0 - G(\alpha_0) \geqq 0$ and $\alpha_1 - G(\alpha_1) \leqq 0$ and so continuity implies $G(\beta) - \beta$ must be 0 for some β in I.

Assume $d \leqq a < b < c$ as in the theorem. The proof for the case $d \geqq a > b > c$ is similar and so is omitted. Write $K = [a, b]$ and $L = [b, c]$.

Proof of T1: Let k be a positive integer. For $k > 1$ let $\{I_n\}$ be the sequence of intervals $I_n = L$ for $n = 0, \cdots, k-2$ and $I_{k-1} = K$, and define I_n to be periodic inductively, $I_{n+k} = I_n$ for $n = 0, 1, 2, \cdots$. If $k = 1$, let $I_n = L$ for all n.

Let Q_n be the sets in the proof of Lemma 1. Then notice that $Q_k \subset Q_0$ and $F^k(Q_k) = Q_0$ and so by Lemma 2, $G = F^k$ has a fixed point p_k in Q_k. It is clear that p_k cannot have period less than k for F; otherwise we would need to have $F^{k-1}(p_k) = b$, contrary to $F^{k+1}(p_k) \in L$. The point p_k is a periodic point of period k for F.

3. Behavior near a periodic point. For some functions F, the asymptotic behavior of iterates of a point can be understood simply by studying the periodic points. For

$$F(x) = ax(1-x) \tag{3.1}$$

a detailed discussion of the points of period 1 and 2 may be found in [**1**] for $a \in [0, 4]$ and we now summarize some of those results. For $a \in [0, 4]$, F: $[0, 1] \to [0, 1]$.

For $a \in [0, 1]$, $x = 0$ is the only point of period 1; in fact, for $x \in [0, 1]$, the sequence $F^n(x) \to 0$ as $n \to \infty$.

For $a \in (1, 3]$, there are two points of period 1, namely 0 and $1 - a^{-1}$, and for $x \in (0, 1)$, $F^n(x) \to 1 - a^{-1}$ as $n \to \infty$.

For $a > 3$ there are also two points of period 2 which we may call p and q and of course $F(p) = q$ and $F(q) = p$. For $a \in (3, 1 + \sqrt{6} \approx 3.449)$ and $x \in (0, 1)$, $F^{2n}(x)$ converges to either p or q while $F^{2n+1}(x)$ converges to the other, except for those x for which there is an n for which $F^n(x)$ equals the point $1 - a^{-1}$ of period 1. There are only a countable number of such points so that the behavior of $\{F^n(x)\}$ can be understood by studying the periodic points.

For $a > 1 + \sqrt{6}$, there are 4 points of period 4 and for a slightly greater than $1 + \sqrt{6}$, $F^{4n}(x)$ tends to one of these 4 unless for some n, $F^n(x)$ equals one of the points of period 1 or 2. Therefore we may summarize this situation by saying that each point in [0, 1] is asymptotically periodic.

For those values of a for which each point is asymptotically periodic, it is sufficient to study only the periodic points and their "stability properties." For any function F a point $y \in J$ with period k is said to be **asymptotically stable** if for some interval $I = (y - \delta, y + \delta)$ we have

$$|F^k(x) - y| < |x - y| \qquad \text{for all} \qquad x \in I.$$

If F is differentiable at the points $y, F(y), \cdots, F^{k-1}(y)$, there is a simple condition that will guarantee this behavior, namely

$$\left| \frac{d}{dx} F^k(x) \right| < 1.$$

By the chain rule

$$\begin{aligned}\frac{d}{dx}F^k(y) &= \frac{d}{dx}F(F^{k-1}(y))\cdot\frac{d}{dx}F^{k-1}(y)\\ &= \frac{d}{dx}F(F^{k-1}(y))\times\frac{d}{dx}F(F^{k-2}(y))\times\cdots\times\frac{d}{dx}F(y)\\ &= \prod_{n=0}^{k-1}\frac{d}{dx}F(y_n),\end{aligned}\tag{3.2}$$

where y_n is the nth iterate, $F^n(y)$. Therefore y is asymptotically stable if

$$\left|\prod_{i=0}^{k-1}\frac{d}{dx}F(y_i)\right| < 1, \qquad \text{where} \qquad y_i = F^i(y).$$

This condition of course guarantees nothing about the limiting behavior of points which do not start "near" the periodic point or one of its iterates. The function in Figure 2 which was studied by Lorenz has the opposite behavior, namely, where the derivative exists we have

$$\left|\frac{d}{dx}F(x)\right| > 1.$$

For such a function every periodic point is "unstable" since for x near a periodic point y of period k, the kth iterate $F^k(x)$ is further from y than x is. To see this, approximate $F^k(x)$ by

$$F^k(y)+\frac{d}{dx}F^k(y)[y-x] = y+\frac{d}{dx}F^k(y)[y-x].$$

Thus for x near y, $|F^k(x)-y|$ is approximately $|x-y|\,|(d/dx)F^k(y)|$. From (3.2) $|(d/dx)F^k(y)|$ is greater than 1. Therefore $F^n(x)$ is further from y than x is.

We do not know when values of a begin to occur for which F in (3.1) has points which are not asymptotically periodic. For $a = 3.627$, F has a periodic point (which is asymptotically stable) of period 6 (approx. $x = .498$). This x is therefore a point of period 3 for F^2 and so Theorem 1 may be applied to F^2. Since F^2 has points which are not asymptotically periodic, the same is true of F.

In order to contrast the situations in this section with other possible situations discussed in the next section, we define the limit set of a point x. The point y is a **limit point** of a sequence $\{x_n\}\subset J$ if there is a subsequence x_{n_i} converging to y. The **limit set** $L(x)$ is defined to be the set of limit points of $\{F^n(x)\}$. If x is asymptotically periodic, then $L(x)$ is the set $\{y, F(y),\cdots, F^{k-1}(y)\}$ for some periodic point y of period k.

4. Statistical properties of $\{F^n(x)\}$. Theorem 1 establishes the irregularity of the behavior of iterates of points. What is also needed is a description of the regular behavior of the sequence $\{F^n(x)\}$ when F is piecewise continuously differentiable (as is Lorenz's function in Figure 2) and

$$\inf_{x\in J_1}\left|\frac{dF}{dx}\right| > 1 \qquad \text{where} \qquad J_1=\left\{x\colon \frac{dF}{dx}\text{ exists}\right\}.\tag{4.1}$$

One approach to describing the asymptotic behavior for such functions is to describe $L(x)$, if possible. A second approach, which turns out to be related, is to examine the average behavior of $\{F^n(x)\}$. The fraction of the iterates $\{x,\cdots, F^{N-1}(x)\}$ of x that are in $[a_1, a_2]$ will be denoted by $\phi(x, N, [a_1, a_2])$. The limiting fraction will be denoted

$$\phi(x,[a_1,a_2]) = \lim_{N\to\infty}\phi(x,N,[a_1,a_2])$$

when the limit exists. The subject of ergodic theory, which studies transformation on general spaces,

motivates the following definition. We say g is the **density of** x (for F) if the limiting fraction satisfies

$$\phi(x,[a_1,a_2]) = \int_{a_1}^{a_2} g(x)dx \quad \text{for all} \quad a_1, a_2 \in J; \quad a_1 < a_2.$$

The techniques for the study of densities use non-elementary techniques of measure theory and functional analysis, so that we shall only summarize the results. But their value lies in the fact that for certain F almost all $x \in J$ have the same density. Until recently the existence of such densities had not been proved, except for the simplest of functions F. The following result has recently been proved:

THEOREM 2. **[5]**. *Let F: $J \to J$ satisfy the following conditions*:

1) *F is continuous.*
2) *Except at one point $t \in J$, F is twice continuously differentiable.*
3) *F satisfies* (4.1).

Then there exists a function g: $J \to [0,\infty)$, such that for almost all $x \in J$, g is the density of x. Also for almost all $x \in J$, $L(x) = \{y: g(y) > 0\}$ which is an interval. Moreover, the set $J_\infty = \{y: g(y) > 0\}$ is an interval, and $L(x) = J_\infty$ for almost all x.

The proof makes use of results in **[8]**. The problem of computationally finding the density is solved in **[9]**.

A detailed discussion of (3.1) is given in **[16]**, describing how $L(x)$ varies as the parameter a in (3.1) varies between 3.0 and 4.0.

A major question left unsolved is whether (for some nice class of functions F) the existence of a stable periodic point implies that almost every point is asymptotically periodic.

Appendix 1: Period 5 does not imply period 3. In this Appendix we give an example which has a fixed point of period 5 but no fixed point of period 3.

Let F: $[1,5] \to [1,5]$, be defined such that $F(1) = 3$, $F(2) = 5$, $F(3) = 4$, $F(4) = 2$, $F(5) = 1$ and on each interval $[n, n+1]$, $1 \leqq n \leqq 4$, assume F is linear. Then

$$F^3([1,2]) = F^2([3,5]) = F([1,4]) = [2,5].$$

Hence, F^3 has no fixed points in $[1,2]$. Similarly, $F^3([2,3]) = [3,5]$ and $F^3([4,5]) = [1,4]$, so neither of these intervals contains a fixed point of F^3. On the other hand,

$$F^3([3,4]) = F^2([2,4]) = F([2,5]) = [1,5] \supset [3,4].$$

Hence, F^3 must have a fixed point in $[3,4]$. We shall now demonstrate that the fixed point of F^3 is unique and is also a fixed point of F.

Let $p \in [3,4]$ be a fixed point of F^3. Then $F(p) \in [2,4]$. If $F(p) \in [2,3]$, then $F^3(p)$ would be in $[1,2]$ which is impossible since then p could not be a fixed point. Hence $F(p) \in [3,4]$ and $F^2(p) \in [2,4]$. If $F^2(p) \in [2,3]$ we would have $F^3(p) \in [4,5]$, an impossibility. Hence p, $F(p)$, $F^2(p)$ are all in $[3,4]$. On the interval $[3,4]$, F is defined linearly and so $F(x) = 10 - 2x$. It has a fixed point 10/3 and it is easy to see that F^3 has a unique fixed point, which must be 10/3. Hence there is no point of period 3.

Appendix 2. Proof of T2 of Theorem 1. Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

(A.1) $$M_n = K \quad \text{or} \quad M_n \subset L, \quad \text{and} \quad F(M_n) \supset M_{n+1}$$

$$\text{if} \quad M_n = K \quad \text{then}$$

(A.2) $$n \text{ is the square of an integer and } M_{n+1}, M_{n+2} \subset L,$$

where $K = [a, b]$ and $L = [b, c]$. Of course if n is the square of an integer, then $n+1$ and $n+2$ are not, so the last requirement in (A.2) is redundant. For $M \in \mathcal{M}$, let $P(M, n)$ denote the number of i's in $\{1, \cdots, n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M^r_n\}_{n=1}^\infty$ to be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} P(M^r, n^2)/n = r. \tag{A.3}$$

Let $M_0 = \{M^r : r \in (3/4, 1)\} \subset M$. Then $\mathcal{M}_0$ is uncountable since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$. For each $M^r \in \mathcal{M}_0$, by Lemma 1, there exists a point x_r with $F^n(x_r) \in M^r_n$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, let $P(x, n)$ denote the number of i's in $\{1, \cdots, n\}$ for which $F^i(x) \in K$. We can never have $F^k(x_r) = b$, because then x_r would eventually have period 3, contrary to (A.2). Consequently $P(x_r, n) = P(M^r, n)$ for all n, and so

$$\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r$$

for all r. We claim that

(A.4) for $p, q \in S$, with $p \neq q$, there exist infinitely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$ or vice versa.

We may assume $\rho(p) > \rho(q)$. Then $P(p, n) - P(q, n) \to \infty$, and so there must be infintely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$.

Since $F^2(b) = d \leqq a$ and F^2 is continuous, there exists $\delta > 0$ such that $F^2(x) < (b+d)/2$ for all $x \in [b-\delta, b] \subset K$. If $p \in S$ and $F^n(p) \in K$, then (A.2) implies $F^{n+1}(p) \in L$ and $F^{n+2}(p) \in L$. Therefore $F^n(p) < b - \delta$. If $F^n(q) \in L$, then $F^n(q) \geqq b$ so

$$|F^n(p) - F^n(q)| > \delta.$$

By claim (A.4), for any $p, q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| \geqq \delta > 0.$$

Hence (2.1) is proved. This technique may be similarly used to prove (B) is satisfied.

Proof of 2.2. Since $F(b) = c$, $F(c) = d \leqq a$, we may choose intervals $[b^n, c^n]$, $n = 0, 1, 2, \cdots$, such that

(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset \cdots \supset [b^n, c^n] \supset \cdots$,

(b) $F(x) \in (b^n, c^n)$ for all $x \in (b^{n+1}, c^{n+1})$,

(c) $F(b^{n+1}) = c^n$, $F(c^{n+1}) = b^n$.

Let $A = \bigcap_{n=0}^\infty [b^n, c^n]$, $b^* = \inf A$ and $c^* = \sup A$, then $F(b^*) = c^*$ and $F(c^*) = b^*$, because of (c).

In order to prove (2.2) we must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in M$, we will assume that if $M_k = K$ for both $k = n^2$ and $(n+1)^2$ then $M_k = [b^{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j-1)$, $M_k = [c^*, c^{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, \cdots, n$. For the remaining k's which are not squares of integers, we assume $M_k = L$.

It is easy to check that these requirements are consistent with (A.1) and (A.2), and that we can still choose M^r so as to satisfy (A.3). From the fact that $\rho(x)$ may be thought of as the limit of the fraction of n's for which $F^{n^2}(x) \in K$, it follows that for any $r^*, r \in (3/4, 1)$ there exist infinitely many n such that $M^r_k = M^{r^*}_k = K$ for both $k = n^2$ and $(n+1)^2$. To show (2.2), let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*$, $c^n \to c^*$ as $n \to \infty$, for any $\varepsilon > 0$ there exists N with $|b^n - b^*| < \varepsilon/2$, $|c^n - c^*| < \varepsilon/2$ for all $n > N$. Then, for any n with $n > N$ and $M^r_k = M^{r^*}_k = K$ for both $k = n^2$ and $(n+1)^2$, we have

$$F^{n^2+1}(x_r) \in M^r_k = [b^{2n-1}, b^*]$$

with $k = n^2+1$ and $F^{n^2+1}(x_r)$ and $F^{n^2+1}(x_{r*})$ both belong to $[b^{2n-1}, b^*]$. Therefore, $|F^{n^2+1}(x_r) - F^{n^2+1}(x_{r*})| < \varepsilon$. Since there are infinitely many n with this property, $\liminf_{n\to\infty}|F^n(x_r) - F^n(x_{r*})| = 0$. $\square$

REMARK. The theorem can be generalized by assuming that $F\colon J \to R$ without assuming that $F(J) \subset J$ and we leave this proof to the reader. Of course $F(J) \cap J$ would be nonempty since it would contain the points a, b, and c, assuming that b, c, and d, are defined.

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References

1. E. N. Lorenz, The problem of deducing the climate from the governing equations, Tellus, 16 (1964) 1–11.
2. ———, Deterministic nonperiodic flows, J. Atmospheric Sci., 20 (1963) 130–141.
3. ———, The mechanics of vacillation, J. Atmospheric Sci., 20 (1963) 448–464.
4. ———, The predictability of hydrodynamic flow, Trans. N.Y. Acad. Sci., Ser. II, 25 (1963) 409–432.
5. T. Y. Li and J. A. Yorke, Ergodic transformations from an interval into itself, (submitted for publication).
6. P. R. Stein and S. M. Ulam, Nonlinear transformation studies on electronic computers, Los Alamos Sci. Lab., Los Alamos, New Mexico, 1963.
7. S. M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1960, p. 150.
8. A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc., 186 (1973) 481–488.
9. T. Y. Li, Finite approximation for the Frobenius-Perron operator — A Solution to Ulam's conjecture, (submitted for publication).
10. Syunro Utida, Population fluctuation, an experimental and theoretical approach, Cold Spring Harbor Symposia on Quantitative Biology, 22 (1957) 139–151.
11. A. Lasota and P. Rusek, Problems of the stability of the motion in the process of rotary drilling with cogged bits, Archium Gornictua, 15 (1970) 205–216 (Polish with Russian and German Summary).
12. N. Metropolis, M. L. Stein and P. R. Stein, On infinite limit sets for transformations on the unit interval, J. Combinatorial Theory Ser. A 15, (1973) 25–44.
13. S. Smale, Differentiable dynamical systems, Bull. A.M.S., 73 (1967) 747–817 (see §1.5).
14. G. Oster and Y. Takahashi, Models for age specific interactions in a periodic environment, Ecology, in press.
15. D. Auslander, G. Oster and C. Huffaker, Dynamics of interacting populations, a preprint.
16. T. Y. Li and James A. Yorke, The "simplest" dynamics system, (to appear).
17. R. M. May, Biological populations obeying difference equations, stable cycles, and chaos, J. Theor. Biol. (to appear).

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QUERIES

Edited by A. C. Zitronenbaum

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.

Reply to Query 20. In this query, A. R. Loch asks about attempts to improve courses in arithmetic and algebra for students who have not mastered this material in school. E. Halsey of the Department of Mathematics of the University of Washington, Seattle, Washington 98195 and J. R. Leitzel and B. R. Waits of the Department of Mathematics of The Ohio State University, Columbus, Ohio 43210, wrote, telling of their experiences in this area. Interested readers are invited to correspond directly with the above individuals.

MATHEMATICAL NOTES

Edited by Richard A. Brualdi

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.

VARIATIONS ON VAN DER WAERDEN'S AND RAMSEY'S THEOREMS

T. C. Brown

1. Van der Waerden's Theorem. The particular variation of van der Waerden's Theorem to be presented here has been discovered independently by any number of people, but a proof of its equivalence to van der Waerden's Theorem has, to the author's knowledge, never appeared in print. (The variation is stated in **[6]**, and a recent application will appear in **[5]**.)

Theorem V (Van der Waerden **[12]**). *For all positive integers k and l there exists $n = n(k, l)$ such that if any set of n consecutive positive integers is partitioned into k subsets, at least one of these subsets contains an arithmetic progression of length l.*

Theorem V′ (Variation). *For all positive integers m and l there exists $p = p(m, l)$ such that if $a_1 < a_2 < \cdots < a_p$ are positive integers such that $a_{j+1} - a_j \leqq m$, $1 \leqq j \leqq p-1$, then $\{a_1, a_2, \cdots, a_p\}$ contains an arithmetic progression of length l.*

Most published proofs of van der Waerden's theorem are carried out by induction on k and l. It would be of interest to find a direct inductive proof of Theorem V′. In this note, however, we shall show that Theorem V implies Theorem V′ and conversely.

To this end, it is convenient to let $V(k, l)$ denote the statement that if the set N of all positive integers is partitioned into k subsets, then at least one of these subsets contains an arithmetic progession of length l. Also, let $V'(m, l)$ denote the statement that if $a_1 < a_2 < \cdots$ are a sequence of positive integers such that $a_{j+1} - a_j \leqq m$, $j = 1, 2, \cdots$, then the set $\{a_1, a_2, \cdots\}$ contains an arithmetic progression of length l.

Clearly Theorem V implies statement $V(k,l)$ for every k and l, and Theorem V′ implies statement $V'(m,l)$ for every m and l. We show now that for every k and l, $V(k,l)$ implies the existence of $n(k,l)$. Indeed, suppose that the integer $n = n(k,l)$ does not exist. Then for every n we have a sequence of length n on k symbols which represents a partition of $\{1,2,\cdots,n\}$ into k subsets such that no subset contains a progression of length l.

Let a_1 be one of the k symbols which is the lst symbol of infinitely many of these sequences. Let a_2 be a symbol which is the 2nd symbol of infinitely many sequences beginning with a_1. Let a_3 be the 3rd symbol of infinitely many sequences beginning with $a_1\, a_2$. In this way we construct an infinite sequence $a_1\, a_2 \cdots$ on k symbols which represents a partition of N into k subsets, none of which contains an arithmetic progression of length l, contradicting $V(k,l)$. (A similar argument shows that $V'(m,l)$ implies the existence of $p(m,l)$. However, this fact will not be used here.)

We are now ready to show that Theorem V′ implies Theorem V. We fix l and demonstrate the existence of $n(k,l)$ by induction on k. The case $k = 1$ is trivial, so we assume that $n(k,l)$ exists and use this to establish $V(k+1,l)$, which, as noted above, implies the existence of $n(k+1,l)$, thus completing the induction.

Hence, let N be partitioned into $k+1$ subsets and consider the $(k+1)st$ subset $\{a_1, a_2, \cdots\}$. We may as well assume it is infinite. If for some m, $a_{j+1} - a_j \leqq m$, $j = 1,2,\cdots$, then by $V'(m,l)$ (which holds since we are assuming Theorem V') the $(k+1)st$ subset contains an arithmetic progression of length l. If no such m exists, then the given partition of N induces partitions of arbitrarily large sets of consecutive positive integers into k subsets only. But then by the existence of $n(k,l)$, at least one of these subsets contains an arithmetic progression of length l. Thus at least one of the $k+1$ subsets into which N was partitioned contains an arithmetic progression of length l. This establishes $V(k+1,l)$ and, as previously remarked, completes the induction.

Conversely, we now show that Theorem V implies Theorem V′. Let m,l be given, and let $p = n(m,l) - (m-1)$. Let $A_0 = \{a_1, a_2, \cdots, a_p\}$, where $a_1 < a_2 < \cdots < a_p$ and $a_{j+1} - a_j \leqq m$, $1 \leqq j \leqq p-1$. We show A_0 contains an arithmetic progession of length l. Define

$$A_1 = \{a_1+1, a_2+1, \cdots, a_p+1\} \setminus A_0,$$

$$A_2 = \{a_1+2, a_2+2, \cdots, a_p+2\} \setminus (A_0 \cup A_1), \cdots,$$

$$A_{m-1} = \{a_1+m-1, a_2+m-1, \cdots, a_p+m-1\} \setminus \{A_0 \cup A_1 \cup \cdots \cup A_{m-2}\}.$$

Then $\{1,2,\cdots,a_p+m-1\}$ is partitioned into the m sets $A_0, A_1, \cdots, A_{m-1}$. Since $a_p + m - 1 \geqq p + m - 1 = n(m,l)$, at least one of these sets, say A_i, contains an arithmetic progression of length l. Since

$$A_i \subset \{a_1+i, a_2+i, \cdots, a_p+i\},$$

the set $A_0 = \{a_1, a_2, \cdots, a_p\}$ also contains such a progression.

This completes the proof of the equivalence of Theorems V and V′.

2. Ramsey's Theorem. Let G be a graph with an infinite number of vertices such that at least two of every three vertices of G are joined by an edge of G. G. Szekeres (**[10]**, **[11]**) showed that such a graph G must contain an infinite complete subgraph. (An infinite complete subgraph of G is an infinite set of vertices of G, *every* two of which are joined by an edge of G.)

P. Turán (**[10]**, **[11]**) generalized this result by showing that for every fixed positive integer d, if G is a graph with an infinite number of vertices such that at least two of every d vertices of G are joined by an edge of G, then G contains an infinite complete subgraph.

This result of Turán can itself be further strengthened to obtain the following result, which is designated Theorem R′ since it is a "variation" on Ramsey's Theorem.

THEOREM R′. *Let G be a graph with an infinite number of vertices such that at least two of every **infinite** set of vertices of G are joined by an edge of G. Then G contains an infinite complete subgraph.*

Ramsey's Thorem is the following:

THEOREM R (Ramsey [9]). *If G is a graph with an infinite number of vertices, then either G contains an infinite complete subgraph or there is an infinite set of vertices of G no two of which are joined by an edge of G.*

Thus Theorem R′ is in fact Ramsey's Theorem itself.

Added in proof: Another paper has recently appeared in which Theorem V′ is mentioned. It is by John R. Rabung, *On applications of van der Waerden's theorem*, Math. Magazine, 48 (1975) 142–148. Rabung's paper contains, amongst other interesting things, an argument essentially identical with the proof above that Theorem V implies Theorem V′.

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References

1. R. L. Graham, and B. L. Rothschild, Ramsey's Theorem for n-parameter sets, Trans. AMS, 159 (1971) 257–292.
2. ———, A short proof of van der Waerden's Theorem on arithmetic progressions, Proc. AMS, 42 (1974) 385–386.
3. A. W. Hales, and R. I. Jewitt, Regularity and positional games, Trans. AMS, 106 (1963) 222–229.
4. A. Y. Khinchin, Three Pearls of Number Theory, Graylock Press, Rochester, N.Y. 1952.
5. N. S. Mendelsohn, The golden ratio and van der Waerden's Theorem, to appear in Proc. Fifth SE Conf. on Combinatorics, Graph Theory, and Computing, 1974.
6. J. R. Rabung, A note on van der Waerden's Theorem on arithmetic progressions, Mathematics Research Center Report 70–6, Naval Research Laboratory, 1970.
7. R. Rado, Studien zur Kombinatorik, Math. Zeit., 36 (1933) 424–480.
8. ———, Note on combinatorial analysis, Proc. London Math. Soc., (2) 48 (1943) 122–160.
9. F. P. Ramsey, On a problem of formal logic, Proc. Lond. Math. Soc., Ser. 2, 30 (1930) 264–286.
10. P. Turán, Matem. és Physikai Lapok, 48 (1941) 436–452.
11. ———, On the theory of graphs, Colloquium Math., 3 (1954) 19–30.
12. B. L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wiskunde, 15 (1927) 212–216.
13. ———, How the proof of Baudet's conjecture was found, *Studies in Pure Mathematics*, Academic Press, New York, 1971, 251–260.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

MONTHLY RESEARCH PROBLEMS, 1969–75

RICHARD K. GUY

This continues earlier updating articles [GK,1113–1122; G,1120–1128] which appeared in 1971 and 1973. Readers are recommended to read the opening paragraphs of these, which give an outline of the policy of this section, about which comments are welcome. In summary, problems should have immediate intuitive appeal, and should be intelligible and interesting to broad bands of the wide

spectrum of MONTHLY readers. Solutions are *not* published, but the present writer would very much like to have copies of any papers written about problems which have appeared here, so that successor articles to this one may be as complete as possible.

References in brackets are to articles in the MONTHLY; mostly from this section and not listed at the end, including references to the two articles mentioned in the opening sentence. References in parentheses are either year numbers of papers listed at the end of this article or are to written communications (wrc) or papers (tbp) for which publication plans are not known to the writer.

The paper of Donelly (1973) [G,1121] on Carmichael's conjecture [GK,1115] has appeared, but neither he nor the reviewer seem to be aware of Klee's work [1969, 288]. Pomerance (1974) gives a sufficient condition for the falsity of the conjecture: if there is an x such that for every prime p, $p-1$ divides $\phi(x)$ implies that p^2 divides x, then $\phi(x)=m$ has this unique solution, where $\phi(x)$ is Euler's totient function. On the other hand, truth of the conjecture would follow from that of a conjecture of Pomerance: if p_i is the ith prime and $k \geqq 2$, then $p_k - 1$ divides the product of $p_i(p_i - 1)$ taken over $1 \leqq i \leqq k-1$. This in turn follows from the conjecture H_2 of Schinzel (1958).

Kronk [1969, 809] asked if there was a hypotraceable graph and Horton [G,1121] answered affirmatively. Thomassen (1974a) has shown that for $v = 34$, 37, 39, 40 and $v \geqq 42$, there is a hypotraceable graph with v vertices, and that for $v = 10$, 13, 15, 16, 18 and $v \geqq 20$ there is a hypohamiltonian graph on v vertices with two non-adjacent trivalent vertices. See also Thomassen (1974b), Doyen and van Diest (1975), Grünbaum (1974), and references therein.

In connexion with the conjecture of Nash-Williams and Plummer [1969, 1045; GK,1116; G,1121] exact references can now be given to Chartrand *et al.* (1974) and to Fleischner (1974a, 1974b). Related work is that of Bhat and Kapoor (1971, 1973), Bondy (1971), Chartrand, Kapoor and Lick (1970) and of Faudree and Schelp (1975, tbp).

Work continues on Ringel's tree-labelling and tree-packing problems which were reposed by Duke [1969, 1128; GK,1116; G,1122] but a complete solution still seems far away. Some further families of trees have been labelled by I. and R. Cahit (1975, tbp) who are also concerned with minimizing or maximizing the sum of the "edge values" when the n vertices of a graph are labelled from 1 to n. Kotzig's (1973) paper has appeared, and contains further references. He defines a graph to be **ringelian** if it has r edges, and if the complete graph, K_{2r+1}, on $2r+1$ vertices can be decomposed into $2r+1$ isomorphs of the given graph. He shows that almost all trees are ringelian. For a very readable introduction to these problems, see Martin Gardner (1972). Considerable generalization has been achieved by Doob (1974, tbp), but he is concerned with *edge*-labelling, as were Stanley (1973) and Stewart [G,1128], rather than vertex-labelling. See also the comments on the problem of Golomb [1974, 499] below. A complete reference to Stanton and Zarnke (1973) is now possible. The paper of Haggard and McWha (1975) has appeared with an addendum (p. 36) explaining how their main theorem will not generalize in the way originally earnested [G,1122].

As mentioned in [G,1122] progress has been made by Preparata and Nievergelt (1974) on the problem of Klee [1970, 63] on "snake-in-the-box" codes. They discuss the application of difference-preserving codes to pattern recognition and classification problems, and construct codes whose information content is asymptotically (in the length of the code-words) of the order of the theoretical upper bounds.

In [GK,1117] we mentioned work by Rohde (1974) and Gerber (1973) on the problem of Ogilvy [1970, 388] on the convergence of complex iterated radicals. Complete references can now be given: Gerber proves the convergence of $z_{n+1} = (z_n - c)^{1/2}$ for c real and arbitrary z_0, while Rohde treats the case $z_0 = 0$ and c a function of n.

In [GK,1117] we also mentioned the solution by Riordan and Stein (1973) of the problem by Gandhi [1970, 505] on Genocchi numbers. A related paper is that of Dumont (1972).

Fischer (1975), in relation to Fejes Tóth's [1970, 869] illumination problem, has shown that if the

intensity of the lamps is a strictly convex function, then equidistant lamps maximize the minimum illumination.

In connexion with the problem of Wills [1971, 47; G,1123] on lattice points and volume/area ratio of convex bodies, complete reference can now be made to McMullen and Wills (1973), to Bokowski and Odlyzko (1973), to Bokowski and Wills (1974) and to Hadwiger and Wills (1973). Other papers are by Diviš (1973), and Odlyzko (1973) who gives further references.

McMullen's solution to Rosenfeld's problem [1971, 49] contained a misprint in [G, 1124]. It should be the **floor** (greatest integer not greater than) of

$$(n+3)(6n^4-18n^3+34n^2-42n+105+45(-1)^n)/1440.$$

This result has also been obtained by a different method and in a different form by Reid (1974).

A further reference to Lind's problem [1971, 179] on polynomials mapping algebraic integers into themselves is Cahen (1973).

To the numerous references relevant to the problem of Higgins and Ballew [1971, 274] should be added Dubuque (1939) who generalized theorems of Frobenius (1903) and Weisner (1925); this was pointed out by Joel Brenner.

The paper of Abbott, Erdös and Hanson (1974) on Singmaster's problem [1971, 385] concerning the number of times an integer occurs as a binomial coefficient has now appeared. See also Singmaster's own papers (1974).

In connexion with Duke's problem [1971, 386] concerning the genus and Betti number of a graph, Ringeisen (1973) has shown that there are graphs with given genus and arbitrarily large maximum genus.

Witsenhausen's paper (1973) improving Rosenthal's bounds, quoted by Bolker [1971, 529] in discussing the zonoid problem, has now appeared.

Chui [1971, 779] posed a problem concerning fields due to point masses. His paper (1973) and that of Newman (1972) have appeared, and so have papers by Ching and Chui (1973).

In connexion with T. C. Brown's problem [1971, 886], Entringer, Jackson and Schatz (1974) prove that there is an infinite binary sequence with no identical adjacent blocks of length three or greater and that every binary sequence of length greater than 18 has identical adjacent blocks of length two or greater and that every infinite binary sequence has arbitrarily long adjacent blocks which are permutations of each other. This contradicts the understanding of Erdös (1961) that Euwe had proved that in any binary sequence there are arbitrarily large identical consecutive blocks.

For a résumé of the confirmation of Herda's conjecture [1971, 888] see his own survey article (1974), the references there and the papers of Ault (1974) and Chakerian (1974). For analogous three-dimensional considerations, see Herda (1975, tbp).

For the settling of two conjectures of Smith and Kumin [1972, 157] see two papers of Owens (1975, tbp) and one of Rowen (tbp). These conjectures were also settled by Joan Hutchinson (1974, 1975).

The papers of Goodey (1974) and Peterson (1973) on Peterson's problem [1972, 505] concerning self-intersections and curves of constant width, have appeared.

Work on the Hadamard maximum determinant problem, discussed by Brenner and Cummings [1972, 626; G,1125] has been done by Payne (1973, 1974). If A is an n by p matrix of ± 1's, $n \geqq p$ then max det (A^TA) is determined for $p \leqq 4$ and for $p=5$, $n \not\equiv 2 \pmod 4$ if the Hadamard conjecture is true. For $p>5$ the maximum is obtained if n is large compared with p and provided certain Hadamard matrices exist. If the complex entries $\exp i\theta$ of the $n \times n$ matrix A are restricted

to the arc $|\theta| \leqq \theta_0$ of the unit circle, can $|\det A| = n^{n/2}$? Brenner (wrc) answers "yes" with $A = \varepsilon J - (\varepsilon - \varepsilon^{-1}) I$ in the two cases $n = 2$, $\theta_0 = \pi/4$, $\varepsilon^8 = 1$; $n = 3$, $\theta_0 = \pi/3$, $\varepsilon^6 = 1$, and says that it seems that A can be found with $\theta_0 < \pi/2$.

Wilansky [1972, 764] made a conjecture concerning the separability of a space. Foland and Kirk (1973/74) confirm one part, and also the other under the generalized continuum hypothesis and a further restriction. Wilansky writes that the problem was also solved by Franklin D. Tall of Toronto.

For further remarks on how to cut all the edges of a polytope [1972, 890] see Grünbaum's own paper (1973) and the references given there.

Meredith and Lloyd (1973) showed that Biggs' footballers of Croam (1972, 1020; G,1125] need not play on Sunday.

Alter [1973, 192] raised Lehmer's question: does $\phi(n)$, Euler's totient function, ever properly divide $n - 1$? Pomerance writes to give further references to Erdös (1961–62) and Meijer (1974); in his own papers (1974–75, 1975) he notes that the problem is still unsolved and likely to remain so, but shows that if N is the number of composite n less than x for which $\phi(n) \mid n - 1$, then

$$N = O(x^{1/2}(\log x)^{3/4}).$$

Papers relevant to Nash's reachability problem [1973, 292] are those of Conway (1972) and van Leeuwen (1974).

Recognition of an article in this section by Math. Reviews was made (MR 48 # 5992) when the problem of Parberry and Graudons [1973, 295] was reviewed.

Erdös and Simonovits (1975), Vaughan (1974) and Ellen Hertz (wrc) have each proved the "problime number theorem" earnested by Hirschhorn [1973, 675]. Lal *et al.* (1973) provided further numerical evidence.

The paper of Hedrick (1974) is relevant to the van der Waerden conjecture, discussed by Merris [1973, 791], who now suggests that one might ask the related question: is

$$n \operatorname{per}(A) \geqq \max \sum_{j=1}^{n} \operatorname{per}(A_{\sigma(j),j}),$$

where the max is over all permutations σ of degree n? In fact it is a consequence of Mirsky (1958, Lemma 4) that the inequality always goes the other way. What about replacing max with min? This is related to a problem of Erdös discussed by Marcus [1960, 219].

Raymond Queneau writes concerning questions on a sequence of Ulam, posed by Recamán [1973, 919]. Muller (1966) computed the first 20000 terms; there is a probabilistic study by Wunderlich (1970) and Queneau observes that the sequence is a special case of an s-additive sequence. His paper (1972) does not answer any of the specific questions raised by Recamán; P. Braffort of Paris has studied s-additive sequences on a computer. Owens (1974) gives the second example, $u_{19} = 62$, $u_{20} = 69$, of two consecutive U-numbers which add to a U-number, $u_{31} = 131$; there are no others with sum $\leqq 99933 = u_{7584}$. David Zeitlin's extensive comments must await later publication.

Roger Eggleton writes concerning Meyer's problem [1973, 920] on equitable coloring, that his proof that any tree T of maximum valence $\Delta(T)$ can be equitably colored with $\lceil \Delta(T)/2 \rceil + 1$ colors is not quite correct, although the theorem is valid. On p. 921, *l*.11, just under the Figure, the tree T is pruned by removing edge xv, leaving a tree T'. Now it is only true that the inductive hypothesis guarantees T' can be equitably colored with $\lceil \Delta(T')/2 \rceil + 1$ colors. If $\lceil \Delta(T')/2 \rceil = \lceil \Delta(T)/2 \rceil$, the argument as given holds. However it is just possible that x is the only vertex of valence $\Delta(T)$ in T,

e.g., if T is as in Figure 1. In this case $\Delta(T') = \Delta(T) - 1$, so if $\Delta(T)$ is odd we have $\lceil \Delta(T')/2 \rceil = \lceil \Delta(T)/2 \rceil - 1$.

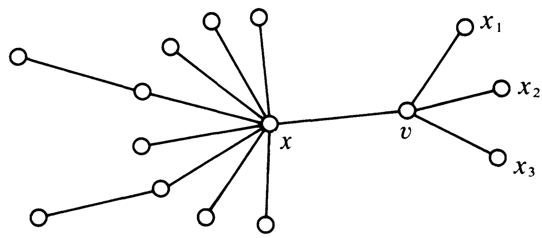

FIG. 1

The way to repair the proof, in the spirit of the argument given, seems to be to show that T' has an equitable coloring with $\lceil \Delta(T)/2 \rceil + 1$ colors. The hypothesis needs to be widened; then with a little extra argument, Eggleton can prove

THEOREM. *If T is a tree with n vertices and maximum valence Δ, then T has an equitable coloring with r colors, provided* $\lceil \Delta/2 \rceil + 1 \leqq r \leqq n$.

Linhart (1974) has solved Fejes Tóth's [1973, 1043] planet exploring problem for the case of 4 satellites; their orbits form the regular mosaic $\{2, 8\}$.

Deakin writes, concerning his conjecture from theoretical genetics [1974, 56] that Feldman (1972) has published the related conjecture that $\bar{w}_{eq}(R)$ is monotonic decreasing in R just if $\bar{w}_{eq}$ is a stable equilibrium for the local value of R. A less precise statement of the conjecture is in Lewontin (1971). Recently Karlin and McGregor (1974, p. 74) state that they have proved some partial results; these support the conjecture in Feldman's rather than Deakin's form. For applications, only stable equilibria are of interest.

S. Baskaran of the Ramanujan Institute writes that he can solve the case of the problem [1974, 156] that Balogun himself solved, using only elementary ideas. Readers will recall that a subgroup H of a group G is called **conjugately pure** in G if two elements of H are conjugate in G only if they are already conjugate in H, and that G is called **hereditarily pure** if every subgroup of G is conjugately pure in G. Then to show that every hereditarily pure finite group is abelian, let G be a minimal counterexample. Then all proper subgroups and proper homomorphic images of G are hereditarily pure and so are abelian. Hence G is non-simple [J. D. Dixon, Problems in group theory, Blaisdell, 1967, problem 1.44, solution p. 81]. If N is any proper non-trivial normal subgroup of G then $gxg^{-1} = x$ for all $g \in G$, $x \in N$ since N is abelian and conjugately pure in G. Further G/N is abelian; it is non-cyclic since G is not abelian by hypothesis. So if H, L are two distinct maximal proper subgroups of G containing N, then both H and L are normal in G; but then $Z(G)$ contains each of them by the above argument applied to H and L in place of N, and hence G itself, a contradiction.

Joseph Zaks of Haifa improves Silver's area-perimeter ratio [1974, 382] by replacing his semicircular arcs by quadrantal ones, as in Figure 2, so that the ratio approaches $(\pi + 2)/2\pi\sqrt{2} \approx 0.57863$, which is better than $(4 + \pi)/4\pi \approx 0.56831$. In turn, Silver used a computer to discover that an even better ratio, 0.5828222, is obtained by using circular arcs of 2.062316 radians (118.162008°).

In answer to Molnar's matrix problem [1974, 383], Morris Newman (wrc) sent the matrices

$$\begin{bmatrix} 3 & 2 & 0 \\ 15 & 11 & 10 \\ 4 & 3 & 3 \end{bmatrix} \begin{bmatrix} -5183 & 2 & 0 & 0 \\ 18 & 3 & 2 & 0 \\ 7 & 15 & 11 & 10 \\ 129 & 4 & 3 & 3 \end{bmatrix} \begin{bmatrix} -11519 & 2 & 0 & 0 & 0 \\ -5760 & -5183 & 2 & 0 & 0 \\ 4 & 18 & 3 & 2 & 0 \\ 12 & 7 & 15 & 11 & 10 \\ 3 & 129 & 4 & 3 & 3 \end{bmatrix}$$

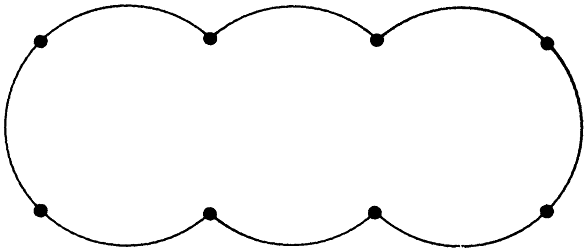

FIG. 2

which give an affirmative answer for all orders greater than 2, a true exception. P. L. Montgomery (wrc) gives the following infinite families of order 3; he notes that $n = 3$ in either of the first two

$$\begin{bmatrix} n-1 & n & n+1 \\ n & n+1 & n+2 \\ n^2-1 & n^2-1 & n^2-2 \end{bmatrix} \quad \begin{bmatrix} n-1 & n & n+1 \\ n & n+1 & n+2 \\ n^2+2n-1 & n^2+2n & n^2+2n \end{bmatrix}$$

$$\begin{bmatrix} n+1 & n & n^2+2n+1 \\ n & n-1 & n^2+n-2 \\ n+1 & n-1 & n^2-2 \end{bmatrix} \quad \begin{bmatrix} n^2+n-2 & n+2 & n+1 \\ n^2 & n+1 & n \\ n^2+2n-1 & n+2 & n \end{bmatrix}$$

gives examples without entries ± 1; he does not know of symmetric examples without entries ± 1, but suspects that there are infinitely many. Harry Appelgate (wrc) sent another such family

$$\begin{bmatrix} n(n+1) & n & n^2+1 \\ n+1 & 0 & n \\ 2-n^2 & 1-n & n(1-n) \end{bmatrix}.$$

By changing the sign of the third row, and interchanging it with another row, another example with non-negative entries is produced. In an extended paper with H. Onishi, Appelgate gives some other infinite families of order 3 matrices, including

$$\begin{bmatrix} 2 & 3 & 0 \\ 18k-82 & 27k-124 & -3 \\ 16k-73 & 24k-110 & -2 \end{bmatrix}.$$

Fejes Tóth writes concerning his covering problem [1974, 632] that G. Wegner has constructed for any $n > 1$ a convex polygon covered with n equal centrosymmetric convex hexagons which cannot be rearranged to cover the polygon without crossing each other.

Edgar writes that what is known about his problem [1974, 758] leads to an easy solution of Pomerance's problem 6036 [1975, 671] but that no light seems to be shed on the $u = 1$ conjecture. There is some overlap between his paper and those of Chinburg and Henriksen (1975, tbp).

Gus Simmons (1974, wrc) soon found (see Roselle [1974, 1097]) a larger graceful subgraph of the complete graph than those of Golomb [1974, 499]. However, John Leech wrote to say that the problem and better solutions were older. See Leech's own paper (1956) and those of Wichmann (1962), Miller (1971) and Golay (1972). It follows from Wichmann's result that Golomb's formula $\lfloor n^2/4 \rfloor + n - 2$ can be improved to $(n^2 + n - 2 - k^2)/3$ where k is the difference between $n + 2$ and the nearest multiple of 6. Even this is not best for $n = 13$, where $e = 58$ can be achieved. An informal but informative exposition of this topic, with applications in a variety of combinatorial situations (race tracks, finite geometries, thickness gauges) is given in Chapter 6 of the fascinating book by O'Beirne (1965). Compare also the discussion of Ringel's problem [1969, 1128; GK,1116; G,1122] earlier in this article.

William Scott writes to say that Hadwiger's problem, discussed by Meier [1974, 630] also appeared as a problem by Fine and Niven (1946). The only new result is the division of a cube into 54 smaller cubes, a result Scott was unable to verify, either in 1946, or now; several of us would be interested to see details of the proof. According to his notes $c(4) \leqq 854$ and $c(5) \leqq 1891$. Scott had also computed a bound for $c(6)$ but the notes are unfortunately lost. Bateman [1947, 42] is quoted as reporting that the problem had appeared in Ripley's "Believe it or not" column.

P. R. Scott's conjecture [1974, 884] has been verified by J. Case (wrc), Tung-Po Lin (wrc) and by Scott himself (1974).

Erdös (oral communication) believes that one could construct a counterexample to the suggestion [1974, 1003] that if X is a non-degenerate, connected metric space with the double midset property, then X is compact.

The telephone problem of van Lint [1975, 55] is marginally related to a more notorious one, of which a generalization has been solved by Lebensold (1973). He shows that n people can share all their information by means of $f(n, k)$ k-person party-line calls, where

$$\begin{aligned} f(n,k) &= \lfloor (n-2)/(k-1) \rfloor + \lfloor (n-1)/k \rfloor + 1, \quad 1 \leqq n \leqq k^2 \\ &= 2\lfloor (n-2)/(k-1) \rfloor, \quad n > k^2. \end{aligned}$$

The original problem had $k = 2$, ordinary 2–person calls. Solutions for this case were given by R. T. Bumby, Baker and Shostak (1972), Hajnal, Milner and Szemerédi (1972), Joel Spencer and by Tijdeman (1971).

A number of correspondents sent counterexamples to the conjecture of Chow [1975, 155] including J. H. Carruth, Maria Klawe, William C. Waterhouse and Kermit Sigmon.

Martin Gardner (wrc) draws my attention to the delightful paper of Roger Penrose (1974), which touches on, among many other things, the problem of Fejes Tóth [1975, 273] on tesselations of the plane.

Carl Pomerance (tbp) writes concerning Eggleton's article [1975, 499] that for $a = b = c = 1$ and for $a = 1$, $b = 2$, $c = 0$, the equation

$$x - a/x = y - b/y + c \tag{1}$$

has no rational solutions. However he can show that (1) does have infinitely many rational solutions for certain broad classes of a, b, c, and believes that this will yield a strict tiling of the plane with rational triangles no two congruent, and such that the set of vertices is discrete. Raymond Killgrove, Marion Smith and Laird Taylor write in similar vein, as does Morris Newman. Later, Eggleton and Pomerance earnest a joint paper. Eggleton wishes to acknowledge support by a Visiting Fellowship at the Weizmann Institute where the original paper was written.

The paper by Bammel and Rothstein (1975), giving the number of 9×9 latin squares [1975, 632] has appeared. David Zeitlin and John Selfridge independently observed that in the known results, the power of 2 occurring in the number of such squares of order n is $\binom{n-2}{2}$ unless $n = 2^q$, when $q - 1$ must be added.

I repeat the plea [G,1126] for help with keeping the section up-to-date by readers' comments, references, preprints and offprints; much of this article is owed to a large number of helpful correspondents, only a few of whom are mentioned in the text. To all of them my grateful thanks.

References

H. L. Abbott, P. Erdös and D. Hanson, On the number of times an integer occurs as a binomial coefficient, this MONTHLY, 81 (1974) 256–261.

Robin Ault, Metric characterization of circles, this MONTHLY, 81 (1974) 149–153.

Brenda Baker and Robert Shostak, Gossips and telephones, Discrete Math., 2 (1972) 191–193.

Stanley E. Bammel and Jerome Rothstein, The number of 9×9 Latin squares, Discrete Math., 11 (1975) 93–95.

V. N. Bhat and S. F. Kapoor, The powers of a connected graph are highly hamiltonian, J. Res. Nat. Bur. Standards, 75B (1971) 63–66.

——— and S. F. Kapoor, Minimally n-hamiltonian graphs, J. Indian Math. Soc., 37 (1973) 277–284.

J. Bokowski and A. M. Odlyzko, Lattice points and the volume/area ratio of convex bodies, Geom. Dedicata, 2 (1973) 249–254.

——— and J. M. Wills, Upper bounds for the number of lattice points of convex bodies, this MONTHLY, 81 (1974) 620–622.

J. A. Bondy, Pancyclic graphs, Proc. 2nd Louisiana Conf. on Combinatorics, Graph Theory and Computing (Baton Rouge, 1971), 167–172; M. R. **48** #3805.

Paul-Jean Cahen, Commutative torsion theory, Trans. Amer. Math. Soc., 184 (1973) 73–86.

I. Cahit and R. Cahit, On the graceful numbering of spanning trees, Information Processing Lett., 3 (1974/75) 115–118.

——— and R. Cahit, Extremal valuations in graphs, submitted to SIAM Journal.

J. Case, (tbp) On a conjecture of Scott's.

G. D. Chakerian, A characterization of curves of constant width, this MONTHLY, 81 (1974) 153–155.

G. Chartrand, S. F. Kapoor and D. R. Lick, n-Hamiltonian graphs, J. Combinatorial Theory, 9 (1970) 308–312.

———, Arthur M. Hobbs, H. A. Jung, S. F. Kapoor and C. St. J. A. Nash-Williams, The square of a block is Hamilton connected, J. Combinatorial Theory, 16B (1974) 290–292.

Ted Chinburg and Melvin Henriksen, Sums of kth powers in the ring of polynomials with integer coefficients, Bull. Amer. Math. Soc., 81 (1975) 107–110; also Acta Arith.

Chin-Hung Ching, On the boundedness of pseudo-differential operators, Proc. Amer. Math. Soc., 41 (1973) 179–182.

——— and Charles K. Chui, Recapturing a holomorphic function on an annulus from its mean boundary values, *ibid*, 120–126.

——— and Charles K. Chui, Analytic functions characterized by their means on an arc, Trans. Amer. Math. Soc., 184 (1973) 175–184.

Charles K. Chui, On approximation in the Bers spaces, Proc. Amer. Math. Soc., 40 (1973) 438–442.

J. H. Conway, Unpredictable iterations, Proc. 1972 Number Theory Conf., Univ. of Colorado, Boulder, 49–52.

B. Diviš, Lattice points on convex curves, Monatsh. für Math., 77 (1973) 389–395.

Harold Donelly, On a problem concerning Euler's phi-function, this MONTHLY; 80 (1973) 1029–1031; M. R. **48** #2043.

Michael Doob, Generalizations of magic graphs, J. Combinatorial Theory, 17B (1974) 205–217.

———, Characterization of regular magic graphs, Abstract 75T — A211, Notices, Amer. Math. Soc., 22 (1975) A614.

Jean Doyen and Viviane van Diest, New families of hypohamiltonian graphs, Discrete Math., 14 (1976).

P. Dubuque, Une généralisation des théorèmes de Frobenius and Weissner, Rec. Math. (Moscou) 5 (47) (1939) 189–196 (Russian with French summary); M.R. 1, 161.

D. Dumont, Sur une conjecture de Gandhi concernant les nombres de Genocchi, Discrete Math., 1 (1972) 321–327.

R. C. Entringer, D. E. Jackson and J. A. Schatz, On non-repetitive sequences, J. Combinatorial Theory, 16A (1974) 159–164.

P. Erdös, Some unsolved problems, Magyar Tud. Akad. Mat. Kutató Int. Közl., 6 (1961) 221–254 (esp. p. 240); M. R. 31 #2106.

———, On a problem of G. Golomb, J. Austral. Math. Soc., 2 (1961–62) 1–8.

——— and M. Simonovits, On a problem of Hirschhorn, this MONTHLY, 83 (1976) 23–26.

Ralph Faudree and R. H. Schelp, The square of a block is panconnected, J. Combinatorial Theory, 19B (1975).

——— and R. H. Schelp, The entire graph of a bridgeless connected plane graph is panconnected, J. London Math. Soc.

J. W. Feldman, Selection for linkage modifications, I. Random mating populations, Theor. Pop. Biol., 3 (1972) 324–346.

N. J. Fine and Ivan Niven, Problem E724, this MONTHLY, 53 (1946) 271; solution 54 (1947) 41–42.

Roland Fischer, Über die optimale Beleuchtung einer geraden Strasse, Monatsh. Math., 79 (1975) 191–199.

H. Fleischner, On spanning subgraphs of a connected bridgeless graph and their application to DT-graphs, J. Combinatorial Theory, 16B (1974) 17–28.

———, The square of every two-connected graph is Hamiltonian, J. Combinatorial Theory, 16B (1974) 29–34.

N. E. Foland and R. B. Kirk, Products of spaces with m-dense subsets, Indiana Univ. Math. J., 23 (1973/74) 155–161; M. R. 48 #1135.

F. G. Frobenius, Über einen Fundamentalsatz der Gruppentheorie, S.-B. Preuss. Akad. Wiss., 1903, 987–991.

Martin Gardner, Mathematical Games: The graceful graphs of Solomon Golomb, or how to number a graph parsimoniously, Scientific Amer., 226 #3 (Mar. 1972) 108–111; *ibid* #4 (Apr.) 104.

Leon Gerber, Complex iterated radicals, Proc. Amer. Math. Soc., 41 (1973) 205–210.

M. J. E. Golay, Notes on the representation of $1,2,\ldots,n$ by differences, J. London Math. Soc. (2), 4 (1971–72) 729–734.

P. R. Goodey, Intersections of circles and curves of constant width, Math. Ann., 208 (1974) 49–58.

B. Grünbaum, Intersecting all edges of centrally symmetric polyhedra by planes, Discrete Math., 5 (1973) 241–246.

———, Vertices missed by longest paths or circuits, J. Combinatorial Theory, 17A (1974) 31–38.

H. Hadwiger and J. M. Wills, Über Eikörper und Gitterpunkte im gewöhnlichen Raum, Geom. Dedicata, 2 (1973) 255–260.

Gary Haggard and Peter McWha, Decomposition of complete graphs into trees, Czechoslovak Math. J., 25 (100) (1975) 31–36.

A. Hajnal, E. C. Milner and E. Szemerédi, A cure for the telephone disease, Canad. Math. Bull., 15 (1972) 447–450.

M. B. Hedrick, The permanent at a minimum on certain classes of doubly stochastic matrices, Bull. Amer. Math. Soc., 80 (1974) 836–838.

H. Herda, Characterization of circles and other closed curves, this MONTHLY, 81 (1974) 146–149.

———, A characterization of spheres among compact 3-bodies, Colloq. Math., 30 (1975).

———, A characterization of spheres among compact 3-bodies satisfying Crofton's theorem, Colloq. Math.

Ellen Hertz, On the asymptotic behavior of the sequence of problimes,

Joan P. Hutchinson, Eulerian graphs and polynomial identities for sets of matrices, Proc. Nat. Acad. Sci. USA, 71 (1974) 1314–1316.

———, Eulerian graphs and polynomial identities for skew-symmetric matrices, Canad. J. Math., 27 (1975) 590–609.

S. Karlin and J. McGregor, Towards a theory of the evolution of modifier genes, Theor. Pop. Biol., 5 (1974) 59–103 (esp. p. 74).

A. Kotzig, On certain vertex-valuations of finite graphs, Utilitas Math., 4 (1973) 261–290.

M. Lal, G. Tiller and H. Wareham, A note on Hirschhorn numbers, Congressus Numerantium IX, Proc. 3rd Manitoba Conf. on Numerical Math., Winnipeg, 1973, 271–276.

Kenneth Lebensold, Efficient communication by phone calls, Studies in Applied Math., 52 (1973) 345–358.

J. Leech, On the representation of $1,2,\ldots,n$ by differences, J. London Math. Soc., 31 (1956) 160–169.

J. van Leeuwen, A partial solution to the reachability problem for vector addition systems, Proc. ACM Sympos. on Theory of Computing, Seattle, 1974, 303–309.

R. C. Lewontin, The effect of genetic linkage on the mean fitness of a population, Proc. Nat. Acad. Sci., U.S.A., 68 (1971) 984–986.

Tung-Po Lin, Maximum area under constraint,

Johann Linhart, Eine extremale Verteilung von Grosskreisen, Elem. Math., 29 (1974) 57–59.

P. McMullen and J. M. Wills, Zur Gitterpunktanzahl auf dem Rand konvexer Körper, Monatsh. Math., 77 (1973) 411–415.

H. G. Meijer, Sets of primes with intermediate density, Math. Scand., 34 (1974) 37–43.

Guy H. J. Meredith and E. Keith Lloyd, The footballers of Croam, J. Combinatorial Theory, 15B (1973) 161–166.

J. C. P. Miller, Difference Bases. Three problems in additive number theory, in Computers and Number Theory, Academic Press, 1971, 299–322.

L. Mirsky, Proofs of two theorems on doubly-stochastic matrices, Proc. Amer. Math. Soc., 9 (1958) 371–374; M. R. 20 #1686.

P. N. Muller, On properties of unique summation sequences, Master's thesis, SUNY at Buffalo, 1966.

D. J. Newman, A lower bound for an area integral, this MONTHLY, 79 (1972) 1015–1016.

T. H. O'Beirne, Puzzles and Paradoxes, Oxford Univ. Press, London, 1965.

A. M. Odlyzko, On lattice points inside convex bodies, this MONTHLY, 80 (1973) 915–918.

Frank W. Owens, Applications of graph theory to matrix theory, Proc. Amer. Math. Soc., 51 (1975) 241–248.

———, A graph theoretic generalization of a theorem by Kostant, (see Abstract 73T–A25, Amer. Math. Soc. Notices, 20 (1973) A–7; corrections, *ibid.* A–548).

———, Sums of consecutive U-numbers, Abstract 74T–A74, Amer. Math. Soc., Notices, 21 (1974) A–298.

Stanley E. Payne, On maximizing det (A^TA), Discrete Math., 10 (1974) 145–158.

———, The construction of Hadamard matrices, Aerospace Research Laboratories, ARL 73–0117, July 1973.

R. Penrose, The rôle of aesthetics in pure and applied mathematical research, Bull. Inst. Math. Appl., 10 (1974) 266–271.

Bruce B. Peterson, Intersection properties of curves of constant width, Illinois J. Math., 17 (1973) 411–420.

Carl Pomerance, On Carmichael's conjecture, Proc. Amer. Math. Soc., 43 (1974) 297–298.

———, On the congruences $\sigma(n) \equiv a$, mod n and $n \equiv a$, mod $\phi(n)$, Acta Arith. 26 (1974–75).

———, On composite n for which $\phi(n) | n-1$, I, Acta Arith., 28 (1975); II *ibid.*

———, On a tiling problem of R. B. Eggleton, Discrete Math., (see Abstract 75T–A220, Notices, Amer. Math. Soc., 22 (1975) A617.

F. P. Preparata and J. Nievergelt, Difference-preserving codes, IEEE Trans. Information Theory, IT–20 (1974) 643–649.

R. Queneau, Sur les suites s-additives, J. Combinatorial Theory, 12 (1972) 31–71.

K. B. Reid, The number of graphs on n vertices with 3 cliques, J. London Math. Soc. (2), 8 (1974) 94–98.

Hanns-Walter Rhode, Complex iterated radicals, this MONTHLY, 81 (1974) 14–21.

R. D. Ringeisen, Graphs of given genus and arbitrarily large maximum genus, Discrete Math., 6 (1973) 169–174.

J. Riordan and P. R. Stein, Proof of a conjecture on Genocchi numbers, Discrete Math., 5 (1973) 381–388.

Louis H. Rowen, Standard polynomials in matrix algebras.

A. Schinzel and W. Sierpiński, Sur certaines hypothèses concernant les nombres premiers, Acta Arith., **4** (1958) 185–208; M. R. 21 #4936.

P. R. Scott, Area-diameter relations for two-dimensional lattices, Math. Mag., 47 (1974) 218–221.

G. J. Simmons, Large graceful subgraphs of the complete graph.

———, Synch-sets: a variant of difference sets, Proc. 5th S. E. Conf. on Combinatorics, Graph Theory and Computing, Boca Raton, Florida, 1974, 625–645.

D. Singmaster, Notes on binomial coefficients, I — A generalization of Lucas' congruence, J. London Math. Soc., (2) 8 (1974) 545–548; II — The least n such that p^e divides an r-nomial coefficient of rank n; *ibid.* 549–554; III — Any integer divides almost all binomial coefficients, *ibid.* 555–560.

———, Repeated binomial coefficients and Fibonacci numbers, Fibonacci Quart., 13 (1975).

R. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, Duke Math. J., 40 (1973), 607–632.

R. G. Stanton and C. R. Zarnke, Labelling of balanced trees, Congressus Numerantium VIII, Proc. 4th S. E. Conf. on Combinatorics, Graph Theory and Computing, Boca Raton, 1973, 479–495.

C. Thomassen, Hypohamiltonian and hypotraceable graphs, Discrete Math., 9 (1974) 91–96.

———, On hypohamiltonian graphs, Discrete Math., 10 (1974) 383–390.

R. Tijdeman, On a telephone problem, Nieuw Arch. voor Wisk., (3) 19 (1971) 188–192.

R. C. Vaughan, The problime number theorem, Bull. London Math. Soc., 6 (1974) 337–340.

Louis Weisner, On the number of elements of a group which have a power in a given conjugate set, Bull. Amer. Math. Soc., 31 (1925) 492–496.

B. Wichmann, A note on restricted difference bases, J. London Math. Soc., 38 (1962) 465–466.

H. S. Witsenhausen, Metric inequalities and the zonoid problem, Proc. Amer. Math. Soc., 40 (1973) 517–520.

M. C. Wunderlich, The improbable behaviour of Ulam's summation sequence, in A. O. L. Atkin and B. J. Birch (eds.) Computers in Number Theory, Proc. 2nd Atlas Conf., Oxford, 1970, Academic Press, 1971, 249–257.

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CLASSROOM NOTES

EDITED BY R. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, WI 53706.

CONVEXITY AND JENSEN'S INEQUALITY

ROBERT R. BURNSIDE

A continuous function, ϕ, convex in $I \subseteq R$ has the property

$$\phi(\Sigma\, \alpha_i x_i) \leqq \Sigma\, \alpha_i \phi(x_i),$$

where $x_i \in I$, $i = 1, 2, \cdots, n$, are given data points and $\alpha_i \in R$ are constrained by $\Sigma\, \alpha_i = 1$ and $\alpha_i \geqq 0$.

We give a simple, direct proof of this form of Jensen's inequality and indicate its use in the derivation of certain classical inequalities.

The proof given rests on the characteristic of a convex function: through each point of its graph, a line may be drawn so that the graph of the function does not lie below the line. The following lemma is evident.

LEMMA. *Let $\psi: x \to ax + b$, where a, b are given real numbers and $b \neq 0$, then $\psi(\Sigma\, \alpha_i x_i) = \Sigma\, \alpha_i \psi(x_i)$, if and only if $\Sigma\, \alpha_i = 1$.*

THEOREM. *Let ϕ be a continuous function, convex in $I \subseteq R$ with $x_i \in I$; then $\phi(\Sigma\, \alpha_i x_i) \leqq \Sigma\, \alpha_i \phi(x_i)$, if and only if $\Sigma\, \alpha_i = 1$ with $\alpha_i \geqq 0$ and $i = 1, 2, \cdots, n$.*

Proof. Let $\xi = \Sigma\, \alpha_i x_i$ and let ϕ be any continuous function, convex in I, with $\phi(\xi) = \psi(\xi)$. The lemma gives

$$\phi(\Sigma\, \alpha_i x_i) = \Sigma\, \alpha_i \psi(x_i), \quad \text{provided} \quad \Sigma\, \alpha_i = 1.$$

But $\psi(\eta) \leqq \phi(\eta)$ for all $\eta \in I$ and hence

$$\Sigma\, \alpha_i \psi(x_i) \leqq \Sigma\, \alpha_i \phi(x_i) \quad \text{if and only if} \quad \alpha_i \geqq 0.$$

Jensen's inequality now follows. Equality holds if the data points x_i are equal or if the graph of ϕ is linear.

Particular choices for ϕ yield simple proofs of certain inequalities. For example;

(i) $\phi: y \to -\ln y$, gives the extended form of the arithmetic mean/geometric mean inequality,

(ii) $\phi: y \to y^{q/p}$, yields the ordering result for power means: if $p < q$, then $M_p \leqq M_q$, where $M_r \equiv (\Sigma\, \alpha_i x_i^r)^{1/r}$, $r \neq 0$ and $x_i > 0$ with $\Sigma\, \alpha_i = 1$, $\alpha_i \geqq 0$.

Alternate proofs of these results may be found in **[1]** and **[2]**.

References

1. M. A. Krasnosel'skii and Ya. B. Rutickii, Convex Functions and Orlicz Spaces, P. Noordhoff, Groningen, 1961.

2. N. D. Kazarinoff, Analytic Inequalities, Holt, Rinehart and Winston, New York, 1961.

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MATHEMATICAL EDUCATION

Edited by Shirley Hill and Paul T. Mielke

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

AN ALTERNATIVE TO INDIVIDUAL INSTRUCTION IN MATHEMATICS

Ronald L. McKeen and Neil A. Davidson

There are many variations of individualized instruction systems ranging from very simple to complex. Many of these systems can be observed in mathematics education today. The simplest of these requires students to read the book, solve the problems, and take the tests. The most complex systems may employ many of the following options:

1. Instructional materials including lists of behavioral objectives and learning hierarchies, programmed texts or booklets (linear, with branching, or mixed), conventional textbooks, UNIPACs or learning activity packages, manipulative materials, games, puzzles, wall charts, remediation or enrichment booklets;
2. Instructional media including computers, video or audio tapes, slides, films, filmstrips, autotutors;
3. Assistance and tutorial help for individual students;
4. Evaluation procedures such as pre- and post-tests, unit tests with alternative forms, diagnosis and remediation prescriptions, individual contracts, and numerous other grading schemes;
5. Support personnel such as teacher aides, clerical assistants;
6. Administrative arrangements involving modular or flexible scheduling, open space settings, team teaching, interdisciplinary activities.

As expected, some systems appear to be much more effective than others. Nevertheless, all these systems aim to achieve two highly important goals: active learning and student pacing. The case for active learning and pacing by students has been made quite effectively **[7, 9]** and need not be reiterated here.

Some educators mistakenly equate *individual* instruction with *individualized* instruction. A teacher who claims to be employing individualized instruction may simply be setting students to work individually. In this paper we shall make a crucial distinction between individual instruction and individualized instruction. In an individual instruction system the learner works by himself with some interaction as needed with a teacher or teacher aide. The system either requires that the learner work by himself or does not facilitate his working with others. In contrast, an individualized instruction system allows the option of strictly individual work, but also allows the choice of other learning modes.

We suggest that *individual* instruction has certain characteristics that prevent it from being an optimal learning experience for many students. Our discussion will focus upon four such characteristics that pertain to systems which employ individual instruction as their predominant mode. However, we do not imply that each of the four characteristics applies equally to all individual instruction systems.

Interpersonal relationships. A basic concern is that individual instruction is not individualized instruction for the learner whose cognitive style demands interpersonal relationships. We shall argue that any scheme of instruction must take into account the basic need for human relationships **[8, 11]** in order to function with optimal effectiveness.

Human beings have a need for personal contact and communication with others. This need for affiliation is both powerful and basic. Moreover, an activity can take on increased significance because it is experienced and shared with others. In terms of learning, Buck **[2]** stated:

"...Let me remind you that student–student interactions are also important in learning, and that at the professional level, much mathematical research springs from discussion *between* mathematicians. Moreover, a test of understanding is often the ability to communicate it to others; and this act itself is often the final and most crucial step in the learning process." [p. 563].

An individual instruction scheme, in which students work by themselves, clearly cannot provide for the interpersonal needs of students, the motivational effects of shared activities, and the learning benefits of discussion.

Intellectual challenge. A second characteristic of many individual instruction systems is a minimal emphasis upon intellectual challenge, inquiry or discovery, and creativity. The importance of these dimensions of the learning process has been emphasized repeatedly by many educators including Bruner **[1]**, Rogers **[11]**, Polya **[10]**, and Shulman **[12]**. In contrast, individual instruction systems often deal with "closed loop" objectives **[6]**, i.e., objectives that are assessable by means of observing specific behaviors resulting from specified stimuli. When the appropriate behavior is not observed, the learner is recycled through the program until the desired behavior is demonstrated. For such "closed loop" objectives a proper task analysis should indicate a logical ordering of prerequisite tasks and reduce the necessity for recycling.

According to Walbesser **[14]**, the use of behavioral objectives has had certain negative effects on mathematics curricula, in that these objectives tend to be low powered with respect to intellectual demands. Such "closed loop" objectives do not properly describe many desirable outcomes of learning such as the ability to discriminate relevant and irrelevant information, the ability to generate and test hypotheses, the ability to develop strategies for problem solving or for constructing logically valid arguments, and the ability to generate new relationships from old information. Moreover, "closed loop" objectives do not appropriately describe desirable outcomes of learning relating to the affective domain **[6]**.

Assessment of learning. In many, but not all, individualized programs there is an emphasis on evaluation (grading) of student learning, and not enough attention is paid to appropriate instructional input. The instructional input for the system is often reading material, including programmed textbooks. The instructor's time is often consumed with the duties of assessing outcomes and bookkeeping. Frequently, descriptions of individual instruction systems focus upon the division of content into discrete units to facilitate the construction of criteria-referenced assessment instruments to be administered when the learner is ready. Much attention is given to the construction of assessment items, alternate forms of unit tests and contracting schemes, whereby students or teachers identify beforehand what percentage of learning tasks the students are to be held accountable for. Little is heard about the appropriate relationship of instructional input to the learning process.

Teachers using individual instruction often complain about the lack of interest in learning shown by students. Such teachers report that students become preoccupied with passing tests, sometimes to the extent of trying to pass the test without attempting to study the material. It appears that students in such systems consider the main focus to be on testing as opposed to learning.

Practical considerations. The fourth characteristic for discussion involves some practical difficulties with individual instruction. Teachers tend to become swamped by an ever-increasing mountain of paperwork. Moreover, unless their class sizes are extraordinarily small, teachers are harried in trying to provide assistance to their students. At any given moment, the teacher is besieged by many requests for help—usually many more requests than he can honor at the time. The result is all too often student frustration with extended waiting time to get needed help.

AN ALTERNATIVE

In the opinion of the authors, the small group instructional method [3, 4] has many characteristics that define it as an effective instructional option for individualization in mathematics. Essentially, the small group instructional tactic calls for students in groups of about four to cooperate in finding solutions to mathematical problems. Students are encouraged to participate actively and to share their ideas with their peers. We shall argue that small group learning shares the main strengths of individual instruction and resolves the main problems inherent in individual instruction.

Small group learning shares a strength of individual instruction in that small group learning is an *active* approach to learning. Students spend their time in class solving problems together in small groups. In addition, small group learning allows for *student pacing*, since each small group moves at its own paçe. Students pursue the solution to a mathematical problem until each member of the group is satisfied with the solution. Admittedly, a student in a small group does not have as complete control of pacing as he would in a totally individual system. Nevertheless, in a group that functions well, the pace is quite comfortable for all the members. The student who is ready to go ahead is not just waiting for others to "catch-up"; he is, instead, reinforcing his own mastery by helping other group members to learn the material.

In small group learning, the four major problems inherent in individual instruction are dealt with as follows:

1. Small group learning affords great opportunity for student *inquiry* or *discovery*. In the small group discovery method, students solve challenging problems, prove theorems, construct examples and counterexamples, state conjectures, and develop techniques for solving various types of problems. In general, students working in small groups can solve more difficult problems than students working in an individual instruction system. This is because a group can afford both emotional support and free exchange of ideas when attacking a challenging problem.

2. Small group learning takes into account some of the *interpersonal needs* of human beings. Instead of working in isolation, students can learn by interacting in a friendly atmosphere with a small number of their peers. The teacher visits each group frequently, and helps to maintain a cooperative and enjoyable group climate. Students tend to become friends with their group members, and the teacher–student relationship tends to be relaxed and informal.

3. In small group instruction, emphasis has been placed mainly on *the facilitation of learning*, not upon the precise assessment of individual performance. Experience has shown that a wide variety of grading schemes are compatible with the small group method.

4. The teacher's problem of *providing assistance* to students is much more manageable with small groups than with individual instruction. There are two basic reasons for this. First, in the small group setting, each student receives help from his peers. The teacher provides help on those questions which cannot be resolved by a discussion involving three or four students. Secondly, in a small group setting, the teacher can interact with several students at once. For example, in a class of twenty-eight students working together in groups of four, the teacher interacts with seven groups. The problems of seven groups can be dealt with much more efficiently and comfortably than the problems of twenty-eight individuals.

References

1. Jerome S. Bruner, Toward a Theory of Instruction, Belknap Press, Cambridge, Mass., 1966.

2. R. C. Buck, Teaching machines and mathematics programs: Statement, this MONTHLY, 69 (1962) 561–564.

3. Neil A. Davidson, The small group-discovery method of mathematics instruction as applied in calculus, Doctoral disseration, University of Wisconsin, 1970. Technical Report No. 168, Wisconsin Research and Development Center for Cognitive Learning, Madison, Wisconsin.

4. Neil A. Davidson, The small group-discovery method as applied in calculus instruction, this MONTHLY, (August–September 1971) 789–791.

5. ———, Ronald McKeen and Theodore Eisenberg, Curriculum construction with student input, The Mathematics Teacher, (March 1973) 271–275.

6. Albert F. Eiss and Mary Blatt Harbeck, Behavioral objectives in the affective domain, National Science Supervisors Assoc., Div. National Science Teachers Assoc., Washington, DC, 1969.

7. Leon Lessinger, Every kid a winner: accountability in education, Science Research Associates College Division, Palo Alto, Cal.; also Simon & Schuster, New York, 1970.

8. A. H. Maslow, Motivation and Personality, Harper & Row, New York, 1954.

9. Gertrude Noar, Individualized Instruction: Every Child a Winner, Wiley, New York, 1972.

10. George Polya, Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving, 2 vols., Wiley, New York, 1962, 1965.

11. Carl Rogers, Freedom to Learn, Merrill, Columbus, Ohio, 1969.

12. Lee Shulman, Psychological controversies in the teaching of science and mathematics, Science Teacher, 35 (September 1968) 27–30.

13. Sherman K. Stein, Mathematics for the captured student, this MONTHLY, (November 1972) 1023–1032.

14. H. H. Walbesser, Behavioral objectives, a cause célèbre, The Arithmetic Teacher, (October 1972) 418, 436–440.

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PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before March 31, 1976.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2564. *Submitted by D. J. Kleitman, Massachusetts Institute of Technology*

Can one cover the vertices of any regular graph of degree 4 (every vertex in it has degree four) by disjoint arcs and stars? (A star consists of a vertex and all the arcs containing it; an arc covers both its ends.) [This problem is due to R. L. Graham and is a variation of a problem of Claude Berge.]

E 2565. *Submitted by D. J. Kleitman, Massachusetts Institute of Technology*

Given a bipartite graph on n and $2n$ vertices that is regular on either set (of degree $2k$ and k

respectively), can one necessarily find n vertices of the second kind such that upon their removal along with all arcs containing them the remaining graph is regular of degree k? [This problem is due to T. Nemetz.]

E 2566. *Proposed by Edvard Kramer, Ljubljana, Yugoslavia*

A triplet (a,b,c) of natural numbers is an *obtuse Pythagorean triplet* if a,b,c are the sides of a triangle ABC with $\measuredangle C = 120°$. Such a triplet is *primitive* if a,b,c have no common factor other than unity.

(i) Show that each positive integer except 1, 2, 4 and 8 can appear as the smallest member of an obtuse Pythagorean triplet.

(ii)* What positive integers can appear in primitive obtuse Pythagorean triplets?

E 2567. *Proposed by J. H. Conway, Cambridge University, England, and R. L. Graham, Bell Laboratories, Murray Hill, New Jersey*

Define polynomials $f_m = f_m(x_1,\ldots,x_m)$ by $f_0 = 1$, $f_1 = x_1$, $f_k = x_k f_{k-1} - f_{k-2}$, $k \geqq 2$. For a fixed $n \geqq 3$, let $y_1, y_2, \ldots$ satisfy $f_n(y_{k+1},\ldots,y_{k+n}) = 1$ for all $k \geqq 0$. Show that $y_{n+k+2} = y_k$ for all $k \geqq 1$.

E 2568. *Proposed by Stroughton Bell, University of New Mexico*

Show that the Bernoulli equation

$$y' + y^2 + xy = 0$$

has exactly two solutions on the entire real line for which y'' is nowhere zero.

E 2569. *Proposed by Harry Dweighter, The City College of the City University of New York*

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?

SOLUTIONS OF ELEMENTARY PROBLEMS

Morley Polygons

E 1030 [1952, 465; 1974, 1110]. *Proposed by Charles Salkind, Polytechnic Institute of Brooklyn*

Consider the polygon formed by the internal trisectors of the angles of a given n-gon, intersecting in neighboring pairs.

(1) Prove that a necessary and sufficient condition that the trisector polygon be regular is that the parent n-gon be regular when $n \geqq 4$.

(2) Prove that the area ratio between the parent and trisector polygons is always irrational.

I. *Counterexample to both* (1) *and* (2) *by Judah Milgram, student, Northwood High School, Silver Spring, Maryland.* Consider the irregular hexagon obtained from a rectangle, whose sides are in the ratio 2: $\sqrt{3}$, by taking the two extra vertices at the midpoints of the longer sides. This parent hexagon gives rise to a regular trisector hexagon whose area is 3/16 that of the parent hexagon. (See Figure 1.)

II. *Counterexample to* (1) *by Judah Milgram.* Consider the quadrilateral obtained from an isosceles right-angled triangle by taking the fourth vertex at the midpoint of the hypotenuse. The trisector 4-gon to which this quadrilateral gives rise is regular (i.e., a square). (See Figure 2.)

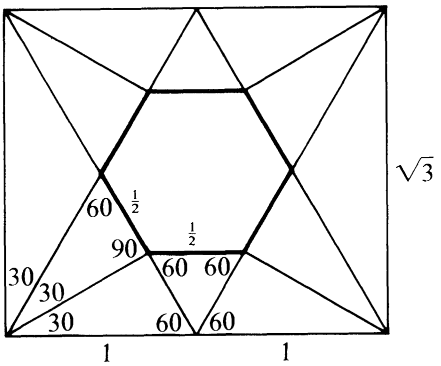

FIG. 1.

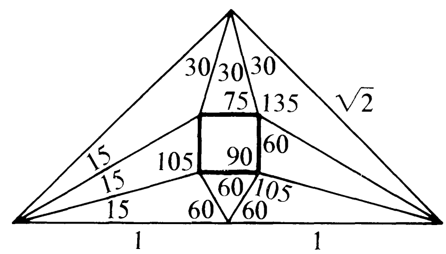

FIG. 2.

III. *Counterexample to* (2) *by Anders Bager, Hjørring, Denmark.* Let the given polygon be a rectangle with sides in the ratio $2\sqrt{3}$: 1. Then the trisector polygon is a rhombus whose area is 5/12 that of the given rectangle. (See Figure 3.)

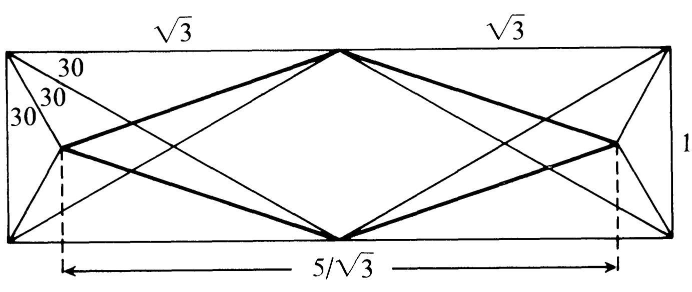

FIG. 3.

Also solved by F. P. Callahan and Peter Ungar who gave counter-examples to (1).

Editor's comment. Part (1) is an attempted generalization to n-gons of Morley's Theorem, which states that the trisector triangle of any triangle is an equilateral (i.e., regular) triangle. (See H. S. M. Coxeter, *Introduction to Geometry*, 2nd edition, Wiley, New York, 1969, pp. 23 ff or R. Penrose, *Morley's trisector theorem*, Eureka No. 16 (Oct. 1953), 6–7.)

A number of respondents (Callahan, Milgram, and Ungar) showed that a *sufficient* condition for the existence of irregular "Morley n-gons" is that n be divisible by three. Their proofs are all essentially based upon an extension of the proof of Penrose for the case $n = 3$. They show that every third angle and every third side of the parent n-gon must be equal, and hence deduce that the parent n-gon can be irregular if n is divisible by three, and will then have angles with three different values. Callahan and Ungar, however, overlooked one exceptional case in their proofs, and so were led to believe that this condition is also *necessary*. That it is not can be seen from counterexample II (Fig. 2) in which $n = 4$.

Murray Klamkin conjectures that (1) may be true if we impose the additional condition that the parent n-gon be cyclic. A general statment concerning the existence of Morely n-gons thus still remains to be given.

Anders Bager conjectures that (2) may be true when the parent n-gon is regular. The area ratio is then given by

$$\left[\sin\frac{(n+4)\pi}{6n}\Big/\sin\frac{(n-2)\pi}{6n}\right]^2$$

for $n \geqq 3$. It remains to be shown that this number is always irrational.

The title of the problem is copied, with thanks, from that of a forthcoming paper by Francis Callahan.

Subgroups of $Z(p^n) \oplus Z(p^n)$

E 2331 [1972, 87]. *Proposed by Albert Baake, Sentinal High School, Missoula, Montana*

Let p be a prime, n a natural number, and let $Z(p^n)$ denote the cyclic group of order p^n. Find all subgroups of a group G which is the direct sum of two copies of $Z(p^n)$.

I. *Editorial comment.* It is perhaps not immediately clear how the question is to be interpreted; in a certain sense, the subgroups of G do not need to be found—they are already there. All the same, there appears to be a consensus that some sort of enumeration of the subgroups of G is required and in this direction the most complete information available to us can be read off, for the particular case in hand, from a general theorem of Yenchien Yeh (*On prime power Abelian groups*, Bull. Amer. Math. Soc., 54 (1948), 323–327; we are indebted to W. R. Scott for this reference). Noting that a finite Abelian p-group is said to be of *type* $(k_1, \cdots, k_n)$ if it is isomorphic to $Z(p^{k_1}) \oplus \cdots \oplus Z(p^{k_n})$, we obtain from Yeh's theorem that there is exactly one subgroup of G isomorphic to $Z(p^{h_1}) \oplus Z(p^{h_1})$ for each h_1, $0 \leqq h_1 \leqq n$, and exactly $p^{h_1 - h_2 - 1}(p+1)$ subgroups isomorphic to $Z(p^{h_1}) \oplus Z(p^{h_2})$ for each h_1 and h_2, $0 \leqq h_2 < h_1 \leqq n$. Z. Z. Uoiea obtained the same count independently of Yeh's result, as did the proposer. It is, of course, an easy matter to obtain the total number of subgroups of G from this count, but a shorter and more direct argument is given below.

II. *Solution (to the problem of the total number of subgroups of G) by Norman Blackburn, University of Illinois at Chicago Circle.* Let f_n denote the number of subgroups of G; f_n is easily calculated by induction on n. For a start, we have $f_0 = 1$. Now if $n > 0$, we simply need to observe that if H is a subgroup of G, and if W is the subgroup $Z(p) \oplus Z(p)$ of G, then either H contains W or H is cyclic. But since $G/W \cong Z(p^{n-1}) \oplus Z(p^{n-1})$, the number of subgroups H which contain W is f_{n-1}, so that

$$f_n = f_{n-1} + \sum_{r=0}^{n} k_r,$$

where k_r is the number of cyclic subgroups of G of order p^r. Now $k_0 = 1$ and if $r > 0$, then G has $p^{2r} - p^{2(r-1)}$ elements of order p^r so that

$$k_r = \frac{p^{2r} - p^{2(r-1)}}{p^r - p^{r-1}} = p^{r-1}(p+1).$$

Hence

$$f_n - f_{n-1} = 1 + \sum_{r=1}^{n} p^{r-1}(p+1) = 1 + (p+1)\frac{p^n - 1}{p-1},$$

and so

$$f_n = 1 + \sum_{k=1}^{n} \left\{ 1 + (p+1)\frac{p^k - 1}{p-1} \right\} = (n+1) + \frac{p+1}{p-1}\,\frac{p(p^n - 1)}{p-1} - n\frac{p+1}{p-1}.$$

Another Solution in Rationals

E 2481 [1974, 660]. *Proposed by Bruce Reznick, California Institute of Technology*

Do the simultaneous equations

$$x^3 + y^3 + z^3 = 1, \qquad x + y + z = 2$$

have a solution in rationals $0 < x < y < z < 1$ other than $x = 1/2$, $y = 2/3$, $z = 5/6$?

Dickson, *History of the Theory of Numbers*, Vol. II, pp. 552 ff.). Umberger sets this up in such a way that the given solution corresponds to (1,0,0,1) and makes a computer search through the 60^4 parameter sets from (0,0,0,0) to (59,59,59,59) and only in one case does the set correspond to a solution in the required range. This solution is the same as that found by Federico.

A Problem of Pappus — Final Appearance

E 2499 [1974, 1026]. *Proposed by S. R. Conrad, B. N. Cardozo High School, Bayside, N.Y.* Construct the triangle ABC, given side a, angle A, and the bisector t_a of angle A.

Editor's comment. The problem is well known and has been around a long time. The title is that given in Nathan Altshiller-Court's *College Geometry* (section 84, 2nd edition, Barnes and Noble, New York, 1952) where a solution may be found and where it is remarked that the problem was considered by Pappus in his *Mathematical Collection.*

The problem has in fact already been posed and solved on two different occasions in this MONTHLY. It appeared as number 310 in the Geometry Section, December 1906 (vol. 13, p. 235) with a solution in April 1907 (vol. 14, pp. 75–76). In 1931 it reappeared as number 3520, the solution (vol. 39, 1932, p. 556) being described as "a slight modification of that suggested in Casey's *Sequel to Euclid*, edition of 1886, p. 80." It is remarked, as it is by a number of the present solvers, that the condition for a solution to exist is $a \leqq 2\ t_a\ \tan \frac{1}{2} A$.

These references were provided by a number of readers who also sighted at least five other published occurrences of the problem.

Complete solutions were received from Nicolas Artemiadis, Günter Bach (Germany), the Bennett College Team, Walter Bluger, Robert Breusch (New Zealand), Paul Bruckman, Jordi Dou (Spain), M. G. Greening (Australia), G. J. Griffith, Norman Gunderson, L. Kuipers, W. Mac Stewart, J. C. Migliore, Simeon Reich (Israel), St. Olaf College Students, Michael Skalsky, L. R. Tanner, William Wernick, Hyman Zimmerberg, and Aleksandras Zujus.

The solution published in the 1907 MONTHLY seems to be as succinct as any. It starts, as with most of the solutions, with the side BC and proceeds to construct A. (Charles Wexler notes that the locus of A for which the bisector of angle A has the constant value t_a is a Conchoid of Nicomedes, so that one wishes to find where this intersects the circular locus of the point A such that angle A is constant.) However, it is feasible to start with the angle A given; Jordi Dou's solution, which is of this type, is based upon the nice observation that the two positions for C on the side AC of the triangle are precisely the two foci of the ellipse whose major axis lies along AC and which is tangent to the bisector of angle A.

Finally, let us mention the observation made by Marc Laframboise that if D is the intersection of BC with the bisector of angle A then $AB \cdot AC = AD^2 + DB \cdot DC$.

A Perfect Number Puzzle

E 2500 [1974, 1026]. *Proposed by P. Richard Herr, University Park, Pennsylvania*

Find all natural numbers n with the property that both n and $\sigma(\sigma(n))$ are perfect numbers, or prove that none exist.

Solution by M. G. Greening, University of New South Wales, Australia.

Case I. If n is an even perfect number then $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$, and hence p, are prime. Since σ is multiplicative we then have $\sigma(\sigma(n)) = 2^p(2^{p+1} - 1)$, which is even. If it is also perfect then $2^{p+1} - 1$, and hence $p + 1$, are prime. Thus $p = 2$, and $n = 6$.

Case II. If n is an odd perfect number then $\sigma(n) = 2n$ gives $\sigma(\sigma(n)) = 6n$. If this is perfect then $6n = 2^{p-1}(2^p - 1)$ with $2^p - 1$ prime. Hence $n = 1$, which is not perfect.

Thus there is only one solution, namely $n = 6$, $\sigma(\sigma(n)) = 28$.

Also solved by 64 other solvers.

Twenty partial solutions were received also, most of these omitting the case n odd.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before March 31, 1976.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6060. *Proposed by Daniel Sokolowsky, Antioch College*

Let $N(S)$ denote the number of elements in a finite set S, and S_n denote the set of integers $\{1,2,\cdots,n\}$. Suppose for $k \geqq 3$ we have $2k$ sets a_i, b_i, $i=1,\cdots,k$ and n elements $w_1,\cdots,w_n$ contained in these $2k$ sets so that the following conditions are satisfied:

(i) a_ib_i has no nonzero entries and $N(a_i \cup b_i) = n-1$, $i=1,\cdots,k$.

(ii) For each j, $j=1,\cdots,n$, $(w_j) = a_{p_1}\cdots a_{p_r}b_{q_1}\cdots b_{q_s}$ for appropriate subsets $\{p_1,\cdots,p_r\}$, $\{q_1,\cdots,q_s\}$ of S_k.

Show that the maximum possible value of n is 2^k-1.

6061. *Proposed by Hung C. Li, Southern Colorado State College*

For any positive semi-definite Hermitian matrix $H(n\times n)$, the set

$$S=\{A \mid \operatorname{tr}(AA^*)H \leqq \lambda\}$$

is convex in A, were A is $n\times m$, X^* is the complex conjugate and transpose of X, and $\operatorname{tr} X$ is the trace of X.

6062.* *Proposed by B. H. Voorhees, University of Alberta*

Consider an infinite sequence of regular n-gons such that each $(n+1)$-gon is contained within the preceding n-gon and is of maximal area consistent with this constraint. Take the first element of this sequence as an equilateral triangle having unit area. Is the limit of this sequence a point or a circle? If it is a circle, determine its area.

6063. *Proposed by H. J. Marcum, Universidad Federal do Rio de Janeiro, Brazil*

Let S be the set of all circles in the plane provided with the Hausdorff metric ρ induced by the usual Euclidean metric d. (i.e., $\rho(A,B)=\max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$, where $d(a,B)=\inf\{d(a,b): b\in B\}$ denotes the distance from the point a to the set B.)

Let $z: S\to R^2$ be the function which to each circle A assigns its center $z(A)$. Prove that $d(z(A),z(B)) \leqq \rho(A,B)$ for all $A,B\in S$.

6064. *Proposed by H. W. Lenstra, Jr., University of Amsterdam, the Netherlands*

For a nonnegative integer m, let $s(m)$ denote the sum of those divisors d of m for which $1\leqq d<m$. Prove that for every integer $t\geqq 1$ there exists m such that $m<s(m)<s^2(m)<\cdots<s^t(m)$. Here $s^2(m)=s(s(m))$, etc.

6065. *Proposed by the late C. W. Anderson, University of California, Berkeley*

Where $\varphi: N\to N$ is Euler's totient function, it is known that the natural density of $\varphi(N)\subset N$ is zero — in symbols, $d[\varphi(N)]=0$. Where $\sigma: N\to N$ is the sum of the divisors function, demonstrate that $d[\sigma(N)]=0$.

SOLUTIONS OF ADVANCED PROBLEMS

Bound for e^z in an Annulus

4052 [1942, 549]. *Proposed by H. S. Wall*

If $\frac{1}{2}<r\leqq 1$ and $|z|\leqq(2r-1)/4r^2$, prove that

$$\left|e^z-\frac{4r^2}{4r-1}\right|<\frac{2r(2r-1)}{4r-1}.$$

Solution by Predrag S. Milenković, student, University of Belgrade, Yugoslavia. Let us introduce the following notations:

$$b=\frac{4r^2}{4r-1} \quad\text{and}\quad C=\frac{2r(2r-1)}{4r-1},$$

and consider a function $f(\theta)=|e^z-b|^2$, where

$$z=ae^{i\theta},\quad 0\leqq a\leqq\frac{2r-1}{4r^2}\quad\text{and}\quad \tfrac{1}{2}<r\leqq 1,$$

(1) $f(\theta)=e^{2a\cos\theta}-2be^{a\cos\theta}\cos(a\sin\theta)+b^2,$

$$f'(\theta)=2a\sin\theta\, e^{a\cos\theta}\left[\frac{b\cos\theta\sin(a\sin\theta)}{\sin\theta}+b\cos(a\sin\theta)-e^{a\cos\theta}\right].$$

From $f'(\theta)=0$, $a\neq 0$ we infer $\sin\theta=0$ or

$$b\left[\frac{\cos\theta\sin(a\sin\theta)}{\sin\theta}+\cos(a\sin\theta)\right]=e^{a\cos\theta}. \tag{2}$$

Since $\sin(a\sin\theta)/\sin\theta$ is finite for $\sin\theta=0$, i.e. for $\theta=k\pi$ ($\pm k=0,1,2,\cdots$), then the extreme values of $f(\theta)$ for $\theta=k\pi$ are

$$f(k\pi)=(e^{\pm a}-b)^2.$$

If we eliminate $e^{a\cos\theta}$ from (1) by means of (2) we obtain the curve that contains the other extreme values if they exist. Doing the specified operations, we obtain the resulting value

$$f_1(\theta)\equiv(ab)^2\left[\frac{\sin(a\sin\theta)}{a\sin\theta}\right]^2\leqq(ab)^2.$$

The proposed inequality will hold if

$$\max[(ab)^2,(e^{\pm a}-b)^2]<C^2,$$

but this is true for every $r>\frac{1}{2}$.

Generalization. If $f(z)$ is a regular function of the complex variable z and $w=be^{i\alpha}$ ($b\in R$, $0\leqq\alpha<\pi$), then the following inequality is valid

$$|e^{f(z)}-w|\leqq\max[|b|\cdot|f(z)-i\alpha|,|e^{\pm|f(z)-i\alpha|}-b|].$$

The proof may be derived in a similar manner.

Iterates of the Zeta-function

5405 [1965, 674]. *Proposed by A. A. Mullin, Fairfax, Virginia*

Let $\zeta(s)$ be the Riemann zeta function. Define $\zeta^2=\zeta(\zeta(\quad))$ and higher "powers" recursively. Prove that the set of all complex s for which $\lim_{n\to\infty}\zeta^n(s)$ exists is precisely countably infinite. Does there exist a nonreal s for which $\lim_{n\to\infty}\zeta^n(s)$ exists? What is the greatest real value of s for which $\lim_{n\to\infty}\zeta^n(s)$ exists?

Solution by Peter Ungar, Courant Institute, New York University. Let s_0 be arbitrary and let

$$s_1=\zeta(s_0),\qquad s_2=\zeta(s_1),\qquad \text{etc.} \tag{1}$$

If the sequence converges to a limit s^* then s^* satisfies

$$s^*=\zeta(s^*). \tag{2}$$

The contracting mapping theorem says that if (2) holds at a point s^* and

$$|\zeta'(s^*)|<1, \tag{3}$$

then s^* has a neighborhood such that if s_0 is in that neighborhood then $s_n\to s^*$ (attracting fixed

point). However, if

$$|\zeta'(s^*)| > 1, \tag{4}$$

then a sequence (1) can converge to s^* only if all except a finite number of its members are equal to s^*.

A look at a table of values of $\zeta(s)$ for real values of s in Jahnke and Emde's tables reveals that one solution of (2) is $s^* \approx -.296$ and $\zeta'(s^*) \approx -.5$. Consequently, the set of points s_0 for which $\lim s_n$ exists includes a disk around the point $-.296$ and the claim that it is countable is incorrect.

On the interval $1 < s$, $\zeta(s)$ is monotone decreasing and it maps this interval into itself. There is one fixed point which according to the table in Jahnke and Emde is at $s^* = 1.83\cdots$. Since $|\zeta'(s^*)| \approx 1.5 > 1$, the only sequences which converge to s^* are those which have only finitely many elements different from s^*. If $s_0 \in (1, \infty)$, this means $s_0 = s^* = 1.83\cdots$, and hence this is the largest real value of s_0 for which the sequence (1) converges.

A study of the appropriate formulas shows that (2) has infinitely many real solutions and infinitely many non-real solutions. All the solutions except the two given above lie in $\operatorname{Re} s < -\frac{1}{2}$ and satisfy (4), but this really tells us very little about the ζ-function. If instead of $\zeta(s)$ the proposer had selected any other meromorphic transcendental function $f(z)$ at random, the chances are that $f(z) - z$ would have infinitely many zeros and that few, if any, would lie in the set S where $|f'(z)| \leqq 1$. There seem to be two reasons for this: first, the set S will tend to be small, and second, the contracting mapping theorem prevents the occurrence of more than one root in any complex component of S, however large.

Density of Deficient Odd Numbers

5967 [1974, 412]. *Proposed by the late C. W. Anderson, University of California, Berkeley*

In 1521, Giardus Ruffus conjectured that most odd numbers are deficient. Show that the density of odd deficient numbers is at least

$$\frac{48 - 3\pi^2}{32 - \pi^2} = 0.831.$$

Solution by the proposer. Let

$$\Sigma(n) = \sigma(n)/n, \tag{1}$$

where $\sigma(n)$ is the sum of the divisors of n. Now

$$\Sigma : N \to [1, \infty). \tag{2}$$

For our solution, we need the average of $\Sigma(n)$ over the odd numbers N_0, viz.

$$\overline{\Sigma(N_0)} = \pi^2/8 \tag{3}$$

which is proved at the end.

Let us designate the inverse images:

$$\begin{aligned} N_1 &= \Sigma_0^{-1}\{[1, \pi^2/8)\} = \{1, 5, 7, 11, 13, 17, \cdots\} \\ N_2 &= \Sigma_0^{-1}\{(\pi^2/8, 2)\} = \{3, 9, 15, 21, 25, \cdots\} \\ N_3 &= \Sigma_0^{-1}\{[2, \infty)\} = \{945, \cdots\}, \end{aligned} \tag{4}$$

and let

$$\alpha = \text{Density } (N_1), \qquad \beta = \text{Density } (N_2), \qquad \gamma = \text{Density } (N_3). \tag{5}$$

It is known that the numbers (densities in the odd numbers) α, β, and γ exist, since the range distribution function of Σ is continuous; it follows that

$$\alpha + \beta + \gamma = 1. \tag{6}$$

If $n \in N_1$, then $3 \nmid n$ and $3^k n \in N_2$ for all $k \geqq 1$, for we have

$$\Sigma(3) = 4/3 > \pi^2/8, \ \pi^2/8 < \Sigma(3^k) < 3/2. \tag{7}$$

Let $A \subset N$, then Density$(nA) = (1/n)$Density(A), provided Density(A) exists. Hence

$$\beta \geqq \frac{1}{3}\alpha + \frac{1}{9}\alpha + \frac{1}{27}\alpha + \cdots = \frac{1}{2}\alpha, \tag{8}$$

and

$$3\beta \geqq \alpha + \beta = 1 - \gamma. \tag{9}$$

Now let

$$A = \overline{\Sigma_0(N_1)}, \qquad B = \overline{\Sigma_0(N_2)}, \qquad C = \overline{\Sigma_0(N_3)}, \tag{10}$$

(averages or inferior limits in the odd numbers). Clearly

$$1 \leqq A \leqq \pi^2/8, \qquad \pi^2/8 \leqq B \leqq 2, \qquad 2 \leqq C. \tag{11}$$

We now write $\overline{\Sigma(N_0)}$ as

$$\overline{\Sigma(N_0)} = \alpha A + \beta B + \gamma C, \tag{12}$$

or, in case the averages A, B, and C do not exist, we take the appropriate inferior limits and (12) becomes

$$\overline{\Sigma(N_0)} \geqq \alpha A + \beta B + \gamma C. \tag{13}$$

In any case, we have from (11) that

$$\overline{\Sigma(N_0)} \geqq \alpha + (\pi^2/8)\beta + 2\gamma = 1 + (\pi^2/8 - 1)\beta - \gamma. \tag{14}$$

Now from (9), we have

$$\overline{\Sigma(N_0)} \geqq \frac{16 + \pi^2}{24} + \frac{32 - \pi^2}{24}\gamma. \tag{15}$$

Using $\overline{\Sigma(N_0)} = \pi^2/8$, and $\gamma = 1 - \alpha - \beta$, we obtain

$$\alpha + \beta \geqq \frac{48 - 3\pi^2}{32 - \pi^2}.$$

It remains to evaluate the average of Σ over N_0 and obtain (3). Let

$$S_N = \Sigma(1) + \Sigma(3) + \Sigma(5) + \cdots + \Sigma(2N - 1). \tag{16}$$

We have

$$S_N = \sum_{\substack{n=1 \\ n \text{ odd}}}^{2N-1} \frac{1}{n} \sum_{d \mid n} d = \sum_{\substack{n=1 \\ n \text{ odd}}}^{2N-1} \sum_{d \mid n} \frac{d}{n} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{2N-1} \sum_{d \mid n} \frac{1}{d} = \sum_{\substack{d=1 \\ d \text{ odd}}}^{2N-1} \sum_{\substack{j=1 \\ j \text{ odd}}}^{[(2N-1)/d]} \frac{1}{d}, \tag{17}$$

whence

$$\frac{1}{N} S_N = \frac{\pi^2}{8} + O\left(\frac{\log N}{N}\right).$$

The Set of Zeros of Entire Functions with Integral $D^k f(0)$

5968 [1974, 412]. *Proposed by Michael Golomb, Purdue University*

Is the set of zeros of all entire functions F, for which $F^{(k)}(0)$ ($k = 0, 1, \cdots$) are integers, the field of complex numbers? Compare problem 5898 [1974, 414].

Solution by the proposer. The function $f(z)$ is called a Hurwitz function (H.f.) if $f(z) = \sum_0^\infty a_k z^k / k!$ is entire and the coefficients $a_k = D^k f(0)$ are complex integers. We show: Given an arbitrary sequence $\{\alpha_n\}$ ($n = 1, 2, \cdots$) in the complex plane for which $\lim |\alpha_n| = \infty$, there is an H.f. whose set of zeros (with multiplicities) is $\{\alpha_n\}$. This is somewhat more than required by the proposal.

First observe that if f is an H.f. for shich $f(0) = 0$ then $\exp(f)$ is an H.f. Now let g be any entire function whose set of zeros is $\{\alpha_n\}$: $g(z) = z^m (1 + \sum_1^\infty b_k z^k)$. Then

$$\log g(z) = m \log z + \sum_{k=1}^{\infty} \frac{c_k}{k!} z^k$$

with the series converging in some neighborhood of 0. Set $c'_k = [c_k]$, where $[c] = [x] + i[y]$ ($[\cdot]$ denotes the largest integer function). Then

$$h(z) = \sum_{k=1}^{\infty} \frac{c'_k - c_k}{k!} z^k$$

is clearly an entire function and

$$\log g(z) + h(z) = m \log z + \sum_{k=1}^{\infty} \frac{c'_k}{k!} z^k.$$

Then

$$f(z) = g(z) \exp h(z) = z^m \exp\left(\sum_{k=1}^{\infty} \frac{c'_k}{k!} z^k\right)$$

is an H.f. and has the same zeros as $g(z)$.

If $\alpha \in \{\alpha_n\}$ implies $\bar{\alpha} \in \{\alpha_n\}$ (and $\bar{\alpha}$ appears with the same frequency as α) then the function g above may be chosen so that the coefficients b_k are real. Then also the c_k and c'_k are real, and the numbers $D^k f(0)$ are real integers.

The Subring of Commutators

5969 [1974, 413]. *Proposed by Charles Small, Queen's University, Canada*

Let R be a ring; for $x, y \in R$ define $[x, y] = xy - yx$; call elements of the form $[x, y]$ commutators. Prove or disprove: the subring of R generated by all commutators is an ideal. Note that "ideal" is unambiguous because $a[x, yb] + ay[b, x] = a[x, y]b = [ax, y]b + [y, a]xb$. For an affirmative solution it would suffice for example to show that any element of the form $a[x, y]$ is a sum of products of commutators.

Solution by Mark Kidwell, Yale University. The statement is false. As a counterexample, let $R = Z[i, j, k]$, the ring of quaternions with integer coefficients. Then the commutator subring S is generated by $[i, j] = 2k$, $[j, k] = 2i$ and $[k, i] = 2j$. Thus $S = 4Z + 2Zi + 2Zj + 2Zk$. Let I be the commutator ideal. I contains $k[i, j] = -2$, which is not contained in S. Thus $S \neq I$.

Also solved by L. N. Childs, W. H. Gustafson, A. A. Jagers (Netherlands), J. P. Jordan, Michael Josephy, O. P. Lossers (Netherlands), Kenneth Mandelberg, J. G. Mauldon, Peter Neumann & Manley Perkel, William Nuesslein, and Barbara L. Osofsky.

Editor's comment. Many solvers offered the free ring on three generators as a counterexample, and some noted that the answer to the general question is equivalent to the answer in this case.

REVIEWS

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with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.

Introductory Modern Algebra. By Elwyn H. Davis. Charles E. Merrill, Columbus, Ohio, 1974. xii + 241 pp. $10.95. (Telegraphic Review, October 1974.)

One glance at the table of contents of this new text and the experienced reader is assailed by a feeling of *déjà vu.* There it is, the old arrangement: groups, rings, and then fields. But wait. Intended for "the average undergraduate mathematics major... either as a prospective high school teacher or as a prospective graduate student" and inspired by the historical development of abstract algebra, the book begins with a lengthy discussion of permutation groups. The principle of "example first, then theory" is followed whenever practical throughout the book, so that, for example, the concept of an abstract group is introduced only after the student has become familiar with group-theoretic notions in the context of permutation groups.

Introductory Modern Algebra was used by one of the reviewers for a one-semester course in which some students had experience with abstract mathematics, while others had none. Most of the students were prospective high school teachers. They were initially enthusiastic about the clarity of the author's style and enjoyed his development of permutation groups. However, their enthusiasm began to wane with the increasing number of computations required to follow the text's treatment of cosets, Lagrange's Theorem, and factor groups. Depending, of course, on the mathematical maturity of the class, a teacher using this text could easily find himself (or herself) spending so much time on permutations that it would be difficult to cover all the other important topics in a one-semester course. Skipping sections is difficult since the text and exercises frequently refer to examples treated in the first chapter. At this stage, students may be left with the impression that permuation groups are the most important things in abstract algebra. Fortunately, this emphasis disappears with the introduction of rings, where integers, 2×2 matrices, and power sets are used as motivating examples. For both groups and rings, homomorphisms and quotient structures are emphasized.

Along the way, the student reading this text is introduced to the usual number-theoretic topics, polynomials (including Lagrange interpolation), complex numbers, field extensions, and geometric constructions. There is no Galois theory — the reader is led to the foothills and then abandoned. There are some applications: plane patterns (with the structure of a salt crystal mentioned) and binary group codes.

The book seems to contain a disturbing number of errors and/or misprints — incorrect references to figures, at least one exercise that is false without additional hypotheses, incorrect answers to

exercises, and so forth. Most of the exercise sets are divided into three sections: a first batch of problems requiring routine solutions, a second group demanding proofs, counterexamples, and the formation of conjectures, and a third section containing (generally) more difficult computations, often linked to problems in the second section. Several important ideas—for example, the conjugation mapping and the center of a group—are introduced in the exercises. Answers to both odd-and even-numbered problems (but not to all of either class) are given. There are appendices on sets, operations, relations, natural numbers (including the well-ordering principle and induction), and a construction of the rational numbers. A twenty-six entry bibliography is given.

In summary, then, this new text covers much (but not all) of the usual classical material in a way that the average student should find understandable. There is not, however, enough material for the better student, especially if he or she is headed for graduate school.

Henry J. Ricardo, Manhattan College, and
Catherine M. Ricardo, Dominican College

A Geometric Introduction to Topology. By C. T. C. Wall. Addison-Wesley, Reading, Massachusetts, 1972. vi + 168 pp. $7.95. (Telegraphic Review, August-Septemper 1972.)

Topology. By Murray Eisenberg. Holt, Rinehart and Winston, New York, 1974. xiv + 427 pp. $16.00. (Telegraphic Review, June-July 1974.)

My opinion of these two books is based on having used them in two one-semester courses which I taught at Tufts University last year. The classes in which they were used had fewer than ten students each, mostly undergraduate mathematics majors. While the objectives of the two books are different, they were both written basically for undergraduates who have never taken topology before.

Eisenberg's book is a traditional introductory general topology text. The book contains enough material so that it could be used for a variety of courses aimed at different audiences. The author presents several well-planned possibilities in the preface. Since the author is very clear in his explanations and gives many examples and detailed proofs, the students can read and understand most of the sections on their own. The ample collection of problems at the end of each section (583 exercises in all) provides reasonable practice for those students who have read and understood the text. While some problems are quite challenging, others are easy enough to provide the students with the necessary sense of confidence and satisfaction. As the author mentions in his preface, the book seems to be most useful when class time is devoted to problem solving and to adding intuitive supplements when necessary for understanding the text. The relatively few mistakes were easily detected by the students. In general, they seemed to feel that through the reading assignments and homework problems they had finally learned how to read a mathematics book and write proofs — a major breakthrough on the part of the students and a definite compliment to Eisenberg's fine work.

Wall's book, on the other hand, is a beautiful introduction to plane topology and should be a "must" for all undergraduate math majors. I wish this book had been available when I was first introduced to the subject! This is a short concise trip through the essence of plane topology from the contemporary functorial point of view which culminates in the Alexander duality and the proofs of the Jordan curve theorem. Many pretty illustrations are used to aid the reader and the lack of mistakes (typographical or otherwise) is impressive.

Unfortunately we could only cover the first part of this text in class because the majority of the students had never actually been exposed to proofs before. Those who felt familiar with the idea of proving an "abstract statement" (the words of the students) found the book fascinating; some of them even initiated an independent non-credit seminar to finish the book after the semester was over. Others, however, found the book less palatable because it did not show them how to solve the problems, i.e., how to plug the information from a problem into the theorems or formulas in the text. For these students this was the first time they had to use all their previous knowledge,calculus and

sometimes even analytic geometry, to solve a problem. The mere appearance in an exercise of the equation $x_1^2 - x_2^2 - x_3^2 = 1$ defining a hyperboloid of two sheets, for example, baffled them completely. The problems, however, are challenging and well chosen. They expand upon the text by giving the students more concrete objects to work with and they help promote a true feeling for what is going on, a feeling for what mathematics is all about. At times there were also problems with Wall's extremely elegant style. I was often asked to paraphrase and thus destroy the beautifully concise statements which appear throughout his book. I believe, however, that this is not a fault of the text, but rather an indication that many mathematics majors approach graduation, and even graduate, without being able to think independently enough to understand or to prove mathematical statements. Wall's book would be a good text to encourage this type of thought if the instructor is patient enough. However, the book is really most effective when it is used with more mature students as Wall himself suggests.

In summary, Eisenberg's book is good for students who need to learn how to prove a theorem; it guides them and provides them with many detailed examples to study. On the other hand, students who have been exposed to some independent thinking already, but who have not learned topology, simply must read Wall.

TADATOSHI AKIBA, Tufts University

Surreal Numbers. By D. E. Knuth. Addison-Wesley, Reading, Massachusetts, 1974. 119 pp. $3.95 Paper. (Telegraphic Review, May 1975.)

This *tour de force*, subtitled "How two ex-students turned on to pure mathematics and found total happiness, a mathematical novelette," by a pen from which we expect the unexpected, is interesting in several ways — as a sophomore-junior textbook in heuristic (Moore style, but with fierce competition replaced by loving cooperation), an intriguing introduction to a significant bit of contemporary mathematics, a case study in creativity, and a collection of suggestive epimathematical comments.

It is cast in the form of a dialogue between two *simpatico* cop-outs, who are saved from boredom on a distant shore by a stone fragment on which has been carved a partial description of a recursive construction starting "In the beginning everything was void..." We witness and are intended to participate in a reconstruction of John Horton Conway's "extraordinal" numbers ("surreal numbers", "enriched reals," or "reals enriched by infinitesimals and infinite numbers," as one can say if he objects, as I do, to the term "nonstandard" for what may become very standard indeed). Conway's construction has yet to be published, (except in outline in "All numbers, great and small," University of Calgary Mathematics Department Research Paper No. 149, February 1972) and Martin Gardner has opined in his February 1975 *Scientific American* column that this may be the first time that new mathematics has been first published embedded in fiction.

Although the dialogue is inevitably a bit forced at times, students (and faculty who recall their youth) can empathize with the pair of liberated beachcombers. In particular, women can identify with the female member of the pair who is in no way inferior to her partner, and everyone who is tired of sexism in the mathematical community can applaud the author for portraying her that way, her partner as completely comfortable living with her in equality, and both of them as sexually stimulated by their joint mathematical endeavors.

Knuth says that he intends to teach "how one might go about developing such a theory" rather than the theory itself, but the book will in fact do both in an enjoyable way. It is also a case study in heuristic, because the author has based it on his own efforts (including false turns) that relied only on "a vague memory of a lunchtime conversation" with Conway. Finally, Knuth uses it as a vehicle for suggesting "important principles, techniques, joys, passions, and philosophy of mathematics." This is explained in a postscript, which includes also some supplementary problems and a proposal that students write an essay which should be graded on expository style as well as mathematical content.

He concludes: "In my opinion the two weaknesses in our present mathematics education are the lack of training in creative thinking and the lack of practice in technical writing. I hope that the use of this little book can help make up for both of these deficiencies." Right on!

But why "pure" in the subtitle and elsewhere? What is pure about this mathematics? Does "pure" refer to the autotelic (purely aesthetic) nature of the couple's endeavor, to their lack of external motivation (ignoring the heterotelic aphrodisiac effects)? If so, the adjective should be applied to them or their inclinations, not to the mathematics, which will be the same whatever the motives of its creators. Does "pure" suggest a lack of applications of the theory? If so, it is certainly out of place since the number system being constructed is undoubtedly the most widely useful intellectual tool consciously devised by man. Does "pure" indicate no possibility of venal uses? Hardly, as the young people themselves point out. Can it refer to the manner in which the theory is constructed, axiomatically and allegedly without intuition or explicit reference to experience? If so, it is misapplied, since the couple (like all mathematicians) constantly refer to experience with familiar numbers and concepts from the "new math," always trying to find analogous properties, relations, and operations. Would it not be the same mathematics if the pair were motivated by making money (instead of merely avoiding boredom) or by the desire for social usefulness or fame, if they were guided by applications and strong intuitions, and proceeded informally by experimentation, observation, and verification; and if they established their results by plausible rather than by "rigorous" proof? Yes, it certainly would be the same, and indeed that is just how they do proceed! Knuth's emphasis is on insight, observation, and heuristic rather than formalism. Indeed we get a picture of mathematics and mathematical activity — unmodified by any objective. Leaving out "pure" throughout would simply remove a slight whiff of romantic elitism. Our young couple would have said just "mathematics" anyway, since they have not had enough university courses to have been corrupted by the philosophically indefensible pure-applied dichotomy and imagine quite correctly that there is just mathematics, any part of which may be applied within or outside mathematics without thereby changing its mathematical character.

KENNETH O. MAY, University of Toronto

Real Analysis: *An Introductory Course.* By J. R. Giles. Wiley, New York, 1972, ix + 171 pp. $13.50. (Telegraphic Review, August–September 1974.)

According to the author, this book is intended to bridge the gap between a typical non-theoretical calculus course and a theoretical approach to a real analysis-advanced calculus course. I used this book as a text for an introductory analysis course for prospective secondary school teachers and found it quite suitable for that purpose. The book is short (120 pages, excluding appendices and solutions to exercises); consequently, there are some notable omissions. Countability is not mentioned, nor are any topological considerations, though I found the ideas of limit points and open and closed sets easily introduced with a couple of definitions and some additional exercises. Lebesgue's theorem on Riemann integration is not mentioned, and quite easily could be added with the introduction of Lebesgue measure zero, which I did. The book is suitable for a leisurely-paced semester course. In two quarters we covered the entire book and then supplemented it with a large selection of material on inequalities from Chapters I and VI of *Convex Functions* by Roberts and Varberg. This material was well motivated because of several exercises on convex functions throughout the book.

The basic aim of the book is to show how the theories involving limits (sequences, series, integration, etc.) are based on the completeness property of the set R of real numbers. After a brief chapter on the structure of N, Q, and R, chapters on sequences, series, continuity, differentiability follow in well-motivated, logical fashion. Completeness and compactness properties in R are discussed thoroughly, and series are motivated by a study of decimal representation of real numbers. Unfortunately, the main theorem in this section (page 34) is not stated correctly, but I

found it useful to ask the students to make the necessary correction. Continuity, uniform continuity, and differentiability are well done and fairly complete, but only in the single variable case. As in the previous chapters, there is in the chapter on integration a substantial collection of challenging excercises, one of which is false (exercise 6, page 112). There is an excellent appendix on the logarithm and exponential mappings, and there are 39 pages of solutions to "selected exercises." Unfortunately, all of the more challenging exercises have been "selected," and extensive solutions appear. I would have preferred less substantial hints.

In my opinion, Giles has accomplished his stated purpose quite well with this book which, except for a small number of misprints, is well written and readable. To cover the material thoroughly in such a short book is an accomplishment which both instructor and student should appreciate.

R. M. MATHSEN, North Dakota State University

Introductory Statistics for the Behavioral Sciences. By Joan Welkowitz, Robert B. Ewen, and Jacob Cohen. Academic Press, New York, 1973. xvi + 271 pp. $9.50; *Workbook to Introductory Statistics for the Behavioral Sciences.* By Robert B. Ewen. Academic Press, New York, 1971. xiv + 155 pp, $2.95.

For the past two years we have used this text and workbook for a one semester introduction to statistics for undergraduates majoring in social and natural sciences. For the most part their last formal mathematics was algebra, although currently, many are first taking a course in finite mathematics.

We feel that, on the whole, Welkowitz, et al., has worked well in this situation, primarily because of its balance between the "why" and the "how" of statistics. The authors' style is very informal and not abstract; their examples concern people, not colored balls in urns. On the other hand, the text avoids being merely a compendium of techniques, and conveys an understanding of some of the underlying principles. For the purist, some algebraic proofs are presented in appendices at the ends of the chapters. Others are in the workbook. Although the examples are slanted somewhat towards psychology, we have encountered no adverse reaction from our diversity of students; in fact, our classes have commented favorably on the readability of the text.

The topics covered are, with one major exception, the traditional ones for a course of this type: descriptive statistics, a (perhaps too) brief discussion of discrete probability, and inferential statistics, including one and two way analysis of variance. An unusual feature is the chapter on power analysis, containing an excellent discussion of the trade-off between sample size (cost) and the conditional probability of making a Type I or Type II error (information). Although no derivation is provided, the chapter gives a concrete answer to the question of sample size. The authors include an unusually cogent discussion of the disadvantages inherent in one tailed tests. We have found it possible to cover essentially the entire text in one semester.

The only major shortcoming we found was the workbook. Although the conceptual problems are very good, we both feel that more computational problems are needed. Some problems depend on answers to prior problems; hence we would like to see at least some of the answers to the problems listed in the workbook. The assumptions underlying some of the tests are omitted. Also omitted is a discussion of the misuse of statistics; we use S. K. Campbell's *Flaws and Fallacies in Statistical Thinking* (reviewed in *J. Am. Stat. Assoc.*, March, 1975) as a supplement. Other minor problems include an occasional confusion between one and two tailed tests, between unbiased estimators and sample moments, and a chapter on linear regression that fails to emphasize strongly enough its use in prediction. We hope these problems will be cured in a forthcoming revision.

H. P. GREENOUGH, Marshall University
J. S. LANCASTER, Marshall University

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook | P = professional reading
S = supplementary reading | L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, L***, *Creative Teaching: Heritage of R.L. Moore*. D. Reginald Traylor. U of Houston, 1972, iv + 469 pp, $8.50 (P). An impressive monument to one of America's most creative and effective mathematics teachers: 200 typed pages of biography, emphasizing the Moore-method and the fifteen year struggle against his enforced retirement, followed by a 250 page list and bibliography of Moore's mathematical Ph.D. descendants. LAS

GENERAL, T(13-14), S, L. *Quatre cours de Mathématiques*. A. Markouchévitch. MIR, 1973, 245 pp. Designed for people who "wish to continue to learn, whose mathematical knowledge stops at secondary school." The four lectures are: remarkable curves: conic sections, cycloids, lemmiscates; area and logarithms; recursive sequences; complex numbers and conformal maps, concluding with the Jovkorski airfoil. PJM

GENERAL, P. *Dissertationen in Mathematik an den Hochschulen der Bundesrepublik Deutschland in der Zeit von 1961 bis 1970: Eine Bibliographie*. P.L. Butzer, E.L. Stark. Teubner, Stuttgart, 1975, xii + 101 pp, 10,80 DM (P). Titles of dissertations in mathematics and a few in related areas. Arranged by university and date. Can be ordered only from Dr. E.L. Stark, Lehrstuhl A für Mathematik, RWTH Aachen, D 51 Aachen, Templergraben 55, West Germany. JAS

GENERAL, P. *Jahrbuch Überblicke Mathematik 1975*. Benno Fuchssteiner, *et al*. Bibliographisches Inst, 1975, 181 pp, (P). Six expository essays, plus reports on the 1974 International Congress of Mathematicians and other miscellaneous items. This is the last *Jahrbuch* to be published by *Überblicke Mathematik*. LAS

GENERAL, P**, L. *Invisible Colleges: Diffusion of Knowledge in Scientific Communities*. Diana Crane. U of Chicago Pr, 1972, x + 213 pp, $9; $2.95 (P). Thesis: the social institution responsible for the growth of scientific knowledge is the "invisible college", a small group of highly productive scientists who monitor the rapidly changing structure of knowledge in their field. Two careful case studies provide data: rural sociology and finite group theory. A perceptive sociological analysis of the way folklore dominates research directions. Should be read by anyone concerned about the financial and intellectual feasibility of research journals, citation indices and related problems of scientific communication. LAS

GENERAL, P. *Transactions of the Twentieth Conference of Army Mathematicians*. US Army Research Office, Durham, N.C., 1974, xxi + 682 pp, $15.25 (P). Papers from a conference at Natick, Massachusetts, May 1974. LAS

BASIC, T(13: 1). *Intermediate Algebra*. Frances S. Mangan. Merrill, 1975, x + 470 pp, $10.95 (P). Intended for self study or a traditional course. Many drill problems with the answers given for all of them. The chapter on sequences and series is confused and confusing. FLW

BASIC, T(13: 1). *Elementary Functions: Trigonometry*. Gus Klentos, Joseph Newmyer, Jr. Merrill, 1974, viii + 414 pp, $10.95 (P). A programmed text designed for use with audiotapes. Introduces trigonometric functions as circular functions. Includes identities, inverse functions, solving triangles, and complex numbers. PJM

BASIC, T(13). *Fundamentals of Trigonometry, Third Edition*. Earl W. Swokowski. Prindle, 1975, viii + 243 pp, $10.50. Changes from second edition (TR, February 1972): prerequisite (functions, etc.) and logarithm chapters revised; more examples in the introduction to circular functions. PJM

PRECALCULUS, T(13: 1), L. *Modern Analytic Geometry, Third Edition*. W.K. Morrill, Samuel M. Selby, Wendell G. Johnson. Intext, 1972, xii + 481 pp, $10. Third edition adds some modern terminology, retains vector approach. More depth than customary in new texts. Part I: introduction, point and plane vectors, straight line, circle, conics, transformation of axes, polar coordinates, transcendental and other curves. Part II: point and space vectors, plane, straight lines in space, surfaces and curves. RBK

PRECALCULUS, T(13: 1). *Analytic Geometry, Second Edition*. Murray H. Protter, Charles B. Morrey, Jr. A-W, 1975, xi + 370 pp, $9.95. This edition contains a new chapter on analytic geometry in four dimensions. A large number of new and challenging problems have been added. A well written and thorough text. CEC

EDUCATION, P, L*. *Games and Puzzles for Elementary and Middle School Mathematics: Readings from the Arithmetic Teacher*. Ed: Seaton E. Smith, Jr., Carl A. Backman. NCTM, 1975, viii + 280 pp, $4.50 (P). Over 100 brief papers, ranging from whole numbers to geometry. LAS

EDUCATION, S(15-16). *Beiträge zum Mathematikunterricht 1975*. Hermann Schroedel, 1975, 271 pp, 19 DM (P). Lectures and discussions from the 1975 annual meeting of the *Bundestagung für Didaktik der Mathematik*. Includes a wide variety of subjects appropriate for kindergarten through junior college. JAS

EDUCATION, T(14-15: 2). *Unified Mathematics, Content, Methods, Materials for Elementary School Teachers*. Arnold L. Fass, Claire M. Newman. Heath, 1975, xii + 465 pp, $10.95. For a unified methods and mathematics course. Includes content from CUPM Level I such as sets, logic, integers, rational numbers, axiomatic approach in mathematics, probability and statistics, geometry and measurement. Exercises and discussion questions at the end of each chapter only. PSJ

EDUCATION, S(16), P. *Mathematik Lehrerausbildung: Lineare Algebra,* Th. Kreutzkamp, W. Neunzig. Teubner, Stuttgart, 1975, 136 pp, (P); *Nichteuklidische Elementargeometrie,* G. Buchmann, 126 pp, (P); *Zahlbereichserweiterungen,* G. Messerle, 119 pp, (P). Books in a series written for prospective teachers in German schools other than the *Gymnasium,* to which the academically ablest students usually go. American specialists in the teaching of mathematics should find them of considerable interest. JD-B

EDUCATION, T(13-14), S. *Initiation mathématique: Activités mathématiques des enfants de cinq à six ans. Suggestions à l'usage des maitres*. Jean and Suzanne Daniau. CEDIC, 1975, 174 pp, 32,00 Fr (P). The title of this book is an accurate description--activities to initiate 5 qnd 6 year old children to mathematics. Preceded by a thorough chapter on intentions and methodological considerations the book starts with discussion of perception of space (up and down, left and right, etc.) and concludes five chapters later with "The child approaches the notion of number." An excellent book for future elementary teachers or mathematicians with small children. PJM

EDUCATION, P. *Statistics at the School Level*. Ed: Lennart Råde. Almqvist & Wiksell, 1975, 242 pp, $19.95. Proceedings of an International Statistical Institute "Round Table Conference" held in Vienna in September 1973 on the teaching of statistics at the secondary school level. LAS

EDUCATION, P. *Research on Mathematical Thinking of Young Children*. Ed: Leslie P. Steffe. NCTM, 1975, v + 202 pp, $3.90 (P). Six empirical studies on the acquisition of logical and quantitative structures in 4 to 8 year old children, followed by a comparative, summary essay. LAS

HISTORY, P, L. *Stability of Motion*. Ed: A.T. Fuller. Halsted Pr, 1975, ix + 228 pp, $19.95. A reprint of E.J. Routh's "brilliant essay" of 1877, "Stability of a given state of motion", together with related papers by Routh, Sturm, Clifford and Bôcher. Routh's test for stability is one of the foundation stones of modern control theory. LAS

HISTORY, P, L. *Kunihiko Kodaira: Collected Works*. Walter L. Bailey, Jr. Princeton U Pr, 1975. *V. I:* xx + 647 pp; *V. II:* x + 493 pp; *V. III:* x + 479 pp, $48.50 set. 70 papers arranged chronologically preceded by a personal and mathematical biography by the editor. LAS

FOUNDATIONS, T(15-16: 1), S, P, L. *Nonarchimedean Fields and Asymptotic Expansions*. A.H. Lightstone, Abraham Robinson. Math. Lib., V. 13. North-Holland, 1975, x + 204 pp, $24.95. The authors show that infinitesimals and infinitely large numbers form a natural background to asymptotics. All necessary resources from mathematical logic are introduced as needed. The arguments are elementary and unusually detailed. The book can be strongly recommended as an introduction to nonstandard analysis. Overpriced at 12¢ per page. LCL

FOUNDATIONS, T(15). *A Bridge to Advanced Mathematics*. Dennis Sentilles. Williams & Wilkins, 1975, xiii + 387 pp, $14.75. Designed to bridge the gap between procedural and conceptual mathematics courses. Part I discusses logic, methods of proof, and the historical development of mathematics. Part II uses topology as a vehicle to learn the "art of proof" and the nature of modern mathematics. More wordy than most mathematics books, which may be helpful to some students. RBK

FOUNDATIONS, P. *Lecture Notes in Mathematics-447: Tableau Systems for First Order Number Theory and Certain Higher Order Theories*. Sue Toledo. Springer-Verlag, 1975, 339 pp, $12.90 (P). Aspects of proof theory that have developed out of Gentzen's work, including constructive consistency proofs for first order number theory and topics in pure second order logic and type theory. CEC

FOUNDATIONS, T*(16-17: 1), S*, P, L*. *An Algebraic Introduction to Mathematical Logic*. Donald W. Barnes, John M. Mack. Grad. Texts in Math., V. 22. Springer-Verlag, 1975, viii + 121 pp, $10.80. A mathematical study of the logic used in mathematics--as opposed to foundational studies based on general logic with all its subsequent complications about precise meanings of words. In this treatment, the metalanguage includes our existing knowledge of mathematics--especially universal algebra. A delightful approach of interest and value to all mathematicians. Propositional and predicate calculus, Zermelo-Fraenkel set theory, non-standard models, decision processes. LCL

FOUNDATIONS, P, L. *Logic Colloquium '73*. Ed: H.E. Rose, J.C. Shepherdson. Stud. in Logic and Found. of Math., V. 80. North-Holland, 1975, viii + 513 pp, $49.95. Most of the papers from a colloquium at Bristol, July 1973, on philosophy of mathematics, proof theory, category theory and the theory of computation. LAS

NUMBER THEORY, T(18: 1), S, P. *Transcendental Number Theory*. Alan Baker. Cambridge U Pr, 1975, x + 147 pp, $13.95. A remarkable amount of material is presented in a comprehensive and systematic account of the recent major advances in this area. Includes an extensive bibliography. CEC

NUMBER THEORY, L. *The Factor Book*. R.L. Hubbard. Hilton Management Serv, 1975, $13.75 (P). Prime factorization for all integers between 1 and 100,000. CEC

NUMBER THEORY, S*(13), L*. *Pascal's Triangle*. V.A. Uspenskii. Trans: David J. Sookne, Timothy McLarnan. U of Chicago Pr, 1974, vii + 35 pp, $2.50 (P). A delightful (but expensive) little monograph suitable for the man on the street. CEC

NUMBER THEORY, P. *Proceedings of the International Conference on Number Theory*. Ed: I.M. Vinogradov, *et al*. Proc. of Steklov Inst. of Math., No. 132. AMS, 1975, v + 298 pp, $30.10 (P). Report of a conference in Moscow, September 1971. LAS

NUMBER THEORY, P. *Lecture Notes in Mathematics-437: Elliptic Functions and Transcendence.* David Masser. Springer-Verlag, 1975, xiv + 143 pp, $7.80 (P).

LINEAR ALGEBRA, T(14-15: 1, 2), S, L. *Linear Algebra and Geometry.* Kam-Tim Leung. Hong Kong U Pr, 1974, x + 309 pp, (P). First course in linear algebra and geometry at University of Hong Kong. Easy to read, symbolism held to a minimum. Features a coordinate-free treatment of linear spaces (real or complex throughout; infinite dimensional case included), linear transformations (including section on the category of linear spaces), affine and projective geometry. Followed by standard material on matrices and determinants (eigenvalues through Jordan forms; inner product spaces and their transformations). LCL

LINEAR ALGEBRA, S(14-16). *Le Livre du Problème, Volume 5: Calcul Barycentrique.* I.R.E.M. de Strasbourg. CEDIC, 1975, 128 pp, 18,00 (P). Barycenters for geometry, physics, etc., are introduced via brief expositions and "do-it-yourself" problems. The last half of the book is devoted to solutions and commentary. An unusual unifying presentation, with a bit of R.L. Moore flavor. JAS

LINEAR ALGEBRA, T(16-17: 1), P, L. *Lectures in Abstract Algebra II. Linear Algebra.* Nathan Jacobson. Grad. Texts in Math., V. 31. Springer-Verlag, 1975, xii + 280 pp, $14.80. Reprint of the 1953 van Nostrand edition. LAS

ALGEBRA, T(15-16: 2), L. *Introduction to Abstract Algebra.* Louis Shapiro. McGraw, 1975, x + 340 pp, $12.95. Suitable for undergraduates. Fairly conventional in coverage but differs from most of its competitors in three ways. The problems are more important to the development. There are many historical references. At the end are three "surveys", which treat the Wedderburn theorems, group representation and Galois theory, but omit many proofs. JD-B

ALGEBRA, S(17-18), P. *Lecture Notes in Mathematics-441: PI-Algebras, An Introduction.* Nathan Jacobson. Springer-Verlag, 1975, 115 pp, $7.80 (P). For a course on ring theory given at Yale in 1973. Presents the theory of algebras with polynomial identity over a commutative coefficient ring, using recent results of Formanek and Rowen, and a detailed account of Amitsur's construction of non-crossed product division algebras. No index, a very short bibliography. JD-B

ALGEBRA, P. *Lecture Notes in Mathematics-444: Prime Spectra in Non-Commutative Algebra.* F. van Oystaeyen. Springer-Verlag, 1975, 128 pp, $7.80 (P). Topics: symmetric localization and pseudo-places of algebras over fields. LCL

ALGEBRA, T(18: 1, 2), P. *Linear Algebraic Groups.* James E. Humphreys. Grad. Texts in Math., V. 21. Springer-Verlag, 1975, xiv + 247 pp, $18.80. A "fairly linear" organization of the Borel-Chevalley theory including developments over the past two decades. LCL

ALGEBRA, T(16-17: 1, 2), S, L. *An Introduction to Group Representation Theory.* R. Keown. Math. in Sci. and Eng., V. 116. Acad Pr, 1975, xi + 331 pp, $21. All representations are real or complex, arguments are computational in nature, emphasis on understanding nontrivial examples which illustrate those ideas and computational methods. More accessible to undergraduates than most books bearing this title. Nominal prerequisites: linear algebra, abstract algebra. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-456: Trivial Extensions of Abelian Categories.* Robert M. Fossum, Phillip A. Griffith, Idun Reiten. Springer-Verlag, 1975, xi + 122 pp, $7.80 (P). A trivial extension of an abelian category is a generalization of the trivial (representing 0 in $H^2(R,M)$) extension of a ring R by a bimodule M. These notes are concerned with the homological algebra of such objects as determined by the category (ring R) and extending endofunctor (bimodule M). PJM

ALGEBRA, P. *Symmetric Closed Categories.* W.J. de Schipper. Math. Centre Tracts, No. 64. Math. Centrum, 1975, 197 pp, Dfl. 21 (P). A closed category is one where the hom sets can be endowed with structure to make them objects in the category. Symmetric closed categories are closed categories with natural isomorphisms $S_{xyz}:(x,(y,z)) \to (y,(x,z))$. PJM

FINITE MATHEMATICS, T(13: 2), L. *Numbers & Mathematics, Second Edition.* Clayton W. Dodge. Prindle, 1975, xvi + 557 pp, $12.95. Emphasis on development of the real and complex numbers, together with chapters on logic and probability; includes a section on proof based on Polya. Brief historical introductions to each chapter. Especially appropriate for prospective teachers. JG

FINITE MATHEMATICS, T(13: 1, 2). *Introduction to Mathematics.* Ramakant Khazanie, Daniel Saltz. Goodyear, 1974, viii + 472 pp, $13.95. Truth tables, the language of sets, the number systems, some elementary plane and analytic geometry, probability, and descriptive statistics. FLW

FINITE MATHEMATICS, T(15: 1). *Modern Mathematical Methods in Technology, V. 2.* S. Fenyö. Appl. Math. and Mech., V. 17. North-Holland, 1975, viii + 326 pp, $39.75. Three chapters which present matrix theory, linear and convex optimization theory, and graph theory for "technologists and engineers." Attractive, with drawings and some applications but no exercises or material on numerical methods. Finding three separate, equivalent texts which together are cheaper should be no problem.DFA

FINITE MATHEMATICS, T(13-14: 1, 2). *Finite Mathematics, An Elementary Approach.* Lawrence G. Gilligan, Robert B. Nenno. Goodyear, 1975, xiii + 463 pp, $12.95. Propositional logic, set algebra, permutations, combinations, probability, matrix algebra, linear programming, two-person games, descriptive statistics, and some examples of models. FLW

FINITE MATHEMATICS, T(13: 2). *Finite Mathematics.* N.A. Weiss, M.L. Yoseloff. Worth, 1975, xv + 628 pp, $12.95; *Instructor's Manual,* 140 pp, (P). Extensive treatment of topics in probability and statistics, as well as chapters on logic, set theory, and linear programming. Many examples for motivation. *Instructor's Manual* has abstracts of each section. JG

CALCULUS, T(13: 1, 2). *The Power of Calculus, Second Edition.* Kenneth L. Whipkey, Mary Nell Whipkey. Wiley, 1975, xii + 369 pp, $10.95. Changes from the first edition include appendix material on basic algebra, more answers and more applied exercises. Also in several places, rigor has been replaced by more intuitive methods, e.g., limits and the definite integral have been made less formal. TAV

CALCULUS, T(13-14: 2, 3). *Calculus With Analytic Geometry.* Earl W. Swokowski. Prindle, 1975, ix + 854 pp, $17.95. Unlike several recent calculus offerings, the book has no gimmick. It appears to be a well organized, clearly written text with a large number of well chosen exercises. The amount of rigor seems appropriate. A well executed text, worthy of consideration. TAV

CALCULUS, T*(14-15: 2). *Advanced Calculus, Pure and Applied.* Peter V. O'Neil. Macmillan, 1975, xix + 622 pp, $15.95. From the preface: "...written partly with a view toward opposing...an unhealthy trend toward excessive abstraction in advanced calculus courses." The style is conversational. Emphasis on problem solving. In addition to typical topics, the book covers calculus of variations, complex functions, Fourier transforms. TAV

CALCULUS, S(13-14). *Problems in Calculus of One Variable.* I.A. Maron. Trans: Leonid Levant. MIR, 1975, 453 pp. A Russian version of a Schaum outline, revised and translated from the 1973 edition. Covers the basic elements of single variable calculus: basic definitions precede problems, some with complete solutions, other with hints and answers. LAS

REAL ANALYSIS, T(17-18: 2), P, L. *Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable.* Edwin Hewitt, Karl Stromberg. Grad. Texts in Math., V. 25. Springer-Verlag, 1975, xi + 476 pp, $16.80. Third printing of this 1965 classic--at nearly double its original price. LAS

REAL ANALYSIS, P. *Lecture Notes in Mathematics-463: Gaussian Measures in Banach Spaces.* Hui-Hsiung Kuo. Springer-Verlag, 1975, vi + 224 pp, $9.90 (P). From a 1974 course at Virginia. Discusses these measures, their equivalence and orthogonality, and gives various results about abstract Wiener space. Includes exercises. DFA

COMPLEX ANALYSIS, P, L. *Potential Theory in Modern Function Theory.* M. Tsuji. Chelsea, 1975, x + 590 pp, $17.50. Reprint of the 1958 original Tokyo edition. LAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-467: The* $\cos\pi\lambda$ *Theorem.* Matts R. Essén. Springer-Verlag, 1975, vii + 112 pp, $7.80 (P). Presents new results of the author and discusses some of A. Baernstein II; includes a paper concerning an inequality for a class of harmonic functions in n-space by C. Borell. DFA

DIFFERENTIAL EQUATIONS, T***(14-15: 1, 2), S, L***. *Differential Equations and Their Applications: An Introduction to Applied Mathematics.* M. Braun. Appl. Math. Sci., V. 15. Springer-Verlag, 1975, xiv + 718 pp, $14.80 (P). An outstanding beginning text in differential equations. The book includes several applications which are entire case studies. It is mathematically sound with lots of good exercises and examples. CEC

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-442: Scattering Theory for the d'Alembert Equation in Exterior Domains.* Calvin H. Wilcox. Springer-Verlag, 1975, 184 pp, $8.60 (P). From lectures in 1974 at Tulane and Stuttgart. Constructs eigenfunction expansions for the Laplacian in Hilbert space and calculates the asymptotic form of solutions of the d'Alembert equation for large values of the time parameter. DFA

DIFFERENTIAL EQUATIONS, P. *Introduction to Spectral Theory: Self-adjoint Ordinary Differential Operators.* B.M. Levitan, I.S. Sargsjan. Trans. Math. Mono., V. 39. AMS, 1975, xi + 525 pp, $48.50. Concerns principally modern spectral theory of one second-order equation and a system of two first-order ones of which Dirac systems are a special case. Studies regular problems, but emphasis is on singular ones. DFA

DIFFERENTIAL EQUATIONS, P. *International Conference on Differential Equations.* Ed: H.A. Antosiewicz. Acad Pr, 1975, xix + 834 pp, $26.50. Proceedings of the conference held at the University of Southern California in September 1974. JAS

DIFFERENTIAL EQUATIONS, P. *Boundary Value Problems of Mathematical Physics, VIII.* Ed: O.A. Ladyzenskaja. Proc. of Steklov Inst. of Math., No. 125. AMS, 1975, vi + 217 pp, $31.30 (P).

NUMERICAL ANALYSIS, P. *Metode Numerice Pentru Rezolvarea Ecuatiilor Diferentiale.* Alexandru I. Schiop. Editura Academiei Romania, 1975, 161 pp, (P). Numerical methods for linear and non-linear differential equations with a chapter on approximation techniques for integral equations. JAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-417: Differential Calculus in Locally Convex Spaces.* H.H. Keller. Springer-Verlag, 1974, 143 pp, $7.40 (P). Investigates various concepts of continuous differentiability, depending on preassigned topologies or convergence structures on spaces of continuous linear mappings. Develops differential calculus for functions between locally convex spaces and establishes relations to pre-existing differential theories. I-CH

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-433: Self-Adjoint Operators.* William G. Faris. Springer-Verlag, 1975, vii + 115 pp, $7.80 (P). Reports recent developments in problem of when the sum of two unbounded self-adjoint operators is self-adjoint. Stimulated by developments in quantum field theory, new results exploit positivity. Chapters: forms and operators; operator domains; self-adjoint extensions; moments; references. Theory illustrated throughout by Schroedinger equation example. RBK

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-323: Perturbations Singulières dans les Problèmes aux Limites et en Controle Optimal.* J.L. Lions. Springer-Verlag, 1973, xii + 645 pp, $15.60 (P). Evolved from lectures (1970-1971) given to doctorate level students at the University of Paris VI. Deals with 'passing to the limit' problem for families of partial differential operators. Investigates classification of singularities, perturbation techniques and regularization processes. Contains serious and powerful applications to optimal control theory. I-CH

FUNCTIONAL ANALYSIS, P. *Invariant Subspaces of Hardy Classes on Infinitely Connected Open Surfaces.* Charles W. Neville. Memoirs No. 160. AMS, 1975, viii + 151 pp, $4.20 (P). Classification, for each admissible surface R, of the closed subspaces of the Hardy class $H^p(R)$ which are invariant under multiplication by each bounded analytic function on R. References. RJA

FUNCTIONAL ANALYSIS, T(18: 1), P. *Volterra Stieltjes-Integral Equations.* Chaim Samuel Hönig. Math. Stud., V. 16. North-Holland, 1975, x + 157 pp, $9.50 (P). Studies Volterra-Stieltjes integral equations with linear constraints (including initial conditions, boundary conditions, periodicity conditions, multiple point conditions, interface conditions, etc.). Solves these equations via Green functions, whose existence is characterized by finding the resolvents of the equations and the integral representations of the linear constraints. I-CH

FUNCTIONAL ANALYSIS, T(18: 1), P*, L. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness.* Michael Reed, Barry Simon. Acad Pr, 1975, xv + 361 pp, $24.50. Two chapters continuing the work begun in Volume I (TR, October 1972) to explicate functional analysis in the context of modern mathematical physics. Extensive historical notes are themselves almost a course in the history of functional analysis. Three chapters originally planned for this volume will be published later as Volume III. LAS

OPTIMIZATION, T*(17-18: 1), S, P. *Optimal Control Theory.* L.D. Berkovitz. Appl. Math. Sci., V. 12. Springer-Verlag, 1974, ix + 304 pp, $9.50 (P). A broad, yet relatively deep, concise and coherent introduction to the mathematical theory of optimal control of processes governed by ODE's. Draws examples from production planning, engineering and mechanics; formulates the control problems mathematically; proves the existence theorems with or without convexity assumptions; derives the maximum principle using a dynamic programming argument. I-CH

OPTIMIZATION, T(14-16: 1), S, L. *Integer Programming: Theory, Applications, and Computations.* Hamdy A. Taha. Acad Pr, 1975, xii + 380 pp, $19.50. Three basic techniques: zero-one enumeration, branch-and-bound method, and cutting-plane method. Applications. Problems. References. Index. RJA

OPTIMIZATION, P. *Lecture Notes in Computer Science-27: Optimization Techniques IFIP Technical Conference.* Ed: G.I. Marchuk. Springer-Verlag, 1975, viii + 507 pp, $16.80 (P). Proceedings of a conference held at Novosibirsk in July 1974. JAS

OPTIMIZATION, T(14-16: 1), S, L. *Nonlinear and Dynamic Programming: An Introduction.* Sven Danø. Springer-Verlag, 1975, 164 pp, $12.50 (P). Intended for students of managerial economics and operations research, this slim monograph provides an applications-oriented survey of nonlinear (esp., quadratic and dynamic programming). Prerequisite: introductory knowledge of linear programming. Occasional exercises in footnotes, with selected answers at back of book. LAS

ANALYSIS, T(18: 1), P. *Lectures on Harmonic Analysis.* Charles N. Kellogg. Math. Ser., No. 11. Texas Tech U, 1975, iii + 169 pp, $7.50 (P). Harmonic analysis on the unit circle (45 pages) and the real line (82 pages), with an introduction to Hardy spaces on the unit circle (41 pages). Exercises of quite varying difficulty follow each of the 26 chapters. Printed from typescript. DFA

ANALYSIS, P. *The Lebesgue-Nikodym Theorem for Vector Valued Radon Measures.* Erik Thomas. Memoirs No. 139. AMS, 1974, ii + 101 pp, $3.20 (P).

ANALYSIS, T(15-16: 1-4). *Advanced Engineering Mathematics, Fourth Edition.* C. Ray Wylie. McGraw, 1975, xii + 937 pp, $16.50. Suitable for postcalculus "applied" courses in: (1) ordinary and partial differential equations; (2) linear algebra, calculus of variations, vector and tensor analysis; (3) elementary complex variable. A rewritten third edition with some new material and 750 new exercises (for a total of 2137). DFA

ANALYSIS, P, L. *Orthogonal Polynomials, Fourth Edition.* Gabor Szegö. AMS, 1975, xiii + 432 pp, $22.70. A reprinting of the third (1966) edition, with "minor changes and new material" including an up-to-date supplementary bibliography. LAS

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-439: Classification Theory of Algebraic Varieties and Compact Complex Spaces.* Kenji Ueno. Springer-Verlag, 1975, xix + 278 pp, $12.10 (P). Notes of lectures on the classification of birational (resp. bimeromorphic) equivalent classes of complete algebraic manifolds (resp. compact complex manifolds). Contains introductory historical notes, several examples, no exercises. Well-written. SG

ALGEBRAIC GEOMETRY, P. *Algebraic Geometry Arcata 1974.* Robin Hartshorne. Proc. of Symp. in Pure Math., V. 29. AMs, 1975, xiv + 642 pp, $40.10. Lectures and seminar talks from the July and August 1974 summer institute at Arcata, California. JAS

GEOMETRY, T(18: 3, 4), S, P*, L**. *A Comprehensive Introduction to Differential Geometry.* Michael Spivak. Publish or Perish, 1975. *V. 4:* v + 561 pp, $12 (P); *V. 5:* v + 661 pp, $13 (P). Volume four generalizes the geometric material of the preceeding volumes to general dimensions and codimensions. Volume five appears to complete the work with chapters on a number of major results and surveys of interconnections with mathematics as a whole, e.g., partial differential equations, imbeddings, rigidity, and the Gauss-Bonnet Theorem. JAS

GEOMETRY, S(17-18), P. *Vorlesungen über Minimalflächen*. Johannes C.C. Nitsche. Grund. math. Wissenschaften, B. 199. Springer-Verlag, 1975, xiii + 775 pp, $84.30. A comprehensive monograph on the theory of two-dimensional real parametric minimal surfaces in three-dimensional Euclidean space. Contains a 56-page bibliography, some references to very recent literature, and a list of about 90 research problems. JD-B

GEOMETRY, S. *A Programmed Study of Intuitive Geometry (With Applications in Metric Units)*. Ruric E. Wheeler, Ed R. Wheeler. Brooks/Cole, 1975, vii + 224 pp, $4.95 (P). An informal branched programmed text intended to supplement the usual preparation and inservice training of elementary teachers. Provides an easily accessible introduction to geometric terms and notation for anyone without a recent high school geometry background. JNC

GEOMETRY, P. *Lecture Notes in Mathematics-55: Riemannsche Geometrie im Grossen*. D. Gromoll, W. Klingenberg, W. Meyer. Springer-Verlag, 1975, vi + 287 pp, $12.10 (P). A second edition, apparently unchanged, of the 1967 original. JAS

TOPOLOGY, S(18), P. *Differentiable Germs and Catastrophes*. Th. Bröcker, L. Lander. London Math. Soc. Lect. Notes, No. 17. Cambridge U Pr, 1975, vi + 179 pp, $11.95 (P). A study of the null set of a differentiable mapping from R^n to R^k, intended to provide a transition from basic analysis to Thom's theory of catastrophes. Rather costly for a typewritten text. LAS

TOPOLOGY, T(16-18: 1), S, P. *Dimension Theory of General Spaces*. A.R. Pears. Cambridge U Pr, 1975, xii + 428 pp, $47.50. Particular emphasis on non-metrizable spaces. Treats covering dimension function, small and large inductive dimension functions. Very thorough treatment. Gives good sense of historical development of the subject. Bibliography. Index. RJA

TOPOLOGY, T(16-17: 1, 2), P. *Selected Topics in Infinite-Dimensional Topology*. Czesław Bessaga, Aleksander Pełczyński. PWN, 1975, 353 pp. Infinite dimensional topology deals with topological properties of spaces which are subspaces of or modelled on infinite dimensional linear spaces. So convex subspaces of Banach or Hilbert spaces and infinite dimensional manifolds are the two main objects of study. Prerequisites: a little general topology and functional analysis (outlined in a chapter on preliminaries). A good text for a second course in topology. PJM

TOPOLOGY, T*(15-17: 1, 2), L. *Topology, An Introduction to the Point-Set and Algebraic Areas*. Donald W. Kahn. Williams & Wilkins, 1975, viii + 211 pp, $12.75. A nice readable text for a wide ranging introductory full-year course. Starts with chapters on set theory and logic, and metric spaces; covers point set topology without frills, and then does some "geometry": manifolds (classification of surfaces), the fundamental group, and covering spaces. An unusually nice balance of topics at a truly introductory level. JAS

TOPOLOGY, T(15-16: 1, 2), S, L. *Introduction to Metric and Topological Spaces*. W.A. Sutherland. Clarendon Pr, 1975, xiii + 181 pp, $14. Point set topology is developed as a generalization of some of the basic ideas of real analysis. A clean and substantial but not sophisticated presentation of core material. Each of the ten chapters ends with a sizeable set of exercises. JAS

TOPOLOGY, T(17-18), S, P, L. *General Topology*. John L. Kelley. Grad. Texts in Math., V. 27. Springer-Verlag, 1975, xiv + 298 pp, $14.80. Unaltered reprint of the well-known 1955 van Nostrand edition. LAS

PROBABILITY, P. *Ergodic Theory, A Seminar*. J. Moser, E. Phillips, S. Varadhan. Courant Inst, 1975, v + 131 pp, $3.50 (P). Lecture notes covering Onstein's theorem on the isomorphism of Bernoulli shifts and various examples of Bernoulli systems. LAS

PROBABILITY, T(16-17: 1, 2), *A First Course in Stochastic Processes, Second Edition*. Samuel Karlin, Howard M. Taylor. Acad Pr, 1975, xvi + 557 pp, $16. An extensive reworking and extension of the first edition. More examples, more exercises and problems, new chapters on Martingales, random sums, stationary processes, and diffusion theory. Still coming: *A Second Course*... . TAV

PROBABILITY, T(16-17: 2), *An Introduction to Stochastic Processes*. M.T. Wasan. Pure and Appl. Math., No. 39. Queen's U, 1975, ix + 598 pp, $15 (P). Photocopied seminar notes, intended for mathematics, statistics and engineering students. Topics introduced by practical examples. Includes Markov chains, Poisson processes, renewal processes, Brownian motion, diffusion processes. An extensive treatment of first passage times for Brownian motion concludes the work. TAV

PROBABILITY, T*(13-15: 1), S, L. *Introduction to Probability Theory with Computing*. J. Laurie Snell. P-H, 1975, vii + 294 pp, $9.95 (P). Probability on finite sample spaces, limit theorems (as approximations to the finite case), and an introduction to finite Markov chains (this last presupposing matrix algebra). Presupposes familiarity with BASIC. FLW

PROBABILITY, P. *Lecture Notes in Mathematics-465: Séminaire de Probabilités IX Université de Strasbourg*. Ed: P.A. Meyer. Springer-Verlag, 1975, iv + 589 pp, $19 (P).

STATISTICS, S**(13-18), P*, L**. *The Sample Survey: Theory and Practice*. Donald P. Warwick, Charles A. Lininger. McGraw, 1975, viii + 344 pp, $9.95. Practical and theoretical aspects of making a sample survey, from survey and questionnaire design to analysis and reporting. Presupposes elementary statistical notions. FLW

STATISTICS, P. *Asymptotic Expansions and the Deficiency Concept in Statistics*. W. Albers. Math. Centre Tracts, No. 58. Math Centrum, 1974, 144 pp, 16 F (P). Technical study of deficiency--difference in sample sizes needed to make a second statistical procedure perform at the same level as the first--in some particular (primarily nonparametric) situations. Investigations are carried out by using asymptotic expansions for the deficiency. Includes some numerical illustrations. RSK

STATISTICS, P. *Stochastic Processes and Related Topics; Statistical Inference and Related Topics.* Ed: Madan Lal Puri. Acad Pr, 1975. *V. I:* xiii + 315 pp, $17.50; *V. II:* xi + 352 pp, $18. Proceedings of the Summer Research Institute on Statistical Inference for Stochastic Processes held in Bloomington, Indiana in August 1974. JAS

STATISTICS, T*(1). *Statistics for Physical Education, Health, and Recreation.* Charles O. Dotson, Don R. Kirkendall. Har-Row, 1974, xi + 315 pp, $10.95. Well-written precalculus text covering the standard topics, but with most examples and problems relevant to the fields in the title. Includes some sample FORTRAN programs and problems based on them. RSK

STATISTICS, T(13-15: 1). *Introduction to Statistics for Psychology and Education.* Jum C. Nunnally. McGraw, 1975, x + 342 pp, $10.95. A non-mathematical treatment which includes correlational analysis, analysis of variance, factorial analysis, and some non-parametric tests. FLW

STATISTICS, T(17), P*. *Discrete Multivariate Analysis: Theory and Practice.* Yvonne M.M. Bishop, Stephen E. Fienberg, Paul W. Holland. MIT Pr, 1975, x + 557 pp, $27.50. Comprehensive treatment of techniques of handling cross-classified data, including both theory and many examples. Main emphasis is on the log-linear model. Includes study of irregularities, such as tables with structural zeros. Good set of references. RSK

STATISTICS, T(17), P**. *Clustering Algorithms.* John A. Hartigan. Wiley, 1975, xiii + 351 pp, $19.95. In the Wiley Series in Probability and Mathematical Statistics. Algorithm centered, chapters give the concept, purpose and a step-by-step description of a main algorithm, illustrate it on a small data set, discuss and evaluate it (sometimes giving alternative strategies), and suggest ways to apply it. FORTRAN programs implementing the algorithms are listed at the end of each chapter, together with excellent sets of annotated references. RSK

STATISTICS, T(13: 1). *How to Use Statistics.* Joe D. Megeath. Canfield Pr, 1975, x + 310 pp, $10.95. Assumes only basic algebra. Presents few derivations and attempts to develop intuitive understanding. Descriptive statistics, probability, estimation, hypothesis testing (including analysis of variance and some chi-square tests), regression analysis, and index numbers. Examples from business. FLW

STATISTICS, T(17: 1). *Statistical Inference.* S.D. Silvey. Chapman and Hall, 1975, 192 pp, $6.95 (P). Reprint, with corrections, of a 1970 Penguin book (TR, February 1972). Limited to estimation, hypothesis testing and decision theory. RSK

STATISTICS, T(14-15: 1, 2). *Probability and Statistical Analysis.* Edgar P. Hickman, James G. Hilton. Intext, 1971, x + 366 pp, $10. Assumes no more than "a very limited knowledge" of calculus. Presents the common probability distributions, and then considers estimation, decision theory, linear models, and time series. Examples are largely from economics. FLW

STATISTICS, T(13: 1). *Statistics for Math Haters.* Elijah P. Lovejoy. Har-Row, 1975, x + 275 pp, $8.95 (P). Designed to teach statistical thinking rather than statistical techniques. As a consequence coverage is minimal, but comprehensible. RSK

STATISTICS, T(14-17: 1). *Experimental Design and Analysis.* Wayne Lee. Freeman, 1975, xvi + 353 pp, $15; *Workbook for Experimental Design and Analysis,* 116 pp, $3 (P). Aimed at behavioral science students, especially in psychology, with a background of introductory statistics. Presents comprehensive and general methods for creating and analyzing designs having factors related by crossing and nesting. Also discusses Latin squares, analysis of covariance, power computations, missing scores, multiple comparisons and trend analysis. *Workbook* contains true-false, multiple-choice, sentence completion and matching questions, in addition to numerical exercises. RSK

STATISTICS, T(15-17: 1), S. *Datenanalyse: Mit statistischen Methoden und Computerprogrammen.* Siegmund Brandt. Bibliographisches Inst, 1975, 342 pp. Contains various statistical tests (e.g., analysis of variance), examples of their applications, and FORTRAN programs using the ideas developed. Appendix on basics of FORTRAN. Appendix of formulas. References. Index. RJA

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-28: Mathematical Foundations of Computer Science.* Ed: A. Blikle. Springer-Verlag, 1975, vii + 484 pp, $16 (P). Proceedings of the third symposium on mathematical foundations of computer science held in Jadwisin near Warsaw in June 1974. JAS

COMPUTER SCIENCE, P. *Nonlinear Programming 2.* Ed: O.L. Mangasarian, R.R. Meyer, S.M. Robinson. Acad Pr, 1975, ix + 361 pp, $19. Proceedings of a symposium at Madison, Wisconsin, April 1974. LAS

COMPUTER SCIENCE, S(15-17), P, L. *Advances in Computers, V. 13.* Ed: Morris Rubinoff, Marshall C. Yovits. Acad Pr, 1975, xi + 248 pp, $19.50. Five survey papers, including poetry generation, mapping, natural language, and artificial intelligence. LAS

COMPUTER SCIENCE, T(16), L*. *Programming Languages, Design and Implementation.* Terrence W. Pratt. P-H, 1975, xiv + 530 pp, $13.50. Part I lays the conceptual framework for Part II which compares and analyses seven languages; viz., Fortran, Algol, PL/I, APL, Lisp, Snobol 4 and Cobol. Generally, the emphasis is environmental--discussing data structures and operations on them, sequence control and storage management. A good buy. RWN

COMPUTER SCIENCE, P. *Fuzzy Sets and Their Applications to Cognitive and Decision Processes.* Ed: Lotfi A. Zadeh, *et al.* Acad Pr, 1975, x + 496 pp, $16. A fuzzy set is a class in which there may be continuous variation in the grades of membership as, say, in the class of *long* objects. This volume contains papers from a U.S.-Japan seminar held at Berkeley in July 1974. The theory of fuzzy sets provides, according to its advocates, a "systematic approach to approximate reasoning." LAS

COMPUTER SCIENCE, T(1), S*. *Basic.* Samuel L. Marateck. Acad Pr, 1975, xii + 404 pp, $7.95 (P). Unique double-page format presents the fundamentals of BASIC programming. Left-hand pages contain textual material and provide a thorough discussion; right-hand pages have sample programs with commentary illustrating the text, including examples of programming errors, and in many cases are self-contained enough to be sufficient. RSK

COMPUTER SCIENCE, T(17-18), P. *Dynamic Information and Library Processing.* Gerard Salton. P-H, 1975, xiv + 523 pp, $19.95. A detailed new look at automating library processes via the concept of the "dynamic library" in which resource input, cooperative operations and user influences are orchestrated into an entirely new environment. LAS

COMPUTER SCIENCE, T(13-15: 1), S, L. *Einführung in die Programmiersprache COBOL.* Klaus-Peter Mickel. Hochschultaschenbucher, B. 745. Bibliographisches Inst, 1975, 206 pp, (P). Basically a COBOL programming manual with a concise introductory chapter on computing fundamentals. Attempts to relate the structure of COBOL to the underlying computer operations. Answers to exercises. References. Index. RJA

COMPUTER SCIENCE, T**(15-17: 1), S, L. *Introduction to Computer Architecture.* Ed: Harold S. Stone, *et al.* SRA, 1975, x + 565 pp, $16.95. Text on design and organization of computing devices from hand calculators to complex computer installations. Treats both the hardware and the appropriate software of various systems. Each chapter contains appropriate problems. Extensive bibliography. Index. RJA

COMPUTER SCIENCE, T(13: 1). *Fundamentals of Computer Science.* Terry M. Walker. Allyn, 1975, xiv + 399 pp, $13.95. Similar to the author's *Introduction to Computer Science: An Interdisciplinary Approach.* General problem solving using flowcharts. Applications to business, education, science and statistics. Interestingly written. Companion language-dependent programming manuals are said to be available. RWN

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-26: GI-4. Jahrestagung.* D. Siefkes. Springer-Verlag, 1975, ix + 748 pp, $20.10 (P). Proceedings of the fourth annual conference of the Gesellschaft für Informatik which took place at the Technischen Universität Berlin in October 1974. Papers on education, formal languages, information theory, computer graphics and much more. JAS

SYSTEMS THEORY, T(15-16: 1). *Mathematical Description of Linear Systems.* Wilson J. Rugh. Dekker, 1975, xi + 177 pp, $13.50. Readable. Some theory. Presumes linear algebra. Continuous and discrete systems are treated in parallel. Many examples are reused through the text. Discusses reachability, observability, realization, identification and stability. Exercises. RWN

APPLICATIONS, P. *Collective Phenomena and the Applications of Physics to Other Fields of Science.* Ed: Norman A. Chigier, Edward A. Stern. Brain Research Pub, 1975, xxvi + 491 pp, $25 (P). Undelivered papers prepared for an "abruptly called off" interdisciplinary international seminar intended to coincide with the 250th anniversary celebration of the Soviet Academy of Sciences. Organized by dissident Soviet scientists in many fields (who were thrown together while pursuing their science despite loss of formal jobs) with the sponsorship of a Nobel-studded international committee. LAS

APPLICATIONS (GEOPHYSICS), P. *The Earth's Density.* K.E. Bullen. Chapman, 1975, xiii + 420 pp, $37.50. A very mathematical chronology of contributions to knowledge about the earth's density. DFA

APPLICATIONS (ENGINEERING), P. *Lecture Notes in Economics and Mathematical Systems-108: Supercritical Wing Sections II: A Handbook.* Frances Bauer, *et al.* Springer-Verlag, 1975, v + 296 pp, $12.10 (P).

APPLICATIONS (INFORMATION THEORY), P. *Foundations of Coding Theory.* Ed: William E. Hartnett. Episteme, V. 1. Reidel, 1974, xiii + 216 pp, $28. Selected and slightly edited papers from the Coding Group which functioned at Parke Mathematical Laboratories from 1957 until 1968. First volume in a new interdisciplinary series. LAS

APPLICATIONS (INFORMATION THEORY), P, L. *A Theory of Indexing.* Gerard Salton. SIAM, 1975, v + 56 pp, $5.25 (P). A survey of various procedures for measuring the informational significance of key or index words that are used to identify documents in a collection, together with a model for the construction of effective indexing vocabularies. LAS

APPLICATIONS (INSURANCE), P. *Claims Provisions for Non-Life Insurance Business.* Inst Math Appl, 1974, viii + 190 pp, £3.50 (P). Proceedings of a symposium held in London on May 22, 1974. Concerned with methods of determining needed reserves to handle future non-life insurance claims. RSK

APPLICATIONS (MECHANICS), P. *Theoretical and Applied Mechanics, V. 23.* Itiro Tani, Toshie Okumura. U of Tokyo Pr (U.S. Distr.: ISBS, Inc.), 1975, ix + 560 pp, $45. Proceedings of the 23rd Japan national congress in 1973. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-34: One-Dimensional Conductors.* Ed: H.G. Schuster. Springer-Verlag, 1975, x + 371 pp, $13.80 (P). Papers from a conference at Saarland, July 1974. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Central Michigan University: Assistant Professors R. St. Andre and D. D. Smith have been promoted to Associate Professors.

Miami University: Assistant Professors D. E. Kullman and Chull Park have been promoted to Associate Professors; Associate Professor E. M. Bolger has been promoted to Professor.

Southeast Missouri State University: Professor H. W. Hager has been appointed head of the Department of Mathematics, succeeding Professor R. J. Michel who has retired after 35 years as Department Head; Associate Professor R. L. Francis has been promoted to Professor; Assistant Professors Mangho Ahuja and J. E. Young have been promoted to Associate Professors.

Southern Oregon State College: Assistant Professors Robert McCoy, L. T. Hill, and R. G. Montgomery have been promoted to Associate Professors.

University of San Francisco: Assistant Professor Daniel Gallin has been promoted to Associate Professor and reappointed Chairman of the Department of Mathematics; Assistant Professor A. B. Cruse has been promoted to Associate Professor.

Dr. D. R. Beuerman, SUNY at Buffalo, has been appointed Assistant Professor and Director, Computation Center, Parks College of St. Louis University.

Associate Professor J. A. Goldstein, Tulane University, has been promoted to Professor.

Associate Professor Gerald Kimble, University of Nevada, Reno, has been promoted to Professor.

Distinguished Professor Morris Marden has retired from the University of Wisconsin, Milwaukee, after 45 years on its faculty. A colloquium and dinner were given in his honor on May 9, 1975. During the 1975–76 academic year, Professor Marden will serve as Visiting Distinguished Professor of Mathematics at the California Polytechnic State University, San Luis Obispo.

Dr. V. N. Murty, Associate Professor of Mathematics and Statistics and Chairman of the Mathematical Sciences Program, Pennsylvania State University (Capitol Campus), has been promoted to Professor.

Dr. C. V. Newsom has been elected Chairman of the Board of the Guggenheim Foundation.

Assistant Professor D. J. Samuelson, Pennsylvania State University at McKeesport, has been promoted to Associate Professor.

Professor Ivar Stakgold, Northwestern University, has been appointed Chairman of the Department of Mathematics at the University of Delaware.

Associate Professor C. O. Wilde, Naval Postgraduate School, has been promoted to Professor.

Dr. Israel E. Glover, Norfolk, Virginia, died on March 12, 1975. He was a member of the Association for twenty-three years.

INTIAL ANNOUNCEMENT OF THE TENTH INTERNATIONAL SYMPOSIUM ON RAREFIED GAS DYNAMICS

The Tenth Rarefied Gas Dynamics Symposium will be held at Aspen, Colorado, in July 1976. For information regarding this Symposium, please write to Dr. E. P. Muntz, Dept. of Aerospace Engineering, University of Southern California, University Park, Los Angeles, CA 90007, or Dr. J. L. Potter, Deputy Director, Technology, ARO, Inc., Arnold Air Force Station, TN 37389.

SECONDARY SCHOOL LECTURESHIP PROGRAM TO INTEREST MORE WOMEN IN MATHEMATICS

The Mathematical Association of America is pleased to announce the establishment of a new lectureship program designed to encourage young women at the high school level to take a serious interest in mathematics and to prepare for careers in mathematics or in mathematically-related fields. Made possible through the generous support of IBM, the program will be in operation for 1975–76 in three pilot regions: New York/New

Jersey Metropolitan Area, Chicago, and the San Francisco Bay Area. Speakers' visits will be geared primarily to a 10th grade audience, and there will be no charge to participating high schools.

In addition to fulfilling the aims of the already existing MAA Secondary School Lecturer Programs which aid students in appreciating the importance of mathematics and promote cooperation between high school teachers and college mathematicians, the new speakers bureau will focus on two further goals:

1. to provide role models for high school girls,
2. to impress upon guidance counselors, parents, curriculum consultants, and others who influence student selection of courses, the critical necessity for students—in particular, women students—to develop a strong mathematics background in order to be eligible to enter many traditional as well as emerging career fields.

The diverse careers represented by speakers in the lectureship program will reflect the multiplicity of ways in which quantitative methods are used in the 1970's. Participants will be drawn from the academic, business, industrial, scientific, and social science worlds.

Dr. Eileen L. Poiani, Associate Professor of Mathematics at Saint Peter's College in Jersey City, New Jersey, has been named the Director of the program. She will coordinate the bureau for the New York/New Jersey area.

Dr. Margariete A. Montague, Department of Mathematical Sciences at Northern Illinois University, De Kalb, and Mrs. Jean J. Pedersen, Assistant Professor at the University of Santa Clara, will serve as the Regional Coordinators for the Chicago and San Francisco Bay regions, respectively.

The MAA is grateful for the opportunity to join in a cooperative endeavor with IBM to interest more American women in mathematics and in careers which use mathematical methods. It is hoped that the program will be continued and expanded in future years.

THE JOURNAL FOR RESEARCH IN MATHEMATICS EDUCATION: SPECIAL RATES FOR MAA MEMBERS

Through an agreement with the National Council of Teachers of Mathematics (NCTM), MAA members may now subscribe to the NCTM's Journal for Research in Mathematics Education (JRME) at the special rate of \$6 per year. JRME appears four times a year, in November, January, March, and May. It contains articles on research in mathematics education at all levels, including reports on empirical studies, summaries of major research studies, and articles about current research in the field.

An MAA member who wishes to subscribe may do so by writing to: JRME, The National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091, stating that he or she is an MAA member and enclosing a check or money order for \$6.

AAAS ANNUAL MEETING TO BE ACCESSIBLE TO THE PHYSICALLY DISABLED SCIENTIST

The Office of Opportunities in Science is working with the AAAS Meetings Office to make the Annual Meeting in Boston, February 18–24, 1976, fully accessible to physically disabled persons who wish to attend. For information and/or suggestions, please contact: Project on the Handicapped in Science, Office of Opportunities in Science, American Association for the Advancement of Science, 1776 Massachusetts Ave., N. W., Washington, D. C. 20036.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTY-FIFTH SUMMER MEETING OF THE ASSOCIATION

The Fifty-fifth Summer Meeting of the Mathematical Association of America was held at Western Michigan University, Kalamazoo, Michigan, from Monday, August 18, to Wednesday, August 20 in conjunction with the meetings of the American Mathematical Society, the Association for Women in Mathematics and Pi Mu Epsilon Fraternity. There were registered 982 persons, including 646 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in Miller Auditorium. Presiding officers at the three Earle Raymond Hedrick Lectures were President H. O. Pollak, First Vice-President Ivan Niven and Secretary Emeritus, H. L. Alder; at the lecture by Dr. R. L. Graham, Prof. E. A. Nordhaus; at the lecture by Professor Vera Pless, Professor John W. Petro; and at the lecture by Professor J. R. Thompson, Professor E. A. Tanis.

FIRST SESSION OF THE ASSOCIATION

Welcome on behalf of the University by Dr. John Bernhard, President, Western Michigan University.

The Earle Raymond Hedrick Lectures: *Geometric Measure Theory and the Calculus of Variations*, Lecture I, by Professor F. J. Almgren, Jr., Princeton University.

A variety of geometric problems in the calculus of variations are now being studied in the context of geometric measure theory. In particular, optimal solutions to these problems are sought among geometric configurations of varying topological type and singularity structure. Particular applications include the shape of crystals, the geometry of compound soap bubbles and immiscible liquids in equilibrium and disequilibrium, and the structure of certain organisms. A variety of examples illustrate the phenomena of general least area problems. In addition to the calculus of variations geometric measure theory has strong ties to many other parts of mathematics.

Panel Discussion: *The Mathematics Accreditation Question*

A panel discussion with Professor Wade Ellis, University of Michigan, Professor Shirley A. Hill, University of Missouri at Kansas City and Professor C. T. Long, Washington State University, with Professor D. T. Finkbeiner II, Kenyon College, as moderator.

Professor Ellis observed that the potential of accreditation is bounded in the dimension of what it can accomplish if ideally performed. In practice the extent of its effectiveness is significantly less. To be effective, accreditation requires well-defined purposes which can be closely approached, carefully formulated criteria which promote achievement of the purposes, continuously monitored procedures which bear completely and exclusively on the criteria, and operating personnel who understand and appreciate not only the quality of the mathematical components of American higher education, but also their diversity.

Professor Hill observed that one of the major service roles of most mathematics departments is in the mathematical preparation of pre-college teachers. Especially here, while problems of accreditation and certification should be kept distinct, in fact the implementation of requirements presents unavoidable interrelationships. If the institution trains teachers, the department should have available at least the minimum that consensus of the mathematical and mathematical education communities (as expressed in official organizational positions) deems essential for the teachers of mathematics. This may go beyond the bare legal minimum expressed in certification requirements.

Professor Long pointed out that the MAA adopted Guidelines for the Evaluation of Collegiate Mathematics Programs in August, 1972. These Guidelines were intended for use by regional accrediting associations, by departments of mathematics, and by college and university administrations. Professor Long summarized the work of the ad hoc committee that has recently reviewed the Guidelines — the use to which they have been put and suggestions for their revisions.

Professor Finkbeiner observed that, in spite of the uniqueness property implicit in the title of this panel discussion, there are many valid questions concerning accreditation standards and procedures and their influence on undergraduate mathematics programs. To sharpen the focus, this panel has considered the question in the form "In what manner should the MAA participate in the processes that accredit undergraduate programs and institutions?" This subject has been before the Association in various forms since 1968, and at other times in the more distant past. The purpose of this Panel has been to report on the present status and to receive guidance from MAA members for the future.

This was followed by a general discussion by the audience.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Almgren

Mathematical Models in Cancer Research, by Professor J. R. Thompson, Rice University.

Using simple cell kinetic models, treatment modifications are proposed for three neoplastic situations. Firstly, it is suggested that in the irradiation of solid tumors by acute external beam dosage, the generally preferred effects of radiation implant therapy might be achieved by increasing the frequency of treatment while decreasing the size of each dose. Next, it is shown how cell synchrony might be utilized in cell cycle specific chemotherapeutic regimes for the treatment of metastasized solid tumors (e.g., melanoma) to give a far greater tumor cell killing effect than obtainable with existing protocols. Finally, it is conjectured that tumor cell competition might be utilized to increase the efficiency of myeloma treatment.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Almgren

Business meeting of the Association; presentation of Lester R. Ford Awards.

Some Recent Applications of Graph Theory, by Dr. R. L. Graham, Bell Telephone Laboratories.

Dr. Graham illustrated that graph theory is a natural and effective tool for attacking many problems arising in a variety of applied fields. His talk discussed several interesting recent applications of graph-theoretical techniques to: (i) design of optimal communication networks, (ii) algorithms for circuit card testing, and (iii) efficient location of facilities.

FOURTH SESSION OF THE ASSOCIATION

Error Correcting Codes: Practical Origins and Mathematical Applications, by Professor Vera Pless, Massachusetts Institute of Technology.

Professor Pless recalled that coding began twenty-five years ago in response to the practical problem in electrical engineering of reliably communicating digitally encoded information. By using concepts from vector spaces, a technique was demonstrated for correcting single errors in the Hamming (7,4) code. This code has fewer redundancy positions than information positions and so represents a considerable saving over the simple-minded way of correcting single errors by repeating the message three times and taking a majority count. On the other hand, the search for codes capable of correcting the most errors possible has yielded configurations basic to the construction of some of the new, finite simple groups. Further, these and other good codes have produced new five-designs.

Panel Discussion: Training of Non-Doctoral Mathematics Students for Non-Academic Employment.

A panel discussion with Dr. David Bossard, Daniel H. Wagner Associates, Professor Alan Karr, Johns Hopkins University, and Dr. Werner Ulrich, Bell Telephone Laboratories, with Dean D. W. Lick, Old Dominion University, as moderator.

Dr. Bossard, a practicing operations analyst, remarked on operations research as a career and academic preparation for it.

Professor Karr observed that a program for non-academic, non-doctoral mathematics students should differ in content from a classical program, but not in level of rigor. Many such students are employed in some aspect of mathematical modeling and a model is a particular kind of axiomatic system. Courses such as probability, statistics, analysis, computing, numerical analysis, optimization and economics, when rigorously presented, convey not only useful technical material but also important aspects of modeling philosophy and the structure of deductive reasoning. The program should include a course in mathematical modeling, preferably using a case study approach; all courses should be designed to emphasize development of the essential abilities of written and verbal exposition.

Dr. Ulrich noted that computer science is becoming an increasing field of application for mathematical sciences and mathematical talents. These can be applied in the areas of computer architecture and design, systems programming, and applications programming. A very large fraction of applications programming work is no longer in numerical analysis or manipulation but rather deals with the manipulation of textual or logical data.

Dean Lick summarized and added to the comments of the panelists. His remarks addressed: (i) present state of affairs for mathematics, (ii) new areas of need for mathematicians, (iii) new employment trends and projections, (iv) needed characteristics in mathematical students, (v) curricular considerations, (vi) general suggestions for action by departments of mathematics.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Shaw Theatre on Sunday, Monday, and Tuesday at 7:00 P.M. The showings included several films not previously shown at an Association meeting. The following films were shown on Sunday:

7:00—7:25 P.M. LOGIC II: A BBC BROADCAST AS PART OF THE OPEN UNIVERSITY FOUNDATION COURSE IN MATHEMATICS (b&w)

Films of the COLLEGE GEOMETRY PROJECT (in color):
7:30—7:43 P.M. DIHEDRAL KALEIDOSCOPES
7:45—8:01 P.M. CURVES OF CONSTANT WIDTH
8:05—8:21 P.M. PROJECTIVE GENERATION OF CONICS

Films of the MAA CALCULUS FILM SERIES:
8:25—8:45 P.M. WHAT IS AREA?
8:50—9:04 P.M. AREA UNDER A CURVE
9:08—9:30 P.M. SHAPES OF THE FUTURE I: SOME UNSOLVED PROBLEMS IN GEOMETRY — TWO DIMENSIONS with Victor Klee.

The following films were shown on Monday:
7:00—8:03 P.M. NIM AND OTHER ORIENTED GRAPH GAMES with Andrew Gleason (b&w)
8:10—8:26 P.M. SPACE FILLING CURVES
8:30—8:45 P.M. THE THEOREM OF THE MEAN
8:50—9:00 P.M. LINEAR PROGRAMMING
9:05—9:30 P.M. FOURIER SERIES – A BBC BROADCAST AS PART OF THE OPEN UNIVERSITY'S COURSE LINEAR MATHEMATICS.

The Tuesday film program was devoted to films produced by the late Professor C. B. Allendoerfer.
7:00—7:25 P.M. GAUSS-BONNET THEOREM
7:30—7:52 P.M. CYCLOIDAL CURVES OR TALES FROM THE WANKLENBURG WOODS
8:00—8:10 P.M. AREA AND PI
8:13—8:20 P.M. ASSOCIATIVE PROPERTY
8:25—8:35 P.M. BINARY OPERATIONS AND COMMUTATIVE PROPERTY
8:37—8:45 P.M. DISTRIBUTIVE PROPERTY
8:50—9:00 P.M. GEOMETRIC CONCEPTS
9:05—9:15 P.M. GEOMETRIC TRANSFORMATIONS

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Sunday at 9:00 A.M. in the Dean's Conference Room in Friedman Hall with 37 members present. Among the items of business transacted were the following:

The Board elected Mr. Jonathon Dreyer, Carleton College, Dr. R. L. Graham, Bell Telephone Laboratories, Professor Morris Kline, New York University, Professor L. F. Meyers, Ohio State University, and Professor Doris W. Schattschneider, Moravian College as Associate Editors of MATHEMATICS MAGAZINE for the period 1976–80.

A report was received from the Committee on High School Contests and the Subcommittee on the USA Mathematical Olympiad. It was reported that 103 students took the USA Mathematical Olympiad on May 6, 1975, and that the eight students who scored the highest on this examination and their parents were honored at an awards ceremony at the National Academy of Sciences and a dinner at the State Department in Washington, D. C. on June 5, 1975. Professor P. D. Lax of New York University delivered an address at the awards ceremony. A grant of $5000 from the International Business Machines Corporation to defray expenses of the awards ceremony was gratefully acknowledged.

It was also reported that the United States entry in the International Mathematical Olympiad held on July 7 and 8 in Burgas, Bulgaria finished in third place. The U.S. team was awarded three of the eight first prizes in the competition and these were awarded to Paul Herdeg, Miller Puckette and Paul Vojta. The first five teams and their scores were announced to be: Hungary 258, E. Germany 249, U.S.A. 247, U.S.S.R. 246 and Great Britain 241. Grants of $1000 from the Johnson and Johnson Foundation, $500 from the Minnesota Mining and Manufacturing Corporation, $4800 from the Spencer Foundation, $500 from the Standard Oil Company of California, and $1000 from the Xerox Corporation to pay travel expenses to Bulgaria were gratefully acknowledged.

A report from the Committee on Secondary School Lecturers was received in which it was announced that the International Business Machines Corporation has granted the Association $7500 to fund a secondary school lectureship program designed to interest more women in mathematics. During the 1975–76 academic year this program will operate in the New York, Chicago, and San Francisco Bay areas and is under the direction of Professor Eileen L. Poiani of St. Peter's College.

At its meeting on January 24, 1975 in Washington, D. C., the Board voted that, beginning January 1, 1976, THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL (TYCMJ) should be an official journal of the Association. Upon recommendation of the Executive Committee, the Board voted to instruct the Secretary to submit to the membership at the business meeting on Sunday, January 25, 1976, in San Antonio, Texas, an amendment providing that an official journal of the Association can be sold to non-members for less than member's dues. This amendment is necessary in order that non-members of the Association are able to subscribe to the TYCMJ at agreed upon rates.

The Board approved the recommendation of the Committee on Sections that the Rocky Mountain Section be partitioned to form a new section to be called the Intermountain Section, consisting of Utah and parts of Idaho.

Professor W. F. Lucas presented a report on the activities of the Committee on Institutes and Workshops. This report included an announcement of a grant of $86,267 from the National Science Foundation for support of a college faculty workshop on modules in applied mathematics. It is currently planned to hold this workshop at Cornell University during the summer of 1976.

The Executive Director reported that the membership of the Association on April 1, 1975 was 19,107 compared to 19,447 a year earlier. As of the same date, there were 382 academic members of the Association compared with 360 a year earlier.

The Board approved holding the Fifty-Sixth Summer Meeting of the Association at the University of Toronto in August, 1976.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting was held on Tuesday morning with President Pollak presiding. He announced with deepest sympathy the death of Professor D. E. Christie on July 18, 1975, and called upon Professor D. T. Finkbeiner, who addressed the meeting as follows:

"The Mathematical Association of America records with sorrow the death of Professor Dan E. Christie of Bowdoin College."

"Dan Christie was graduated *summa cum laude* from Bowdoin in 1937, attended Cambridge University on a Henry Fellowship, and received his Ph.D. degree from Princeton. He returned to Bowdoin in 1942 as a member of the faculty, serving that college with distinction as teacher, scholar, and wise counsellor throughout his career."

"Dan Christie's legacy to mathematics is reflected from many facets — his intellect, his integrity, the students and colleagues whom he inspired, the summer institutes and seminars that he organized, his personal demonstration that small colleges can foster a high level of scholarly work in mathematics, and his devoted service in the councils of this Association, including CUPM, the Publications Committee, and two terms on the Board of Governors."

"His dedication quickened our efforts, his wisdom guided our deliberations, and his friendship lightened our days. We mark his passing in sadness, and we speak our gratitude for the time he shared with us."

At the conclusion of Professor Finkbeiner's remarks, President Pollak asked the audience to stand in memory of Professor D. E. Christie.

The tenth set of Lester R. Ford Awards were presented by President Pollak to authors of expository articles published in the MONTHLY during 1974. These awards, in the amount of $100 each, were presented for six articles (for further details on these Awards, see the August-September issue of this MONTHLY, page 787).

The following schedule of future meetings was announced: San Antonio, Texas, January 24–26, 1976; University of Toronto, August 22–28, 1976; St. Louis, Missouri, January 29–31, 1977.

The Secretary called attention to the excellent arrangements which had been provided for this meeting. He singled out Professor Yousef Alavi, Chairman of the Arrangements Committee and Publicity Co-Director and Professor A. Bruce Clarke, Chairman of the Program Committee and Chairman of the host department as two persons to whom the Association was particularly grateful for coordinating the details of this meeting.

It was announced that the Association has an excess of income over expenditures during 1974 of $38,511.64. The Secretary emphasized that there are continuing efforts to increase the number of individual and academic members of the Association, but much of the success depends upon individual members asking their colleagues and their institutions to join the Association.

The Secretary then moved that the Past-President of the Association be added to the membership of the

Finance Committee by replacing the last sentence of Article III, Section 5 by "This Committee (the Finance Committee) shall consist of six members including the President, the Past-President (for a term of two years), the Secretary and the Treasurer," changing the parenthetical expression in Article IV, Section 1 (c) to read "(other than the President, the Past-President, the Secretary and the Treasurer)," and replacing the first sentence of Article IV, Section 4 by "In the absence of the President, the First Vice-President (or in his or her absence, the Second Vice-President) shall have and exercise the powers of the President, except that the Past-President shall preside at meetings of the Finance Committee (or in his or her absence the senior member, in terms of service, of the two elected members of the Finance Committee)." The motion carried.

The Secretary next moved an amendment to the By-Laws which would permit the Board of Governors to annually elect two members of the Board. Specifically, the motion was to delete "and shall elect annually two Governors for terms of three years" from Article IV, Section 1 (a), to revise Article IV, Section 1 (c) to read, "The Board shall elect annually two Governors for terms of three years and at appropriate times by ballot and for terms $\cdots$," and to replace the second sentence of Article IV, Section 1 (g), by "At least two nominations shall be made for each office to be filled in the case of the Second Vice-President, Governors (except Section Governors), and members of the Finance Committee." This motion also carried.

Finally, the Secretary addressed the meeting as follows:

"After the Washington, D.C. meeting of the Association in January, 1975, Professor H. L. Alder retired as Secretary of the Association, a position he had held for 15 years. At that meeting, Professor Alder was elected by the Board of Governors as an honorary life member of the Association and was given the title Secretary-Emeritus. Also, there was a luncheon in Professor Alder's honor and his many accomplishments as Secretary were reviewed by various speakers. However, one of the tasks on which Henry worked the hardest during the last two years of his tenure went unmentioned. I am speaking, of course, of training his replacement."

"I hope that the many lessons Professor Alder taught during this apprentice period were not wasted on a poor student, but, at this point, I can only report that Henry approached the task at hand with all of the good humor, kindness, and thoroughness with which he has approached so many tasks on behalf of the MAA. Our association of the past two years has been a pleasure for me and this seems an appropriate opportunity to thank Henry Alder for his past, present, and, I hope, continuing help and advice to the Association and to me."

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. There were two sets of Colloquium Lectures. Professor E. R. Kolchin of Columbia University, gave one set, entitled "Differential Algebraic Groups" on Tuesday at 1:00 P.M. and on Wednesday, Thursday, and Friday at 8:30 A.M. Professor E. M. Stein of Princeton University delivered the second set, entitled, "Singular Integrals, Old and New" at 2:15 P.M. on Tuesday and at 8:30 A.M. on Wednesday, Thursday, and Friday.

The AMS Committee on Employment and Educational Policy presented two panel discussions. The first was held on Monday at 4:30 P.M. and was an open meeting on the state of the job market with panelists Professors R. D. Anderson and C. W. Curtis, with Professor W. H. Fleming acting as moderator. The second, entitled "The Role of Applications in Ph.D. Programs," was held at 8:00 P.M. on Thursday with Professor R. D. Anderson, Professor Lipman Bers, and Dr. H. O. Pollak as panelists and with Professor W. H. Fleming acting as moderator.

Invited addresses were given as follows:

Nearly Trivial Outer Automorphisms of Finite Groups, Professor E. C. Dade, University of Illinois, Wednesday, August 20, 9:45 A.M.

On the Classification of Kleinian Groups, Professor Bernard Maskit, SUNY at Stony Brook, Wednesday, August 20, 11:00 A.M.

The Singular Cardinals Problem, Professor J. H. Silver, University of California, Berkeley, Thursday, August 21, 11:00 A.M.

Aspects of Value Distribution Theory in Several Complex Variables, Professor W. F. Stoll, University of Notre Dame, Thursday, August 21, 11:00 A.M.

The Continuous Cohomology of Groups and Classifying Spaces, Professor J. D. Stasheff, Temple University, Thursday, August 21, 1:30 P.M.

Algebraic Cycles on Algebraic Varieties, Professor David Mumford, Harvard University, Friday, August 22, 9:45 A.M.

Ergodic Properties of Elementary Mappings of the Unit Interval, Dr. R. L. Adler, IBM T. J. Watson Research Center, Friday, August 22, 11:00 A.M.

The Pi Mu Epsilon Fraternity held its sessions for contributed papers on Tuesday at 3:00 P.M. and on Wednesday at 10:00 A.M. in Room 1118, Rood Hall. A banquet was held Tuesday at 6:30 P.M. in Room # 157

adjacent to the University Student Center cafeteria. After this banquet, Professor J. S. Frame of Michigan State University spoke at 8:00 P.M. on *Matrix Functions; a Powerful Tool,* in Room 1104 of Rood Hall. A Dutch treat breakfast meeting for Pi Mu Epsilon members was held on Wednesday at 8:00 A.M. in Room # 157. The Pi Mu Epsilon Governing Council Luncheon was held on Tuesday.

The Association for Women in Mathematics held a panel discussion, Noether to Now — The Woman Mathematician, on Tuesday at 3:30 P.M. The moderator for this panel was Professor Alice T. Schafer and panelists were Professors Lenore Blum, Vivienne Mayes, M. Susan Montgomery, Mary Ellen Rudin, and Jane Cronin Scanlon.

The Mathematicians Action Group held a panel discussion, *Unemployment: An Exchange of Experiences* at 4:30 P.M. on Wednesday. They also held a business meeting at 10:00 A.M. on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements consisted of Yousef Alavi, Chairman; P. T. Bateman, J. M. Calloway, Gary Chartrand, A. B. Clarke, S. F. Kapoor, D. R. Lick, J. W. Petro, J. H. Powell, D. P. Roselle, G. L. Walker, and A. H. Wright.

Registration headquarters were located in the Miller Auditorium Upper Lobby. Dormitory rooms and dining facilities were in the Goldsworth Valley Residence Halls. Book and educational media exhibits were displayed on the second lobby level of Miller Auditorium from noon to 4:30 on Monday, 8:30 A.M. to 4:30 P.M. on Tuesday and Wednesday, and 8:30 A.M. to noon on Thursday.

A picnic was offered on Wednesday at 6:00 P.M. and a beer party was held following the picnic at the Holiday Inn-West. Tours of the Upjohn pharmaceutical plant, the area wineries, and the Kellogg cereal company were conducted during the meeting.

DAVID P. ROSELLE, *Secretary*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting on August 17, 1975 at Western Michigan University, elected to membership the twenty-sixth set of applicants for academic membership (for election of the other twenty-five sets, see the April, 1975 issue of this MONTHLY, pages 448–9, and the references cited there). Approval of election was given to the following applicants for academic membership:

Abilene Christian College, Abilene, Texas
Bishop College, Dallas, Texas
Bowdoin College, Brunswick, Maine
Bryan College, Dayton, Tennessee
Columbus College, Columbus, Georgia
Dawson College, Montreal, Quebec, Canada
Georgia State University, Atlanta, Georgia
Pima Community College, Tucson, Arizona
State University College at Brockport, Brockport, New York
University of Southern Mississippi, Hattiesburg, Mississippi.

DAVID P. ROSELLE, *Secretary*

APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The spring meeting of the Allegheny Mountain Section of the MAA was held at Duquesne University in Pittsburgh, Pennsylvania, on April 25 and 26, 1975.

A social hour and buffet dinner preceded the activities on Friday evening. The main speaker on Friday evening was Dr. Thomas Saaty, of the University of Pennsylvania, who spoke on "Mathematics in the Social Sciences." A panel discussion for students on "Job Opportunities in Business" was chaired by Kathleen Taylor of Duquesne University. Panel members included Ben Mount, of Bettis Atomic Laboratory, Westinghouse Electric; Jane Barbustiak of the Aluminum Company of America; and Newton Huntly of Marsh and McClennan.

Short talks by students included:

Automorphisms of 2-groups of order less than 2^7, by Diane Herrmann, Allegheny College.

Toward proving the non-existence of a simple group of order 86,400, by Debby Bergstrand, Allegheny College.

Mathematical Studies — A unified undergraduate program, by Brad Osgood, Carnegie Mellon University.

On the convergence of $\Sigma \log(n \sin 1/n)$, by David Carrouthers, Westminster College.

Computer model for electron diffusion, by Debbie Schmalsteig, representing the Industrial Mathematics Class of the University of Pittsburgh.

An analysis of two statistical methods employed in scientific and social science research, by David Miller, Duquesne University.

The game of life, by L. Kachmar, Duquesne University.

Faculty presentations included:

Finite groups in which the centralizer of M of an involution is solvable and $O_2(M)$ is extra special, by J. Richard Lundgren, Allegheny College.

Information science: Information theory to measure content in documents, by Jack Belzer, University of Pittsburgh.

Dear Abby, 'How do you square a circle?' by Henry Gould, West Virginia University.

Quadratic loss in a model of political participation, by R. W. Hoyer, West Virginia University.

Some combinatorial problems, by Charles Cunkle, Slippery Rock State College.

Selected MAA films were also shown.

Invited presentations on Saturday included:

Shape, by Steve Armentrout, Pennsylvania State University.

Spline functions, by Ralph Carlson, Grove City College.

Shortest connecting networks: An example of applied geometry, by H. O. Pollak, Director of Research at Bell Laboratories and President, MAA.

Professor Charles Cable, Chairman of the Section, presided at the Business Meeting. The Secretary's report included the list of Putnam Examination winners from the Section, who are D. R. Strick, of the University of Pittsburgh; L. W. Riddle, of Carnegie-Mellon University; Deborah Bergstrand, of Allegheny College; G. R. Bradley, of Allegheny College; and R. G. Caflisch, of West Virginia University.

A report of the executive committee's activities, and recognition of Steve Modelezsky of Shady Side Academy, who participated in the Mathematics Olympiad for the U.S.A., was given by the Secretary. Steve reported on his activities as a member of the team and thanked the Section for its financial support of the team. It was announced that Earle Myers of the University of Pittsburgh was elected Sectional Governor 1975–78.

Reports from Francis Hall and I. D. Peters on the High School Mathematics Contests for Pennsylvania and West Virginia, respectively, were given. Lindsey Military Institute was first in West Virginia and Shady Side Academy of Pittsburgh was first in Western Pennsylvania. Three of the four team members from Shady Side were recognized.

Professor E. Myers reported on the Visiting Lecturer Program sponsored by the Section.

Professor R. Ayoub, Governor of the Section, and H. O. Pollak, President of the MAA, reported on the activities of the MAA.

Elected chairman of the Section was James Derr of West Virginia. Albert Rabenstein, of Washington and Jefferson College, was elected Second Vice Chairman. Elevated to First Vice Chairman was Robert McDermot of Duquesne University.

M. R. WOODARD, *Secretary-Treasurer*

APRIL MEETING OF THE WISCONSIN SECTION

The Forty-Third Annual Meeting of the Wisconsin Section of the MAA was held at the University of Wisconsin—Superior on Saturday, April 19, 1975, with 62 registrations.

The program featured invited addresses by Professor L. A. Steen, Saint Olaf College, entitled "From Catastrophe Theory to Salmon Hatcheries: Memoirs of a Mathematical Missionary"; by Professor R. P. Boas, Northwestern University, entitled "Convergence and Divergence"; and by Professor J. D. E. Konhauser, Macalester College, entitled "Curves of Constant Width—Overworked or Not?"

The following papers were presented:

1. *$n \times n$ matrices over $GF(p)$ of order p, p prime*, by C. B. Hanneken, Marquette University.
2. *Small group-discovery vs. expository teaching in calculus*, by N. J. Loomer, Ripon College.
3. *Is the Plimpton* 322 *a Cuneiform tablet dealing with Pythagorean triples*?, by Don Voils, UW-Oshkosh.
4. *A classroom application of the Schröder-Bernstein Theorem*, by D. W. Bange, UW-LaCrosse.
5. *Applications of conformal mapping with computer graphics*, by D. T. Piele and Bob Manulik (student), UW-Parkside.
6. *A history of the work on stabilizing automorphisms in finite groups*, by Bruce Staal, UW-Stevens Point.
7. *The relationship between cognitive and mathematical structures*, by John Moyer, Marquette University.
8. *Category theory and the undergraduate curriculum*, by K. A. Beres, Ripon College.
9. *Some remarkable subsets of the real line*, by Brian Scott, UW-Madison.
10. *A short course in abstract algebra*, by Jim Sobota, UW-LaCrosse.

Chairman C. H. Johnson presided at the annual business meeting. Professor Edwin Wilde gave the governor's report and Secretary-Treasurer Ray Wagner gave the High School Contest Committee report showing that 23,357 students from 314 high schools participated in the preliminary examination and 1103 students from 220 schools participated in the final contest examination.

Professor Wilde also reported for an ad hoc committee which suggested 8 possible projects for the Wisconsin Section.

A motion was passed to pay the student membership in MAA for the top winner from the state in the Putnam Mathematical Competition and to apply for help from the MAA Committee on Sections to do the same for the next two winners.

Professor R. L. Hall, UW-Milwaukee, was elected Chairman and Professor F. G. Florey, UW-Superior, was elected Vice-Chairman.

R. D. WAGNER, *Secretary-Treasurer*

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CALENDAR OF FUTURE MEETINGS

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

OHIO

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, May 1–2, 1976.

SEAWAY, College of St. Rose, Albany, April 30–May 1, 1976.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, North Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A&M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Statler-Hilton Hotel, New York City, December 28–29, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 22–23, 1976.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, March 31–April 2, 1976.

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SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Sheraton-Palace Hotel, San Francisco, December 3–5, 1975 (SIAM-SIGNUM 1975 Fall Meeting).

THE MATHEMATICAL ASSOCIATION OF AMERICA
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